Exact multiplicity of positive solutions for a class of semilinear equations on a ball

Philip Korman
Institute for Dynamics and
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

Abstract

We study exact multiplicity of positive solutions for a class of Dirichlet problems on a ball. We consider nonlinearities generalizing cubic, allowing both $f(0) = 0$ and non-positone cases. We use bifurcation approach. We first prove our results for a special case, and then show that the global picture persists as we vary the roots.

Key words: Multiplicity of solutions, positivity for linearized equation, Crandall-Rabinowitz bifurcation theorem.

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1 Introduction

We study positive solutions of the Dirichlet problem for the semilinear elliptic equation on the unit ball

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < 1, \;\; u = 0 \;\text{ on }\; |x| = 1, \tag{1.1}$$

i.e, a Dirichlet problem on a ball in R^n, depending on a positive parameter λ. In view of the classical results of B. Gidas, W.-M. Ni and L. Nirenberg [4] any positive solution of (1.1) is radial, i.e. $u = u(r)$, where $r = |x|$, and the problem (1.1) takes the form

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0 \quad r \in (0, R), \;\; u'(0) = u(R) = 0. \tag{1.2}$$

Exact multiplicity for (1.1) is a notoriously difficult question, and few results are available (even for the radial solutions). One could compare this problem to finding the exact number of critical points for some implicitly defined function $y = y(x)$. Unless one has some strong information about $y(x)$, it is hard to exclude the possibility of "wiggles" for some range of x.

Recently together with Y. Li and T. Ouyang (see [8] and the references there to our earlier papers) we developed a bifurcation theory approach to the exact multiplicity question. Instead of studying the solution at some particular λ, we take a more global approach, and study the solution curves. We then follow these curves to some corner in (λ, u) "plane", where some extra information is available (usually it is either where $\lambda = 0$, or where $u = \infty$). This way we can often prove uniqueness of the solution curve. If, moreover, we can prove that this solution curve does not turn, we obtain a uniqueness result. If the solution curve admits exactly one turn, we get an exact multiplicity result. To continue solutions we use a bifurcation theorem of M.G. Crandall and P.H. Rabinowitz, which is recalled below. The crucial ingredient of our approach is to prove positivity of any nontrivial solution of the corresponding linearized problem

$$(1.3) \quad w'' + \frac{n-1}{r} w' + \lambda f'(u) w = 0 \quad r \in (0, R), \;\; w'(0) = w(R) = 0.$$

This information is used to compute the direction of bifurcation, and to prove monotonicity of some solution branches.

Next we state a bifurcation theorem of Crandall-Rabinowitz [1].

Theorem 1.1 *[1] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) =$ span $\{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

We assume that the function $f \in C^2(\bar{R}_+)$ satisfies the following conditions

(f1) $f(0) \le 0$,

(f2) $f(b) = f(c) = 0$ for some $0 < b < c$, and $f(u) < 0$ on $(0, b)$; $\int_0^c f(u) du > 0$,

(f3) $f''(u)$ changes sign only once on $(0,c)$, i.e. there exists an $\alpha \in (0,c)$ such that
$$f''(u) \geq 0 \text{ on } (0,\alpha), \quad f''(u) \leq 0 \text{ on } (\alpha,c).$$

Define θ to be the smallest positive number such that $\int_0^\theta f(u)du = 0$. We set $G(u) = \frac{uf'(u)}{f(u)}$, and after T. Ouyang and J. Shi [12] we define $\rho = \alpha - \frac{f(\alpha)}{f'(\alpha)}$, provided that $\alpha > b$. (If $\alpha \leq b$ then the assumptions (f4) and (f5) below are vacuous. In that case it easy to prove that any nontrivial solution of the linearized equation (2.2) is positive, see Lemma 2.1 below, and then our result follows along the lines of [8].) Our next assumption is

(f4) If $\rho > \theta$ then
$$\begin{gathered} G(u) < 1 \text{ on } (0,b), \\ G(u) \text{ is decreasing on } (\theta,\rho), \\ G(u) > G(\rho) \text{ on } (b,\theta), \\ G(u) < G(\rho) \text{ on } (\rho,c). \end{gathered}$$

If $\theta \geq \rho$ these assumptions are vacuous. Finally, another technical assumption

(f5) If $\rho > \theta$ then for any $t \in [0, c-b]$ the function
$$f''(u)(c-t-u)(c-u)^2 - 2f'(u)t(c-u) - 2tf(u)$$
changes sign exactly once on $(0,c)$.

Our motivating example was $f(u) = u(u-b)(c-u)$ with positive constants b and c. In case $c \leq 2b$ it is easy to check that the problem (1.1) has no positive solutions. We will therefore assume that $c > 2b$. Then the function $f(u)$ satisfies all of the above conditions, so that our main result, Theorem 2.1 below, applies. In particular it implies the following theorem.

Theorem 1.2 *Assume that $c > 2b > 0$. There exists a critical value $\lambda_0 > 0$ so that the problem*

$$\Delta u + \lambda u(u-b)(c-u) = 0 \text{ for } |x| < 1, \quad u = 0 \text{ on } |x| = 1. \tag{1.4}$$

has no positive solutions for $\lambda < \lambda_0$, exactly one positive solution for $\lambda = \lambda_0$, and exactly two for $\lambda > \lambda_0$.

This application is not a new result, it is included in P. Korman, Y. Li and T. Ouyang [8] in the case of two spatial dimensions, and in T. Ouyang and J. Shi [12] for the general case. What is novel here is the technique of the proof, which allows one to avoid a very difficult step of proving positivity for the linearized equation (except in a relatively easy special case). In case of cubic the easier case is when c is close to $2b$. We then vary c, and show that the global picture persists. In case $f(0) < 0$ our result appears to be new. In particular it implies uniqueness of solution for large λ. In case of one space dimension we obtain an exact multiplicity result.

2 Multiplicity of Solutions

By the classical theorem of Gidas, Ni and Nirenberg [4] positive solutions of (2.1) are radially symmetric, which reduces (1.1) to

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \text{ for } 0 < r < 1, \; u'(0) = u(1) = 0. \tag{2.1}$$

We shall also need the corresponding linearized equation

$$w'' + \frac{n-1}{r}w' + \lambda f'(u)w = 0 \text{ for } 0 < r < 1, \; w'(0) = w(1) = 0. \tag{2.2}$$

For multiplicity results it is important to know if $w(r)$ is of one sign on $(0, 1)$, i.e. we could then assume that $w(r) > 0$ on $(0, 1)$. The following lemma provides a condition for the positivity of w.

Lemma 2.1 *Assume that $f(u)$ satisfies the conditions (f1-f5). Assume that*

$$\theta \geq \rho. \tag{2.3}$$

Then we may assume that $w(r) > 0$ on $(0, 1)$.

Proof. Multiplying (2.1) by u' and integrating over $(0, 1)$, we conclude that

$$u(0) > \theta. \tag{2.4}$$

We can therefore find $0 < r_1 < r_2 < 1$, so that $u(r_1) = \theta$ and $u(r_2) = \rho$. By the uniqueness result in [13] we know that $w(0) \neq 0$, and hence we may assume that $w(0) > 0$. By T. Ouyang and J. Shi [12], see also J. Wei [14], $w(r)$ cannot vanish on $[0, r_2]$. By the results of M.K. Kwong and L. Zhang [9] $w(r)$ cannot vanish on $(r_1, 1)$ either. (In formula (3.4) of [9] a function

$\bar{w}(r)$ was defined. This function satisfies the linearized equation (2.2) and $\bar{w}(r_1) = 0$. In Lemma 15 of the same paper M.K. Kwong and L. Zhang show that $\bar{w}$ has no zeroes on $(r_1, 1)$. Since w and $\bar{w}$ are solutions of the same equation, it follows by the Sturm's comparison theorem that $w(r)$ has no zeroes on $(r_1, 1)$.) The proof follows.

We recall that solution (λ, u) of (2.1) is called singular if (2.2) possesses a nontrivial solution. The following lemma was proved in Korman [5].

Lemma 2.2 *Let (λ, u) be a singular solution of (2.1). Then*

$$\int_0^1 f(u) w r^{n-1} dr = \frac{1}{2\lambda} u'(1) w'(1). \tag{2.5}$$

We shall say that a singular solution is regular singular if $u'(1) < 0$, irregular singular if $u'(1) = 0$. By the Hopf's boundary lemma we have $u'(1) < 0$ in case $f(0) \geq 0$. Using Lemma 2.2, it is easy to show that at a regular singular solution the Crandall-Rabinowitz theorem applies, and hence solution set is locally a simple curve, see [8] for more details. In case of irregular singularity one may have a more complicated solution set (including pitchfork bifurcation and symmetry breaking), which we do not discuss in this work.

Next we study the linearized eigenvalue problem corresponding to any solution of (2.2):

$$\varphi'' + \frac{n-1}{r}\varphi' + \lambda f'(u)\varphi + \mu\varphi = 0 \ \text{ on } \ (0,1), \ \ \varphi'(0) = \varphi(1) = 0. \tag{2.6}$$

Comparing this to (2.2), we see that at any singular solution of (2.1) $\mu = 0$ is an eigenvalue, corresponding to an eigenfunction $\varphi = w$. We shall need the following generalization of Lemma 2.2.

Lemma 2.3 *Let $\varphi > 0$ be a solution of (2.6) with $\mu \leq 0$. (I.e. φ is a principal eigenfunction of (2.6).) Then*

$$\int_0^1 f(u) \varphi r^{n-1} dr \geq \frac{1}{2\lambda} u'(1) \varphi'(1). \tag{2.7}$$

Proof. The function $v = r u_r - u_r(1)$ satisfies

$$\begin{aligned} \Delta v + \lambda f'(u) v + \mu v &= \mu v - 2\lambda f(u) - \lambda f'(u) u'(1) \ \text{ for } \ |x| < 1, \\ v &= 0 \ \text{ on } \ |x| = 1. \end{aligned} \tag{2.8}$$

Comparing (2.8) with (2.6), we conclude by the Fredholm alternative

$$\mu \int_0^1 v\varphi r^{n-1}dr - 2\lambda \int_0^1 f(u)\varphi r^{n-1}dr - \lambda u'(1) \int_0^1 f'(u)\varphi r^{n-1}dr = 0. \tag{2.9}$$

Integrating (2.6)

$$-\lambda \int_0^1 f'(u)\varphi r^{n-1}dr = \varphi'(1) + \mu \int_0^1 \varphi r^{n-1}dr.$$

Using this in (2.9), we have

$$2\lambda \int_0^1 f(u)\varphi r^{n-1}dr = \mu \int_0^1 ru_r\varphi r^{n-1}dr + u'(1)\varphi'(1),$$

and the proof follows.

We now define the Morse index of any solution of (2.1) to be the number of negative eigenvalues of (2.6). The following lemma is based on K. Nagasaki and T. Suzuki ([11]).

Lemma 2.4 *Assume that (λ, u) is a regular singular solution of (2.1) such that $w'(1) < 0$ and*

$$\int_0^1 f''(u)w^3 r^{n-1}dr < 0. \tag{2.10}$$

Then at (λ, u) a turn to "the right" in (λ, u) "plane" occurs, and as we follow the curve in the direction of decreasing $u(0, \lambda)$, the Morse index is increased by one.

Proof. To see that the turn is to the right, we observe that the function $\tau(s)$, defined in Crandall-Rabinowitz theorem, satisfies $\tau(0) = \tau'(0) = 0$ and

$$\tau''(0) = -\frac{\lambda \int_0^1 f''(u)w^3 r^{n-1}dr}{\int_0^1 f(u)wr^{n-1}dr}, \tag{2.11}$$

see [8] for more details. By our assumption the numerator in (2.11) is negative, while by Lemma 2.2 the denominator is positive. It follows that $\tau''(0) > 0$, and hence $\tau(s)$ is positive for s close to 0, which means that the turn is to the right.

At a turning point one of the eigenvalues of (2.6) is zero. Assume it is the ℓ-th one, and denote $\mu = \mu_e$. Here $\mu = \mu(s)$, and $\mu(0) = 0$. We now write (2.6) in the corresponding PDE form and differentiate this equation in s

$$\Delta\varphi_s + \lambda f'(u)\varphi_s + \lambda' f'(u)\varphi + \lambda f''(u)u_s\varphi + \mu'\varphi + \mu\varphi_s = 0 \quad \text{for } |x| < 1, \ \ \varphi_s = 0 \ \text{ on } |x| = 1. \tag{2.12}$$

At (λ, u) the Crandall-Rabinowitz theorem applies, and hence we have: $\mu(0) = 0$, $\varphi(0) = w$, $\lambda'(0) = 0$, and $u_s(0) = -w$ (considering the chosen parameterization). Here w is a solution of the linearized equation (2.2). The equation (2.12) becomes

$$\Delta\varphi_s - \lambda f''(u)w^2 + \lambda f'(u)\varphi_s + \mu'(0)w = 0. \tag{2.13}$$

Multiplying (2.2) by φ_s, (2.13) by w, subtracting and integrating, we have

$$\mu'(0) = \frac{\lambda \int_0^1 f''(u)w^3 r^{n-1}dr}{\int_0^1 w^2 r^{n-1}dr} < 0.$$

It follows that across the turning point one of the positive eigenvalues crosses into the negative region, increasing the Morse index by one.

The following lemma is due to T. Ouyang and J. Shi [12]. For completeness we sketch an independent geometric proof.

Lemma 2.5 *Assume that conditions $(f1 - f3)$ hold. Then the function $f(u) = \dfrac{f(u)}{u-\rho}$ is nonincreasing for all $u > \rho$. Moreover, ρ is the smallest number with such property.*

Proof. The lemma asserts that the line joining the points $(\rho, 0)$ and $(u, f(u))$ rotates clockwise as we increase u. The proof follows, given the concavity properties of $f(u)$.

In the following lemma we adapt an argument of E.N. Dancer [3].

Lemma 2.6 *Assume that conditions $(f1 - f4)$ hold. Then*

$$\frac{\rho f'(\rho)}{f(\rho)} > 1. \tag{2.14}$$

Proof. By the definition of ρ

$$f'(\alpha)(\alpha - \rho) - f(\alpha) = 0.$$

Using this and our last condition in $(f4)$,

$$G(\rho) > G(\alpha) = 1 + \frac{\rho f'(\alpha)}{f(\alpha)} > 1,$$

proving the lemma.

We now state the main result of this paper.

Theorem 2.1 *Assume that conditions (f1-f5) are satisfied. If $f(0) = 0$ then there is a critical λ_0, such that for $\lambda < \lambda_0$ the problem (2.1) has no nontrivial solutions, it has exactly one nontrivial solution for $\lambda = \lambda_0$, and exactly two nontrivial solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^+(r,\lambda)$ strictly monotone increasing in λ and $\lim_{\lambda\to\infty} u^+(r,\lambda) = c$ for $r \in [0,1)$. For the lower branch, $\lim_{\lambda\to\infty} u^-(r,\lambda) = 0$ for $r \neq 0$, while $u^-(0,\lambda) > \theta$ for all $\lambda > \lambda_0$.*

If $f(0) < 0$ then the problem (2.1) admits exactly one infinite curve of positive solutions, which has exactly one turn (to the right). Any two solutions on the curve corresponding to the same λ are strictly ordered. The upper branch of this curve has the same properties as $u^+(r,\lambda)$ in the $f(0) \geq 0$ case. In addition to this curve only the following curves of positive solutions are possible (we call them submarines): they exist over bounded λ-intervals, say $\underline{\lambda} < \lambda < \overline{\lambda}$, and lose positivity at $\underline{\lambda}$ and $\overline{\lambda}$ (i.e. at $\overline{\lambda}$ we have $u'(1,\overline{\lambda}) = 0$ and $u(r,\lambda) < 0$ near $r = 1$ for $\lambda > \overline{\lambda}$, and similarly at $\underline{\lambda}$).For large λ there exists exactly one positive solution. No submarines are possible if any nontrivial solution of the linearized problem (2.2) is of one sign (e.g. in the one-dimensional case).

Proof. We begin by noticing that existence of solutions under our conditions follows by the Theorem 1.5 in P.L. Lions [10], see also [12]. Indeed the result in [10] implies existence of a critical $\bar{\lambda}$, so that for $\lambda \geq \bar{\lambda}$ there exists a maximal positive solution of (1.1), while for $\lambda > \bar{\lambda}$ there exists at least two positive solutions. Next we remark that once properties of the infinite curve of solutions are established, uniqueness of solutions for large λ in case $f(0) < 0$ will follow similarly to [5]. Indeed, we showed in [5] that for large λ the solutions must be uniformly close to c. Since solutions of (1.1) are globally parameterized by $u(0)$, see [2], it follows in view of the above existence result from [10], that there is an infinite solution curve with exactly one turn, and no submarines are possible for large λ. Also we showed in [7]

that no submarines are possible if any non-trivial solution of the linearized problem (2.2) is of one sign.
We now consider two cases.

Case (i) $\theta \geq \rho$. By Lemma 2.1 any solution of the linearized problem (2.2) is positive. This implies that only turns to the right are possible. In case $f(0) \geq 0$ the theorem follows the same way as in P. Korman, Y. Li and T. Ouyang [8], see also T. Ouyang and J. Shi [12] and J. Wei [14]. In case $f(0) < 0$ the theorem follows from the Theorem 2.4 in Korman [5]. In both cases we can compute the direction of bifurcation since we can prove that

$$\int_0^1 f''(u) w^3 r^{n-1} dr < 0. \tag{2.15}$$

Only conditions (f1 - f3) are needed for both cases.

Case (ii) $\theta < \rho$. We define a family of functions $f_t(u)$, depending on a positive parameter t,

$$f_t = f(u) h_t(u), \quad \text{where} \quad h_t(u) = \frac{c - t - u}{c - u}.$$

Clearly both conditions (f1) and (f2) are satisfied for f_t. (With the obvious modification that the second zero of $f_t(u)$ is now $c - t$.) An easy computation shows that the condition (f5) ensures that the condition (f3) is satisfied as well. As for (f4), we compute

$$G_t(u) = G(u) + \frac{u h_t'}{h_t} < G(u) < 1 \quad \text{for } u \in (0, b),$$

so that the first condition in $(f4)$ is satisfied, and

$$G_t' = G' + \frac{h_t'' h_t u + h_t' h_t - h_t'^2 u}{u^2} < G'.$$

The last inequality means that G_t is a sum of G and a decreasing function. We shall show next that f_t satisfies the other conditions in $(f4)$ as well.

We define θ_t by $\int_0^{\theta_t} f_t(u) du = 0$, $\theta_0 = \theta$. We know that $f_t''(u)$ can change sign only once at some $u = \alpha_t$. As before we define $\rho_t = \alpha_t - \frac{f(\alpha_t)}{f'(\alpha_t)}$. If $t > \tau$ then

$$f_t(u) - f_\tau(u) = \frac{\tau - t}{c - u} f(u) \begin{cases} > 0 & \text{on} \quad (0, b) \\ < 0 & \text{on} \quad (b, c), \end{cases}$$

It easily follows that θ_t is increasing in t. Since $f_t(u) = f(u) - \frac{t}{c-u} f(u) \equiv f(u) - th(u)$, and $h''(u) > 0$ for $u \in (b, \alpha)$, it follows easily that α_t moves monotonically to the left, i.e. α_t is decreasing in t. We claim that ρ_t is decreasing in t as well. Since it is easier to work with derivatives rather than differences, let us assume just for simplicity of presentation that f is of class C^3, which will imply that the derivative $\frac{d\alpha}{dt} \equiv \alpha_t$ is defined, and by the above $\alpha_t < 0$. We also write $f_t(\alpha_t) \equiv \bar{f}(\alpha(t), t)$, and $\rho = \alpha - \dfrac{\bar{f}(\alpha, t)}{\bar{f}_u(\alpha, t)}$. Differentiate this (using that $\bar{f}_{uu}(\alpha(t), t) = 0$)

$$\rho_t = \alpha_t - \frac{\left(\bar{f}_u \alpha_t + \bar{f}_t\right) \bar{f}_u - \bar{f}\left(\bar{f}_{uu}\alpha_t + \bar{f}_{ut}\right)}{\bar{f}_u^2} = \frac{-\bar{f}_t \bar{f}_u + \bar{f}\bar{f}_{ut}}{\bar{f}_u^2}. \tag{2.16}$$

Since $\bar{f}_t = -\frac{f}{c-u}$, $\bar{f}_u = f' - \frac{tf}{(c-u)^2} - \frac{tf'}{c-u}$, and $\bar{f}_{ut} = -\frac{f}{(c-u)^2} - \frac{f'}{c-u}$, we compute

$$-\bar{f}_t \bar{f}_u + \bar{f}\bar{f}_{ut} = -\frac{tf^2}{(c-u)^3} - \frac{tff'}{(c-u)^2} - \frac{f^2}{(c-u)^2},$$

which is negative when $u \in (b, \alpha)$. By (2.16) we see that ρ_t is decreasing in t, as claimed. It follows that as we increase t the interval (θ_t, ρ_t) shrinks with both of its ends moving toward each other. Since $G_t'(u)$ is a sum of $G(u)$ and a decreasing function, it follows that G_t is decreasing on (θ_t, ρ_t), verifying the second condition in (f4). Likewise, the other two conditions in (f4) are preserved, when a decreasing function is added to G.

As we increase t the points θ_t and ρ_t will move together. Hence we must reach a point such that $\theta_{t_0} = \rho_{t_0}$. Notice that for all $0 \le t \le t_0$ the function f_t satisfies all of the conditions (f1 - f4). For $t = t_0$ by Case (i) the result is true. Our goal is to show that the same picture persists as we decrease t from t_0 back to zero, where $f_0 = f$, giving the desired result.

At $t = t_0$ solution of the linearized equation is positive, and hence the inequality (2.15) holds. This is precisely the "condition A" of E. N. Dancer [2]. It follows by [2] that near the turning point the solution curve looks the same, and in particular the turning point itself is perturbed only slightly. We denote the turning point by u_t, and by w_t the corresponding eigenvector. It is known that simple eigenvalues of elliptic operators vary continuously, and the same is true for the corresponding eigenvectors, see M. G. Crandall and P. H. Rabinowitz [1]. It follows that the linearized equation at u_t has an eigenvalue μ_t close to zero with corresponding eigenfunction z_t which is close to w_{t_0}. (We assume that all eigenfunctions are normalized to have the slope of say -1 at $r=1$.) But eigenvalues of elliptic operator are discrete, and zero is an eigenvalue of the linearized equation at u_t (since u_t is a turning point). It follows that $\mu_t = 0$ and $z_t = w_t$. We conclude that the eigenfunction corresponding to the turning point varies continuously. It follows that this eigenfunction has to stay positive. (Indeed, uniqueness theorem for ODE precludes $w_t'(\xi) = 0$ for any $\xi \in [0,1]$, which would have to occur in order for w to cease being positive. For $\xi \in (0,1]$ this is a standard result, while $\xi = 0$ case is covered in L.A. Peletier and J. Serrin [13].)

We show next that having $w_t > 0$ is enough to show that the entire solution curve has the desired shape. For simplicity we drop the subscript t when working with f_t. By the Crandall-Rabinowitz Theorem near the turning point u_t the solution set has two branches $u^-(r,\lambda) < u^+(r,\lambda)$, $r \in [0,1)$, $\lambda > \lambda_t$. By the Crandall-Rabinowitz Theorem we also conclude

$$u_\lambda^+(r,\lambda) > 0 \text{ for } \lambda \text{ close to } \lambda_t \text{ (for all } r \in [0,1)). \tag{2.17}$$

Arguing like in P. Korman, Y. Li and T. Ouyang [8] we show that the same inequality holds for all $\lambda > \lambda_t$ (until a possible turn), see also T. Ouyang and J. Shi [12] and J. Wei [14]. We claim next that solutions $u^+(r,\lambda)$ are stable, i.e. all eigenvalues of (2.6) are negative. Indeed, let $\mu \le 0$ be an eigenvalue of (2.6) and $\varphi > 0$ the corresponding eigenvector. The equation for u_λ is

$$\begin{gathered} u_\lambda'' + \tfrac{n-1}{r} u_\lambda' + \lambda f'(u) u_\lambda + f(u) = 0 \text{ for } r \in (0,1], \\ u_\lambda'(0) = u_\lambda(1) = 0. \end{gathered} \tag{2.18}$$

¿From the equations (2.6) and (2.18) we obtain

$$r^{n-1}(\varphi' u_\lambda - u_\lambda' \varphi)|_0^1 = -\mu \int_0^1 \varphi u_\lambda r^{n-1} dr + \int_0^1 f(u) \varphi r^{n-1} dr. \tag{2.19}$$

The right hand side in (2.19) is positive by our assumptions, inequality (2.17), and Lemma 2.2, while the quantity on the left is zero, a contradiction.

We show next that for $\lambda > \lambda_t$ both branches $u^+(r,\lambda)$ and $u^-(r,\lambda)$ have no critical points. Indeed, if we had a critical point on the upper branch $u^+(r,\lambda)$ at some $\overline{\lambda} > \lambda_t$, then by the Crandall-Rabinowitz Theorem solution of the linearized equation would be positive at $\lambda = \overline{\lambda}$. But then we know precisely the structure of solution set near $(\overline{\lambda}, u^+(r,\overline{\lambda}))$, namely it is a parabola-like curve with a turn to the right. This is impossible, since solution curve has arrived at this point from the left. Turning to the lower branch $u^-(r, \lambda)$, we know by Lemma 2.3 that each solution on this branch has Morse index of one, until a possible critical point. At the next possible turning point one of the eigenvalues becomes zero, which means that the Morse index of the turning point is either zero or one. If Morse index is zero, it means that solutions of the corresponding linearized equation are of one sign, and we obtain a contradiction the same way as on the upper branch. If Morse index $= 1$, it means that zero is a second eigenvalue, i.e. $w_t(r)$ changes sign exactly once, but that is ruled out by the results of M. K. Kwong and L. Zhang [9]. Indeed, condition (f4) implies that for any $v \in (\theta_t, \rho_t)$ the horizontal line $y = \gamma$, with $\gamma \equiv G(v)$ intersects $y = G(u)$ exactly once on the entire interval $(0, c)$. It follows that for each $v \in (\theta_t, \rho_t)$ there exists a constant $\gamma > 1$ such that

$$\gamma f(u) - uf'(u) \begin{cases} < 0 & \text{for all} \quad u < v \\ > 0 & \text{for all} \quad u > v, \end{cases} \tag{2.20}$$

i.e. the conclusion of Lemma 7 in M. K. Kwong and L. Zhang [9] holds. (Recall that if $w_t(r)$ is not of one sign, its first zero, call it r_0 has the property that $u(r_0) \in (\theta_t, \rho_t)$. We then use that $G(\rho_t) > 1$ by Lemma 2.6.) We now select $v = u(r_0)$ and the corresponding γ so that the inequality (2.20) holds. By Lemma 8 in the same paper [9], $w_t(r)$ cannot have its second zero at $r = 1$, which is a contradiction, and the claim follows. (We are using the inequality (2.30) of [9].)

We now define S as the set of points t in $[0, t_0]$, such that our solution curve has exactly one critical point, at which the condition $\int_0^1 f''(u) w^3 r^{n-1}\, dr < 0$ holds. Our previous discussion shows that the set S is open in $[0, t_0]$. We claim that S is also closed in $[0, t_0]$. Notice that all eigenvalues of (2.6) are simple (as easily follows by the uniqueness result for initial value problems). It follows that the eigenvector w_t can be chosen to be continuous, and if we have a sequence of $\{t_n\} \in S$ with $t_n \to t$, then $w_t > 0$. As above, the solution curve at t has the desired properties, i.e. $t \in S$. It follows that $S = [0, t_0]$, and at $t = 0$ we obtain the conclusion of our theorem.

Remark. It is easy to check that the theorem applies in case of cubic, $f = u(u - b)(c - u)$. Our deformation was based on this case, since for a cubic $f_t = u(u - b)(c - t - u)$. One could try other choices of $h_t(u)$, possibly relaxing or even removing our condition $(f5)$.

We now give an example involving $f(0) < 0$. We consider $f(u) = (u + a)(u - b)(c - u)$, with some positive constants a, b and c. Writing

$$G(u) = u\,(\ln f(u))' = 3 - \frac{a}{u+a} + \frac{b}{u-b} - \frac{c}{c-u},$$

we see that $G(u) < 1$ on $(0, b)$ (the third and the fourth terms are less than -1, and the second is negative). We observe also

$$G' = \frac{a}{(u+a)^2} - \frac{b}{(u-b)^2} - \frac{c}{(c-u)^2} < \frac{1}{4u} - \frac{b}{(u-b)^2} - \frac{c}{(c-u)^2},$$

where we have optimized the first term on the right in a. We shall have $G'(u) < 0$ on (b, c), provided we assume

$$\frac{b}{(u-b)^2} + \frac{c}{(c-u)^2} > \frac{1}{4u} \quad \text{for all } u \in (b, c). \tag{2.21}$$

It is easy to see that this is not a very restrictive condition on b and c. Next we claim that condition (f2) is verified, provided $c > 2b+a$. Indeed, defining $g(u) \equiv u(u - b - a)(c + a - u)$ we recognize our nonlinearity as a shifted function $(u+a)(u-b)(c-u) = g(u+a)$. For $g(u)$ the condition (f2) is satisfied since $c+a > 2(b+a)$. When we shift $g(u)$ to the left, we keep the area of the positive hump fixed, while decreasing the area of the negative hump, so that (f2) continues to hold. Alternatively, by rescaling $u = av$, and examining the resulting equation, we easily see that condition (f2) is satisfied for all $a > 0$, provided $c > 3b$. Finally, the function $f_t(u) = (u+a)(u-b)(c-t-u)$ changes concavity only once at $\alpha_t = \frac{-a+b+c-t}{3}$. (The possibility of $\alpha_t < 0$ poses no problem. Then $f''(u) < 0$ for all $u > 0$, and the exact multiplicity result is even easier to prove, see [5].) We have proved the following theorem.

Theorem 2.2 *With positive constants a, b and c assume that either $c > 2b + a$, or $c > 3b$ holds, and the condition (2.21) is satisfied. Then for the problem*

$$\Delta u + \lambda(u+a)(u-b)(c-u) = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{on } |x| = 1,$$

the second statement of the Theorem 2.1 applies.

Remark. These conditions on a, b and c are not optimal, as we sacrificed some generality for the sake of simplicity. We shall now give some completely explicit conditions on the coefficients, sacrificing some more generality. Minimizing the quantity on the left in (2.21), and maximizing the one on the right, we see that (2.21) is satisfied, provided that

$$(1+\frac{c^{\frac{2}{3}}}{b^{\frac{2}{3}}})^2\frac{bc^{\frac{2}{3}}+cb^{\frac{2}{3}}}{c^{\frac{2}{3}}(c-b)^2} > \frac{1}{4b}.$$

Setting $c=\alpha b$, we simplify this to

$$(1+\alpha^{\frac{2}{3}})^2(1+\alpha^{\frac{1}{3}}) > \frac{1}{4}(\alpha-1)^2.$$

Denoting α_0 to be solution of $(1+\alpha^{\frac{2}{3}})^2(1+\alpha^{\frac{1}{3}}) = \frac{1}{4}(\alpha-1)^2$, $\alpha_0 \simeq 140.74$, we see that we can start with arbitrary $b>0$, then take $c=\alpha b$ with either $\alpha \in (3,\alpha_0)$ and arbitrary $a>0$, or $\alpha \in (2,\alpha_0)$, and $a<c-2b$.

Similarly to varying c we can also vary the root b, in particular sending it to zero. I.e. we can cover the $f'(0)=0$ case. For simplicity we state the result for the case of cubic, although one can easily generalize.

Theorem 2.3 *Consider the problem*

$$\Delta u + \lambda u^2(c-u) = 0 \quad \textit{for } |x|<1, \quad u=0 \quad \textit{on } |x|=1,$$

with any constant $c>0$. Then the first statement of the Theorem 2.1 applies.

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