HYSTERESIS IN URYSOHN-VOLTERRA SYSTEMS

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ABSTRACT. This paper deals with the hysteresis operator coupled to the system of Urysohn-Volterra equations. The local solutions of the system as well as the global solutions have been obtained.

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1. INTRODUCTION

The theory of hysteresis operators developed in the past fifteen years (see [10, 6, 1, 12, 13]) has proved to be a powerful tool for solving mathematical problems in various branches of applications such as solid mechanics, ferromagnetism, phase transitions and many others. In this paper we deal with the system of nonlinear integral equations with hysteresis, namely

$$y(t) = f(t) + \int_0^t F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds,\ 0 \le t \le T < \infty, \tag{1}$$

where $\mathcal{W}$ and S denote a hysteresis operator and a superposition operator of the form $S[y](t) = g(y(t))$, respectively. More precisely, we assume f and F to be given n-vector-valued functions, while y is the unknown n-vector function. Such problems arise naturally in connection with physical models described by systems of $PDEs$ with hysteretic constitutive laws. Typically, space discretization or Galerkin-type approximations lead to systems of $ODEs$ which can equivalently be written as integral equations of the form (1), see [1, 6, 10, 12, 13, 14, 15]. Eq.(1) is known as Urysohn-Volterra equation (see [9]).

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In [4, 5, 16, 17] the local and the global solutions of nonlinear integral equation without hysteresis have been obtained. None of the authors, however, has provided even the local solution of nonlinear integral equation with hysteresis, by the way see [7, 8].

For the case where the hysteresis operator is coupled to the initial value problem, see [11, 12] and for the case where the hysteresis operator is coupled to the system of partial differential equations, see [14, 15]. But our case where the hysteresis operator coupled to the system of nonlinear integral equations seems to have been neglected in literature.

The main object of this paper is to give sufficient conditions in order to guarantee the existence of the unique local solutions of Eq.(1) as well as the unique global solutions. This paper is organized as follows: In §2 the basic concepts are stated and in §3 we have proved the local existence and uniqueness of solutions provided the nonlinearities are of Volterra type and locally Lipschitz continuous in the space of continuous functions. Finally, in §4, time-dependent Lipschitz constants are allowed and it is shown that the contraction mapping principle can be used in spaces of continuous functions with controlled growth for proving the existence and uniqueness of global solutions in time with an explicit growth condition.

2. Preliminaries

In this section, we state some results needed in the proof of our main results. Let $C([0,T];\mathbb{R}^n)$ be a Banach space of all continuous functions $\phi:[0,T]\to\mathbb{R}^n$ endowed with the sup-norm $\|\phi\| = \sup\limits_{0\le\theta\le T}\|\phi(\theta)\|_{\mathbb{R}^n}$.

Definition 1. $(Rate\ independent\ functionals)$
A functional $\mathcal{H}:C([0,T];\mathbb{R}^{\mathrm{n}})\to\mathbb{R}$; is called rate independent if and only if $\mathcal{H}[u\circ\psi]=\mathcal{H}[u]$ holds for all $u\in C([0,T];\mathbb{R}^{\mathrm{n}})$ and all admissible time transformations, i.e., continuous increasing functions $\psi:[0,T]\to[0,T]$ satisfying $\psi(0)=0$ and $\psi(T)=T$.

Definition 2. $(Volterra-operator)$
Let X be a Banach space. An operator $F:C([0,T];X)\to C([0,T])$ is called a Volterra-operator if for all $s\in[0,T]$ and for all $u,v\in C([0,T];X)$ with $u(\sigma)=v(\sigma)$ for all $\sigma\in[0,s]$ implies $(Fu)(\sigma)=(Fv)(\sigma)$ for all $\sigma\in[0,s]$.

Definition 3. (*Hysteresis operator*)
An operator $\mathcal{W}: C([0,T];\mathrm{I\!R^n}) \to \mathrm{C([0,T])}$, *is called a hysteresis operator on* $C([0,T];\mathrm{I\!R^n})$ *if and only if*

$$\mathcal{W}[u](t) = \mathcal{H}[u_t], \quad t \in [0,T], \quad u \in C([0,T];\mathrm{I\!R^n}),$$

where u_t *is defined by*

$$u_t(\sigma) = \begin{cases} u(\sigma) & 0 \le \sigma \le t \\ u(t) & t < \sigma \le T, \end{cases}$$

holds for some rate independent functional $\mathcal{H}$ *on* $C([0,T];\mathrm{I\!R^n})$.

Lemma 1. [8]
Let $F: C([0,T];\mathrm{I\!R^n}) \to \mathrm{C([0,T])}$ *be a Volterra-operator. Assume that* F *is Lipschitz continuous on every bounded subset of* $C([0,T];\mathrm{I\!R^n})$. *Then for every* $C > 0$ *there exists* $L > 0$ *such that*

$$|(Fy_2)(s) - (Fy_1)(s)| \le L \sup_{0 \le \sigma \le s} \|y_2(\sigma) - y_1(\sigma)\|_{\mathrm{I\!R^n}}, \tag{2}$$

holds for all $0 \le s \le T$ *and all* $y_i \in C([0,T];\mathrm{I\!R^n})$ *with* $\|y_i\| \le C$, $i = 1,\ 2$.

Remark 1. *By definition hysteresis operators possess the Volterra property. This is actually what is needed here; The rate-independence itself does not play any role.*

3. Local solution

Our aim in this section is to prove the existence and uniqueness of solutions of Eq.(1), at least in a subinterval of $[0,T]$, say $[0,\tau]$, with $\tau \le T$. In order to achieve this goal, we shall apply the classical successive approximations method. But, first, let us state a set of conditions sufficient to guarantee the existence and uniqueness of the solution of (1) as follows:
(**A1**) $f:\ [0,T] \to \mathrm{I\!R^n}$ is continuous,
(**A2**) $F: \Delta_T \times D \to \mathrm{I\!R^n}$ continuous, $\Delta_T := \{(t,s) | 0 \le s \le t \le T\}$ and $D := \{(y,w) |\ y \in B,\ |w| \le r_2\}$, where $B := \{y |\ y \in \mathrm{I\!R^n},\ \|\mathrm{y}\| \le \mathrm{r_1}\}$,
(**A3**) F satisfies in $\Delta_T \times D$ the Lipschitz condition

$$\|F(t,s,y_2,w_2) - F(t,s,y_1,w_1)\| \le L_1\{\|y_2 - y_1\| + |w_2 - w_1|\}, \tag{3}$$

for some $L_1 > 0$,
(**A4**) if $b = \sup\limits_{0\le t\le T} \|f(t)\|_{\mathbb{R}^n}$, then $b < r$, $r = \min\{r_1,\ r_2\}$,
(**A5**) $M = \sup\{\|F(t,s,y,w)\| \mid (t,s,y,w) \in \Delta_T \times D\}$,
(**A6**) $\mathcal{W} \circ S:\ C([0,T];\mathbb{R}^n) \to \mathrm{C}([0,\mathrm{T}])$ satisfies for all $s \in [0,T]$ and for all $y_i \in B,\ i = 1,\ 2$, the Lipschitz condition

(4) $$|\mathcal{W}[S[y_2]](s) - \mathcal{W}[S[y_1]](s)| \le L_2 \sup_{0\le\sigma\le s} \|y_2(\sigma) - y_1(\sigma)\|_{\mathbb{R}^n},$$

for some $L_2 > 0$. Now, consider the Picard iteration

(5)
$$\begin{aligned} y_0(t) &= f(t), \\ y_k(t) &= f(t) + \int_0^t F(t,s,y_{k-1}(s),\mathcal{W}[S[y_{k-1}]](s))ds, k \ge 1, \end{aligned}$$

on $[0,T]$ or at least some subinterval $[0,\tau]$, $\tau \le T$. Clearly, by our assumptions, the iteration (5) is well-defined.

Theorem 1. *Let assumptions* $(A1) - (A6)$ *be satisfied and let* $\tau = \min\left\{T,\ \frac{r-b}{M}\right\}$. *Then Eq. (1) has a unique continuous solution on* $[0,\tau]$. *Moreover, the sequence of successive approximations (5) converges uniformly to this solution.*

Proof : The proof goes as in the case of nonlinear Volterra integral equation, (see [5, 16, 17]).

4. Global solution

In this section, we present our main result by proving the existence and uniqueness of global solutions of (1). Our approach is based on the contraction mapping principle which has been used for proving the existence and uniqueness of global solutions in time with an explicit growth condition. To facilitate our discussions, let us first state the following assumptions.
(**A7**) $F : \Delta \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ continuous, $\Delta := \{(t,s)|0 \le s \le t \le T\}$,
(**A8**) there exists a continuous nonnegative function $m_0(t,s)$ defined on Δ, such that

(6)
$$\|F(t,s,y_2,w_2) - F(t,s,y_1,w_1)\| \le m_0(t,s)\{\|y_2 - y_1\| + |w_2 - w_1|\},$$

in $\Delta \times \mathbb{R}^n \times \mathbb{R}$, and

$$F(t,s,0,0) = 0 \ \ in \ \Delta, \tag{7}$$

(**A9**) $\mathcal{W} \circ S: \ C([0,T];\mathbb{R}^n) \to \mathrm{C([0,T])}$ satisfies the Lipschitz condition

$$|\mathcal{W}[S[y_2]](s) - \mathcal{W}[S[y_1]](s)| \leq L \sup_{0\leq\sigma\leq s} \|y_2(\sigma) - y_1(\sigma)\|_{\mathbb{R}^n}, \tag{8}$$

for some $L > 0$, and

$$\mathcal{W}[S[0]](s) = 0 \ \ in \ 0 \leq s \leq T. \tag{9}$$

We would like to determine a scalar continuous increasing function h : $[0,T] \to \mathbb{R}_+ = (0,\infty)$, such that for the solution of Eq.(1)

$$\sup_{0\leq\sigma\leq t} \|y(\sigma)\|_{\mathbb{R}^n} \leq K \ h(t), \ \ t \in [0,T], \tag{10}$$

holds, where K is a positive constant. To achieve this goal, we define $C_h \equiv C_h([0,T];\mathbb{R}^n) := \{\mathrm{y}|\ \mathrm{y}: \ [0,\mathrm{T}] \to \mathbb{R}^n$ continuous, y satisfying an inequality of the form (10)$\}$ to be the underlying space for our problem. Obviously C_h is a linear space. Define

$$\|y\|_h = \sup_{0\leq t\leq T} \frac{\|y(t)\|_{\mathbb{R}^n}}{h(t)} \ \forall y \in C_h, \tag{11}$$

then it is easy to see that $y \mapsto \|y\|_h$ is a norm on C_h. Moreover $(C_h, \ \|.\|_h)$ is a Banach space (see [3]).

Now, we are in a position to state our main result.

Theorem 2. *Let assumptions* $(A7) - (A9)$ *be satisfied, let* h *be a continuous increasing positive function satisfying*

$$(1+L)\int_0^t m_0(t,s)\ h(s)\ ds \leq \alpha\ h(t), \ t \in [0,T], \tag{12}$$

with some constant α, $0 < \alpha < 1$, *and let* $f \in C_h([0,T];\mathbb{R}^n)$ *be a given function.*

Then Eq.(1) has a unique continuous solution satisfying (10).

Proof : Define

$$(\mathcal{K}y)(t) = f(t) + \int_0^t F(t,s,y(s),\mathcal{W}[S[y]](s))ds, \ 0 \leq t \leq T. \tag{13}$$

Solving (1) is equivalent to finding a fixed point of $\mathcal{K}$.
Clearly $\mathcal{K}y : [0,T] \to \mathrm{IR}^{\mathrm{n}}$ is continuous for any $y \in C_h$. We claim that $\mathcal{K}C_h \subset C_h$, i.e., $\mathcal{K}$ maps C_h into itself. To establish this claim, we show that the second term of the right hand side of (13) is in C_h. With the aid of $(A7)$ and $(A8)$, we have

(14)

$$\begin{aligned} \|\int_0^t F(t,s,y(s),\mathcal{W}[S[y]](s))ds\|_{\mathrm{IR}^{\mathrm{n}}} &\le \int_0^t m_0(t,s)\,[\|y(s)\|_{\mathrm{IR}^{\mathrm{n}}} + |\mathcal{W}[S[y]](s)|]\,ds \\ &\le (1+L)\int_0^t m_0(t,s) \sup_{0\le\sigma\le s} \|y(s)\|_{\mathrm{IR}^{\mathrm{n}}} ds \\ &\le (1+L)K \int_0^t m_0(t,s)\, h(s)ds, \end{aligned}$$

for some $K > 0$, depending upon $y \in C_h$. Then by (12) the second term of the right hand side of (13) will be in C_h. Hence, since $f \in C_h$, our claim is true.

Now, we need to show that $\mathcal{K}$ is a contraction mapping on C_h. Let $y_i \in C_h$, $i = 1,\ 2$. Then, by the aid of $(A7)$ and $(A8)$,

(15)

$$\|(\mathcal{K}y_2)(t) - (\mathcal{K}y_1)(t)\|_{\mathrm{IR}^{\mathrm{n}}} \le (1+L)\int_0^t m_0(t,s) \sup_{0\le\sigma\le s} \|y_2(\sigma) - y_1(\sigma)\|_{\mathrm{IR}^{\mathrm{n}}}\, ds$$

Thus

(16)

$$\begin{aligned} \|\mathcal{K}y_2 - \mathcal{K}y_1\|_h &\le \sup_t \left\{ \frac{(1+L)}{h(t)} \int_0^t m_0(t,s)h(s)\, ds \right\} \|y_2 - y_1\|_h \\ &\le \alpha\, \|y_2 - y_1\|_h, \end{aligned}$$

by the monotonicity increasing of h and (12). Since $0 < \alpha < 1$, the operator $\mathcal{K}$ is contraction on the space C_h. We are in a position now to apply the Banach Fixed Point Theorem. Then $\mathcal{K}$ has a unique fixed point and so Eq.(1) has a unique solution $y \in C_h([0,T];\mathrm{IR}^{\mathrm{n}})$. This completes the proof of Theorem 2.

Remark 2. *Let us point out that (7) and (9) do not really restrict the generality. Indeed, Eq.(1) can be written in the form*

$$y(t) = f(t) + \int_0^t F(t,s,0,0)ds + \int_0^t \left[F(t,s,y(s),\mathcal{W}[S[y]](s)) - F(t,s,0,0)\right]\, ds,$$

which presents an integrand satisfy condition (7) and (9). Of course the last equation is equivalent to (1).

Remark 3. *In order to construct explicitly such a function h, a solution of inequality (12), define*

$$\tilde{m}(t) = \sup_{0\le s\le \sigma\le t} m_0(\sigma, s) \tag{17}$$

and consider the function

$$m(t) = \int_t^{t+1} \tilde{m}(s)\, ds. \tag{18}$$

Then $m(t)$ satisfies

$$m_0(t,s) \le \tilde{m}(t) \le m(t), \quad 0 \le s \le t \le T. \tag{19}$$

Consequently, inequality (12) will be verified if the stronger inequality

$$(1+L)\, m(t) \int_0^t h(s)\, ds \le \alpha\, h(t), \quad t \in [0,T], \tag{20}$$

is satisfied. But inequality (20) has the solution

$$h(t) = \tilde{\alpha}^{-1} m(t) \exp\left\{\tilde{\alpha}^{-1} \int_0^t m(s)\, ds\right\}, \tag{21}$$

$\tilde{\alpha} = \frac{\alpha}{1+L}$, *(see [2]). It is clear that $h(t)$ is monotone increasing and $h(t) > 0$ if $m_0(t,s)$ does not vanish identically in any set $0 \le s \le t \le \tau$, $\tau > 0$. Indeed, if we take (6) into account, we see that the last requirement on $m_0(t,s)$ is not a restriction at all.*

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