NONEXISTENCE OF GLOBAL SOLUTIONS OF A QUASILINEAR BI-HYPERBOLIC EQUATION WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract

In this work, the nonexistence of the global solutions to a class of initial boundary value problems with dissipative terms in the boundary conditions is considered for a quasilinear bi-hyperbolic equation. The nonexistence proof is achieved by the use of a lemma due to O. Ladyzhenskaya and V.K. Kalantarov and by the usage of the so called generalized convexity method. In this method one writes down a functional which reflects the properties of dissipative boundary conditions and represents the norm of the solution in some sense, then proves that this functional satisfies the hypotheses of Ladyzhenskaya-Kalantarov lemma. Hence from the conclusion of the lemma one deduces that in a finite time t_2, this functional and hence the norm of the solution blows up.

1 INTRODUCTION

Initial-boundary value problems written for hyperbolic quasilinear partial differential equations emerged in several applications to physics, mechanics and engineering sciences. Natures of the solutions to these equations have been investigated by several means.

The nonexistence of global solutions of quasilinear hyperbolic equations with no dissipative terms in the boundary conditions are investigated for example by J.L. Lions [1], R.T. Glassey [2], H.A. Levine [4 ,5, 6], and O.A. Ladyzhenskaya and V. K. Kalantarov [8] and many others. Levine has a survey article [7] with many relevant references (See also Straughan [9]).

In [4] Levine studied the initial value problem for the following “abstract” wave equation with dissipation

$$Pu_{tt} + A_1 u_t + Au = F(u)$$

in a Hilbert space where P, A_1 and A are positive linear operators defined on some dense subspace of the Hilbert space and F is a gradient operator with potential $\mathcal{F}$. It is assumed that $(u, F(u)) \geq \mathcal{F}(u)$ for all u in the domain of F. The global nonexistence result he proved is the following: If the energy is initially negative then the solution can not be global. This is the same result that he proved in the case that $A_1 = 0$ (See [6]).

However, the functional used for the investigation of initial-boundary value problems with no dissipative terms in the boundary conditions can not be directly applied to the problems with the dissipative terms in the boundary conditions. Whenever damping is present, one must allow for the possibility that the data restrictions could be more severe than without damping [10]-[14].

The tool used in this work is a lemma due to O.A. Ladyzhenskaya and V.K. Kalantarov [8]. Part (b) of the Lemma was introduced also by H. A. Levine in [3,6]. From now on we will call it the LK Lemma.

The most crucial point in the application of this tool is to find a functional that represents the dissipation on the boundary and satisfies the conditions of the LK Lemma. This method is known as the "generalized convexity" method.

Let us begin by stating LK Lemma [8] without proof.

LEMMA

If a function

$$\psi(t) \in C^2, \qquad \psi(t) \geq 0$$

satisfies the inequality

$$\psi''(t)\psi(t) - (1+\gamma)\psi'(t)^2 \geq -2C_1\psi(t)\psi'(t) - C_2\psi(t)^2$$

for some real numbers $\gamma > 0$, $C_1,\ C_2 \geq 0$, then the following hold

a) If

$$\psi(0) > 0, \quad \psi'(0) > -\gamma_2\gamma^{-1}\psi(0), \quad C_1 + C_2 > 0 \tag{1}$$

where

$$\gamma_1 = -C_1 + \sqrt{C_1^2 + \gamma C_2}, \quad \gamma_2 = -C_1 - \sqrt{C_1^2 + \gamma C_2},$$

then there exists a positive real number,

$$t_1 < t_2 = \frac{1}{2\sqrt{C_1^2 + C_2^2}} \ln \frac{\gamma_1 \psi(0) + \gamma \psi'(0)}{\gamma_2 \psi(0) + \gamma \psi'(0)}$$

such that $\psi(t) \longrightarrow +\infty$ as $t \longrightarrow t_1$.

b) If $\psi(0) > 0$, $\psi'(0) > 0$ and $C_1 = C_2 = 0$ then there exists a positive real number $t_1 \leq t_2 = \psi(0)/(\gamma\psi'(0))$, such that $\psi(t) \longrightarrow +\infty$ as $t \longrightarrow t_1$.

2 THE IBV PROBLEM

Let us now consider the initial-boundary value problem:

$$\begin{gathered} w_{tt} + \Delta^2 w = f(-\Delta w) + \Delta w, \quad (t, x) \in (0, T) \times \bar{\Omega}, \\ -\Delta w = 0, \quad \frac{\partial w_t}{\partial \nu} - \Delta^2 w = 0, \quad (t, x) \in (0, T) \times \partial\Omega, \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x) \quad x \in \Omega. \end{gathered} \tag{2}$$

where Ω is a bounded domain in R^n with a sufficiently smooth boundary $\partial\Omega := \Gamma$, $T > 0$ is an arbitrary real number, and ν is the outward normal of the boundary Γ.

Let us define a functional $\psi(t)$ by

$$\psi(t) = \int_\Omega |\nabla w|^2 dx + \int_0^t \int_\Gamma \left(\frac{\partial w}{\partial \nu}\right)^2 dx dt + \int_\Gamma \left(\frac{\partial w_0}{\partial \nu}\right)^2 dx, \tag{3}$$

and let

(4)

$$E(0) = \int_\Omega |\nabla w_1|^2 dx + \int_\Omega |\nabla \Delta w_0|^2 dx + \int_\Omega (\Delta w_0)^2 dx - 2\int_\Omega \mathcal{F}(\Delta w_0) dx \leq 0.$$

be the initial energy of the system. Then by the use of the LK Lemma, the following theorem will be proved:

THEOREM. Let the function $f(u)$ with its primitive $\mathcal{F}(u) = \int^u f(s)ds$ have the following properties:

$$f(0) = 0, \quad sf(s) \geq 2(2\gamma + 1)\mathcal{F}(s), \quad \forall s \in R^1 \tag{5}$$

for some real number $\gamma > 0$, and let $w_0(x)$ and $w_1(x)$ be two functions such that

1) For ψ in the above (3), and its derivative ψ', the inequality (1) in LK Lemma holds.

2) The initial energy $E(0)$ in (4) is nonpositive.

If $t_2 > 0$ is the number given in the LK Lemma, then there exists a positive real number $t_1 < t_2$ such that $\psi(t) \longrightarrow +\infty$ as $t \longrightarrow t_1$.

PROOF. Differentiating (3) with respect to t one has

$$\psi'(t) = 2\int_\Omega \nabla w \cdot \nabla w_t dx + 2\int_0^t \int_\Gamma \frac{\partial w}{\partial \nu}\frac{\partial w_t}{\partial \nu} dx dt + \int_\Gamma \left(\frac{\partial w_0}{\partial \nu}\right)^2 dx. \tag{6}$$

A further differentiation with respect to t gives

$$\psi''(t) = 2\int_\Omega |\nabla w_t|^2 dx + 2\int_\Omega \nabla w \cdot \nabla w_{tt} dx + 2\int_\Gamma \frac{\partial w}{\partial \nu}\ \frac{\partial w_t}{\partial \nu} dx.$$

Using the Green-Gauss theorem, the partial differential equation in (2) on the boundary, we convert the second and the third integrals in the above to get

$$\psi''(t) = 2\int_\Omega |\nabla w_t|^2 dx - 2\int_\Omega (\Delta w) w_{tt}\ dx.$$

Substituting w_{tt} as in (2) and using the inequality in (5) one obtains

$$\psi''(t) \geq 2\int_\Omega |\nabla w_t|^2 dx - 2\int_\Omega |\nabla \Delta w|^2 dx -$$

$$(7) \qquad 2\int_\Omega (\Delta w)^2 dx + 4(2\gamma+1)\int_\Omega \mathcal{F}(-\Delta w)dx.$$

To make a better estimate for $\psi''(t)$, let us multiply both sides of the equation in (2) by $-2\Delta w_t$, and integrate over Ω to get

$$\frac{\partial E(t)}{\partial t} = -2\int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 dx$$

where

$$E(t) = \int_\Omega |\nabla w_t|^2 dx + \int_\Omega |\nabla \Delta w|^2 \ dx + \int_\Omega (\Delta w)^2 dx - 2\int_\Omega \mathcal{F}(-\Delta w)dx$$

can be regarded as the total energy of the system.

Hence

$$E(t) = E(0) - 2\int_0^t \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 dxdt$$

¿From the second hyphothesis of the theorem, the initial energy $E(0)$ is nonpositive. Therefore $E(t) < 0, \quad \forall t > 0$.

Adding

$$2(2\gamma+1)\left(E(t) - E(0) + 2\int_0^t \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 dxdt\right) = 0$$

to the right hand side of (7) and omitting some of the positive terms one gets the estimate

$$(8) \qquad \psi''(t) \geq 4(\gamma+1)\left(\int_\Omega |\nabla w_t|^2 \ dx + a\int_0^t \int_\Gamma \left(\frac{\partial w_t}{\partial \nu}\right)^2 dxdt\right).$$

If we summarize,

$$\psi(t) = A1 + B1 + C, \quad \psi'(t) = 2\ A2 + 2\ B2 + C, \quad \psi''(t) \geq 4(\gamma+1)(A3+B3),$$

where $A1,\ B1,\ C,\ A2,\ B2,\ A3,\ B3$ represent the corresponding integrals in (3), (6) and (8).

In LK lemma

$$\begin{aligned}\psi''(t)\psi(t)-(1+\gamma)[\psi'(t)]^2 &\geq \\ 4(1+\gamma)[(A3+B3)(A1+B1+C)-(A2+B2+C/2)^2] &\geq \\ 4(1+\gamma)[H-\psi(t)\psi'(t)/2],\end{aligned}$$

where

$$H=(A3+B3)(A1+B1)-(A2+B2)^2\geq 0$$

by Cauchy-Schwarz inequality. Therefore the hypotheses of LK Lemma are satisfied with $C_1=1+\gamma$, and $C_2=0$. Hence from the conclusion of the Lemma, the Theorem is proved.

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