On a Spectral Criterion for Almost Periodicity of Solutions of Periodic Evolution Equations

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Abstract

This paper is concerned with equations of the form: $u' = A(t)u + f(t)$, where $A(t)$ is (unbounded) periodic linear operator and f is almost periodic. We extend a central result on the spectral criteria for almost periodicity of solutions of evolution equations to some classes of periodic equations which says that if u is a bounded uniformly continuous mild solution and P is the monodromy operator, then their spectra satisfy $e^{isp_{AP}(u)} \subset \sigma(P) \cap S^1$, where S^1 is the unit circle. This result is then applied to find almost periodic solutions to the above-mentioned equations. In particular, parabolic and functional differential equations are considered. Existence conditions for almost periodic and quasi-periodic solutions are discussed.

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Abbreviated title: A Spectral Criterion for Almost Periodicity

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1 Introduction and Preliminaries

Let us consider the following linear evolution equations

$$\frac{dx}{dt} = A(t)x + f(t), \ \forall t \in \mathbf{R} \tag{1}$$

and

$$\frac{dx}{dt} = A(t)x, \ \forall t \in \mathbf{R}, \tag{2}$$

where $x \in \mathbf{X}$, $\mathbf{X}$ is a complex Banach space, $A(t)$ is a (unbounded) linear operator acting on $\mathbf{X}$ for every fixed $t \in \mathbf{R}$ such that $A(t) = A(t+1)$ for all $t \in \mathbf{R}$, $f : \mathbf{R} \to \mathbf{X}$ is an almost periodic function. In the paper we always assume that Eq.(1) is well-posed in the sense of [P], i.e. one can associate with it an evolution operator $(U(t,s))_{t\geq s}$ which satisfies the conditions below.

The asymptotic behavior of solutions to well-posed evolution equations has been of particular interest for the last two decades. We refer the reader to the books [Hal1], [He], [LZ], [Na], [Ne] and the surveys [Bat], [V3] and the references therein for more complete information on the subject. In this direction, one of the most interesting topics to be discussed in recent works is to find spectral conditions for almost periodicity of solutions (see e.g. [LZ], [HMN], [Pr] and recent papers [AB], [AS] , [Bas], [RV], [V1], [V2], [V3] , [NM]).

A central result of this theory is as follows: let Eq.(1) be autonomous, i.e $A(t) = A, \ \forall t$. Then the spectrum $sp_{AP}(u)$ of a bounded uniformly continuous mild solution to Eq.(1) satisfies the inclusion $sp_{AP}(u) \subset \sigma(A) \cap i\mathbf{R}$ (see e.g. [LZ], [Bas], [AB]). It is interesting that starting from this inclusion one can not only find spectral conditions for almost periodicity for a bounded uniformly continuous solution of Eq.(1) but also establish a stability theory of solutions to Eq.(1) (see [Bas]).

As is known, an ordinary periodic differential equation with finite dimensional phase space can be transformed into an equation with constant coefficients (see e.g. [DK, chap. V]). Unfortunately, this is not the case for infinite dimensional equations. In fact, it is known [Hal2, Chap. 8], [He, Chap.7] that even for very simple infinite dimensional equations the above procedure fails, i.e. there are no Floquet transformations which reduce the periodic equations to equations with bounded coefficient operators although, for some particular cases one can show the existence of transformations which reduce the periodic equations under consideration to equations with (unbounded) coefficient operators (see e.g. [CLM], and also [Ku] for more information on Floquet theory for partial differential equations). Hence, in the infinite dimensional case, it is not relevant to treat periodic equations as autonomous ones especially for the almost periodicity of solutions (see e.g. [V1] for additional information).

The main purpose of this paper is to extend the above spectral inclusion to some classes of periodic evolution equations such as equations which monodromy operators either depend analytically on time or are compact. These classes of equations can be found among those having analytic solutions (see e.g. [KT1], [KT2], [M1], [M2] for basic results) and parabolic 1-periodic equations $dx/dt + Ax = B(t)x + f(t)$ such that A has compact resolvent (see e.g. [He, p.196]). The spectral inclusions will be then used to find spectral conditions for the almost periodicity of a bounded uniformly continuous solution to the underlying equation. Our proofs of main results are mainly based on analyzing difference equations corresponding to Eq.(1). Our main results are Theorems 3, 4, 5 in Section 2. In Section 3 we find three classes of equations satisfying the conditions of Theorems 3, 4, 5 respectively. Our results in these classes are natural generalizations of well - known ones from ordinary differential equations.

Below we recall some notions and results of the spectral theory of uniformly continuous and bounded functions on the real line. Throughout the paper we shall denote by $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ the set of integers, real and complex numbers, respectively. For every $z \in \mathbf{C}$, $\Re z$ will stand for the real part of z . $L(\mathbf{X})$ will denote the space of bounded linear operators from $\mathbf{X}$ to itself . $\sigma(A), \rho(A)$ will denote the spectrum and resolvent set of an operator A . $BUC(\mathbf{R}, \mathbf{X})$ will stand for the space of all uniformly continuous and bounded functions on the real line with sup-norm. $C_0(\mathbf{R}, \mathbf{X})$ will denote the subspace of $BUC(\mathbf{R}, \mathbf{X})$ consisting of functions v such that $\lim_{t\to\infty} v(t) = 0$. The subspace of $BUC(\mathbf{R}, \mathbf{X})$ consisting of all almost periodic functions in the sense of Bohr will be denoted by $AP(\mathbf{X})$ which can be defined to be the smallest closed subspace of $BUC(\mathbf{R}, \mathbf{X})$ containing the functions of the form $\{e^{i\eta\cdot}x, \eta \in \mathbf{R}, x \in \mathbf{X}\}$ (see e.g. [LZ]). We denote by $\mathcal{F}$ the Fourier transform, i.e.

$$(\mathcal{F}f)(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t)dt \tag{3}$$

$(s \in \mathbf{R}, f \in L^1(\mathbf{R}))$. Then the spectrum of $u \in BUC(\mathbf{R}, \mathbf{X})$ is defined to be the following set

$$sp(u) := \{\xi \in \mathbf{R} : \forall \epsilon > 0 \exists f \in L^1(\mathbf{R}), supp\mathcal{F}f \subset (\xi-\epsilon, \xi+\epsilon), f*u \neq 0\} \quad (4)$$

where

$$f * u(s) := \int_{-\infty}^{+\infty} f(s-t)u(t)dt.$$

It coincides with the set $\sigma(u)$ consisting of $\xi \in \mathbf{R}$ such that the Fourrier-Carleman transform of u

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t}u(\lambda)dt, & (\Re\lambda > 0); \\ -\int_0^\infty e^{\lambda t}u(-t)dt, & (\Re\lambda < 0) \end{cases} \quad (5)$$

has a holomorphic extension to a neighborhood of $i\xi$ (see e.g. [Pr, Proposition 0.5, p.22]). We collect some main properties of the spectrum of a function, which we will need, in the following theorem for the reader's convenience.

Theorem 1 *Let* $f, g_n \in BUC(\mathbf{R}, \mathbf{X})$ *such that* $\lim_{n\to\infty} \|g_n - f\| = 0$. *Then*

i) $\sigma(f)$ *is closed ,*

ii) $\sigma(f(\cdot + h)) = \sigma(f)$,

iii) If $\alpha \in \mathbf{C}\backslash\{0\}$ $\sigma(\alpha f) = \sigma(f)$,

iv) If $\sigma(g_n) \subset \Lambda$ *for all* $n \in \mathbf{N}$ *then* $\sigma(f) \subset \overline{\Lambda}$.

Proof For the proof we refer the reader to [Pr, Proposition 0.4, p. 20 and Theorem 0.8, p. 21] .

Similarly, one defines $sp_{AP}(u)$ as follows.

Definition 1 Let u be a bounded uniformly continuous function. Then

$$\begin{aligned} sp_{AP}(u) := \{&\lambda \in \mathbf{R} : \forall \epsilon > 0 \;\; \exists \phi \in L(\mathbf{R}^1) \text{such that} \\ &supp\mathcal{F}\phi \subset (\lambda-\epsilon, \lambda+\epsilon), \phi * f \text{ is not almost periodic}\}. \end{aligned} \quad (6)$$

By definition, similarly to $sp(u)$, $sp_{AP}(u)$ has the following properties:

Proposition 1 *Let* u, v *be bounded uniformly continuous functions on the real line. Then*

i) $sp_{AP}(u+v) \subset sp_{AP}(u) \cup sp_{AP}(u)$

ii) $sp_{AP}(u(\cdot + h)) = sp_{AP}(u)$.

2 Main Results

We begin this section by considering the *1-periodic strongly continuous process* associated with the given 1-periodic evolution equation (1). We recall this concept in the following definition.

Definition 2 A family of bounded linear operators $(U(t,s))_{t\geq s}, (t,s \in \mathbf{R})$ from a Banach space $\mathbf{X}$ to itself is called *1-periodic strongly continuous evolutionary process* if the following conditions are satisfied

i) $U(t,t) = I$ for all $t \in \mathbf{R}$,

ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,

iii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbf{X}$,

iv) $U(t+1, s+1) = U(t,s)$ for all $t \geq s$.

v) $\|U(t,s)\| < Ne^{\omega(t-s)}$ for some positive N, ω independent of $t \geq s$.

Below if it does not cause any danger of confusion, for the sake of simplicity, we shall use the terminology *periodic evolutionary process* to designate *1-periodic strongly continuous evolutionary process.*

This notion comes naturally from the well-posed evolution equations (see e.g. [P]). And from now on we always assume that Eq.(1) is well-posed and there exisits a unique evolutionary process $(U(t,s))_{t\geq s}, (t,s \in \mathbf{R})$ associated with it having the properties listed in the above definition. Recall that a $\mathbf{X}$ valued function u defined on the real line is said to be a mild solution to Eq.(1) if it satisfies the following integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi, \ \forall t \geq s, t, s \in \mathbf{R}, \tag{7}$$

In connection with periodic evolutionary processes the following will be used as a basic property.

Lemma 1 *Let $(U(t,s)_{t\geq s})$ be a periodic evolutionary process and f be an almost periodic function. Then the function*

$$t \mapsto \int_{t-h}^t U(t,\xi)f(\xi)d\xi,$$

is almost periodic for every fixed $h \geq 0$.

Proof For the proof see [NM].

To deal with periodic evolution equations, the notion of monodromy operator will play the key role. And we shall try to state our conditions in terms of spectral properties of monodromy operators. Recall that for the evolutionary process $(U(t,s))_{t\geq s}, (t,s \in \mathbf{R})$ associated with the 1-periodic equation (1) the monodromy operators are defined to be $P(t) := U(t,t-1), t \in \mathbf{R}$. Note that the map $t \mapsto P(t)$ is also 1-periodic, and usually one takes $P(0)$ as the monodromy operator because the above family of monodromy operators has "similar properties" which are stated in the following

Lemma 2 ([He, Lemma 7.2.2 , p. 197]) *$P(t+1) = P(t)$ for all t; characteristic multipliers are independent of time, i.e. the nonzero eigenvalues of $P(t)$ coincide with those of $P(s)$. Moreover, $\sigma(P(t))\backslash\{0\}$ is independent of t .*

As usual, in this paper we will characterize conditions for almost periodicity of solutions in terms of spectral properties of the monodromy operators. Here, we will allow the spectrum of the monodromy operators to intersect the unit circle because otherwise the problem is trivial as is shown in the following

Theorem 2 *Let $(U(t,s)_{t\geq s})$ be an 1-periodic evolutionary process. Then the following assertions are equivalent:*

i) $\sigma(P(0)) \cap S^1 = \oslash$,

ii) For every bounded function f Eq.(7) has a unique bounded mild solution,

iii) For every almost periodic function f Eq.(7) has a unique almost periodic mild solution.

Proof For the proof see [NM].

2.1 The case of analytic monodromy operators

In this subsection we often assume that Eq.(1) satisfy the following hyperthesis

Definition 3 Eq.(1) is said to satisfy *Condition H* if the monodromy operators $P(t)$ of the strongly continuous 1-periodic evolutionary process $(U(t,s)_{t\geq s})$ associated with its homogeneous equation (2), as a function of t, is analytic in t .

Note that Condition H is satisfied automatically, if $(U(t,s))_{t\geq s}$ is a C_0-semigroup, and in general, it is not required that $(U(t,s))_{t\geq s}$ be norm continuous.

Since the map $t \mapsto P(t)$ is 1-periodic, Condition H will guarantee the following:

Lemma 3 *Let Eq.(1) satisfy condition H. Then the Fourier series*

$$\sum_{k=-\infty}^{+\infty} P_k$$

associated with $P(t)$ *is absolutely convergent and*

$$P(t) = \sum_{k=-\infty}^{+\infty} P_k e^{2\pi kit} \tag{8}$$

is convergent uniformly in t *.*

Proof For the proof see e.g. [Ka, p.32].

Another consequence of condition H which we will use below is concerned with a spectral estimate. To make it more clear we introduce the function space $H(\mathbf{S},\mathbf{X})$ consisting of all $\mathbf{X}$ valued bounded holomorphic functions w on $\mathbf{S} := \{z \in \mathbf{C} : |\Re z| < 1\}$.

The following results are obvious from the theory of holomorphic functions and will be needed in the sequel.

Lemma 4 *With the above notations the following assertions hold true:*

i) $H(\mathbf{S},\mathbf{X})$ *is a Banach space with sup-norm,* *ii) Let $\mu \mapsto V(\mu) \in L(H(\mathbf{S},\mathbf{X}))$ be any holomorphic function defined on a neighborhood $U(z)$ of $z \in S$. Then for every given $g \in H(\mathbf{S},\mathbf{X})$, the function*

$$\lambda \mapsto [V(\lambda)g](\lambda) \tag{9}$$

is holomorphic.

Examples The function

$$h(\lambda) = \int_{-1}^{0} e^{-\lambda t} u(t)dt \tag{10}$$

for any continuous function $u(\cdot)$ is an element of $H(\mathbf{S},\mathbf{X})$. More generally,

$$g(\lambda) := \sum_{k=-\infty}^{+\infty} P_k \int_{-1}^{0} e^{(\lambda - 2\pi k i)\xi} u(\xi)d\xi, \tag{11}$$

where P_k is defined by (8), is also an element of $H(\mathbf{S},\mathbf{X})$.

In the paper we shall need the following spectral property of multiplication operators.

Lemma 5 *Let $\mathbf{R} \ni t \mapsto Q(t) \in L(\mathbf{X})$ be norm continuous and periodic such that*

$$\sigma(Q(t))\backslash\{0\} = \sigma(Q(s))\backslash\{0\} \ \forall t, s \in \mathbf{R}.$$

Then

$$\sigma(\hat{Q})\backslash\{0\} = \sigma(Q(t))\backslash\{0\}, \forall t,$$

where $\hat{Q} \in L(BUC(\mathbf{R},\mathbf{X})), [\hat{Q}v](t) := Q(t)v(t), \ \forall t \in \mathbf{R}, v \in BUC(\mathbf{R},\mathbf{X})$.

Proof Consider the equation $(\lambda - \hat{Q})u = v$ for given $v \in BUC(\mathbf{R},\mathbf{X}), \lambda \neq 0$. It is equivalent to the equation $(\lambda - Q(t))u(t) = v(t), t \in \mathbf{R}$. If $\lambda \in \rho(\hat{Q})\backslash\{0\}$, and for a given $y \in \mathbf{X}$, then take $v(t) = y \in \mathbf{X}, \ \forall t$. Thus there is a unique $u \in BUC(\mathbf{R},\mathbf{X})$ which satisfies the first equation, i.e.

$$\lambda u(t) - Q(t)u(t) = y, \ \forall t.$$

Moreover,

$$\|u(t)\| \leq \sup_{\xi} \|u(\xi)\| \leq \|R(\lambda,\hat{Q})v\| \leq \|R(\lambda,\hat{Q})\| \sup_{\xi} \|v(\xi)\| = \|R(\lambda,\hat{Q})\| \|y\|.$$

Thus the second equation has a unique solution for every fixed $t \in \mathbf{R}$. This shows that $\lambda \in \rho(Q(t))\backslash\{0\}$. Conversely, let $\lambda \in \rho(Q(t))\backslash\{0\}$. Then, by assumption, for a given $v \in BUC(\mathbf{R},\mathbf{X})$ and $t \in \mathbf{R}$ the second equation has a unique solution $u(t) = R(\lambda, Q(t))v(t)$. It may be seen that the operator $R(\lambda, Q(t))$ is continuous and periodic in t. Hence, $u \in BUC(\mathbf{R},\mathbf{X})$. Moreover,

$$\sup_t \|u(t)\| \leq \sup_t \|R(\lambda, Q(t))v(t)\| \leq \sup_t \|R(\lambda, Q(t))\| \sup_t \|v(t)\| \leq r\|v\|,$$

where, by the above remark, $\sup_t \|R(\lambda, Q(t))\| < \infty$. This shows that $\lambda \in \rho(\hat{Q})\backslash\{0\}$. $\Box$

In light of this lemma below we often identify the multiplication operator $\hat{Q}$ in case $Q(t) =$ constant with the operator Q itself if there is no danger of confusion.

For a fixed $\lambda_0 \in S$ we consider the space of bounded sequences $Q(\lambda_0) := \{g : \{\lambda_0 + 2\pi ki, k \in \mathbf{Z}\} \mapsto \mathbf{X}, \sup_z \|g(z)\| < \infty\}$. We denote by $T_Q^{2\pi}$ the translation on $Q(\lambda_0)$ defined as

$$[T_Q^{2\pi} g](z) = g(z + 2\pi i), \quad \forall \quad z \in \{\lambda_0 + 2\pi ki, k \in \mathbf{Z}\}.$$

We shall denote by T^t the translation group on $H(\mathbf{S},\mathbf{X})$, i.e.

$$T^t w(\lambda) := w(it + \lambda), \quad \forall t \in \mathbf{R} \ , \lambda \in \mathbf{S}.$$

We shall denote by Q_k the operator acting on $BUC(\mathbf{R},\mathbf{X})$ defined as $Q_k v(s) := e^{2\pi iks} P_k v(s), \ s \in \mathbf{R}$.

Lemma 6 *Let Eq.(1) satisfy condition H. Then the following estimates hold true:*

$$\sigma(\sum_{k=-\infty}^{+\infty} P_k T^{2k\pi}) \subset \sigma(P(0)) \bigcup \{0\} \ , \tag{12}$$

$$\sigma(\sum_{k=-\infty}^{+\infty} Q_k T^{2k\pi}) \subset \sigma(P(0)) \bigcup \{0\} \ , \tag{13}$$

$$\sigma(\sum_{k=-\infty}^{+\infty} P_k T_Q^{2k\pi}) \subset \sigma(P(0)) \bigcup \{0\} \ . \tag{14}$$

Proof Consider the function

$$f(\lambda) := \sum_{k=-\infty}^{+\infty} P_k \lambda^k ,$$

where $\lambda \in S^1$. Since $P(t)$ is analytic with respect to $t \in \mathbf{R}$, $f(\lambda)$ is holomorphic in a neighbourhood of S^1 in the complex plane. Thus applying [S,Theorem 1] to function $f(\lambda)$ and operator $T^{2\pi}$ we obtain

$$\sigma(f(T^{2\pi})) \subset \bigcup_{\lambda\in\sigma(T^{2\pi})} \sigma(f(\lambda)).$$

Since $(T^t)_{t\in\mathbf{R}}$ is the translation group,

$$\sigma(T^{2\pi}) \subset S^1 := \{z \in \mathbf{C} : \|z\| = 1\}.$$

Thus, in view of Lemma 2,

$$\begin{aligned} \sigma(f(T^{2\pi})) &= \sigma(\sum_{k=-\infty}^{+\infty} P_k T^{2\pi k}) \\ &\subset \bigcup_{\lambda\in S^1} \sigma(\sum_{k=-\infty}^{+\infty} P_k \lambda^{2\pi k}) \\ &\subset \bigcup_{t\in\mathbf{R}} \sigma(\sum_{k=-\infty}^{+\infty} P_k e^{2\pi i k t}) = \\ &= \bigcup_{t\in\mathbf{R}} \sigma(P(t)) \subset \sigma(P(0)) \bigcup \{0\}. \end{aligned}$$

Hence (12) holds true. (13), (14) are proved in a similar manner. □

Now we consider first the homogeneous equation (2). Thus, if u is a mild solution to the homogeneous equation, then by (7)

$$u(t) = P(t)u(t-1), \ \forall t \ . \tag{15}$$

We are going to prove the following

Theorem 3 *Let Eq.(1) satisfy condition H and $u(\cdot)$ is a bounded uniformly continuous mild solution of the homogeneous equation. Then the inclusion*

$$e^{i\ sp(u)} \subset \sigma(P(0)) \cap S^1 \tag{16}$$

holds true.

Proof Let us first compute the Carleman transform of $u(\cdot)$. For $\Re\lambda > 0$,

$$\begin{aligned}
\hat{u}(\lambda) &= \int_0^{\infty} e^{-\lambda t} u(t)dt = \int_0^{\infty} e^{-\lambda t} P(t)u(t-1)dt \\
&= \int_0^{\infty} e^{-\lambda t} \sum_{k=-\infty}^{+\infty} e^{2\pi ikt} P_k u(t-1)dt \\
&= \sum_{k=-\infty}^{+\infty} P_k \int_0^{\infty} e^{-\lambda t} e^{2\pi ikt} u(t-1)dt \\
&= \sum_{k=-\infty}^{+\infty} P_k e^{-(\lambda-2\pi ki)} \int_{-1}^{\infty} e^{-(\lambda-2\pi ki)t} u(t)dt \\
&= e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \int_{-1}^{\infty} e^{-(\lambda-2\pi ki)t} u(t)dt \\
&= e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \int_{0}^{\infty} e^{-(\lambda-2\pi ki)t} u(t)dt \\
&\quad + e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \int_{-1}^{0} e^{-(\lambda-2\pi ki)t} u(t)dt \ .
\end{aligned}$$

On the other hand, for $\Re\lambda < 0$,

$$\begin{aligned}
\hat{u}(\lambda) &= -\int_0^{\infty} e^{\lambda t} \sum_{k=-\infty}^{+\infty} e^{-2\pi ikt} P_k u(-t-1)dt \\
&= -\int_0^{\infty} \sum_{k=-\infty}^{+\infty} P_k e^{(\lambda-2\pi ik)t} u(-t-1)dt
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=-\infty}^{+\infty} P_k \int_1^{\infty} e^{(\lambda-2\pi ik)(\xi-1)} u(-\xi)d\xi \\
&= -e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \int_0^{\infty} e^{(\lambda-2\pi ik)\xi} u(-\xi)d\xi + \\
&\quad +e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \int_0^{1} e^{(\lambda-2\pi ik)\xi} u(-\xi)d\xi.
\end{aligned}
$$

Finally, for $\Re\lambda \neq 0$

$$
\begin{aligned}
\hat{u}(\lambda) &= e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \hat{u}(\lambda - 2\pi ki) + \\
&\quad +e^{-\lambda} \sum_{k=-\infty}^{+\infty} P_k \int_{-1}^{0} e^{-(\lambda-2\pi ki)\xi} u(\xi)d\xi.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
e^{\lambda}\hat{u}(\lambda) &= \sum_{k=-\infty}^{+\infty} P_k \hat{u}(\lambda - 2\pi ki) + \\
&\quad + \sum_{k=-\infty}^{+\infty} P_k \int_{-1}^{0} e^{-(\lambda-2\pi ki)\xi} u(\xi)d\xi. \qquad (17)
\end{aligned}
$$

Now , if $\zeta \in \mathbf{R}$ such that $e^{i\zeta} \in \rho(P(0)) \cap S^1$, then the function

$$
w(\lambda) := [R(e^{\lambda}, f(T^{2\pi}))g](\lambda),
$$

where

$$
g(\lambda) := \sum_{k=-\infty}^{+\infty} P_k \int_{-1}^{0} e^{-(\lambda-2\pi ki)\xi} u(\xi)d\xi,
$$

in view of Examples and Lemma 4, is holomorphic on an open neighborhood $U(i\zeta) \subset S$ such that $e^{U(i\zeta)} \subset \rho(P(0))$. We show now that $w(\cdot)$ is a holomorphic extension of $\hat{u}$ at $i\zeta$, i.e.

$$
w(\lambda) = \hat{u}(\lambda),\ \forall \lambda \in U(i\zeta),\ \Re\lambda \neq 0.
$$

To this purpose, we fix a $\lambda_0 \in U(i\zeta)$ such that $\Re(\lambda_0) \neq 0$. Since

$$e^{U(i\zeta)} \subset \rho(f(T_Q^{2\pi}))$$

and for every $\lambda \in Q(\lambda_0)$

$$e^{\lambda_0}\hat{u}(\lambda) = \sum_{k=-\infty}^{+\infty} P_k \hat{u}(\lambda - 2\pi ki) + g(\lambda) \tag{18}$$

we see that

$$\hat{u}(\cdot)|_{Q(\lambda_0)} = R(\lambda_0, f(T_Q^{2\pi}))g|_{Q(\lambda_0)}$$

and Eq.(17) is uniquely solvable in $Q(\lambda_0)$. On the other hand, if we put

$$v(\lambda) = [R(\lambda_0, f(T^{2\pi}))g](\lambda), \lambda \in S,$$

then

$$e^{\lambda_0}v(\lambda) = \sum_{k=-\infty}^{+\infty} P_k w(\lambda - 2\pi ki) + g(\lambda)$$

for all $\lambda \in S$ and in particular for $\lambda \in Q(\lambda_0)$.

Since $e^{\lambda_0} \in e^{U(i\zeta)}$, from the unique solvability of (17), we have

$$\hat{u}(\lambda) = v(\lambda) \ , \ \forall \lambda \in Q(\lambda_0) \ ,$$

in particular,

$$\hat{u}(\lambda_0) = v(\lambda_0) = [R(\lambda_0, f(T^{2\pi}))g](\lambda_0) = w(\lambda_0).$$

Hence, $\hat{u}(\lambda_0)$ has a holomorphic extension at $i\zeta$ where $e^{i\zeta} \in \rho(P(0))$, i.e. $\zeta \notin sp(u)$. $\square$

Now we consider the inhomogeneous equation

$$\frac{du}{dt} = A(t)u(t) + f(t)$$

with the same assumptions as for the homogeneous equation except that f is almost periodic with respect to t . We will use the so-called factorization technique developed in [AB]. We recall that the main idea of the method is as follows. Let us denote by $(T_t)_{t\in\mathbf{R}}$ the translation group on $BUC(\mathbf{R}, \mathbf{X})$, i.e.

$(T_t u)(s) := u(t+s), \forall \xi \in \mathbf{R}$. Since $(T_t)_{t\in\mathbf{R}}$ leaves $AP(\mathbf{R},\mathbf{X})$ invariant there is an induced C_0-group on the quotient space $\mathbf{Y} := BUC(\mathbf{R},\mathbf{X})/AP(\mathbf{R},\mathbf{X})$ given by

$$\overline{T}_t = p(u) = p(T_t u) \tag{19}$$

where $p : BUC(\mathbf{R},\mathbf{X}) \mapsto \mathbf{Y}$ is the quotient mapping. Its generator is denoted by $\tilde{B}$. For $g \in \mathbf{Y}$ we denote by $\mathcal{M}(g) := \overline{span}\{\overline{S}(t)g : t \in \mathbf{R}\}$ the smallest closed subspace of $\mathbf{Y}$ which is invariant under $(\overline{T}_t)_{t\in\mathbf{R}}$. Using the notation $\tilde{B}_g$ for the part of $\tilde{B}$ in $\mathcal{M}(g)$ we have

$$isp_{AP}(u) = \sigma(\tilde{B}_{p(u)}). \tag{20}$$

For more details see [AB].

For $\mu \in \mathbf{R}, P \in L(\mathbf{X})$ let us denote by M_μ the multiplication operator in $BUC(\mathbf{R},\mathbf{X})$ defined as

$$M_\mu v(t) := e^{i\mu t} P v(t), \ \forall t \in \mathbf{R}.$$

Then, since M_μ leaves the space $AP(\mathbf{X})$ invariant it is clear that if $w \in p(v), \ \ M_\mu w \in p(M_\mu v)$. This remark allows us, by abuse of notations for the sake of simplicity, to write

$$p(e^{2\pi ik\cdot} P_k T^{2\pi k} R(\lambda, B)u) = e^{2\pi ik\cdot} P_k T^{2\pi k} \ p(R(\lambda, B)u).$$

Moreover, since the multiplication operator $\hat{P}$ (recall that $\hat{P}v(t) := P(t)v(t), \ \forall t, v$) leaves $AP(\mathbf{X})$ invariant it induces an operator $\tilde{\hat{P}}$ on the quotient space $\mathbf{Y} := BUC(\mathbf{R},\mathbf{X})/AP(\mathbf{R},\mathbf{X})$. From the periodicity of the resolvent $R(\lambda, P(s))$ with respect to s , it may be seen that $R(\lambda, \hat{P})$ leaves $AP(\mathbf{X})$ invariant. By this one can show easily that $\sigma(\tilde{\hat{P}}) = \sigma(\hat{P})$. Thus

$$\rho(\tilde{\hat{P}})\backslash\{0\} = \rho(P(0))\backslash\{0\} \subset \rho(\sum_{k=-\infty}^{\infty} Q_k T^{2\pi k}) = \rho(\sum_{k=-\infty}^{\infty} \tilde{Q}_k T^{2\pi k}),$$

where $Q_k = e^{2\pi ik\cdot} P_k$.

Theorem 4 *Let Eq.(1) satisfy condition H, u be a bounded uniformly continuous mild solution to the inhomogeneous equation (1) with f almost periodic. Then*

$$e^{i \ sp_{AP}(u)} \subset \sigma(P(0)) \cap S^1. \tag{21}$$

Proof Put $u_s := T_s u,\ s \in \mathbf{R}$. Then

$$\begin{aligned} u_s(t) = u(t+s) &= U(t+s, t+s-1)u(t+s-1) + \int_{t+s-1}^{t+s} U(t+s,\xi)f(\xi)d\xi \\ &= P(t+s)u_s(t-1) + \psi_s(t). \end{aligned}$$

Let us compute the Carleman transform of $u_s(\cdot)$. For $\Re\lambda > 0$,

$$\begin{aligned} \hat{u}_s(\lambda) &= \int_0^\infty e^{-\lambda t}u_s(t)dt = \int_0^\infty e^{-\lambda t}P(t+s)u_s(t-1)dt + \int_0^\infty e^{-\lambda t}\psi_s(t)dt \\ &= \int_0^\infty e^{-\lambda t}\sum_{k=-\infty}^{+\infty} e^{2\pi ik(t+s)}P_k u_s(t-1)dt + \hat{\psi}_s(\lambda) \\ &= \sum_{k=-\infty}^{+\infty} e^{2\pi iks}P_k \int_0^\infty e^{-\lambda t}e^{2\pi ikt}u_s(t-1)dt + \hat{\psi}_s(\lambda) \\ &= \sum_{k=-\infty}^{+\infty} e^{2\pi iks}P_k \int_0^\infty e^{-(\lambda-2\pi ik)t}u_s(t-1)dt + \hat{\psi}_s(\lambda) \\ &= \sum_{k=-\infty}^{+\infty} e^{2\pi iks}P_k \int_{-1}^\infty e^{-(\lambda-2\pi ik)(t+1)}u_s(t)dt + \hat{\psi}_s(\lambda) \\ &= \sum_{k=-\infty}^{+\infty} e^{2\pi iks}e^{-(\lambda-2\pi ki)}P_k \int_{-1}^\infty e^{-(\lambda-2\pi ki)t}u_s(t)dt + \hat{\psi}_s(\lambda) \\ &= e^{-\lambda}\sum_{k=-\infty}^{+\infty} e^{2\pi iks}P_k \int_{-1}^\infty e^{-(\lambda-2\pi ki)t}u_s(t)dt + \hat{\psi}_s(\lambda) \\ &= e^{-\lambda}\sum_{k=-\infty}^{+\infty} e^{2\pi iks}P_k \int_0^\infty e^{-(\lambda-2\pi ki)t}u_s(t)dt \\ &\quad + e^{-\lambda}\sum_{k=-\infty}^{+\infty} e^{2\pi iks}P_k \int_{-1}^0 e^{-(\lambda-2\pi ki)t}u_s(t)dt + \hat{\psi}_s(\lambda)\ . \end{aligned}$$

Hence, we have

$$\hat{u}_s(\lambda) = e^{-\lambda}\sum_{k=-\infty}^{\infty} e^{2\pi iks}P_k\hat{u}_s(\lambda-2\pi ki) + g_s(\lambda) + \hat{\psi}_s(\lambda),$$

where

$$\begin{aligned}\psi_s(t) &= \psi(t+s), \psi(t) = \int_{t-1}^{t} U(t,\xi)f(\xi)d\xi \\ g_s(\lambda) &= e^{-\lambda} \sum_{k=-\infty}^{\infty} e^{2\pi iks} P_k \int_{-1}^{0} e^{(\lambda-2\pi ik)\xi} u_s(\xi)d\xi.\end{aligned}$$

This equation is also valid for $\Re\lambda < 0$.

As $\hat{u}_s(\lambda) = (R(\lambda, B)u)(s)$, where B is the generator of the group $(T_t)_{t\in\mathbf{R}}$, from the above equality

$$(R(\lambda, B)u)(s) = e^{-\lambda} \sum_{k=-\infty}^{\infty} e^{2\pi iks} P_k (T^{2\pi k} R(\lambda, B)u)(s) + g_s(\lambda) + (R(\lambda, B)\psi)(s).$$

By Lemma 1, $\psi(t)$ is almost periodic. Hence, since

$$R(\lambda, B)\psi = \int_0^\infty e^{-\lambda t} T_t \psi dt \in AP(\mathbf{X}), \quad \forall \ \Re\lambda > 0 \ ,$$

and similarly for $\Re\lambda < 0$, we have

$$p(R(\lambda, B)\psi) = 0.$$

Thus

$$p(R(\lambda, B)u) = e^{-\lambda} \sum_{k=-\infty}^{\infty} p(e^{2\pi ik\cdot} P_k T^{2\pi k} R(\lambda, B)u) + h(\lambda),$$

where

$$\begin{aligned}h(\lambda) &= pg.(\lambda) = \\ &= e^{-\lambda} \sum_{k=-\infty}^{\infty} P_k \int_{-1}^{0} e^{(\lambda-2\pi ik)\xi} p(e^{2\pi ik\cdot} u.(\xi))d\xi\end{aligned}$$

which is, obviously, a holomorphic function.

In general, from here we can follow the proof of Theorem 2 with some remarks concerning the procedure "passage to the quotient space". We have

$$R(\lambda, \tilde{B})p(u) = e^{-\lambda} \sum_{k=-\infty}^{\infty} T^{2\pi k} \tilde{Q}_k R(\lambda, \tilde{B})p(u) + h(\lambda).$$

for all λ with $\Re\lambda \neq 0$.

Now let $\lambda = i\zeta$ such that $e^{i\zeta} \in \rho(P(0))$. We are going to show that $\zeta \notin sp_{AP}(u)$. To this purpose, it is equivalent to show that $R(\lambda, \tilde{B})p(u)$ has an holomorphic extension at $i\zeta$ (see [AB, p.371]). This will be done in the same way as in the proof of Proposition 2. In fact,

$$w(\lambda) := R(e^{\lambda}, j(T^{2\pi}))h(\lambda),$$

where

$$j(\lambda) := \sum_{k=-\infty}^{\infty} \tilde{Q}_k \lambda^k.$$

is an analytic extension at $\lambda = i\zeta$ of $R(\lambda, \tilde{B})p(u)$. This means that $\lambda \notin sp_{AP}(u)$. $\square$

2.2 The case of compact monodromy operators

In case where $P(t)$ is compact, we can drop condition H, i.e. analyticity of the monodromy operator $P(t)$ with respect to t , to get the spectral criteria (16) and (21). Indeed, it can be done as follows. Let us denote by

$$\sigma_1 = \{\lambda \in \sigma(P(0)) : |\lambda| \geq 1\}.$$

Thus, σ_1 is finite. As is known that there exists a partial Floquet representation (see e.g. [He, p.198-200]), by a periodic transform we can reduce the equation

$$\frac{du}{dt} = A(t)u(t) + f(t)$$

to the system of equations of the form

$$\begin{cases} dy/dt = Cy + F(t)E_1(t)f(t) \\ dx_2/dt = A_2(t)x_2 + E_2(t)f(t) \end{cases} \tag{22}$$

where $F(t), E_1(t), E_2(t)$ are 1-periodic. Moreover, letting $P_2(t)$ denote the monodromy operators associated with the second component equation of (22), we have

$$\sigma(P_2(0)) = \{\lambda \in \sigma(P(0)) : |\lambda| < 1\}.$$

This, by Theorem 2, as $E_2(t)f(t)$ is almost periodic implies that there exists a unique mild almost periodic solution x_2 . Thus we are in a position to prove the following.

Theorem 5 *Let $P(0)$ be compact. Then for every bounded uniformly continuous mild solution to the homogeneous equation (inhomogeneous equation, respectively) the inclusion*

$$e^{i\ sp(u)} \subset \sigma(P(0)) \cap S^1$$

$$(e^{i\ sp_{AP}(u)} \subset \sigma(P(0)) \cap S^1\ , \textit{respectively})$$

holds true.

Proof We prove only the assertion for the inhomogeneouos equations. The assertion for the homogeneous equations can be carried out in a similar manner. First we recall the property that

$$sp_{AP}(f+g) = sp_{AP}(f) \cup sp_{AP}(g)$$

for f, g bounded uniformly continuous . Now suppose that $(y(t), x_2(t))$ is the bounded uniformly continuous mild solution of the decoupled equation (22) which by a 1-periodic transformation corresponds to $u(t)$. Obviously,

$$sp_{AP}(y(\cdot) + x_2(\cdot)) \subset sp_{AP}(y(\cdot)).$$

On the other hand from the first component equation of (22) (see e.g. [Bas], [AB])

$$isp_{AP}(y(\cdot)) \subset i\mathbf{R} \cap \sigma(C).$$

Thus

$$e^{i\ sp_{AP}(y(\cdot))} \subset \sigma(P(0)) \cap S^1\ . \tag{23}$$

As $u(\cdot) = Q(\cdot)(y(\cdot) + x_2(\cdot))$ for some 1-periodic transformation $Q(t)$ the following estimate holds

$$sp_{AP}(u) \subset \overline{\{\lambda + 2\pi k, \lambda \in sp_{AP}(y), k \in \mathbf{Z}\}}$$

(see e.g. [V1, Lemma 4.3]). Hence, by (24)

$$\begin{aligned} e^{i\ sp_{AP}(u)} &\subset e^{i\ \overline{\{\lambda+2\pi k, \lambda \in sp_{AP}(y), k\in\mathbf{Z}\}}} \\ &\subset \overline{e^{i\ sp_{AP}(y)}} \subset \sigma(P(0)) \cap S^1. \end{aligned}$$

□

2.3 Applications to Almost Periodicity of Bounded Solutions

Below we shall derive from the spectral inclusions proved in Theorems 3, 4, 5 some consequences by a standard argument. To this purpose we recall now the concept of *total ergodicity.* A function $u \in BUC(\mathbf{R}, \mathbf{X})$ is said to be *totally ergodic* if

$$M_\eta u := \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} e^{i\eta s} u(s+\cdot)ds$$

exists in $BUC(\mathbf{R}, \mathbf{X})$ for all $\eta \in \mathbf{R}$. Now we are in a position to state our main results on the spectral criteria for almost periodicity of solutions to Eq.(1).

Corollary 1 *Let f be almost periodic. Moreover, let Eq.(1)either satisfy condition H and $\sigma(P(0))\cap S^1$ be countable or its monodromy operators $P(t)$ be compact for some t_0. Then every bounded uniformly continuous mild solution to Eq.(1) is almost periodic provided one of the following conditions holds*

i) $\mathbf{X}$ *does not contain any subspace isomorphic to* c_0*,*

ii) Range of $u(\cdot)$ *is relatively compact,*

iii) u *is totally ergodic.*

Proof The corollary is an immediate consequence of Theorems 3, 4 and [LZ,Theorem 4, p.92] and [RV, Section 3].

3 Examples

In this section we will give examples in which the conditions of Theorems 3, 4, 5 are satisfied. Moreover, we will discuss the relations of our results with other well known ones. In particular, our assertions on the existence of quasi - periodic solutions for parabolic and functional differential equations seem to be new. Note that the conditions of Theorems 3, 4 are automatically fulfilled in the trivial case of nonautonomousness, i.e. the homogeneous equation of

Eq.(1) is autonomous and generates a C_0-semigroup. Nontrivial examples of nonautonomous evolution equations will be mainly found in the class of evolution equations having holomorphic solutions. For information on this topics we refer the reader to the following basic results [KT1], [KT2], [M1] [M2].

Let Δ be a complex convex open neighborhood of the closed real interval $[0,1]$ and $\{A(t), t \in \Delta\}$ be a family of operators in $\mathbf{X}$ such that

(a) For each $t \in \Delta$, $A(t)$ is densely defined closed linear operator in $\mathbf{X}$;

(b) For any compact set $K \subset \Delta$ and any $\epsilon > 0$ there exist constants λ_0 and M such that

(b-1) the sector
$$\Sigma_\epsilon := \Sigma(\lambda_0; -\frac{\pi}{2} - \theta + \epsilon, \frac{\pi}{2} + \theta - \epsilon)$$
is contained in the resolvent set $\rho(A(t))$ of $A(t)$ for t in some neighborhood of K ,

(b-2) $\|(z - A(t))^{-1}\| \leq M/|z|$ for $z \in \Sigma_\epsilon, t \in K$,

(b-3) for each fixed $z \in \Sigma_\epsilon$, $(z - A(t))^{-1}$ is holomorphic for t in some neighborhood of K .

We will use the following notations in the sequel:

$$\Omega_\theta := \{(s,t); s,t \in \Delta, |arg(t-s)| < \theta\}, \; for \; 0 < \theta < \pi/2. \tag{24}$$

Proposition 2 *Let $\{A(t), t \in \mathbf{R}\}$ be a 1-periodic family of operators in $\mathbf{X}$, i.e. $A(t+1) = A(t)$, $\forall t$ such that on the interval $[0,1]$ it has an extension to the neighborhood Δ and conditions (a), (b) are satisfied. Then there exists an 1-periodic evolutionary process $(U(t,s))_{t\geq s}$ associated with the equation*

$$\frac{dx}{dt} = A(t)x, \quad \forall t \in \mathbf{R},$$

which has the monodromy operators $P(t) := U(t,t-1)$ analytic in $t \in \mathbf{R}$.

Proof The proposition is an immediate consequence of [KT1], [KT2]. In fact, since $U(t,s)$ is analytic in $(s,t), s \neq t$ the monodromy operator $P(t) := U(t,t-1)$ is analytic in $t \in \mathbf{R}$.□

Let us consider the equation

$$\frac{dx}{dt} + Ax = B(t)x + f(t) \tag{25}$$

where $A(\cdot)$ is sectorial in $\mathbf{X}$, $0 \leq \alpha < 1$ and $t \mapsto B(t) : \mathbf{R} \mapsto L(\mathbf{X}^\alpha, \mathbf{X})$ is Hölder continuous. Moreover, suppose that A has compact resolvent. Then as shown in [He, Chap. 7], the homogeneous equation corresponding to Eq.(25) has an evolution operator $T(t,s)$ which is compact as an operator on $\mathbf{X}^\beta$ for each $\beta < 1,\ t > s$. Observe that in this case if $dim\mathbf{X} = \infty$ there is no Floquet representation, i.e. there does not exist a bounded operator B such that $T(1,0) = e^B$. In fact, using the spectral mapping theorem one sees easily that for this representation to exist it is necessary that $0 \in \rho(T(1,0))$. As is known, in this case there exists partial Floquet representation (see e.g. [He, Chap.7]). Hence we have the following:

Corollary 2 *Let Eq.(25) satisfy the above mentioned conditions. Then every bounded uniformly continuous solution is almost periodic provided one of the following conditions holds*

i) $\mathbf{X}$ *does not contain any subspace isomorphic to* c_0*,*

ii) Range of $u(\cdot)$ *is relatively compact,*

iii) u *is totally ergodic.*

Proof This corollary is an immediate consequence of Corollary 1.□

Remarks It is shown in [DM, Theorem 6.11, p.81] that the inhomogeneous periodic equation (25) with f 1-periodic has a 1-periodic solution if and only if it has a bounded solution. This is a generalization of the well-known result from ordinary differential equations. However, the assertion does not carry over to the mentioned bounded solution itself. In view of Corollary 2, we claim that the bounded solution itself is almost periodic. Corollary 2 is in fact a generalization of a well - known result for ordinary differential equations [F, Corollary 6.5, p.101].

To illustrate another usefulness of Theorem 5 we consider now the existence of quasi-periodic solutions to the homogeneous equation of Eq.(25). It turns out that we do not need any further geometrical condition on the Banach space $\mathbf{X}$ to get the quasi-periodicity of a bounded solution.

Corollary 3 *Let* $\mathbf{X}$ *be any complex Banach space and* $u(\cdot)$ *be a bounded solution to the homogeneous equation corresponding to Eq.(25). Then there are real numbers* $\lambda_1 < \cdots < \lambda_n$ *in* $[0, 2\pi)$ *and 1-periodic continuous functions* $u_1, u_2, \cdots u_n$ *such that*

$$x(t) = \sum_{k=1}^{n} e^{i\lambda_k t} u_k(t), t \in \mathbf{R}. \tag{26}$$

Proof In this case, the boundedness of solution $x(\cdot)$ implies the uniform continuity (see [He, Theorem 7.1.3 iii), p.190]). From Theorem 5 it follows that $e^{i\ sp(x(\cdot))} \cap S^1$ consists of only finitely many points. Hence,

$$sp(x(\cdot)) \subset \cup_{k=1}^{n} \{\lambda_k + 2\pi p,\ p \in \mathbf{Z}\}.$$

This yields that the spectrum $sp(x(\cdot))$ is discrete. And then by [AS, Theorem 3.5], the almost periodicity of x follows. By the Approximation Theorem (see [LZ, Chap.2]) we see that there is a sequence of functions of the form (26)

$$u^{(m)}(t) = \sum_{k=1}^{n} e^{i\lambda_k t} u_k^{(m)}(t), t \in \mathbf{R} \tag{27}$$

which converges uniformly in t to x . The proof is complete if the following lemma holds true.

Lemma 7 *With above notations if* $u^{(m)}(t)$ *converges to* 0 *uniformly in* t *, then* $u_k^{(m)}(t)$ *converges uniformly to* 0 *in* t *as well.*

Proof of Lemma 7 We will prove the lemma by induction. The lemma holds true for $n = 1$. Now suppose that the lemma holds true for $n = p - 1$. We shall show that the lemma holds true for $n = p$ as well. In fact, without loss of generality we can assume that $\lambda_1 = 0$, otherwise we can multiply both sides of (27) by $e^{-i\lambda_1 t}$. Now set

$$
\begin{aligned}
w^{(m)}(t) &:= u^{(m)}(t) - u^{(m)}(t-1) \\
&= [u^{(m)}(t) + \sum_{k=2}^{p} e^{i\lambda_k t} u_k^{(m)}(t)] - [u^{(m)}(t-1) + \sum_{k=2}^{p} e^{i\lambda_k (t-1)} u_k^{(m)}(t-1)] .
\end{aligned}
$$

Since $u_k^{(m)}(t)$ is 1-periodic, we have

$$
\begin{aligned}
w^{(m)}(t) &= \sum_{k=2}^{p} e^{i\lambda_k t}(1 - e^{-i\lambda_k}) u_k^{(m)}(t)] \\
&= \sum_{k=2}^{p} e^{i\lambda_k t} w_k^{(m)}(t), \qquad (28)
\end{aligned}
$$

where $w_k^{(m)}(t) := (1 - e^{-i\lambda_k}) u_k^{(m)}(t)]$ is 1-periodic. Obviously, since $u^{(m)}(t)$ converges to 0 uniformly in t , $w^{(m)}(t)$ converges to 0 uniformly in t . By induction assumption, $w_k^{(m)}(t)$ converges to 0 uniformly in t . As $\lambda_k \in [0, 2\pi)$, this yields that $u_k^{(m)}(t)$ converges to 0 uniformly in t .□

We now state a similar assertion to Corollary 3 for functional differential equations which proof is the same as that of Corollary 3. To this end, let us consider linear homogeneous functional differential equations of the form

$$
\frac{dx}{dt} = L(t)x_t, \quad \forall t \in \mathbf{R}, x \in \mathbf{R}^n, \qquad (29)
$$

where $L(t)$ is a bounded linear operator from $C[-r, 0]$ to $\mathbf{R}^n$ for every fixed t and depends continuously and 1-periodically on the time t , $x_t(\theta) := x(t + \theta)$, $\forall \theta \in [-r, 0], t \in \mathbf{R}$ and $r > 0$ is given. For more details on this subject see e.g. [Hal2], [HL].

Corollary 4 *Let $x(\cdot)$ be a bounded solution to Eq.(29). Then $x(\cdot)$ is quasi-periodic. More precisely, there are real numbers $\lambda_1, \cdots, \lambda_k$ and periodic functions $u_1(\cdot), \cdots, u_k(\cdot)$ with period $\tau > r$ such that*

$$
x(t) = \sum_{j=1}^{k} e^{i\lambda_j t} u_j(t), \ \forall t \in \mathbf{R}.
$$

Proof Consider the solution operator $T(t,s), t \geq s; t,s \in \mathbf{R}$ (see [Hal2] for more details). As is known, for sufficiently large $t-s$ (for instance $t-s > r$) the operator $T(t,s)$ is compact. Thus we are in a position to apply Theorem 5 in a similar manner as in Corollary 3 to assert that there are $\lambda_1, \cdots, \lambda_k$ and $C[-r,0]$-valued periodic functions $v_1(\cdot), \cdots, v_k(\cdot)$ with period $\tau > r$ such that

$$x_t = \sum_{j=1}^{k} e^{i\lambda_j t} v_j(t), \quad \forall t \in \mathbf{R}.$$

Hence,

$$\begin{aligned} x(t) &= x_t(0) = \sum_{j=1}^{k} e^{i\lambda_j t} v_j(t)(0), \\ &= \sum_{j=1}^{k} e^{i\lambda_j t} u_j(t), \quad \forall t \in \mathbf{R}, \end{aligned}$$

where $u_j(t) := v_k(t)(0),\ j = 1, \cdots, k$.□

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