ON THE SIGN DEFINITENESS OF LIAPUNOV FUNCTIONALS AND STABILITY OF A LINEAR DELAY EQUATION

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Abstract. In this paper we give a new definition of the positive-definiteness of the Liapunov functional involved in the stability and asymptotic stability investigation. Using this notion we prove Liapunov type theorems and apply these results to the scalar equation $\dot{x}(t) = b(t)x(t - r(t))$, where $b(t)$ and $r(t)$ may be unbounded.

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1. Introduction

This paper deals with stability of delay equations in terms of Liapunov functionals.

We consider the system

$$\dot{x}(t) = f(t, x_t), \ f(t, 0) \equiv 0, \tag{1.1}$$

where $x \in R^n$, $t \in R$, $f : R_+ \times C([-r(t), 0], R^n) \to R^n$, $x_t(\tau) = x(t + \tau)$, $\tau \in [-r(t), 0]$, $C([-r(t), 0], R^n)$ is the space of continuous functions on the interval $[-r(t), 0]$ at any point t (see, for example, [6,17])

It is supposed that there exists a solution $x(t, t_0, \varphi)$ of (1.1) with initial data (t_0, φ), $t_0 \in R$, $\varphi \in B_t(0, H)$ and it depends continuously on initial data. ($B_t(0, H) = \{\varphi \in C([-r(t), 0], R^n), ||\varphi|| < H\}$, $||\varphi|| = \max_{\tau\in[-r(t),0]} |\varphi(\tau)|$).

Definition 1.1: The zero solution of (1.1) is called *stable* if for each $\varepsilon > 0$ and each $t_0 \geq 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $||\varphi|| < \delta$ implies $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \geq t_0$.

The zero solution is called *uniformly stable* if δ in Definition 1.1 does not depend on t_0.

Definition 1.2: The zero solution of (1.1) is called *attractive* if for each $t_0 \geq 0$ there exists $\varepsilon_1 = \varepsilon_1(t_0)$ such that $||\varphi|| < \varepsilon_1$ implies $x(t, t_0, \varphi) \to 0$ as $t \to \infty$.

The zero solution is called *asymptotically stable* if it is stable and attractive.

Let $V : R_+ \times C \to R$ $(V(t,0) \equiv 0)$ be a continuous, Lipschitzian with respect to the second argument functional. $\dot{V}(t,\varphi)$ denotes the derivative of $V(t,\varphi)$ with respect to (1.1).

Stability and asymptotic stability conditions have been studied in numerous papers [1]-[20]. It is well-known [6,17] that conditions

$$u(|\varphi(0)|) \le V(t,\varphi), \tag{1.2}$$

$$\dot{V}(t,\varphi) \le 0, \tag{1.3}$$

guarantee stability of the zero solution of (1.1), where $u : R_+ \to R_+$ is continuous and $u(\alpha) > 0$ for all $\alpha > 0,\ u(0) = 0$ (functions with these properties are called *Hahn functions*).

To study asymptotic stability the following requirements were proposed in [8]

$$u(|\varphi(0)|) \le V(t,\varphi) \le v(||\varphi||), \tag{1.4}$$

$$\dot{V}(t,\varphi) \le -p(t)w(|\varphi(0)|), \tag{1.5}$$

where v, w are Hahn functions, $p(t)$ is an integrally positive function, i.e.,

$$\sum_{i=1}^{\infty} \int_{c_i}^{d_i} p(t)dt = \infty \tag{1.6}$$

holds for any set of intervals $(c_i, d_i),\ i \in Z_+$ if $d_i - c_i > \sigma > 0,\ c_{i+1} > d_i$.

In fact, there are numerous papers and books (see,[1-4],[6],[8-10],[13-17]) dealing with improvements of requirements for V and $\dot{V}$. The purpose of this paper is to weaken condition (1.2) and to study asymptotic stability for (1.1) with unbounded right hand side allowing that $p(t)$ in (1.5) is vanishing at infinity and the second inequality in (1.4) is not fulfilled.

2. Stability

To investigate stability we need the following condition: for any $t_0 \ge 0$ there exists $T > 0$ such that $t_0 - r(t_0) \le t - r(t)$ for $t > t_0 + T$.

Theorem 2.1: Suppose that for equation (1.1) there exist a Liapunov functional $V(t,\varphi)$, a number $H > 0$ and Hahn functions u, v such that

(i) $u(|\varphi(0)|) \le V(t,\varphi)$, provided that $t \in R_+$ and φ satisfies $|\varphi(0)| = \max\limits_{\tau \in [-r(t),0]} |\varphi(\tau)|,\ ||\varphi|| < H$;

(ii) $\dot{V}(t,\varphi) \le 0$, provided that $t \in R_+$ and $V(t,\varphi) > 0,\ ||\varphi|| < H$;

Then the zero solution of (1.1) is stable. If, in addition, $r'(t) \le 1$ and

(iii) $V(t,\varphi) \le v(||\varphi||)$ for $t \in R_+,\ ||\varphi|| < H$;

then the zero solution of (1.1) is uniformly stable.

Proof: Let us suppose that the equilibrium is not stable at some initial time t_0. Without loss of generality we can assume $t_0 = 0$. Then there is $\varepsilon > 0$ such that there exists sequences $\{t_n\}_{n=1}^{\infty},\ \{\varphi_n\}_{n=1}^{\infty},\ ||\varphi_n|| \to 0$ $(n \to \infty)$ such that $|x(t_n, 0, \varphi_n)| = \varepsilon$. It is easy to see that the sequences $\{t_n\}_{n=1}^{\infty},\ \{\varphi_n\}_{n=1}^{\infty}$ can be chosen so that $|x(t_n, 0, \varphi_n)| < \varepsilon$ for all $t \in [-r(0), t_n)$. Note that continuous dependence of the solutions on the initial data guarantees that $t_n \to \infty$ if $n \to \infty$. So we can choose n_0 such that $-r(0) < t - r(t)$ for $t \geq t_{n_0}$. It is obvious that $|x(t_n, 0, \varphi_n)| \leq \max_{\theta \in [-r(t_n), 0]} |x(t_n + \theta, 0, \varphi_n)| = \varepsilon$ for $n > n_0$. Hence, condition (i) guarantees that $V(t_n, x_{t_n}(\varphi_n)) \geq u(\varepsilon) > 0$. Since $||\varphi_n|| \to 0$ if $n \to \infty$ and the functional V is continuous we can indicate a sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n > 0$ $V(0, \varphi_n) \leq \alpha_n$ and $\alpha_n \to 0$ if $n \to \infty$. On the other hand, according to (ii), the functional V does not increase along the solution $x(t, 0, \varphi_n)$ at any point t if $V(t, x_t(\varphi_n)) > 0$. If there exist a point $0 < t'_n < t_n$ such that $V(t'_n, x_{t'_n}(\varphi_n)) \leq 0$ then $V(t, x_t(\varphi_n)) \leq 0$ for all $t \in [t'_n, t_n]$ by (ii). So we can conclude that $V(t_n, x_{t_n}(\varphi_n)) \leq \alpha_n \to 0$ if $n \to \infty$. Obtained contradiction proves the stability.

Uniform stability can be proved in a similar way.

Remark 2.1: Condition (i) extends the class of functionals which are admissible for stability investigation because functionals with sign changes can be used (see also [5]).

3. Asymptotic stability

Let $q : R_+ \to R_+$ be a continuous function and let I be a set of the form

$$I = \bigcup_{i=1}^{\infty} (c_i, d_i), \quad c_i > d_i, \; d_i \to \infty, \quad \text{if} \quad i \to \infty$$

and there is $\sigma > 0$ such that $\int_{c_i}^{d_i} q(t)dt > \sigma$ for any $i \in Z_+$

Definition 3.1: A function $p : R_+ \to R_+$ is said to be I — *integrally positive with respect to* $q(t)$ if for any $\sigma' > 0$, k whenever $I' = \bigcup\limits_{i=k}^{\infty} (c', d')$ with $c_i < c'_i$, $d'_i < d_i$, $\int_{c'_i}^{d'_i} q(t)dt > \sigma'$, then $\int_{I'} p(t)dt = \infty$.

A function $p(t)$ is called I*-integrally positive* if $q(t)$ in Definition 3.1 is a constant function (see [12] for ordinary differential equations).

Definition 3.2: A delay function $r(t)$ is said to be *embedded into* I if there are points $t_i \in (c_i, d_i)$ such that $t_i - r(t_i) \in (c_i, d_i)$ for all $i \in Z_+$.

Note that the class of the functions I-integrally positive with respect to $q(t)$ is wider then that of integrally positive functions, and includes, for example, functions which decrease like $1/t$ on some appropriate sequence of intervals and have arbitrary behavior outside of these intervals.

In what follows the condition $r'(t) \leq \delta < 1$ is assumed. Let β be an increasing Hahn function and $\beta(s) > s$.

Theorem 3.1: If for equation (1.1) there are a Liapunov functional $V(t, \varphi)$, a number $H > 0$, Hahn functions u, v, w and a set I such that:

(i) $||f(t, \varphi)|| \leq F(t)$ for $t \in I$, $||\varphi|| \leq H$;

(ii) $V(t, \varphi) \leq v(||\varphi||)$ for $t \in I$, $||\varphi|| \leq H$;

(iii) $u(|\varphi(0)|) \leq V(t, \varphi)$, provided that $t \in R_+$ and φ satisfies $\max\limits_{\tau \in [-r(t), 0]} |\varphi(\tau)| \leq \beta|\varphi(0)|)$, $||\varphi|| \leq H$;

(iv) $\dot{V}(t, \varphi) \leq -p(t)w(|\varphi(0)|)$ provided that $t \in R_+$, $V(t, \varphi) > 0$, $||\varphi|| \leq H$;

(v) $p(t)$ is I—integrally positive with respect to $F(t)$;

(vi) $r(t)$ is embedded into I;

then the zero solution of (1.1) is asymptotically stable.

Proof: It remains to prove the attractivity. Without loss of generality we can consider the initial time $t_0 = 0$. Let us take $\varepsilon \in (0, H)$ and $\sigma > 0$ such that for every φ, $||\varphi|| < \sigma$ the inequality $|x(t,\varphi)| \le \varepsilon$ is satisfied. Let $S(t) = \{\varphi : V(t,\varphi) \le 0, ||\varphi|| < \varepsilon\}$. First we investigate asymptotic behavior of the solutions $x(t,\varphi)$, $||\varphi|| < \sigma$ which enter into $S(t')$ at some t' and remain (by condition (iv)) in $S(t)$ if $t > t'$. Assume that there is a solution $x(t,\varphi)$ such that $V(t', x_{t'}) \le 0$ for some t', but $x(t,\varphi)$ does not tend to zero when $t \to \infty$. Then there are a sequence $\{t_n\}$, $t_n \to \infty$ and $0 < \varepsilon_1 < \varepsilon$, for which $|x(t_n,\varphi)| > \varepsilon_1$. Take $\Delta > 0$ such that $\beta(s) - s > \Delta$ for $s > \varepsilon_1$. Since (iii) holds and $V(t_n, x_{t_n}) \le 0$ we have that for every point t_n there is t_n^1 such that $t_n - r(t_n) \le t_n^1 \le t_n$ and, besides, $|x(t_n^1,\varphi)| > |x(t_n,\varphi)| + \Delta$. Also, for every t under sufficiently large n (namely, such that $t_n - r(t_n)$ is sufficiently large) there is t_n^2 such that $t_n^1 - r(t_n^1) \le t_n^2 \le t_n^1$ and $|x(t_n^2,\varphi)| > |x(t_n,\varphi)| + 2\Delta$. Thus, we have $|x(t_n^i,\varphi)| > i\Delta > \varepsilon$ for sufficiently large n and i, which is a contradiction showing that if $|\varphi| < \sigma$ and $x(t,\varphi)$ enters into $S(t')$ at some t' then $|x(t,\varphi)| \to 0$ if $t \to \infty$.

Consider now the solutions of (1.1) which remain in the set $S_+(t) = \{\varphi : V(t,\varphi) > 0, ||\varphi|| < \varepsilon\}$. It follows from condition (iv) of the theorem that the functional $V(t, x_t)$ is not increasing for any such solution $x(t,\varphi)$ because $V(t,x_t) > 0$ if $t \ge 0$.

We prove that $V(t,x_t) \to 0$ if $t \to \infty$. Choose $\bar{t}_i \in (c_i, d_i)$ such that $\bar{t}_i - r(\bar{t}_i) \in (c_i, d_i)$. First we show that there exist intervals $(\bar{t}_{i_k} - r(\bar{t}_{i_k}), \bar{t}_{i_k})$ such that $|x(t,\varphi)| \to 0$ if $t \in \bigcup\limits_{k=1}^{\infty} (\bar{t}_{i_k} - r(\bar{t}_{i_k}), \bar{t}_{i_k})$. Suppose the contrary. Then there exists $\bar{\varepsilon} > 0$ such that for every $n \in Z_+$ there is a point $\hat{t}_n$ such that $|x(\hat{t}_n,\varphi)| > \bar{\varepsilon}$ and $\hat{t}_n \in (\bar{t}_n - r(\bar{t}_n), \bar{t}_n)$. Let (c_n', d_n') be the maximal connected interval such that $|x(t,\varphi)| > \bar{\varepsilon}/2$ for $t \in (c_n', d_n')$ and $c_n \le c_n', d_n' \le d_n$. From (i) and requirements for I it follows that $\int_{c_n'}^{d_n'} F(t)dt > \min\{\bar{\varepsilon}/2, \sigma\}$, where σ is taken from the description of I.

Since (iv) holds and $p(t)$ is I-integrally positive with respect to $F(t)$ we have

$$\sum_{n=1}^{\infty} \int_{c_n'}^{d_n'} \dot{V}(t,x_t)dt = -\infty.$$

This contradicts the assumption that $V(t,x_t) > 0$ if $t > 0$.

Thus it is proved that $||x_{\bar{t}_{i_k}}|| \to 0$ when $k \to \infty$ at least for some subsequence $\{\bar{t}_{i_k}\}$ of the sequence $\{\bar{t}_i\}$. From condition (ii) of the theorem it follows that $V(\bar{t}_{i_k}, x_{\bar{t}_{i_k}}) \to 0$ if $k \to \infty$. Hence, $V(t,x_t) \to 0$ when $t \to \infty$.

Suppose that $|x(t,\varphi)|$ does not tend to zero if $t \to \infty$. Then, taking into account that $||x_{\bar{t}_{i_k}}|| \to 0$ when $k \to \infty$, it is easy to see that there are $\varepsilon_1 > 0$ and points $t_n \to \infty$, $n \to \infty$ such that $|x(t_n,\varphi)| = \varepsilon_1$ and $|x(t,\varphi)| < \varepsilon_1$ for $t \in [t_n - r(t_n), t_n]$. By (iii), $V(t_n, x_{t_n})$ does not tend to zero.

This contradiction shows that $x(t,\varphi) \to 0$ if $t \to \infty$. So asymptotic stability is proved.

In the sequel we construct Liapunov functionals meeting conditions in Theorem 3.1.

4. The equation $\dot{x}(t) = ax(t) + bx(t-r)$.

We shall consider the scalar linear equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(t - r(t)), \quad t > 0. \tag{4.1}$$

For the sake of simplicity, at first we are dealing with the case when $a(t) \equiv a,\ b(t) \equiv b,\ r(t) \equiv r$. So (4.1) looks like

$$\dot{x}(t) = ax(t) + bx(t-r), \quad t > 0. \tag{4.2}$$

The region of asymptotic stability for (4.2) is well-known [6]. This result was obtained by the method of the characteristic equation. On the other hand, the way for obtaining a near result with a Liapunov functional was given in [6]. However, in the process of constructing one must find a solution of a similar delay equation. This does not allow to generate the scheme for more complicated equations. We show that the use of condition (i) of Theorem 2.1 overcomes this difficulty.

Usually, V has the form

$$V(t, x_t) = x^2(t) + \int_{t-r}^{t} K(u)x^2(u)du, \quad K(u) \geq 0,$$

which immediately provides (1.2) and condition (i) of Theorem 2.1 but the stability conditions are far from the best. There is also considered (see, for example, [1]-[3]) functional $V(t, x_t)$ in the form

$$V(t, x_t) = V_1(t, x_t) + V_2(t, x_t),$$

where

$$V_1(t, x_t) = \frac{1}{2}(x(t) + \int_{t-r}^{t} K(u)x(u)du)^2$$

Since

$$(\int_{t-r}^{t} K(u)x(u)du)^2 \leq \int_{t-r}^{t} |K(u)|du \int_{t-r}^{t} |K(u)|x^2(u)du$$

where the function $K(u)$ is not necessarily of a constant sign, $V_1(t, x_t)$ does not satisfy (1.2), in general. So the additional term

$$V_2(t, x_t) = \int_{-r}^{0} \int_{t+s}^{t} D(s, u, x_u)duds, \quad D(s, u, x_u) \geq 0,$$

is used to satisfy (1.2). To this end strict relations between D and K have to be required.

Now we suggest the condition

$$\sup_{t \geq 0} \int_{t-r}^{t} |K(t, u)|du < 1, \tag{4.3}$$

guaranteeing that $V_1(t, x_t) \geq u(|x(t)|)$ for any $x(t)$ such that $|x(t)| = \max_{t-r \leq u \leq t} |x(u)|$, i.e. condition (i) of Theorem 2.1, is fulfilled without any relation between K and D.

We have

$$\dot{V}_1(t, x_t) = (x(t) + \int_{t-r}^{t} K(t, u)x(u)du)$$

$$(\int_{t-r}^{t} \frac{\partial K(t, u)}{\partial t} x(u)du + (a + K(t, t))x(t) + (b - K(t, t-r))x(t-r))$$

Let $K(t,t-r)=b$. To have $\dot V(t,x_t)$ as a quadratic of $x(t)$ and $\int_{t-r}^t K(t,u)x(u)du$ we require that the function $K(t,u)$ is a solution of the equation

$$\frac{\partial K(t,u)}{\partial t}=\phi(t)K(t,u),$$

where $\phi(t)$ is still undefined function.

For the case of constant coefficients a, b, consider a constant function $\phi(t)$. Then

$$K(t,u)=be^{\alpha(u-t+r)},\quad \alpha-\text{const.}$$

Now (4.3) (i.e. condition (i) of Theorem 2.1) is guaranteed by the inequality $|b|\alpha^{-1}(e^{\alpha r}-1)<1$. This is satisfied if $\alpha<0$ and

$$|b|<\frac{|\alpha|}{1-e^{-|\alpha|r}} \tag{4.4}$$

Now we try to choose the best α. Denote

$$x_1(t)=x_1(t,x_t)=b\int_{t-r}^t e^{\alpha(u-t+r)}x(u)du,\quad a_1=a+be^{\alpha r}.$$

Then

$$\dot V_1(t,x_t)=a_1x^2(t)+(a_1-\alpha)x(t)x_1(t)-\alpha x_1^2(t).$$

To ensure the condition $\dot V(t,x_t)<0$ let us take

$$D(s,u,x_u)=pe^{\alpha s}x^2(u),$$

where $p>0$ is constant. Then

$$\dot V_2(t,x_t)=px^2(t)\int_{-r}^0 e^{\alpha s}ds-pe^{\alpha t}\int_{t-r}^t e^{\alpha u}x^2(u)du$$

By the Cauchy - Schwarz integral inequality we have

$$\int_{t-r}^t e^{\alpha u}x^2(u)du\geq\frac{(\int_{t-r}^t e^{\alpha u}x(u)du)^2}{\int_{t-r}^t e^{\alpha u}du}=\frac{e^{\alpha t-2\alpha r}}{b^2\int_{-r}^0 e^{\alpha s}ds}x_1^2(t),$$

whence

$$\dot V_2(t,x_t)\leq\frac{p(1-e^{-\alpha r})}{\alpha}x^2(t)-\frac{p\alpha e^{-2\alpha r}}{b^2(1-e^{-\alpha r})}x_1^2(t),$$

$$\dot V(t,x_t)\leq(a_1+\frac{p(1-e^{-\alpha r})}{\alpha})x^2(t)+(a_1-\alpha)x(t)x_1(t)-\alpha(1+\frac{pe^{-2\alpha r}}{b^2(1-e^{-\alpha r})})x_1^2(t)$$

Clearly, $\dot V(t,x_t)\leq 0$ if there exists p such that

$$b^2(e^{\alpha r}-e^{2\alpha r})\leq p\leq\frac{\alpha a_1}{e^{-\alpha r}-1}, \tag{4.5}$$

and

$$b^2(e^{\alpha r}-1)+\frac{\alpha a_1}{e^{\alpha r}-1}-|b|(a_1+\alpha)\leq 0, \tag{4.6}$$

By virtue of (4.5) and (4.6), such a p exists if

$$a_1 \leq 0, \quad b^2(e^{\alpha r} - e^{2\alpha r}) \leq \frac{\alpha a_1}{e^{-\alpha r} - 1}, \tag{4.7}$$

and

$$b^2(e^{\alpha r} - 1) + \frac{\alpha a_1}{e^{\alpha r} - 1} - |b|(a_1 + \alpha) \leq 0, \tag{4.8}$$

Inequality (4.8) may be reduced to

$$(b + a)(-b + \frac{\alpha}{e^{\alpha r} - 1}) \leq 0 \text{ for } b > 0$$

and

$$(b(2e^{\alpha r} - 1) + a)(b + \frac{\alpha}{e^{\alpha r} - 1}) \leq 0 \text{ for } b \leq 0.$$

Analyzing above inequalities one can see that conditions (4.4),(4.7), (4.8) hold everywhere on the domain $-a \geq |b|$ if $|\alpha|$ is taken sufficiently large. In case when $-b \geq |a|$, requirements (4.7),(4.8) are satisfied if $\alpha = \frac{1}{2}(a + b)$ and (4.4) holds for the pair (a, b).

Remark 4.1: It should be noted that the functional $V(t, x_t)$ satisfies (1.2). In fact, we have in the above example

$$V(t, x_t) = (x(t) + b\int_{t-r}^{t} e^{\alpha(u-t+r)}x(u)du)^2 + \int_{-r}^{0}\int_{t+s}^{t} ke^{\alpha s}x^2(u)duds$$

We will show that $V(t, x_t) \to 0$ implies $x(t) \to 0$. If

$$\int_{-r}^{0}\int_{t+s}^{t} D(s, u, x_u)dsdu = \int_{-r}^{0}\int_{t+s}^{t} ke^{\alpha s}x^2(u)duds \to 0 \tag{4.9}$$

then

$$x_1(t) = b\int_{t+s}^{t} e^{\alpha(u-t+r)}x(u)du \to 0 \tag{4.10}$$

In fact, since $k, e^{\alpha(s)}, x^n(u)$ are bounded, (4.9) implies

$$\int_{t-r}^{t} ke^{-\alpha r}(x^n(u))^2du \to 0 \tag{4.11}$$

Using (4.11) and the Cauchy-Schwarz inequality it is easy to see that (4.10) is valid.

Note that the main step here is that (4.9) implies (4.11). If coefficients of the delay equation are not constant (unbounded, in general) we have to choose k as an unbounded, in general, function. In such a case (4.9) does not imply (4.11) and (1.2) is not true. This situation is realized, for example, if $a \equiv 0$, b depends on t and $\limsup_{t\to\infty} b(t) = \infty$.

5. The equation $\dot{x}(t) = b(t)x(t - r)$

Let us consider the equation

$$\dot{x}(t) = b(t)x(t - r) \tag{5.1}$$

and modify the form of V_1 and V_2. We set $V_1(t,x_t)=\frac{1}{2}(x(t)+x_1(t))^2$, where

$$x_1(t)=e^{-\alpha(t-r)}\int_{t-r}^{t} b(u+r)e^{\alpha(u)}x(u)du,$$

and the function $\alpha(t)$ is defined by

$$\alpha(t)=c\int_0^{t+r} b(u+r)du, \quad b(t)\le 0, \quad c>0 \tag{5.2}$$

Denote $B(t)=\int_t^{t+r} b(u+r)du$. We have

$$\dot V_1(t,x_t)=b(t+r)(e^{cB(t)}x^2(t)+(e^{cB(t)}-c)x(t)x_1(t)-cx^2(t)).$$

Let

$$V_2(t,x_t)=p\int_{-r}^{0}\int_{t+s}^{t} b(|s-u|+r)e^{-\alpha(|s-u|-r)}b(u+r)e^{2\alpha(u)-\alpha(u-r)}x^2(u)duds$$

Then

$$\begin{aligned}\dot V_2(t,x_t)&=p(x^2(t)b(t+r)e^{\alpha(t)+cB(t)}\int_{-r}^{0} b(|s-t|+r)e^{-\alpha(|s-t|-r)}-\\ &b(t+r)e^{-\alpha(t-r)}\int_{t-r}^{t} b(u+r)e^{\alpha(u)+cB(u)}x^2(u)du)\le\\ &pb(t+r)(x^2(t)e^{\alpha(t)+cB(t)}\int_t^{t+r} b(u+r)e^{-\alpha(u-r)}du+\\ &\frac{e^{\alpha(t-r)}}{\int_{t-r}^{t}|b(u+r)|e^{\alpha(u-r)}du}(\int_{t-r}^{t} b(u+r)e^{\alpha(u)}x(u)du)^2)=\\ &pb(t+r)(\frac{1}{c}e^{cB(t)}(e^{cB(t)}-1)x^2(t)+\frac{c}{e^{-cB(t-r)}-1}x_1^2(t)).\end{aligned}$$

Suppose that function $B(t)$ satisfies the inequality

$$b_s=\sup_{t\ge -r}|B(t)|=\sup_{t\ge -r}\int_t^{t+r}|b(u+r)|du<\infty \tag{5.3}$$

Then

$$\begin{aligned}\dot V(t,x_t)&\le b(t+r)((e^{-cb_s}+\frac{p}{c}e^{-cb_s}(e^{-cb_s}-1))x^2(t)+\\ &(e^{-cb_s}-c)x(t)x_1(t)+c(\frac{pe^{-cb_s}}{1-e^{-cb_s}}-1)x_1^2(t)).\end{aligned} \tag{5.4}$$

Evidently, $\dot V(t,x_t)\le 0$ if there exists p such that

$$e^{cb_s}-1\le p\le \frac{c}{1-e^{-cb_s}} \tag{5.5}$$

and

$$4e^{-2cb_s}p^2-4pe^{-cb_s}(\frac{c}{e^{cb_s}-1}+1-e^{-cb_s})+(c+e^{-cb_s})^2\le 0 \tag{5.6}$$

Inequalities (5.5), (5.6) hold with some p if the estimate $b_s\le \ln 4$ is true and $c=\frac{1}{2}$. Everywhere below (see (5.2)) we set $c=\frac{1}{2}$.

Theorem 5.1: Suppose that $b(t) \le 0$,

$$\sup_{t\ge 0}\int_{t-r}^{t} |b(u+r)|du \le \ln 4.$$

and

$$\sup_{t\ge 0}\int_{t-r}^{t} |b(u+r)|e^{\alpha(u)-\alpha(t-r)}du < 1, \tag{5.7}$$

Then the zero solution of (5.1) is uniformly stable.

Remark 5.1: Condition (5.7) is sufficient for condition (i) of Theorem 2.1 (see (4.3)).

Let us find sufficient conditions for the asymptotic stability using Theorem 3.1. Show, at first, that the functional V satisfies condition (ii) of Theorem 3.1. In fact,

$$V_1(t,x_t) \le \frac{1}{2}||x_t||^2(1+\int_{t-r}^{t} |b(u+r)|e^{\alpha(u)-\alpha(t-r)}du)^2 \le$$

$$\frac{1}{2}||x_t||^2(1+\int_{t-r}^{t} |b(u+r)|du)^2 \le \frac{(1+b_s)^2}{2}||x_t||^2,$$

$$V_2(t,x_t) \le pb_s^2 e^{\frac{b_s}{2}}||x_t||^2.$$

Furthermore, it is easy to see that (5.7) implies condition (iii) with $\beta(s) = s(1+\delta_1)/(2\delta_1)$, where δ_1 is chosen from the inequality

$$\sup_{t\ge 0}\int_{t-r}^{t} |b(u+r)|e^{\alpha(u)-\alpha(t-r)}du < \delta_1 < 1$$

Besides, from (5.3)-(5.6) we can conclude that $b_s < \ln 4$ implies the existence of a constant $c_1 > 0$ such that

$$\dot V(t,x_t) \le c_1 b(t+r)x^2(t).$$

Thus we have proved the following

Theorem 5.2: Suppose that all requirements of Theorem 5.1 are satisfied and there is a set I such that r is embedded into I and $|b(t+r)|$ is I-integrally positive with respect to $|b(t)|$. Then the zero solution of (5.1) is asymptotically stable.

Note here that the estimate $b_s \le \ln 4$ is not the best one [19] and requirement (5.7) is not needed [2] if $b(t)$ is bounded.

6. The equation $\dot x(t) = b(t)x(t - r(t))$.

The results can be extended to the case $r = r(t)$. Consider the equation

$$\dot x(t) = b(t)x(t-r(t)), \tag{6.1}$$

where $r'(t) \le \delta < 1$. Take again $V_1(t,x_t) = (x(t)+x_1(t))^2$, where

$$x_1(t) = e^{-\alpha(t-r(t))}\int_{t-r(t)}^{t} \beta(u)e^{\alpha(u)}x(u)du,$$

$$\beta(t-r(t))=\frac{b(t)}{1-r'(t)},\ \alpha(t)=\frac{1}{2}\int_0^t \xi(s)ds,\ \xi(t-r(t))=\frac{\beta(t)}{1-r'(t)}.$$

By the assumption on $r'(t)$, the functions β, ξ are well defined and negative, provided $b(t)<0$.

We have

$$\dot{V}_1(t,x_t)=\beta(t)(e^{\frac{1}{2}\int_{t-r(t)}^t \xi(\tau)d\tau}x^2(t)+(e^{\frac{1}{2}\int_{t-r(t)}^t \xi(\tau)d\tau}-\frac{1}{2})x(t)x_1(t)-\frac{1}{2}x_1^2(t)),$$

$$\int_{-r(0)}^{t-r(t)}\xi(\tau)d\tau=\int_0^t \beta(\tau)d\tau \tag{6.2}$$

Let

$$V_2(t,x_t)=p\int_{t-r(t)}^t F_1(s)\int_s^t F_2(u)x^2(u)duds,$$

where $F_1(s)=\xi(s)e^{-\alpha(s)}, F_2(u)=\beta(u)e^{2\alpha(u)-\alpha(u-r(u))}$. Using (6.2) and previous integral estimates we have

$$\dot{V}_2(t,x_t)\le p\beta(t)(2x^2(t)e^{\frac{1}{2}\int_{t-r(t)}^t \xi(\tau)d\tau}(e^{\frac{1}{2}\int_{t-r(t)}^t \xi(\tau)d\tau}-1)+\frac{x_1^2(t)}{2(e^{-\frac{1}{2}\int_{t-r(t)}^t \beta(\tau)d\tau}-1)}).$$

Since

$$\sup_{t\ge 0}\int_{t-r(t)}^t |\beta(s)|ds=\sup_{t\ge 0}\int_{t-r(t)}^t |\xi(s-r(s))|(1-r'(s))ds=$$

$$\sup_{t\ge 0}\int_{t-r(t)-r(t-r(t))}^{t-r(t)}|\xi(s)|ds\ge\sup_{t\ge 0}\int_{t-r(t)}^t |\xi(s)|ds$$

and

$$\sup_{t\ge t_1}\int_{t-r(t)}^t |b(s)|ds=\sup_{t\ge t_1}\int_{t-r(t)-r(t-r(t))}^{t-r(t)}|\beta(s)|ds\ge\sup_{t\ge 0}\int_{t-r(t)}^t |\beta(s)|ds,$$

we have

$$b_s=\sup_{t\ge t_1}\int_{t-r(t)}^t |b(s)|ds$$

where the point t_1 is defined from the equation $t_1-r(t_1)=0$.

Now we want to apply Theorem 3.1. According to condition (v), function $|\beta(t)|$ is to be I-integrally positive with respect to $|b(t)|$ for some I. But β in not known explicitly, so consider the set

$$I_r=\bigcup_{i=1}^{\infty}(c_i-r(c_i),d_i-r(d_i)).$$

Since $\int_{c_i}^{d_i}|b(s)|ds=\int_{c_i-r(c_i)}^{d_i-r(d_i)}|\beta(s)|ds$, one can state that, for given $I=\bigcup_{i=1}^{\infty}(c_i,d_i)$, the function $|\beta(t)|$ is I_r-integrally positive with respect to $|b(t)|$ if and only if $|b(t)|$ is I-integrally positive with respect to $|b(t-r(t))|(1-r'(t))$.

Thus, if we demand that $r(t)$ is embedded into I_r and $|b(t)|$ is I-integrally positive with respect to $|b(t-r(t))|(1-r'(t))$ then conditions (v) and (vi) of Theorem 3.1 will be satisfied.

A possible way to satisfy condition (i) of Theorem 2.1 is to require

$$\sup_{t\geq t_2}\int_{t-r(t)}^{t}|b(u-r(u))|(1-r'(u))e^{\alpha_1(u,t-r(t))}du<1, \tag{6.3}$$

where $\alpha_1(u,t-r(t))=\frac{1}{2}\int_{t-r(t)}^{u}b(s)ds$ and the point t_2 satisfies $t_2-r(t_2)-r(t_2-r(t_2))=0$.

If we follow the scheme of section 5 we can obtain the next conclusion:

Theorem 6.1: If $b(t)\leq 0,\ b_s\leq \ln 4$ and (6.3) holds, then $x\equiv 0$ is uniformly stable. If, in addition, $b_s\neq \ln 4$, and there is I such that $r(t)$ is embedded into I_r and $|b(t)|$ is I-integrally positive with respect to $|b(t-r(t))|(1-r'(t))$; then $x\equiv 0$ is asymptotically stable.

Remark 6.1: The best estimate for b_s by using Liapunov-Krasovskii functionals is $b_s<1$ (see [1]-[3],[16]). On the other hand, in [7] it was proved by Liapunov-Razumikhin method that for bounded $r(t)$ the zero solution of (6.1) is stable if $b_s\leq 3/2$. Here, we guarantee stability in a more general situation but only under $b_s\leq \ln 4$.

Let us give an example which illustrates the conditions of Theorem 6.1.

Example 6.1: Consider equation (6.1) with $r(t)=\lambda_1 t,\ 0\leq\lambda_1<1,\ b(t)=-(\frac{1}{t+1}+c(t))$, where

$$c(t)=\begin{cases}\gamma t\sin t^2, & t\in T,\\ 0, & t\notin T,\end{cases}$$

$\gamma\geq 0,\ T=\bigcup_{n\geq n_1}(\sqrt{2\pi k},\sqrt{\pi(2k+1)}),\ k=[\frac{\lambda^{1-4n}}{2\pi}],\ \lambda=1-\lambda_1,\ n_1$ is such that

$$\frac{\lambda^{2-4n}}{2\pi}<[\frac{\lambda^{1-4n}}{2\pi}]<\frac{\lambda^{-4n}}{2\pi}-\frac{1}{2},\quad n\geq n_1$$

Clearly, $c(t)$ is a continuous, unbounded, nonnegative function with one "peak" on every interval $(\lambda^{1-2n},\lambda^{-2n}),\ n\geq n_1$ and the area of each "peak" is equal to γ.

Let us check conditions of theorem 6.1. We have $t_1=t_2=0$ and

$$b_s\leq\gamma+\sup_{t\geq 0}\int_{\lambda t}^{t}\frac{1}{u+1}du=\gamma+\sup_{t\geq 0}\ln\frac{t+1}{\lambda t+1}=\gamma-\ln\lambda.$$

Hence $b_s\leq\ln 4$ if $\gamma-\ln\lambda\leq\ln 4$. Condition (6.3) is also fulfilled if

$$\sup_{t\geq t_2}\int_{t-r(t)}^{t}|b(u-r(u))|(1-r'(u))e^{\int_{t-r(t)}^{u}b(s)ds}du\leq\gamma+$$

$$\sup_{t\geq 0}\int_{\lambda t}^{t}\frac{1}{u+1/\lambda}e^{-\int_{\lambda t}^{u}\frac{1}{s+1}ds}du<\gamma+1-\lambda<1.$$

Finally, $|b(t)|$ is I-integrally positive with respect to $|b(t-r(t))|(1-r'(t))$ for $I=\bigcup_{n\geq n_1}(\lambda^{1-2n},\lambda^{-2n})$.

Thus the zero solution is asymptotically stable if $1>\lambda>1/4$ and $\gamma<\min\{\lambda,\ln(4\lambda)\}$.

Remark 6.2: Note that Example 6.1 was considered in [7],[18] under $\gamma=0$, where the asymptotic stability was proved for $1>\lambda>1/e$.

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