On the asymptotic behavior of the pantograph equations

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Dedicated to Professor J. Kato on his 60th birthday

1. Introduction

Our aim is studing the asymptotic behaviour of the solutions of the equation

$$\dot{x}(t) = -a(t)x(t) + a(t)x(pt)\,, \tag{1.1}$$

where $a(t)$ is a nonnegative continuous scalar function on $\mathbf{R}_+ := [0\,,\infty)$ and $0 < p < 1$ is a constant. This equation is a special case of the so called pantograph equations arising in industrial applications [5,11]. The only solution of equation (1.1) with initial data $x(0) = x_0$ is $x(t) \equiv x_0$. However, if $t_0 > 0$ and $\varphi(t)$ is a given continuous function on $[pt_0\,, t_0]$ then the solution $x(t)$ with $x(s) = \varphi(s)$ for $s \in [pt_0\,, t_0]$ is defined for $t \to \infty$ and it differs from any constant solution if φ is not constant.

Equation (1.1) can be transformed to the equation

$$\dot{y}(t) = -a_1(t)y(t) + a_1(t)y(t-h)\,, \tag{1.2}$$

by $y(t) = x(e^t)$, where $p = e^{-h}$ and $a_1(t) = a(e^t)e^t$ or to the equation

$$\dot{z}(t) = -a_2(t)z(t) + a_2(t)z(p(t))\,, \tag{1.3}$$

with a given retardation $p(t)$ choosing the transformation $z(t) = x(g(t))$, where $g(t)$ satisfies the equation $pg(t) = g(p(t))$ and $a_2(t) = a(g(t))\dot{g}(t)$. Therefore some results

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can be concluded for the equation (1.1) from results for (1.2), (1.3) or their generalizations.

T. Krisztin [9] investigated the equation

$$\dot{x}(t) = f(t, x_t)$$

with infinite delay. The application of his result for (1.1) gives that if $\int_{pt}^{t} a(s)\,ds$ is bounded on $\mathbf{R}_+$ then all solution of (1.1) tends to a constant as $t \to \infty$.

N. G. De Bruijn [3, 4] studied linear scalar equation

$$w(t)\dot{x}(t) = -c(t)x(t) + d(t)x(t-1) + r(t)\,.$$

From his results it can be proved that if

$$\sum_{n=1}^{\infty} \exp\{-\int_{p^{-(n-1)}}^{p^{-n}} a(u)\,du\} = \infty$$

then every solution of (1.1) has a finite limit if $t \to \infty$. On the other hand if $a(t)$ is twice continuously differentiable and there exists a continuous nonincreasing positive function Φ such that $\int_1^\infty \Phi(s)\,ds < \infty$ and for $w(t) := 1/a(t)$ the conditions $w(t),\ |w'(t)|,\ |w''(t)| < e^t\Phi(t)$ hold, then there exists a continuous periodic function ψ of period 1 and a positive constant c such that

$$|x(t) - \psi\left\{\frac{\log t}{\log 1/p} - \int_1^{t^{1/\log(1/p)}} 1/a(s)\,ds\right\}| < c\int_{\log t/\log(1/p)-1}^{\infty} \Phi(s)\,ds\,.$$

The scalar equation

$$\dot{x}(t) = -ax(t) + bx(pt)\,,$$

(where a and b are constants, $a > 0$) is also studied. The exact asymptotic behaviour of the solutions as $t \to \infty$ is known [1, 2, 8]. In the special case $a = b$ the following assertion is proved. For any solution $x(t)$ there exists an infinitely many times differentiable, periodic function ψ of period 1 such that

$$|x(t) - \psi(\log t)| \to 0 \qquad \text{as} \qquad t \to \infty. \tag{1.5}$$

We give an extension of the last results for (1.1). We need some light monotonicity like conditions for $a(t)$ that restrict too fast changes of $a(t)$. Our condition works for the function $a(t) = t^\alpha$ if $\alpha > -1$, or for the function $a(t) = a + \sin bt$, if the constants $a,\ b$ satisfies $|b| < (a-b)^2$. We show by an example that ψ may be non-constant function. In the proof of the results we need to know the decay rate of solutions. This argument works for more general equation. Therefore in the second part of the article we study the equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(p(t))$$

and give conditions such that

$$|x(t)| \leq \frac{C}{t^k} \qquad t \in [t_0, \infty),$$

(or a similar) estimate is true. Estimation of such type was given by J. Kato [6,7] and his results were sharpened by T. Krisztin [10]. However these results and our ones in this paper cannot be compared and methods are different, too.

2. An asymptotic estimate of the solutions

Let us consider the equation

$$\dot{x}(t) = -a(t)x(t) + b(t)x(p(t)), \tag{2.1}$$

where $a(t),\ b(t),\ p(t)$ are continuous functions on $\mathbf{R}_+$, $p(t) \leq t$ and $\lim_{t\to\infty} p(t) = \infty$. Let us define the function

$$m(t) := \inf\{s :\ p(s) > t\}$$

on $\mathbf{R}_+$. Then $t \leq m(t)$, $p(m(t)) = t$ and $m(t)$ is increasing. Let be given $t_0 \geq 0$ such that $p(t_0) < t_0$ and introduce the qualities

$$q_{-1} = \inf\{p(s) : s \geq t_0\},\ q_0 = t_0,\ q_n = m(q_{n-1}), \qquad n = 1, 2, \dots\ , \qquad q_\infty = \lim_{n\to\infty} q_n$$

and the intervals

$$I_n := [q_{n-1}, q_n], \qquad n = 0, 1, 2, \dots\ .$$

Then $\cup_{n=1}^{\infty} I_n = [t_0, q_\infty)$, and $p(q_\infty) = q_\infty$, if $q_\infty < \infty$. Moreover we also have $p(I_{n+1}) \subset \cup_{k=0}^{n} I_k$. For a given function $\rho : \mathbf{R}_+ \to (0, \infty)$ having bounded differential on finite intervals let us introduce the numbers

$$\rho_n := \max_{t \in I_n} \rho(t) \int_{q_{n-1}}^{t} \exp\{\int_t^s a(z)\, dz\} \rho^{-2}(s) \dot\rho(s)\, ds \qquad n = 1, 2, \ldots .$$

Since

$$\int \frac{a(s) \exp\{\int_t^s a(z)\, dz\}}{\rho(s)}\, ds = \frac{\exp\{\int_t^s a(z)\, dz\}}{\rho(s)} + \int \frac{\dot\rho(s) \exp\{\int_t^s a(z)\, dz\}}{\rho^2(s)}\, ds \tag{2.2}$$

and $a(t)$ is nonnegative, we get for $t \in I_n$ that

$$\begin{aligned} 0 \le & \rho(t) \int_{q_{n-1}}^{t} \frac{a(s) \exp\{\int_t^s a(z)\, dz\}}{\rho(s)}\, ds + \rho(t) \frac{\exp\{\int_t^{q_{n-1}} a(z)\, dz\}}{\rho(q_{n-1})} = \\ & = 1 + \rho(t) \int_{q_{n-1}}^{t} \frac{\dot\rho(s) \exp\{\int_t^s a(z)\, dz\}}{\rho^2(s)}\, ds\,. \end{aligned} \tag{2.3}$$

Hence $0 \le 1 + \rho_n$ for all $n = 1, 2, \ldots$.

Theorem 1. *Suppose that there exists a differentiable function $\rho : [q_{-1}, q_\infty) \to (0, \infty)$ such that $\dot\rho(t)$ is locally bounded,*

$$|b(t)|\rho(t) \le a(t)\rho(p(t)) \qquad t \in [t_0, q_\infty) \tag{2.4}$$

and

$$P := \prod_{n=1}^{\infty} (1 + \rho_n) < \infty.$$

If $M_0 := \max_{t \in I_0} \rho(t)|x(t)|$, then

$$|x(t)| \le \frac{M_0 P}{\rho(t)}, \qquad t \in [t_0, q_\infty).$$

Proof. Introduce the function

$$y(t) := x(t)\rho(t).$$

Then $y(t)$ satisfies the equation

$$\dot y(t) = -\left(a(t) - \frac{\dot\rho(t)}{\rho(t)}\right) y(t) + \frac{b(t)\rho(t)}{\rho(p(t))} y(p(t))$$

which is equivalent to

$$\frac{d}{dt}\left(\frac{y(t)\exp\{\int_{t_0}^{t} a(s)ds\}}{\rho(t)}\right) = \frac{y(p(t))b(t)\exp\{\int_{t_0}^{t} a(s)ds\}}{\rho(p(t))} \ .$$

Integrating this equality on $[q_n,\, t]$ and using (2.4) we get

$$\begin{aligned}\frac{y(t)\exp\{\int_{t_0}^{t} a(s)ds\}}{\rho(t)} \le \frac{y(q_n)\exp\{\int_{t_0}^{q_n} a(s)ds\}}{\rho(q_n)} + \\ \int_{q_n}^{t} \frac{a(s)\exp\{\int_{t_0}^{s} a(z)dz\}}{\rho(s)} |y(p(s))| ds \ .\end{aligned}$$

Let $m_n := \max_{t\in I_n} |y(t)| \ \ n = 0, 1, 2, 3, \ldots$ and $M_n := \max\{m_0,\, m_1,\, \ldots,\, m_n\}$. Since $|y(p(t))| \le M_n$ for $t \in I_{n+1}$, we have

$$|y(t)| \le M_n \frac{\rho(t)\exp\{\int_{t}^{q_n} a(s)\, ds\}}{\rho(q_n)} + M_n\rho(t)\int_{q_n}^{t} \frac{a(s)\exp\{\int_{t}^{s} a(z)\, dz\}}{\rho(s)}\, ds$$

for all $t \in I_{n+1}$, $n = 0, 1, 2, \ldots$. Using the formula (2.3) we get

$$|y(t)| \le M_n\left(1 + \rho(t)\int_{q_n}^{t} \frac{\dot\rho(s)\exp\{\int_{t}^{s} a(z)\, dz\}}{\rho^2(s)}\, ds\right) \le M_n(1 + \rho_{n+1})$$

for all $t \in I_{n+1}$. Hence $M_{n+1} \le M_n(1 + \rho_{n+1})$ for all $n = 0, 1, 2, \ldots$ that implies

$$M_n \le M_0 \prod_{k=1}^{n+1}(1 + \rho_k) \le M_0 \prod_{k=1}^{\infty}(1 + \rho_k)\, .$$

This inequality is equivalent to the assertion of the theorem.

Corollary 1. *Suppose there exist* $0 < Q \le 1$, $0 < p_1 \le p_2 < 1$, $m > 0$, $0 \le \alpha < 1$ *and* $t_0 > 0$ *such that*

$$(1-\alpha)\log\frac{1}{p_2}\log\frac{1}{p_1} > \log\frac{1}{Q}\left(\log\frac{1}{p_1} - \log\frac{1}{p_2}\right)$$

$$p_1 t \le p(t) \le p_2 t\, , |b(t)| \le a(t)Q\, , mt^{-\alpha} \le a(t) \qquad t \in [p_1 t_0, \infty)\, .$$

If $k = \log Q/\log p_1$, $x : [p_1 t_0, \infty) \to \mathbf{R}$ *is a solution of (2.1) on* $[t_0, \infty)$ *then*

$$|x(t)| \le \frac{MC}{t^k} \qquad t \in [t_0, \infty)\, , \tag{2.5}$$

where $C = \prod_{n=0}^{\infty}(1 + \frac{k}{mt_0^{1-\alpha}p_1^k}\left(\frac{p_2^{k+1-\alpha}}{p_1^k}\right)^n)$ *and* $M = \max_{t\in[p_1t_0,t_0)} t^k|x(t)|$.

Proof. First of all we remark that the product in the definition of C exists since $p_2^{k+1-\alpha}/p_1^k < 1$ by the definition of k. The relations $p(m(t)) = t$ and $p_1t \le p(t) \le p_2t$ imply that $\frac{t}{p_2} \le m(t) \le \frac{t}{p_1}$. Hence $\frac{t_0}{p_2^n} \le q_n \le \frac{t_0}{p_1^n}$. Also

$$|b(t)| \le a(t)Q = a(t)p_1^k \le a(t)\left(\frac{p(t)}{t}\right)^k$$

and hence (2.4) is valid with $\rho(t) = t^k$. Moreover

$$\begin{aligned}
\rho_{n+1} &\le \max_{t\in I_{n+1}} kt^k \int_{q_n}^t \frac{e^{\int_t^s mz^{-\alpha}\,dz}}{s^{k+1}}\,ds \le \\
&\le \max_{t\in I_{n+1}} kt^k e^{-m(1-\alpha)^{-1}t^{1-\alpha}} \int_{t_0p_2^{-n}}^t \frac{e^{m(1-\alpha)^{-1}s^{1-\alpha}}}{s^{k+1}}\,ds = \\
&= \max_{t\in I_{n+1}} kt^k e^{-m(1-\alpha)^{-1}t^{1-\alpha}} \int_{(t_0p_2^{-n})^{1-\alpha}}^{t^{1-\alpha}} \frac{e^{m(1-\alpha)^{-1}u}}{(1-\alpha)u^{(k+1-\alpha)(1-\alpha)^{-1}}}\,du \le \\
&\le \frac{k}{1-\alpha}\left(\frac{t_0}{p_1^{n+1}}\right)^k\left(\frac{t_0}{p_2^n}\right)^{-(k+1-\alpha)} \max_{t\in I_{n+1}} \int_{(t_0p_2^{-n})^{1-\alpha}}^{t^{1-\alpha}} e^{m(1-\alpha)^{-1}(u-t^{1-\alpha})}\,du \le \\
&\le \frac{k}{t_0^{1-\alpha}mp_1^k}\left(\frac{p_2^{k+1-\alpha}}{p_1^k}\right)^n
\end{aligned}$$

So, Theorem 1 implies the assertion.

Corollary 2. *Suppose that* $p(t) = \sqrt[k]{t}$, $a \le a(t)$ *and* $|b(t)| \le \theta a(t)$ *hold on the interval* $[t_0,\infty)$, *where* $t_0 > 1$, $k > 1$, $0 < a$, $0 < \theta < 1$ *are constants. Then there exists a positive constant* C *such that for any solution of (2.1) on* $\left(\sqrt[k]{t_0},\infty\right)$ *we have*

$$|x(t)| \le C\,M(\log t)^{\dfrac{\log\theta}{\log k}} \qquad (t\in[t_0,\infty))\,,$$

where $M = \max\{|x(t)| : t\in[\sqrt[k]{t_0},\ t_0]\}$.

Proof. Apply again Theorem 1 by $\rho(t) = \log^\alpha t$, where $\alpha = -\log\theta/\log k$. Then $q_{-1} = \sqrt[k]{t_0}$, $q_n = t_0^{k^n}$ for $n = 0,1,2,\dots$ and

$$\rho(t)\int_{q_{n-1}}^t \exp\int_t^s a(z)dz\,\dot\rho(s)\rho^{-2}(s)ds \le k^{\alpha n}\log^\alpha t_0 \int_{t_0^{k^{n-1}}}^t e^{-a(t-s)}\frac{\alpha}{s\log^{\alpha+1}s}\,ds \le$$

$$\leq \frac{\alpha k^{\alpha}}{t_0^{k^{n-1}} k^{n-1} \log t_0} \frac{1 - e^{-a(t - t_0^{k^{n-1}})}}{a}$$

Hence

$$\rho_n \leq \frac{\alpha k^{\alpha}}{t_0^{k^{n-1}}\ k^{n-1}\, a\ \log t_0}$$

that is $\prod_{n=1}^{\infty}(1+\rho_n) < \infty$ and the assertion follows with

$$C = \prod_{n=1}^{\infty}(1 + \frac{\alpha k^{\alpha}}{t_0^{k^{n-1}}\ k^{n-1}\, a\ \log t_0}).$$

Remark. If $a \leq a(t)$ and $|b(t)| \leq \theta a(t)$, where $0 < a$ and $0 < \theta < 1$ then Corollary 1 and Corollary 2 imply that

$$\text{if} \quad p(t) = pt, \quad 0 < p < 1 \quad \text{then} \quad |x(t)| \leq MC_1 t^{-\dfrac{\log\theta}{\log p}}, \tag{2.6}$$

$$\text{if} \quad p(t) = \sqrt[k]{t}, \quad 1 < k \qquad \text{then} \quad |x(t)| \leq MC_2 (\log t)^{\dfrac{\log\theta}{\log k}} \tag{2.7}$$

for $t \geq t_0$. T. Krisztin [10] applied his results for the cases $pt \leq p(t)$ and $\sqrt[k]{t} \leq p(t)$ and he gave the conditions

$$\text{if} \quad p(t) \geq pt, \quad 0 < p < 1 \quad \text{then} \quad |x(t)| \leq MC_3 t^{p(p-1)\log\mu}, \tag{2.8}$$

$$\text{if} \quad p(t) \geq \sqrt[k]{t}, \quad 1 < k \qquad \text{then} \quad |x(t)| \leq MC_4 (\log t)^{-\dfrac{\log\mu}{\log p}} \tag{2.9}$$

for $t \geq t_0$, where $\mu \in \left(1, \frac{1}{\theta}\right)$. It is easy to see, that (2.6) and (2.7) are sharper than (2.8) and (2.9). On the other hand we required that $p(t)$ is far from t, and the assumptions that $p(t) = pt$ and $p(t) = \sqrt[k]{t}$ cannot be changed to $p(t) \geq pt$ and $p(t) \geq \sqrt[k]{t}$. Therefore, our results and the ones in Krisztin's paper are independent.

Corollary 3. *Suppose there exist* $0 < p < 1$, $Q > 1$ *such that* $p(t) = pt$, $|b(t)| \leq a(t)Q$ *on* $[pt_0, \infty)$. *If* $k = -\log Q/\log p$ *and* $M = \max_{t\in[pt_0,t_0)} t^{-k}|x(t)|$, *then* $\frac{|x(t)|}{t^k} \leq M$ *on* $[t_0, \infty)$.

Proof. Now, we have $q_n = t_0/p^n$ and (2.4) is valid with $\rho(t) = t^{-k}$. Then $\dot{\rho}(t) \leq 0$, so $1 + \rho_n \leq 1$ and Theorem 1 can be applied.

3. The asymptotic behavior

Consider the equation

$$\dot{x}(t) = -c(t)x(t) + c(t)x(pt), \tag{3.1}$$

where $0 < p < 1$.

Let $c(t)$ be nonnegative, continuously differentiable on $\mathbf{R}_+$. Then the solutions are twice differentiable and $y(t) = \dot{x}(t)$ satisfies the equation

$$\dot{y}(t) = -\left(c(t) - \frac{\dot{c}(t)}{c(t)}\right) y(t) + pc(t)y(pt)\,.$$

Now, we apply the above results to this equation.

Theorem 3. *Suppose that $c(t)$ is continuously differentiable on $\mathbf{R}_+$ and there exist $t_0 > 0$, $0 < k \leq 1$, $m > 0$ and $0 \leq \alpha < 1$ such that*

$$mt^{-\alpha} \leq c(t) \qquad \dot{c}(t) \leq c^2(t)(1 - p^{1-k}) \qquad (t \in [t_0, \infty))\,.$$

Let $x(t)$ be a solution of (3.1) on $[pt_0, \infty)$ then

$$|\dot{x}(t)| \leq \frac{CM}{t^k} \qquad\qquad (t \in [\frac{t_0}{p}, \infty))\,,$$

where $C = \prod_{n=0}^{\infty} \left(1 + \frac{kp^{(n+1)(1-\alpha)-k}}{mt_0^{1-\alpha}}\right)$ and $M = \sup_{t \in [t_0, t_0/p)} t^k |\dot{x}(t)|$.

Proof. Since $a(t) = c(t) - \frac{\dot{c}(t)}{c(t)}$, $b(t) = c(t)p$, $Q = p^k$ the condition $b(t) \leq a(t)Q$ is equivalent to $\dot{c}(t) \leq c^2(t)(1 - p^{1-k})$. Corollary 1 is applicable with $p_1 = p_2 = p$ and replacing t_0 by $\frac{t_0}{p}$.

Now let us transform equation (3.1) in a different way. Let $y(t) = \dot{x}(t)/c(t)$, then

$$\dot{y}(t) = -c(t)y(t) + pc(pt)y(pt).$$

Theorem 4. *Suppose that there exist $t_0 > 0$, $0 < k \leq 1$, $m > 0$ and $0 \leq \alpha < 1$ such that*

$$mt^{-\alpha} \leq c(t) \qquad p^{1-k}c(pt) \leq c(t) \qquad (t \in [t_0, \infty))$$

Let $x(t)$ be a solution of (3.1) on $[pt_0, \infty)$ then

$$|\dot{x}(t)| \leq \frac{CMc(t)}{t^k} \qquad (t \in [\frac{t_0}{p}, \infty))\,,$$

where C is the same as in Theorem 3 and $M = \sup_{t\in[t_0,t_0/p)} t^k \frac{|\dot{x}(t)|}{c(t)}$.

Proof. Use Corollaries 1 and 3 as in Theorem 3.

Note that it is only a technical detail that we estimate the derivative on the interval $[t_0/p, \infty)$ in Theorems 3 and 4. If we choose an initial function so that the solution is continuously differentiable at the point t_0, then we can prove a similar estimate using M as the supremum of the appropriate function on the interval $[pt_0, t_0)$ and a little bit different C's. We will use this comment later.

Note also, that it is easy to see that if x is a solution of (3.1), $M_0 = \sup_{t\in[pt_0,t_0)} x(t)$ and $m_0 = \inf_{t\in[pt_0,t_0)} x(t)$, then

$$m_0 \leq x(t) \leq M_0 \qquad (t \in [t_0, \infty))\,.$$

Definition. We say that the function $x(t)$ is asymptotically logarithmically periodic, if $x(e^t)$ is asymptotically periodic, i.e. there is a periodic function $\phi(t)$ such that $|x(e^t) - \phi(t)| \to 0$ as $t \to \infty$.

Theorem 5. *Suppose that all the conditions of Theorem 3 are satisfied and $k > \alpha$. Then all the solutions of equation (3.1) are asymptotically logarithmically periodic.*

Proof. Let $\phi : [pt_0, t_0] \to \mathbf{R}$ be given and consider $x(t) = x(t, t_0, \phi)$, the solution of (3.1) starting at t_0 with the initial function ϕ. To simplify our notation let us assume that $t_0 = 1$, for other t_0's the proof is similar. Let M and C be the constants appearing in Theorem 3 and hence we have $|\dot{x}(t)| \leq CM/t^k$ on the interval $[1/p, \infty)$.

Let us transform the equation by replacing $t = e^s$ and $x(t) = y(\ln(t)) = y(s)$. From (3.1) we obtain

$$\dot{y} = c(e^s)e^s(-y(s) + y(s + \ln(p))) = c(e^s)e^s(-y(s) + y(s - h)), \tag{3.2}$$

where $h = -\ln(p)$ (here we use that $t_0 = 1$ and hence the solution y corresponding to x starts from $\ln(t_0) = 0$). We also have $\dot{x}(t) = \dot{y}(\ln(t))/t$ for $t \geq 1/p$ and hence $|\dot{y}(s)| \leq CMe^{s(1-k)}$ for $s \geq h$. Then we use the equation to have

$$|y(s) - y(s - h)| \leq \frac{CMe^{-sk}}{c(e^s)} \qquad \forall s \geq h$$

Using that $k > \alpha$ it is easy to prove that the sequence

$$\frac{e^{-(s+ih)k}}{c(e^{s+ih})} \leq e^{-(s+ih)k} e^{(s+ih)\alpha}/m \leq e^{-ih(k-\alpha)}/m \qquad (s \in [0,h])$$

is summable. Therefore

$$|y(s+lh) - y(s+nh)| \leq CM \sum_{i=n+1}^{l} \frac{e^{-(s+ih)k}}{c(e^{s+ih})} \qquad \forall s \in [0,h] \text{ and } l \geq n \geq 0 \qquad (3.3)$$

and hence the function sequence $z_n(s) := y_{nh}(s) = y(s+nh)$ (for $s \in [-h,0]$) is a Cauchy-sequence in the supremum norm. Thus it converges to a function χ. Consider $\chi(s)$ to be an h-periodic function, and then we have $|y(s) - \chi(s)| \to 0$ as $s \to \infty$. Therefore all solutions of (3.2) are asymptotically h-periodic, which means that all solutions of (3.1) are asymptotically logarithmically periodic.

Theorem 6. *Suppose that all the conditions of Theorem 4 are satisfied. Then all the solutions of equation (3.1) are asymptotically logarithmically periodic.*

Proof. The proof is very similar to that of Theorem 5. The only difference is that after the transformation we have

$$|y(s) - y(s-h)| \leq CMe^{-sk} \qquad \forall s \geq 0$$

since the $c(t)$ in the estimate on $\dot{x}(t)$ and the $c(e^s)$ coming from (3.2) cancel each other. The rest of the proof is the same.

In this section we established conditions under which we can prove asymptotical logarithmical periodic behavior of the solutions of equation (3.1). Both the conditions of Theorem 3 and 4 are reasonable, they require $c(t)$ not to be too small or decrease too fast. Clearly, all constant functions are solutions of equation (3.1), and asymptotic logarithmic periodicity includes the special case of the solutions being asymptotically constant. We now show by an example that there is an equation of the form (3.1) which has an asymptotically non-constant solution.

Let $c(t) = 1$, $t_0 = 1$, $k = 1$, $m = 1$, $\alpha = 0$ in Theorem 6. Let $\phi : [p,1] \to \mathbf{R}$ be given (it will be specified later, but it satisfies the condition that the solution is continuously differentiable at t_0 and hence we have an estimate on the derivative on the interval

$[1,\infty)$). We do the same transformation as we did in the proof of Theorem 6. Then we have

$$|y(s) - y(s-h)| \le CMe^{-s} \qquad \forall s \ge 0.$$

By induction we get

$$|y(s+lh) - y(s-h)| \le \frac{CMe^{-s}}{1-e^{-h}} .$$

Let $\psi(s) := \phi(e^s) = \phi(t)$ for $t \in [p,1]$. Define s_{max} and s_{min} so that $\psi(s_{max})$ is a maximum and $\psi(s_{min})$ is a minimum of ψ in the interval $[-h,0]$. The above inequality gives (as a special case) that

$$y(s_{max}+lh) \ge y(s_{max}) - CM\frac{e^{-(s_{max}+h)}}{1-e^{-h}}$$

and

$$y(s_{min}+lh) \le y(s_{min}) + CM\frac{e^{-(s_{min}+h)}}{1-e^{-h}} .$$

Putting these together we obtain

$$y(s_{max}+lh) - y(s_{min}+lh) \ge (y(s_{max}) - y(s_{min})) - CM\frac{e^{-(s_{max}+h)} + e^{-(s_{min}+h)}}{1-e^{-h}}$$

Now we define ϕ a little more precisely. Let ϕ be strictly increasing on the interval $[p,(1+p)/2]$ and decreasing on $[(1+p)/2,1]$, hence $s_{min} = 0$, $s_{max} = \ln((1+p)/2)$ and $\psi(s_{max}) - \psi(s_{min}) > 0$. Then we have

$$\frac{e^{-s_{max}} + e^{-s_{min}}}{1-e^{-h}} = \frac{3+p}{1-p^2}$$

and hence

$$y(s_{max}+lh) - y(s_{min}+lh) \ge \psi(s_{max}) - \psi(s_{min}) - \frac{CMp(3+p)}{1-p^2} \ge \gamma > 0$$

if we choose p small enough. This shows that y at the shifts of s_{max} and s_{min} differs by a fixed positive constant and hence y cannot tend to a constant. Since $x(t) = y(\ln(t))$, we also proved that x does not tend to a constant.

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