MARACHKOV TYPE STABILITY RESULTS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction: finite delay

This paper is concerned with systems of functional differential equations with either finite or infinite delay. We give conditions on the system and on a Liapunov function to ensure that the zero solution is asymptotically stable. Section 2 is devoted to finite delay, Section 3 to infinite delay, and Section 4 to examples.

The remainder of this section introduces the problem for the finite delay case.

We consider a system of functional differential equations with finite delay written as

$$x'(t) = f(t, x_t), \qquad ' = d/dt, \tag{1}$$

where $f : [0, \infty) \times \mathcal{C}_H \to \mathbf{R}^m$ is continuous and takes bounded sets into bounded sets and $f(t, 0) = 0$. Here, $(\mathcal{C}, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^m$ with the supremum norm, h is a non-negative constant, $\mathcal{C}_H$ is the open H-ball in $\mathcal{C}$, and $x_t(s) = x(t + s)$ for $-h \le s \le 0$. Standard existence theory shows that if $\phi \in \mathcal{C}_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0 + \alpha)$ satisfying (1) for $t > t_0$, $x_t(t_0, \phi) = \phi$ and α some positive constant; if there is a closed subset $B \subset \mathcal{C}_H$ such that the solution remains in B, then $\alpha = \infty$. Also, $|\cdot|$ will denote the norm in $\mathbf{R}^m$ with $|x| = \max_{1\le i\le m}|x_i|$.

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We are concerned here with asymptotic stability in the context of Liapunov's direct method. Thus, we are concerned with continuous, strictly increasing functions $W_i : [0,\infty) \to [0,\infty)$ with $W_i(0) = 0$, called wedges, and with Liapunov functionals.

Definition 1: A continuous functional $V : [0,\infty) \times \mathcal{C}_H \to [0,\infty)$ which is locally Lipschitz in ϕ is called a Liapunov functional for (1) if there is a wedge W with

(i) $W(|\phi(0)|) \le V(t,\phi)$, $V(t,0) = 0$, and

(ii) $V'_{(1)}(t,x_t) = \limsup_{\delta\to 0} \frac{1}{\delta}\{V(t+\delta, x_{t+\delta}(t_0,\phi)) - V(t, x_t(t_0,\phi))\} \le 0.$

Remark: A standard result states that if there is a Liapunov functional for (1), then $x = 0$ is stable. Definitions will be given in the next section.

The classical result on asymptotic stability may be traced back to Marachkov [17] through Krasovskii [15;pp. 151-154]. It may be stated as follows.

Theorem MK: Suppose there are a constant M, wedges W_i, and a Liapunov functional V (so $W_1(|\phi(0)|) \le V(t,\phi)$ and $V(t,0) = 0$) with

(i) $V'_{(1)}(t,x_t) \le -W_2(|x(t)|)$ and

(ii) $|f(t,\phi)| \le M$ if $t \ge 0$ and $\|\phi\| < H$.

Then $x = 0$ is asymptotically stable.

Condition (ii) is troublesome, since it excludes many examples of considerable interest. And there are several results which reduce or eliminate (ii). For example, we showed [3] that if

(iii) $V(t,\phi) \le W_2(|x|) + W_3(|x_t|_2)$,

where $|\cdot|_2$ is the L^2-norm, then uniform asymptotic stability would result. Other alternatives may be found in [1,4,6,8], for example.

In an earlier paper [7] we gave a very general theorem and proof which had the following result as a corollary.

Theorem A: Suppose there is a Liapunov functional V, wedges W_i, positive constants K and J, a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \le K$ such that

(i) $V(t_n,\phi) \le W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t,x_t) \le -W_3(|x(t)|)$ if $t_n - h \le t \le t_n$, and

(iii) $|f(t,\phi)| \le J(t+1)\ln(t+2)$ for $t \ge 0$ and $\|\phi\| < H$.

Then $x = 0$ is AS.

Our first result here generalizes that slightly, but more importantly, it gives a simple and instructive proof that shows exactly what is happening so that the infinite delay case follows exactly. It is a simple exercise to see that the statement of Theorem 1 below still holds if condition (ii) of that theorem is replaced by condition (ii) of Theorem A.

2. Stability for finite delay

We now define the terminology to be used here.

Definition 2: The solution $x = 0$ of (1) is:

(a) *stable* if for each $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\|\phi\| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$;

(b) *uniformly stable (US)* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \geq 0, \|\phi\| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$;

(c) *uniformly equi-asymptotically stable (UEAS)* if it is uniformly stable and if there is a $K > 0$ and for each $[\mu > 0, t_0 \geq 0]$ there is a $T > 0$ such that $[t \geq t_0 + T,\ \|\phi\| < K]$ implies that $|x(t, t_0, \phi)| < \mu$.

Lemma: Let $F : [0, \infty) \to [1, \infty)$ be continuous and increasing. Then

$$\int_1^\infty (1/F(t))dt = \infty$$

if and only if

$$\sum_{i=1}^{\infty} (1/F(t_0 + ih)) = \infty$$

for $t_0 \geq 0$.

Theorem 1: Suppose there is a $V : [0, \infty) \times C_H \to [0, \infty)$, wedges W_i, and a continuous increasing function $F : [0, \infty) \to [1, \infty)$ such that

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$,

(ii) $V'_{(1)}(t, x_t) \leq -W_3(|x(t)|)$,

(iii) $|f(t, \phi)| \leq F(t)$ on $[0, \infty) \times C_H$, and

(iv) $\int_1^\infty (1/F(t))dt = \infty$.

Then the zero solution of (1) is uniformly equi-asymptotically stable.

Proof: A classical result yields uniform stability. For the $H > 0$, find $K > 0$ so that $[t_0 \geq 0,\ \|\phi\| < K,\ t \geq t_0]$ implies that $|x(t, t_0, \phi)| < H$.

Let $\mu < K$ and $t_0 \geq 0$ be given. We must find $T > 0$ so that $[\|\phi\| < K,\, t \geq t_0 + T]$ implies that $|x(t, t_0, \phi)| < \mu$. For an arbitrary such ϕ, let $x(t) := x(t, t_0, \phi)$ and $V(t, x_t) =:$ $V(t)$.

Next, for this μ find $\delta \leq 1$ of US. Define

$$I_n := I_n(t_0) = [t_0 + (n-1)h, t_0 + nh].$$

By the US, if there is an n with $|x(t)| < \delta$ on I_n, then $|x(t)| < \mu$ for $t \geq t_0 + nh$. Thus, until t enters such an I_n, if ever, for each n there is a $t_n \in I_n$ with $|x(t_n)| \geq \delta$. It follows readily that there is an $\alpha_n \in (0, h]$ with $|x(t)| \geq \delta/2$ on $[t_n, t_n + \alpha_n]$ and let α_n be maximal with this property.

If $\alpha_n < h$ (and consequently $|x(t_n + \alpha_n)| = \delta/2$), then we now obtain a lower estimate of α_n. Integrating (1) yields

$$\delta/2 \leq |x(t_n) - x(t_n + \alpha_n)| = |\int_{t_n}^{t_n+\alpha_n} f(s, x_s)ds| \leq F(t_n + \alpha_n)\alpha_n$$

or

$$\alpha_n \geq \delta/[2F(t_n + \alpha_n)] \geq \delta/[2F(t_n + h)] \geq \delta/[2F(t_{n+2})]$$

Hence we have $\alpha_n \geq \delta/[2F(t_{n+2})]$ for this case, and also (supposing that $F(t) \geq 1/2h$) in the case when $\alpha_n = h$.

Next, integration of (ii) and $t_n + \alpha_n \leq t_{n+2}$ yield

$$\begin{aligned} V(t_{n+2}) - V(t_n) \leq V(t_n + \alpha_n) - V(t_n) &\leq -\int_{t_n}^{t_n+\alpha_n} W_3(|x(s)|)ds \\ &\leq -W_3(\delta/2)\alpha_n \\ &\leq -W_3(\delta/2)\delta/[2F(t_{n+2})] \end{aligned}$$

Hence,

$$V(t_{2n}) \leq W_2(K) - W_3(\delta/2)(\delta/2)\sum_{i=1}^{n} 1/F(t_{2i}).$$

There is an $n = n(t_0)$ with the right-hand-side negative. For this n let $T = 2nh$. This completes the proof.

If we consider the paragraph after Theorem MK with the result of [3], the reader naturally believes that it may be possible to strengthen the conclusion of Theorem 1 to uniform asymptotic stability. The following proposition shows that this can not be done.

Proposition. There is a function f such that all conditions of Theorem 1 are satisfied, but the zero solution is not uniformly asymptotically stable.

Proof: Let ψ be a continuously differentiable function with the following properties:

1. $\psi(n) = 1/(n+1)$ for $n = 0, 1, \ldots,$
2. $\psi(t) = 0$ unless $t \in [n - (1/2(n+1)), n + (1/2(n+1))]$,
3. $\psi(t)$ is increasing on the interval $t \in [n-(1/2(n+1)), n]$ and decreasing on the interval $t \in [n, n + (1/2(n+1))]$,
4. $|\psi'(t)| \leq C$ for some $C > 0$.

Clearly there is such a ψ. Now let $h = 1$ and we define the right hand side of the equation on the interval $[n, n+1)$:

$$f(t, x_t) := \begin{cases} \dfrac{\psi'(t)}{\psi(n)} x(n), \text{ if } |x(t)| \leq ||x_t|| \dfrac{n+1}{n+2} \text{ or } t \in [n, n + \dfrac{1}{2}) \\ \dfrac{(n+1)(||x_t|| - |x(t)|)}{|x(t)|} \dfrac{\psi'(t)}{\psi(n)} x(n), \text{ otherwise} \end{cases}$$

It is easy to see that f is continuous and satisfies the local Lipschitz condition in its second variable; to prove this, one needs to use the fact that when the second definition holds, then

$$\frac{(n+1)(||x_t|| - |x(t)|)}{|x(t)|} \leq 1.$$

Also, $|f(t, \phi)| \leq C(t+1)||\phi||$.

Next note that the supremum norm of the solution is non-increasing, because of the second part of the definition.

It is also clear that all functions of the form $c\psi(t)$ are solutions. Now let us start a solution at t_0 by an initial function ϕ, and let n be the smallest integer not smaller than t_0. Then for $t \geq n$ we have

$$x(t, t_0, \phi) = \frac{x(n, t_0, \phi)}{\psi(n)} \psi(t)$$

Obviously, after t reaches the next integer after t_0 we will always have the first part of the definition in effect, and the solution is a constant times ψ.

Now we define a Liapunov functional

$$V(t, \phi) := ||\phi|| + \int_t^\infty |x(s, t, \phi)| ds$$

First we prove that V exists as well as the upper bound on V. Let n be the smallest integer larger than t. Then (using the fact that $||x_t||$ is non-increasing and the properties of ψ)

$$\begin{aligned}
V(t,\phi) &= ||\phi|| + \int_t^n |x(s,t,\phi)|ds + \int_n^\infty |x(s,t,\phi)|ds \\
&\leq ||\phi|| + \int_{n-1}^n ||\phi||ds + \int_n^\infty \frac{||\phi||}{\psi(n)}\psi(s)ds \\
&\leq 2||\phi|| + (n+1)||\phi|| \sum_{i=n}^\infty \frac{1}{(i+1)^2} \\
&\leq 2||\phi|| + \frac{(n+1)||\phi||}{n} \leq 4||\phi||
\end{aligned}$$

We also have $|x(t)| \leq V(t,x_t)$ and $V'(t,x_t) \leq -|x(t)|$ using the fact that $|x(s,t,\phi)| \to 0$ as $s \to \infty$. Therefore all conditions of Theorem 1 are satisfied, and hence the solutions are equi-asymptotically stable.

All that is left to be proved is that the solutions are not uniformly asymptotically stable. Suppose for contradiction that there is a $K > 0$ and for all $\mu > 0$ there is a T such that if $t_0 \geq 0$, $t \geq t_0 + T$, and $||\phi|| < K$ then $|x(t,t_0,\phi)| < \mu$. Then let $\mu < K/2$ be given and let T be fixed. Choose $\phi(s) = K/2$ $(s \in [-1,0])$ with n large enough so that $n/(n+[T]+1) > 2\mu/K$, and $t_0 = n$. Then we know from the previous notes that

$$x(t,t_0,\phi) = \frac{K}{2\psi(n)}\psi(t).$$

Choosing $t = t_0 + [T] + 1 > t_0 + T$ we find that

$$x(t_0+[T]+1, t_0, \phi) = \frac{K\psi(n+[T]+1)}{2\psi(n)} = \frac{Kn}{2(n+[T]+1)} > \mu$$

which is a contradiction to our assumption. This contradiction shows that the solutions are not uniformly asymptotically stable.

Note that the above example can easily be modified so that $F(t)$ in Theorem 1 can be t^α for any $\alpha > 0$ and still UAS does not hold.

3. Introduction and stability for infinite delay

Seifert[19] seems to have been the first to clearly show the importance of a fading memory in the study of stability for a system with infinite delay. That concept is now central in the study of stability, boundedness, and periodicity. The fading memory is deduced from the differential equation itself and then is reflected in the Liapunov functional used in the stability investigation. Thus, in a formal presentation the fading memory properties frequently are first seen in the wedges on the Liapunov functional in the form of a weighted norm.

Let $g : (-\infty, 0] \to [1, \infty)$ be a continuous nonincreasing function with $\lim_{t\to-\infty} g(t) = +\infty$. Then

$$(C, |\cdot|_g)$$

is the Banach space of continuous functions $\phi : (-\infty, 0] \to R^n$ for which

$$\sup_{t\le 0} |\phi(t)|/g(t) =: |\phi|_g$$

exists. For $H > 0$, $(C_H, |\cdot|_g)$ is that subset of C with $|\phi|_g < H$.

If $A > 0$ and if $x : (-\infty, A] \to R^n$ is continuous, then for $0 \le t \le A$, x_t is that element of C defined by

$$x_t(s) = x(t + s), -\infty < s \le 0,$$

provided that $|x_t|_g$ exists.

Let $f : [0, \infty) \times C_H \to R^n$ and consider the system

$$x'(t) = f(t, x_t). \tag{2}$$

We suppose the usual conditions (continuity and local Lipschitz condition on f), that imply that for each (t_0, ϕ) in $[0, \infty) \times C_H$ there is a solution x, having value $x(t, t_0, \phi)$, satisfying (2) on an interval $[t_0, \alpha)$ with $x_{t_0} = \phi$; moreover, we suppose that if $H_1 < H$ and if $|x(t)| \le H_1$ for all t for which x is defined, then $\alpha = \infty$.

There are many existence theorems for (2) showing exactly what is needed for the conditions in the above paragraph to be satisfied. Sawano [18] gives one for bounded continuous initial functions, while Hino-Murakami-Naito [14; p. 36] have one for initial functions in C. But existence of solutions is closely tied to the existence of a Liapunov function, as is discussed extensively in Burton [2], especially Theorem 4. If the system is continuous in the g norm, if the g function is unbounded, if the Liapunov function is mildly unbounded in the g norm, and if the derivative along (2) is non-positive, that is sufficient for existence of solutions.

Our interest here is purely in stability and our result will hold whenever the above type of existence obtains. Systems of this kind are extensively discussed in the literature and the reader is referred to Hale-Kato [12], Haddock-Krisztin-Terjéki [10] for phase space theory, Burton-Feng [5] for continuous dependence, Hering [13] for stability and Liapunov functions, and Hino-Murakami-Naito [14] for an in depth treatment of the subject of infinite delay problems.

In setting up phase spaces for infinite delay problems, fairly stringent translation conditions frequently emerge which require g to decrease almost exponentially. See, for example, Chapter 1 of Hino-Murakami-Naito [14] or Hale-Kato [12;p. 24]. The paper by Haddock [9] is devoted in large part to spaces where g is exponential. In this paper we also use exponential g's, but we also show asymptotic stability when this condition does not hold. We now introduce the properties used here.

Definition 3. Let $\phi \in C$ and define

$$\tilde{\phi}(s) := \begin{cases} \phi(s), \text{ if } s \leq 0 \\ \quad 0, \text{ if } s > 0 \end{cases}$$

We say that $(C, |\cdot|_g)$ is a fading memory space if for all $\phi \in C$ we have $|\tilde{\phi}_t|_g \to 0$ as $t \to \infty$.

Definition 4. We say that g satisfies the *exponential condition* if for each $\delta > 0$ there is an $h > 0$ such that $0 \leq t_1 < t_2$ and $t_2 - t_1 \geq h$ imply that

$$g(u - t_2) \geq g(u - t_1)2H/\delta \qquad \text{for} \qquad u \leq t_1. \tag{3}$$

This condition makes it possible to prove an exact counterpart of Theorem 1 for the infinite delay case. It will play an important role in the paper and the following proposition explains its properties.

Proposition 1. The following conditions are equivalent.

1. There is an $\alpha \in (0, 1)$ and $l > 0$ such that

$$\sup_{s \leq 0} \frac{g(s)}{g(s - l)} \leq \alpha$$

2. g satisfies the exponential condition.
3. The space $(C, |\cdot|_g)$ is a fading memory space.

Proof:

1 implies 2: Let δ be given. Choose the natural number $n > 0$ so that $\alpha^n \leq \delta/2H$ and let $h = nl$. If $0 \leq t_1 < t_2$ and $t_2 - t_1 \geq h$, then choosing $s = u - t_1$ and using the property that g is monotone non-increasing we obtain

$$\frac{g(u-t_1)}{g(u-t_2)} = \frac{g(s)}{g(s-(t_2-t_1))} \leq \frac{g(s)}{g(s-nl)} = \frac{g(s)}{g(s-l)} \cdots \frac{g(s-(n-1)l)}{g(s-nl)} \leq \alpha^n \leq \frac{\delta}{2H}$$

which gives the desired result.

2 implies 3: let ϕ be given. Then

$$|\tilde{\phi}_t|_g = \sup_{s\leq 0} \frac{|\phi(s)|}{g(s-t)} \leq \sup_{s\leq 0} \frac{|\phi(s)|}{g(s)} \sup_{s\leq 0} \frac{g(s)}{g(s-t)} = |\phi|_g \sup_{s\leq 0} \frac{g(s)}{g(s-t)} < \frac{\delta|\phi|_g}{2H}$$

if $t \geq h$, where h is chosen from the exponential condition on g for δ. Letting δ tend to 0 we get condition 3.

3 implies 1: We apply the condition of the fading memory space to $\phi(s) = (g(s), 0, 0, ..., 0) \in \mathbf{R}^n$ and we obtain

$$|\tilde{\phi}_t|_g = \sup_{s\leq 0} \frac{g(s)}{g(s-t)} \to 0$$

as $t \to \infty$, which clearly implies condition 1.

Stability definitions from Section 2 carry over by replacing $\|\phi\|$ by $|\phi|_g$, but to be more precise we say (asymptotic) stability in the g-norm. The goal of this section is to prove Theorem 1 for system (2) making only the change of $W_2(\|\phi\|)$ into $W_2(|\phi|_g)$. In particular, here is our result.

THEOREM 2: Let $G : (-\infty, 0] \to [1, \infty)$ be a function such that $G(s) \leq cg(s)$ ($c > 0$ constant) and $|\tilde{G}_t|_g \to 0$ as $t \to \infty$. Suppose there is a $V : [0, \infty) \times C_H \to [0, \infty)$, wedges W_i, and a continuous increasing function $F : [0, \infty) \to [1, \infty)$ such that

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(|\phi|_g)$,

(ii) $V'_{(2)}(t, x_t) \leq -W_3(|x(t)|)$,

(iii) $|f(t, \phi)| \leq F(t)$ on $[0, \infty) \times C_H$, and

(iv) $\int_1^\infty (1/F(t))dt = \infty$.

Then the zero solution of (2) is uniformly equi-asymptotically stable in the G-norm.

Proof: We follow the proof of Theorem 1. Uniform stability in the g-norm follows immediately. Since $c|\phi|_G \geq |\phi|_g$, we also have uniform stability in the G-norm. Let $H_1 < H$ and we find $K > 0$ with $W_1(H_1) = W_2(K)$. Then for $|\phi|_g < K$, $t_0 \geq 0$, and $x(t) := x(t, t_0, \phi)$, since $V'_{(2)}(t, x_t) \leq 0$, by (i) if $t \geq t_0$ we have

$$W_1(|x(t)|) \leq V(t, x_t) \leq V(t_0, \phi) \leq W_2(|\phi|_g) < W_2(K)$$

so

$$|x(t)| < W_1^{-1}(W_2(K)) = H_1,$$

and hence $x(t)$ is defined on the interval $[t_0, \infty)$.

Let $\mu < K$ and $t_0 \geq 0$ be given. We must find $T > 0$ so that $[|\phi|_g < K, t \geq t_0 + T]$ implies that $|x(t, t_0, \phi)| < \mu$.

Pick $\delta = W_2^{-1}(W_1(\mu))$. We will now find an $h > 0$ such that if $|x(t)| < \delta$ on an interval $[t_1, t_2]$ with $t_0 \leq t_1$ and $t_2 - t_1 \geq h$, then $|x(t)| < \mu$ for $t \geq t_2$. The reader will readily verify that if we can do this, then the remainder of the proof is identical to that of Theorem 1.

Now for the given δ, find h such that $|\tilde{G}_s|_g < \delta/H$ for $s \geq h$. Let $t_0 \leq t_1 < t_2$, $t_2 - t_1 \geq h$, $|x(t)| < \delta$ on $[t_1, t_2]$. We then have

$$\begin{aligned}
|x_{t_2}|_g &= \sup_{s \leq 0} \frac{|x(s + t_2)|}{g(s)} = \sup_{u \leq t_2} \frac{|x(u)|}{g(u - t_2)} \\
&\leq \max\left\{ \sup_{u \leq t_1} \frac{|x(u)|}{g(u - t_2)}, \sup_{t_1 \leq u \leq t_2} |x(u)| \right\} \\
&\leq \max\left\{ \sup_{u \leq t_1} \frac{|x(u)|}{G(u - t_1)} \frac{G(u - t_1)}{g(u - t_2)}, \sup_{t_1 \leq u \leq t_2} |x(u)| \right\} \\
&\leq \max\left\{ |x_{t_1}|_G |\tilde{G}_{t_2 - t_1}|_g, \sup_{t_1 \leq u \leq t_2} |x(u)| \right\} < \max\left\{ H\frac{\delta}{H}, \delta \right\} = \delta
\end{aligned}$$

Since $V'_{(2)}(t, x_t) \leq 0$, for $t \geq t_2$ we have

$$W_1(|x(t)|) \leq V(t, x_t) \leq V(t_2, x_{t_2}) \leq W_2(|x_{t_2}|_g) < W_2(\delta)$$

so

$$|x(t)| < W_1^{-1}(W_2(\delta)) = \mu,$$

as required.

The remainder of the proof is identical to that of Theorem 1.

Note that if we have a fading memory space, then we can state the following simplified version of the above theorem, which does not need G.

THEOREM 3: Suppose that g satisfies (3) and conditions (i)-(iv) of Theorem 2 hold. Then $|\tilde{g}_t|_g \to 0$ (as $t \to \infty$) and hence choosing $G(s) = g(s)$ all conditions of Theorem 2 are satisfied and (2) is UEAS in the g-norm.

REMARK. If g does not satisfy (3), then we still have the task of constructing a G for which $|\tilde{G}_t|_g \to 0$ holds. For a particular example of (2), one may construct a Liapunov functional V without any reference to a function g. (The reader should follow our subsequent Example 2 to see how this develops.) From the properties of V we then construct the wedges and g. Two questions then arise. First, what conditons must g satisfy to ensure existence of solutions? We refer the reader to [2], Theorem 4 for a typical answer. Next, what conditions are needed for G to satisfy $|\tilde{G}_t|_g \to 0$? We formalize one result concerning this question as follows.

LEMMA: If $G(s)/g(s) \to 0$ as $s \to -\infty$, then $|\tilde{G}_t|_g \to 0$ as $t \to \infty$.

Proof: Let $\delta > 0$ be given and $h > 0$ large enough so that $G(s)/g(s) < \delta/2$ for $s \leq -h$. Now choose $T > 0$ large enough so that $g(s) \geq 2\max_{-h\leq u\leq 0} G(u)/\delta$ for $s \leq -T$. Then if $t \geq T$ we obtain

$$|\tilde{G}_t|_g = \sup_{s\leq 0} \frac{G(s)}{g(s-t)} \leq \sup_{s\leq -h} \frac{G(s)}{g(s)} + \sup_{-h\leq s\leq 0} \frac{G(s)}{g(s-t)} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

As a consequence of this Lemma we can always choose $G(s) \equiv 1$ and then prove uniform equi-asymptotic stability in the supremum norm using Theorem 2. This is a very useful consequence of our theorems, because in practical examples initial functions are frequently bounded.

There are two final remarks concerning the conditions in Theorem 2. First, (supposing that $G(s) \leq cg(s)$) if either $G(s)/g(s) \to 0$ as $s \to -\infty$, or g satisfies the exponential condition, then $|\tilde{G}_t|_g \to 0$ as $t \to \infty$. Next, the opposite direction is not true. One can construct g and G so that they do not satisfy either of the above conditions, but $|\tilde{G}_t|_g \to 0$ holds.

4. Examples

Lemma: Let x be a solution of (2) and t_0 be fixed. Then

$$\limsup_{t\to t_0+0} \frac{|x_t|_g - |x_{t_0}|_g}{t-t_0} \le |x'(t_0)|$$

Proof. Let $t > t_0$ be arbitrarily fixed. Then

$$\begin{aligned} |x_t|_g = \sup_{s\le 0} \frac{|x(t+s)|}{g(s)} &= \max\left\{ \sup_{s\le -(t-t_0)} \frac{|x(t+s)|}{g(s)}, \max_{s\in[-(t-t_0),0]} \frac{|x(t+s)|}{g(s)} \right\} \\ \le \max\left\{ \sup_{s\le 0} \frac{|x(t_0+s)|}{g(s)}, \max_{s\in[-(t-t_0),0]} |x(t+s)| \right\} &= \max\left\{ |x_{t_0}|_g, \max_{s\in[t_0,t]} |x(s)| \right\} \end{aligned}$$

There are two cases:

1. If $|x_{t_0}|_g < \max_{s\in[t_0,t]} |x(s)|$, then using $|x_{t_0}|_g \ge |x(t_0)|$ we get

$$\begin{aligned} \frac{|x_t|_g - |x_{t_0}|_g}{t-t_0} &\le \frac{\max_{s\in[t_0,t]} |x(s)| - |x(t_0)|}{t-t_0} \\ &= \frac{|x(\theta_t)| - |x(t_0)|}{\theta_t - t_0} \frac{\theta_t - t_0}{t-t_0} \le \left| \frac{x(\theta_t) - x(t_0)}{\theta_t - t_0} \right| \end{aligned}$$

   where $\theta_t \in [t_0, t]$ is a point where $|x(s)|$ takes it's maximum on the interval $[t_0, t]$. Note that $|x_{t_0}|_g \ge |x(t_0)|$ and $|x_{t_0}|_g < \max_{s\in[t_0,t]} |x(s)|$ implies that $\theta_t > t_0$ and hence the above expression is valid.

2. If $|x_{t_0}|_g \ge \max_{s\in[t_0,t]} |x(s)|$, then

$$\frac{|x_t|_g - |x_{t_0}|_g}{t-t_0} \le \frac{|x_{t_0}|_g - |x_{t_0}|_g}{t-t_0} = 0$$

   In this case we define $\theta_t = t$.

   Therefore, in both cases we have

$$\frac{|x_t|_g - |x_{t_0}|_g}{t-t_0} \le \left| \frac{x(\theta_t) - x(t_0)}{\theta_t - t_0} \right|$$

for some $\theta_t \in (t_0, t]$. Then letting $t \to t_0 + 0$ we also have $\theta_t \to t_0 + 0$ and hence we obtain the desired result.

Note that $|x_t|_g$ may not be differentiable everywhere. The main problem is that when $|x_{t_1}|_g > |x(t_1)|$, $x'(t) > 0$ on some interval $[t_1, t_2]$, and at some point $t_3 \in (t_1, t_2)$ we have

$|x_{t_3}|_g = x(t_3)$. When $t < t_3$ then $|x_t|_g$ is monotone non-increasing, but when $t_3 < t$ we have $|x_t|_g = |x(t)|$ and hence it is strictly increasing. Therefore $|x_t|_g$ has a break point at t_3 so it is not differentiable at t_3.

Example 1. Consider the equation

$$x' = -a(t)x + f(t, x_t) \tag{4}$$

where

$$1 \leq a(t) \leq k(t+1)\ln(t+2)$$

($k > 0$ constant) and

$$|f(t, x_t)| \leq b(t)|x_t|_g, \int_0^\infty b(t)dt < \infty,$$

where g satisfies the exponential condition. Then the conditions of Theorem 2 (with $G = g$) are satisfied.

Proof. Define

$$V(t, x_t) = [2|x(t)| + |x_t|_g] \exp\left(-3\int_0^t b(s)ds\right).$$

Then

$$\begin{aligned} V'_{(4)}(t, x_t) \exp\left(3\int_0^t b(s)ds\right) &\leq -6|x(t)|b(t) - b(t)|x_t|_g - 2a(t)|x| + a(t)|x| + b(t)|x_t|_g \\ &\leq [-6b(t) - a(t)]|x| \\ &\leq -|x|. \end{aligned}$$

We then have $\alpha > 0$ with

$$\alpha|x| \leq V(t, x_t) \leq 2|x| + |x_t|_g$$

and

$$V' \leq -\alpha|x|.$$

The conditions of Theorem 2 are satisfied. Moreover, the conditions for existence of solutions are also satisfied, as may be seen in Theorem 4 of [2].

Example 2. Consider the scalar equation

$$x' = -a(t)x - \int_{-\infty}^{t} D(t,s)h(x(s))ds \tag{5}$$

with $h : R \to R$, h, a, and D continuous, and if $-\infty < s < t < \infty$ then

$$D(t,s) \geq 0,\ D_s(t,s) \geq 0,\ D_{st}(t,s) \leq 0, \tag{6}$$

and

$$1 \leq a(t) \leq k(t+1)\ln(t+2) \text{ and } xh(x) > 0 \text{ for } x \neq 0. \tag{7}$$

In order to make the equation be defined for bounded initial function we need the following conditions: let

$$\int_{-\infty}^{t} \left[D(t,s) + D_s(t,s)(t-s+1)^2 + |D_{st}(t,s)|(t-s)^2\right] ds \text{ be bounded and continuous} \tag{8}$$

and

$$\lim_{s\to-\infty}(t-s)D(t,s) = 0 \text{ for fixed } t. \tag{9}$$

These will imply that

$$\text{there is a } B > 0 \text{ with } \int_{-\infty}^{t} D_s(t,s)ds \leq B. \tag{10}$$

Using (8) we obtain that equation (5) satisfies condition (iii) of Theorem 2. Now we define a Liapunov functional as

$$V(t,x(\cdot)) = 2\int_0^x h(s)ds + \int_{-\infty}^{t} D_s(t,s)\left(\int_s^t h(x(v))dv\right)^2 ds \tag{11}$$

so that along a bounded solution of (5) we have

$$\begin{aligned} V'(t,x(\cdot)) = {} & 2h(x)\left[-\int_{-\infty}^{t} D(t,s)h(x(s))ds - a(t)x\right] \\ & + \int_{-\infty}^{t} D_{st}(t,s)\left(\int_s^t h(x(v))dv\right)^2 ds \\ & + 2h(x)\int_{-\infty}^{t} D_s(t,s)\int_s^t h(x(v))dv\, ds. \end{aligned}$$

If we integrate the last term by parts and use (8) and (9) we get

$$V'(t,x(\cdot)) = \int_{-\infty}^{t} D_{st}(t,s)\Big(\int_{s}^{t} h(x(v))dv\Big)^2 ds \\ + 2h(x)\big[-a(t)x\big] \le -xh(x) \tag{12}$$

Since h has the sign of x the derivative satisfies condition (ii) of Theorem 2.

Next, lets take care of existence. We need a place to start so lets ask that

$$h(x) = x^n,$$

where n is an odd integer. Let g be given, and consider

$$\begin{aligned} |\int_{s}^{t} x^n(u)du| &= |\int_{s-t}^{0} x^n(u+t)du| = |\int_{s-t}^{0} (x^n(u+t)/g^n(u))g^n(u)du| \\ &\le \sup_{-\infty<u\le 0} |x(u+t)/g(u)|^n| \int_{s-t}^{0} g^n(u)du| \\ &\le (|x_t|_g)^n| \int_{s-t}^{0} g^n(u)du|. \end{aligned} \tag{13}$$

If g satisfies the condition that

$$\int_{-\infty}^{t} D_s(t,s)(\int_{s-t}^{0} g^n(u)du)^2 ds < M < \infty. \tag{14}$$

then

$$2\int_{0}^{x} h(s)ds \le V \le 2\int_{0}^{x} h(s)ds + M(|x_t|_g)^{2n} \tag{15}$$

Condition (14) will make the Liapunov function satisfy condition (i) of Theorem 2 for bounded initial functions, but we will need an additional condition for unbounded initial functions. To make (5) defined for initial functions from a $(C, |\cdot|_G)$ space, we have

$$\int_{-\infty}^{t} D(t,s)h(x(s))ds = \int_{-\infty}^{0} D(t,u+t)h(x(u+t))du$$

and $|x(u+t)|/G(u) \le H$ implies $|x(u+t))| \le HG(u)$ and so we need to strengthen (8) to get that

$$\int_{-\infty}^{0} D(t,u+t)h(HG(u))du \text{ is bounded and continuous.} \tag{16}$$

In addition, the derivative of the Liapunov function must be defined so we need also strengthen (8) by

$$-\int_{-\infty}^{t} D_{st}(t,s)\left(\int_{s-t}^{0} h(HG(v))dv\right)^{2} ds < \infty. \tag{17}$$

Finally, (9) must be strengthened to

$$D(t,s)\int_{s-t}^{0} h(HG(v))dv \to 0 \text{ as } s \to -\infty. \tag{18}$$

Now condition (16), (17) and (18) are mainly conditions on D and G: they can be satisfied by either decreasing D and D_{st}, or by choosing a "small" G. If we do the later, and choose a G so that $G(s) \le cg(s)$ and $|\tilde{G}_t|_g \to 0$ as $t \to \infty$, then all conditions of Theorem 2 are satisfied, so we have equi-asymptotic stability in the G-norm.

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