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Central Limit Theorem for a Class of Linear Systems

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Abstract

We consider a class of interacting particle systems with values in $[0,\infty)^{\mathbb{Z}^d}$, of which the binary contact path process is an example. For $d \geq 3$ and under a certain square integrability condition on the total number of the particles, we prove a central limit theorem for the density of the particles, together with upper bounds for the density of the most populated site and the replica overlap.

Key words: central limit theorem, linear systems, binary contact path process, diffusive behavior, delocalization.

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1 Introduction

We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \; ; \; x \in \mathbb{N}\}$. For $x = (x_1, .., x_d) \in \mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\eta = (\eta_x)_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\eta| = \sum_{x\in\mathbb{Z}^d} |\eta_x|$. Let $(\Omega, \mathscr{F}, P)$ be a probability space. We write $P[X] = \int X \; dP$ and $P[X : A] = \int_A X \; dP$ for a r.v.(random variable) X and an event A.

1.1 The binary contact path process (BCPP)

We start with a motivating simple example. Let $\eta_t = (\eta_{t,x})_{x\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$, $t \geq 0$ be binary contact path process (BCPP for short) with parameter $\lambda > 0$. Roughly speaking, the BCPP is an extended version of the basic contact process, in which not only the presence/absence of the particles at each site, but also their number is considered. The BCPP was originally introduced by D. Griffeath [4]. Here, we explain the process following the formulation in the book of T. Liggett [5, Chapter IX]. Let $\tau^{z,i}$, ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. mean-one exponential random variables and $T^{z,i} = \tau^{z,1} + ... + \tau^{z,i}$. We suppose that the process (η_t) starts from a deterministic configuration $\eta_0 = (\eta_{0,x})_{x\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ with $|\eta_0| < \infty$. At time $t = T^{z,i}$, η_{t-} is replaced by η_t randomly as follows: for each $e \in \mathbb{Z}^d$ with $|e| = 1$,

$$\eta_{t,x} = \begin{cases} \eta_{t-,x} + \eta_{t-,z} & \text{if } x = z + e, \\ \eta_{t-,x} & \text{if otherwise} \end{cases} \quad \text{with probability } \tfrac{\lambda}{2d\lambda+1},$$

(all the particles at site z are duplicated and added to those on the site $z = x + e$), and

$$\eta_{t,x} = \begin{cases} 0 & \text{if } x = z, \\ \eta_{t-,x} & \text{if } x \neq z \end{cases} \quad \text{with probability } \tfrac{1}{2d\lambda+1}$$

(all the particles at site z disappear). The replacement occurs independently for different (z, i) and independently from $\{\tau^{z,i}\}_{z,i}$. A motivation to study the BCPP comes from the fact that the projected process

$$\left(\eta_{t,x} \wedge 1\right)_{x\in\mathbb{Z}^d}, \quad t \geq 0$$

is the basic contact process [4].

Let

$$\kappa_1 = \frac{2d\lambda - 1}{2d\lambda + 1} \quad \text{and} \quad \overline{\eta}_t = (\exp(-\kappa_1 t)\eta_{t,x})_{x\in\mathbb{Z}^d}.$$

Then, $(|\overline{\eta}_t|)_{t\geq 0}$ is a nonnegative martingale and therefore, the following limit exists almost surely:

$$|\overline{\eta}_\infty| \stackrel{\text{def}}{=} \lim_t |\overline{\eta}_t|.$$

Moreover, $P[|\overline{\eta}_\infty|] = 1$ if

$$d \geq 3 \text{ and } \lambda > \tfrac{1}{2d(1-2\pi_d)}, \tag{1.1}$$

where π_d is the return probability for the simple random walk on $\mathbb{Z}^d$ [4, Theorem 1]. It is known that $\pi_d \leq \pi_3 = 0.3405...$ for $d \geq 3$ [7, page 103].

We denote the density of the particles by:

$$\rho_{t,x} = \frac{\eta_{t,x}}{|\eta_t|}\mathbf{1}\{|\eta_t| > 0\}, \quad t > 0, x \in \mathbb{Z}^d. \tag{1.2}$$

Interesting objects related to the density would be

$$\rho_t^* = \max_{x\in\mathbb{Z}^d} \rho_{t,x}, \text{ and } \mathscr{R}_t = \sum_{x\in\mathbb{Z}^d} \rho_{t,x}^2. \tag{1.3}$$

ρ_t^* is the density at the most populated site, while $\mathscr{R}_t$ is the probability that a given pair of particles at time t are at the same site. We call $\mathscr{R}_t$ the *replica overlap*, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \le \mathscr{R}_t \le \rho_t^*$. These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of ρ_t^* or $\mathscr{R}_t$ indicate that the most of the particles are concentrated on small number of "favorite sites" (*localization*), whereas small values of them imply that the particles are spread out over a large number of sites (*delocalization*).

As a special case of Corollary 1.2.2 below, we have the following result, which shows the diffusive behavior and the delocalization of the BCPP under the condition (1.1):

Theorem 1.1.1. *Suppose (1.1). Then, for any $f \in C_b(\mathbb{R}^d)$,*

$$\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\left(x/\sqrt{t}\right)\rho_{t,x} = \int_{\mathbb{R}^d} f\,d\nu \quad \text{in } P(\,\cdot\,||\overline{\eta}_\infty|>0)\text{-probability,}$$

where $C_b(\mathbb{R}^d)$ stands for the set of bounded continuous functions on $\mathbb{R}^d$, and ν is the Gaussian measure with

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \frac{\lambda}{2d\lambda+1}\delta_{ij}, \quad i,j = 1,..,d.$$

Furthermore,

$$\mathscr{R}_t = \mathscr{O}(t^{-d/2}) \quad \text{as } t \nearrow \infty \text{ in } P(\,\cdot\,||\overline{\eta}_\infty|>0)\text{-probability.}$$

1.2 The results

We generalize Theorem 1.1.1 to a certain class of linear interacting particle systems with values in $[0,\infty)^{\mathbb{Z}^d}$ [5, Chapter IX]. Recall that the particles in BCPP either die, or make binary branching. To describe more general "branching mechanism", we introduce a random vector $K = (K_x)_{x\in\mathbb{Z}^d}$ which is bounded and of finite range in the sense that

$$0 \le K_x \le b_K \mathbf{1}_{\{|x|\le r_K\}} \text{ a.s. for some non-random } b_K, r_K \in [0,\infty). \tag{1.4}$$

Let $\tau^{z,i}$, ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. mean-one exponential random variables and $T^{z,i} = \tau^{z,1} + \ldots + \tau^{z,i}$. Let also $K^{z,i} = (K_x^{z,i})_{x\in\mathbb{Z}^d}$ ($z \in \mathbb{Z}^d$, $i \in \mathbb{N}^*$) be i.i.d. random vectors with the same distributions as K, independent of $\{\tau^{z,i}\}_{z\in\mathbb{Z}^d, i\in\mathbb{N}^*}$. We suppose that the process $(\eta_t)_{t\ge0}$ starts from a deterministic configuration $\eta_0 = (\eta_{0,x})_{x\in\mathbb{Z}^d} \in [0,\infty)^{\mathbb{Z}^d}$ with $|\eta_0| < \infty$. At time $t = T^{z,i}$, η_{t-} is replaced by η_t, where

$$\eta_{t,x} = \begin{cases} K_0^{z,i}\eta_{t-,z} & \text{if } x = z, \\ \eta_{t-,x} + K_{x-z}^{z,i}\eta_{t-,z} & \text{if } x \ne z. \end{cases} \tag{1.5}$$

The BCPP is a special case of this set-up, in which

$$K = \begin{cases} 0 & \text{with probability } \frac{1}{2d\lambda+1} \\ \left(\delta_{x,0} + \delta_{x,e}\right)_{x\in\mathbb{Z}^d} & \text{with probability } \frac{\lambda}{2d\lambda+1}, \text{ for each } 2d \text{ neighbor } e \text{ of } 0. \end{cases} \tag{1.6}$$

A formal construction of the process $(\eta_t)_{t\geq 0}$ can be given as a special case of [5, page 427, Theorem 1.14] via Hille-Yosida theory. In section 1.3, we will also give an alternative construction of the process in terms of a stochastic differential equation.

We set

$$\kappa_p = \sum_{x\in\mathbb{Z}^d} P[(K_x-\delta_{x,0})^p], \quad p=1,2, \tag{1.7}$$

$$\overline{\eta}_t = (\exp(-\kappa_1 t)\eta_{t,x})_{x\in\mathbb{Z}^d}. \tag{1.8}$$

Then,

$$(|\overline{\eta}_t|)_{t\geq 0} \text{ is a nonnegative martingale.} \tag{1.9}$$

The above martingale property can be seen by the same argument as in [5, page 433, Theorem 2.2 (b)]. For the reader's convenience, we will also present a simpler proof in section 1.3 below. By (1.9), following limit exists almost surely:

$$|\overline{\eta}_\infty| \stackrel{\text{def}}{=} \lim_t |\overline{\eta}_t|. \tag{1.10}$$

To state Theorem 1.2.1, we define

$$G(x) = \int_0^\infty P_S^0(S_t = x)dt, \tag{1.11}$$

where $((S_t)_{t\geq 0}, P_S^x)$ is the continuous-time random walk on $\mathbb{Z}^d$ starting from $x\in\mathbb{Z}^d$, with the generator

$$L_S f(x) = \frac{1}{2}\sum_{y\in\mathbb{Z}^d}\left(P[K_{x-y}]+P[K_{y-x}]\right)\left(f(y)-f(x)\right). \tag{1.12}$$

As before, $C_{\mathrm{b}}(\mathbb{R}^d)$ stands for the set of bounded continuous functions on $\mathbb{R}^d$.

Theorem 1.2.1. *Suppose (1.4) and that*

$$\text{the set } \{x\in\mathbb{Z}^d\ ;\ P[K_x]\neq 0\} \text{ contains a linear basis of } \mathbb{R}^d, \tag{1.13}$$

$$\sum_{y\in\mathbb{Z}^d} P[(K_y-\delta_{y,0})(K_{x+y}-\delta_{x+y,0})] = 0 \text{ for all } x\in\mathbb{Z}^d\backslash\{0\}. \tag{1.14}$$

Then, referring to (1.7)–(1.12), the following are equivalent:

(a) $\frac{\kappa_2}{2}G(0) < 1$,

(b) $\sup_{t\geq 0} P[|\overline{\eta}_t|^2] < \infty$,

(c) $\lim_{t\to\infty}\sum_{x\in\mathbb{Z}^d} f\left((x-mt)/\sqrt{t}\right)\overline{\eta}_{t,x} = |\overline{\eta}_\infty|\int_{\mathbb{R}^d} f\,d\nu$ *in* $\mathbb{L}^2(P)$ *for all* $f\in C_{\mathrm{b}}(\mathbb{R}^d)$,

where $m=\sum_{x\in\mathbb{Z}^d} xP[K_x]\in\mathbb{R}^d$ *and* ν *is the Gaussian measure with*

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \sum_{x\in\mathbb{Z}^d} x_i x_j P[K_x], \quad i,j=1,..,d. \tag{1.15}$$

Moreover, if $\frac{\kappa_2}{2}G(0) < 1$, *then, there exists* $C \in (0,\infty)$ *such that*

$$\sum_{x,\widetilde{x}\in\mathbb{Z}^d} f(x-\widetilde{x})P[\overline{\eta}_{t,x}\overline{\eta}_{t,\widetilde{x}}] \le Ct^{-d/2}|\eta_0|^2 \sum_{x\in\mathbb{Z}^d} f(x) \tag{1.16}$$

for all $t > 0$ *and* $f : \mathbb{Z}^d \to [0,\infty)$ *with* $\sum_{x\in\mathbb{Z}^d} f(x) < \infty$.

The main point of Theorem 1.2.1 is that the condition (a), or equivalently (b), implies the central limit theorem (c) (See also Corollary 1.2.2 below). This seems to be the first result in which the central limit theorem for the spatial distribution of the particle is shown in the context of linear systems. Some other part of our results ((a) $\Rightarrow$ (b), and Theorem 1.2.3 below) generalizes [4, Theorem 1]. However, this is merely a by-product and not a central issue in the present paper.

The proof of Theorem 1.2.1, which will be presented in section 3.1, is roughly divided into two steps:

(i) to represent the two-point function $P[\eta_{t,x}\eta_{t,\widetilde{x}}]$ in terms of a continuous-time Markov chain on $\mathbb{Z}^d \times \mathbb{Z}^d$ via the Feynman-Kac formula (Lemma 2.1.1 and Lemma 2.1.4 below),

(ii) to show the central limit theorem for the “weighted" Markov chain, where the weight comes from the additive functional due to the Feynman-Kac formula (Lemma 2.2.2 below).

The above strategy was adopted earlier by one of the authors for branching random walk in random environment [10]. There, the Markov chain alluded to above is simply the product of simple random walks on $\mathbb{Z}^d$, so that the central limit theorem with the Feynman-Kac weight is relatively easy. Since the Markov chain in the present paper is no longer a random walk, it requires more work. However, the good news here is that the Markov chain we have to work on is “close" to a random walk. In fact, we get the central limit theorem by perturbation from that for a random walk case.

Some other remarks on Theorem 1.2.1 are in order:

1) The condition (1.13) guarantees a reasonable non-degeneracy for the transition mechanism (1.5). On the other hand, (1.14) follows from a stronger condition:

$$P[(K_x - \delta_{x,0})(K_y - \delta_{y,0})] = 0 \text{ for } x, y \in \mathbb{Z}^d \text{ with } x \neq y, \tag{1.17}$$

which amounts to saying that the transition mechanism (1.5) updates the configuration by “at most one coordinate at a time". A typical examples of such K's are given by ones which satisfy:

$$P(K=0) + \sum_{a\in\mathbb{Z}^d\setminus\{0\}} P\left(K = (\delta_{x,0} + K_a\delta_{x,a})_{x\in\mathbb{Z}^d}\right) = 1.$$

These include not only BCPP but also models with asymmetry and/or long (but finite) range.

Here is an explanation for how we use the condition (1.14). To prove Theorem 1.2.1, we use a certain Markov chain on $\mathbb{Z}^d \times \mathbb{Z}^d$, which is introduced in Lemma 2.1.1 below. Thanks to (1.14), the Markov chain is stationary with respect to the counting measure on $\mathbb{Z}^d \times \mathbb{Z}^d$. The stationarity plays an important role in the proof of Theorem 1.2.1– see Lemma 2.1.4 below.

2) Because of (1.13), the random walk (S_t) is recurrent for $d = 1,2$ and transient for $d \geq 3$.

Therefore, $\frac{\kappa_2}{2}G(0) < 1$ is possible only if $d \geq 3$. As will be explained in the proof, $\frac{\kappa_2}{2}G(0) < 1$ is equivalent to

$$P_S^0\left[\exp\left(\frac{\kappa_2}{2}\int_0^\infty \delta_0(S_t)dt\right)\right] < \infty.$$

3) If, in particular,

$$P[K_x] = \begin{cases} c > 0 & \text{for } |x| = 1, \\ 0 & \text{for } |x| \geq 2, \end{cases} \tag{1.18}$$

then, $(S_t)_{t\geq 0} \stackrel{\text{law}}{=} (\widehat{S}_{2dct})_{t\geq 0}$, where $(\widehat{S}_\cdot)$ is the simple random walk. Therefore, the condition (a) becomes

$$\frac{\kappa_2}{4dc(1-\pi_d)} < 1. \tag{1.19}$$

By (1.6), the BCPP satisfies (1.13)–(1.14). Furthermore, $\kappa_2 = 1$ and we have (1.18) with $c = \frac{\lambda}{2d\lambda+1}$. Therefore, (1.19) is equivalent to (1.1).

4) The dual process of (η_t) above (in the sense of [5, page 432]) is given by replacing the linear transform in (1.5) by its transpose:

$$\eta_{t,x} = \begin{cases} \sum_{y\in\mathbb{Z}^d} K_{y-x}^{z,i}\eta_{t-,y} & \text{if } x = z, \\ \eta_{t-,x} & \text{if } x \neq z. \end{cases} \tag{1.20}$$

As can be seen from the proofs, all the results in this paper remain true for the dual process.

5) The central limit theorem for discrete time linear systems is discussed in [6].

We define the density and the replica overlap in the same way as (1.2)–(1.3). Then, as an immediate consequence of Theorem 1.2.1, we have the following

Corollary 1.2.2. *Suppose (1.4), (1.13)–(1.14) and that $\frac{\kappa_2}{2}G(0) < 1$. Then, $P[|\overline{\eta}_\infty|] = 1$ and for all $f \in C_b(\mathbb{R}^d)$,*

$$\lim_{t\to\infty}\sum_{x\in\mathbb{Z}^d} f\left((x-mt)/\sqrt{t}\right)\rho_{t,x} = \int_{\mathbb{R}^d} f\,d\nu \quad \textit{in } P(\,\cdot\,||\overline{\eta}_\infty| > 0)\textit{-probability,}$$

where $m = \sum_{x\in\mathbb{Z}^d} xP[K_x] \in \mathbb{R}^d$ and ν is the same Gaussian measure defined by (1.15). Furthermore,

$$\mathscr{R}_t = \mathscr{O}(t^{-d/2}) \quad \textit{as } t \nearrow \infty \textit{ in } P(\,\cdot\,||\overline{\eta}_\infty| > 0)\textit{-probability.}$$

Proof: The first statement is immediate from Theorem 1.2.1(c). Taking $f(x) = \delta_{x,0}$ in (1.16), we see that

$$P[\sum_{x\in\mathbb{Z}^d}\overline{\eta}_{t,x}^2] \leq Ct^{-d/2}|\eta_0|^2 \quad \text{for } t > 0.$$

This implies the second statement. □

For $a \in \mathbb{Z}^d$, let η_t^a be the process starting from $\eta_0 = (\delta_{a,x})_{x\in\mathbb{Z}^d}$. As a by-product of Theorem 1.2.1, we have the following formula for the covariance of $(|\overline{\eta}_\infty^a|)_{a\in\mathbb{Z}^d}$. For BCPP, this formula was obtained by D. Griffeath [4, Theorem 3].

Theorem 1.2.3. *Suppose (1.4), (1.13)–(1.14) and that $\frac{\kappa_2}{2}G(0) < 1$. Then,*

$$P[|\overline{\eta}^a_\infty||\overline{\eta}^b_\infty|] = 1 + \frac{\kappa_2 G(a-b)}{2-\kappa_2 G(0)}, \quad a, b \in \mathbb{Z}^d.$$

The proof of Theorem 1.2.3 will be presented in section 3.2. We refer the reader to [11] for similar formulae for discrete time models.

1.3 SDE description of the process

We now give an alternative description of the process in terms of a stochastic differential equation (SDE), which will be used in the proof of Lemma 2.1.1 below. We introduce random measures on $[0,\infty) \times [0,\infty)^{\mathbb{Z}^d}$ by

$$N^z(dsd\xi) = \sum_{i \geq 1} \mathbf{1}\{(T^{z,i}, K^{z,i}) \in dsd\xi\}, \quad N^z_t(dsd\xi) = \mathbf{1}_{\{s \leq t\}} N^z(dsd\xi). \tag{1.21}$$

Then, N^z, $z \in \mathbb{Z}^d$ are independent Poisson random measures on $[0,\infty) \times [0,\infty)^{\mathbb{Z}^d}$ with the intensity

$$ds \times P(K \in d\xi).$$

The precise definition of the process $(\eta_t)_{t \geq 0}$ is then given by the following stochastic differential equation:

$$\eta_{t,x} = \eta_{0,x} + \sum_{z \in \mathbb{Z}^d} \int N^z_t(dsd\xi) \left(\xi_{x-z} - \delta_{x,z}\right) \eta_{s-,z}. \tag{1.22}$$

By (1.4), it is standard to see that (1.22) defines a unique process $\eta_t = (\eta_{t,x})$, $(t \geq 0)$ and that (η_t) is Markovian.

Proof of (1.9): Since $|\overline{\eta}_t|$ is obviously nonnegative, we will prove the martingale property. By (1.22), we have

$$|\eta_t| = |\eta_0| + \sum_{z \in \mathbb{Z}^d} \int N^z_t(dsd\xi)(|\xi| - 1)\eta_{s-,z},$$

and hence

(1)
$$|\overline{\eta}_t| = |\eta_0| - \kappa_1 \int_0^t |\overline{\eta}_s| ds + \sum_{z \in \mathbb{Z}^d} \int N^z_t(dsd\xi)(|\xi| - 1)\overline{\eta}_{s-,z}.$$

We have on the other hand that

$$\kappa_1 \int_0^t |\overline{\eta}_s| ds = \sum_{z \in \mathbb{Z}^d} \int_0^t ds \int P(K \in \xi)(|\xi| - 1)\overline{\eta}_{s,z}.$$

Plugging this into (1), we see that the right-hand-side of (1) is a martingale. □

2 Lemmas

2.1 Markov chain representations for the point functions

We assume (1.4) throughout, but not (1.13)–(1.14) for the moment. To prove the Feynman-Kac formula for two-point function, we introduce some notation.
For $x, y, \widetilde{x}, \widetilde{y} \in \mathbb{Z}^d$,

$$\begin{aligned}\Gamma_{x,\widetilde{x},y,\widetilde{y}} &\stackrel{\text{def}}{=} P[(K_{x-y} - \delta_{x,y})\delta_{\widetilde{x},\widetilde{y}} + (K_{\widetilde{x}-\widetilde{y}} - \delta_{\widetilde{x},\widetilde{y}})\delta_{x,y}] \\ &\quad + P[(K_{x-y} - \delta_{x,y})(K_{\widetilde{x}-y} - \delta_{\widetilde{x},y})]\delta_{y,\widetilde{y}}, \end{aligned} \tag{2.1}$$

$$V(x) \stackrel{\text{def}}{=} \sum_{y,\widetilde{y} \in \mathbb{Z}^d} \Gamma_{x,0,y,\widetilde{y}} = 2\kappa_1 + \sum_{y \in \mathbb{Z}^d} P[(K_y - \delta_{y,0})(K_{x+y} - \delta_{x+y,0})]. \tag{2.2}$$

Note that

$$V(x - \widetilde{x}) = \sum_{y,\widetilde{y} \in \mathbb{Z}^d} \Gamma_{x,\widetilde{x},y,\widetilde{y}}. \tag{2.3}$$

Remark: The matrix Γ introded above appears also in [5, page 442, Theorem 3.1], since it is a fundamental tool to deal with the two-point function of the linear system. However, the way we use the matrix will be different from the ones in the existing literature.

We now prove the Feynman-Kac formula for two-point function, which is the basis of the proof of Theorem 1.2.1:

Lemma 2.1.1. *Let $(X, \widetilde{X}) = ((X_t, \widetilde{X}_t)_{t \geq 0}, P^{x,\widetilde{x}}_{X,\widetilde{X}})$ be the continuous-time Markov chain on $\mathbb{Z}^d \times \mathbb{Z}^d$ starting from $(x, \widetilde{x})$, with the generator*

$$L_{X,\widetilde{X}} f(x, \widetilde{x}) = \sum_{y,\widetilde{y} \in \mathbb{Z}^d} \Gamma_{x,\widetilde{x},y,\widetilde{y}} \left(f(y, \widetilde{y}) - f(x, \widetilde{x}) \right),$$

where $\Gamma_{x,\widetilde{x},y,\widetilde{y}}$ is defined by (2.1). Then, for $(t, x, \widetilde{x}) \in [0, \infty) \times \mathbb{Z}^d \times \mathbb{Z}^d$,

$$P[\eta_{t,x}\eta_{t,\widetilde{x}}] = P^{x,\widetilde{x}}_{X,\widetilde{X}} \left[\exp\left(\int_0^t V(X_s - \widetilde{X}_s) ds \right) \eta_{0,X_t} \eta_{0,\widetilde{X}_t} \right], \tag{2.4}$$

where V is defined by (2.2).

Proof: We first show that $u(t, x, \widetilde{x}) \stackrel{\text{def}}{=} P[\eta_{t,x}\eta_{t,\widetilde{x}}]$ solves the integral equation

(1) $$u(t, x, \widetilde{x}) - u(0, x, \widetilde{x}) = \int_0^t (L_{X,\widetilde{X}} + V(x - \widetilde{x})) u(s, x, \widetilde{x}) ds.$$

By (1.22), we have

$$\eta_{t,x}\eta_{t,\widetilde{x}} - \eta_{0,x}\eta_{0,\widetilde{x}} = \sum_{y \in \mathbb{Z}^d} \int N^y(ds d\xi) F_{x,\widetilde{x},y}(s-, \xi, \eta),$$

where

$$F_{x,\widetilde{x},y}(s,\xi,\eta)$$
$$= (\xi_{x-y}-\delta_{x,y})\eta_{s,\widetilde{x}}\eta_{s,y}+(\xi_{\widetilde{x}-y}-\delta_{\widetilde{x},y})\eta_{s,x}\eta_{s,y}+(\xi_{x-y}-\delta_{x,y})(\xi_{\widetilde{x}-y}-\delta_{\widetilde{x},y})\eta^2_{s,y}$$

Therefore,

$$\begin{aligned}
u(t,x,\widetilde{x})-u(0,x,\widetilde{x}) &= \sum_{y\in\mathbb{Z}^d}\int_0^t ds\int P[F_{x,\widetilde{x},y}(s,\xi,\eta)]P(K\in\xi) \\
&= \int_0^t \sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{x,\widetilde{x},y,\widetilde{y}}u(s,y,\widetilde{y})ds \\
&\overset{(2.3)}{=} \int_0^t\left(\sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{x,\widetilde{x},y,\widetilde{y}}(u(s,y,\widetilde{y})-u(s,x,\widetilde{x}))+V(x-\widetilde{x})u(s,x,\widetilde{x})\right)ds \\
&= \int_0^t (L_{X,\widetilde{X}}+V(x-\widetilde{x}))u(s,x,\widetilde{x})ds.
\end{aligned}$$

We next show that

(2) $\quad \sup_{t\in[0,T]}\sup_{x,\widetilde{x}\in\mathbb{Z}^d}|u(t,x,\widetilde{x})|<\infty$ for any $T\in(0,\infty)$.

We have by (1.4) and (1.22) that, for any $p\in\mathbb{N}^*$, there exists $C_1\in(0,\infty)$ such that

$$P[\eta^p_{t,x}]\le C_1\sum_{y:|x-y|\le r_K}\int_0^t P[\eta^p_{s,y}]ds,\quad t\ge 0.$$

By iteration, we see that there exists $C_2\in(0,\infty)$ such that

$$P[\eta^p_{t,x}]\le e^{C_2 t}\sum_{y\in\mathbb{Z}^d}e^{-|x-y|}(1+\eta^p_{0,y}),\quad t\ge 0,$$

which, via Schwarz inequality, implies (4).
The solution to (1) subject to (2) is unique, for each given η_0. This can be seen by using Gronwall's inequality with respect to the norm $\|u\|=\sum_{x,\widetilde{x}\in\mathbb{Z}^d}e^{-|x|}|u(x,\widetilde{x})|$. Moreover, the RHS of (2.4) is a solution to (1) subject to the bound (2). This can be seen by adapting the argument in [8, page 5,Theorem 1.1]. Therefore, we get (2.4). □

Remark: The following Feynman-Kac formula for one-point function can be obtained in the same way as Lemma 2.1.1:

$$P[\eta_{t,x}]=e^{\kappa_1 t}P^x_X[\eta_{0,X_t}],\ (t,x)\in[0,\infty)\times\mathbb{Z}^d, \tag{2.5}$$

where κ_1 is defined by (1.7) and $((X_t)_{t\ge 0},P^x_X)$ is the continuous-time random walk on $\mathbb{Z}^d$ starting from x, with the generator

$$L_X f(x)=\sum_{y\in\mathbb{Z}^d}P[K_{x-y}]\left(f(y)-f(x)\right).$$

Lemma 2.1.2. *We have*

$$\sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{x,\widetilde{x},y,\widetilde{y}}=\sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{y,\widetilde{y},x,\widetilde{x}},\tag{2.6}$$

if and only if (1.14) holds. In addition, (1.14) implies that

$$V(x)=2\kappa_1+\kappa_2\delta_{x,0}.\tag{2.7}$$

Proof: We let $c(x)=\sum_{y\in\mathbb{Z}^d}P[(K_y-\delta_{y,0})(K_{x+y}-\delta_{x+y,0})]$. Then, $c(0)=\kappa_2$ and,

$$\begin{aligned}\sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{x,\widetilde{x},y,\widetilde{y}} &= 2\kappa_1+c(x-\widetilde{x}),\quad \text{cf. (2.2)–(2.3)},\\ \sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{y,\widetilde{y},x,\widetilde{x}} &= 2\kappa_1+\delta_{x,\widetilde{x}}\sum_{y\in\mathbb{Z}^d}c(y).\end{aligned}$$

These imply the desired equivalence and (2.7). □

We assume (1.14) from here on. Then, by (2.6), $(X,\widetilde{X})$ is stationary with respect to the counting measure on $\mathbb{Z}^d\times\mathbb{Z}^d$. We denote the dual process of $(X,\widetilde{X})$ by $(Y,\widetilde{Y})=((Y_t,\widetilde{Y}_t)_{t\geq 0},P^{x,\widetilde{x}}_{Y,\widetilde{Y}})$, that is, the continuous time Markov chain on $\mathbb{Z}^d\times\mathbb{Z}^d$ starting from $(x,\widetilde{x})$, with the generator

$$L_{Y,\widetilde{Y}}f(x,\widetilde{x})=\sum_{y,\widetilde{y}\in\mathbb{Z}^d}\Gamma_{y,\widetilde{y},x,\widetilde{x}}\left(f(y,\widetilde{y})-f(x,\widetilde{x})\right).\tag{2.8}$$

Thanks to (2.6), $L_{X,\widetilde{X}}$ and $L_{Y,\widetilde{Y}}$ are dual operators on $\ell^2(\mathbb{Z}^d\times\mathbb{Z}^d)$.

Remark: If we additionally suppose that $P[K_x^p]=P[K_{-x}^p]$ for $p=1,2$ and $x\in\mathbb{Z}^d$, then, $\Gamma_{x,\widetilde{x},y,\widetilde{y}}=\Gamma_{y,\widetilde{y},x,\widetilde{x}}$ for all $x,\widetilde{x},y,\widetilde{y}\in\mathbb{Z}^d$. Thus, $(X,\widetilde{X})$ and $(Y,\widetilde{Y})$ are the same in this case.

The relative motion $Y_t-\widetilde{Y}_t$ of the components of $(Y,\widetilde{Y})$ is nicely identified by:

Lemma 2.1.3. $((Y_t-\widetilde{Y}_t)_{t\geq 0},P^{x,\widetilde{x}}_{Y,\widetilde{Y}})$ *and* $((S_{2t})_{t\geq 0},P^{x-\widetilde{x}}_S)$ *(cf. (1.12)) have the same law.*

Proof: Since $(Y,\widetilde{Y})$ is shift invariant, in the sense that $\Gamma_{x+v,\widetilde{x}+v,y+v,\widetilde{y}+v}=\Gamma_{x,\widetilde{x},y,\widetilde{y}}$ for all $v\in\mathbb{Z}^d$, $((Y_t-\widetilde{Y}_t)_{t\geq 0},P^{x,\widetilde{x}}_{Y,\widetilde{Y}})$ is a Markov chain. Moreover, its jump rate is computed as follows. For $x\neq y$,

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}\Gamma_{y+z,z,x,0} &= P[K_{x-y}]+P[K_{y-x}]+\delta_{x,0}\sum_{z\in\mathbb{Z}^d}P[(K_{y+z}-\delta_{y,z})(K_z-\delta_{0,z})]\\ &\stackrel{(1.14)}{=} P[K_{x-y}]+P[K_{y-x}].\end{aligned}$$

□

To prove Theorem 1.2.1, the use of Lemma 2.1.1 is made not in itself, but via the following lemma. It is the proof of this lemma, where the duality of $(X,\widetilde{X})$ and $(Y,\widetilde{Y})$ plays its role.

Lemma 2.1.4. *For a bounded $g : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$,*

$$\begin{aligned}
&\sum_{x,\widetilde{x}\in\mathbb{Z}^d} P[\overline{\eta}_{t,x}\overline{\eta}_{t,\widetilde{x}}]g(x,\widetilde{x}) \\
&\quad = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[\exp\left(\kappa_2\int_0^t \delta_0(Y_s-\widetilde{Y}_s)ds\right)g(Y_t,\widetilde{Y}_t)\right]. \qquad (2.9)
\end{aligned}$$

In particular, for a bounded $f : \mathbb{Z}^d \to \mathbb{R}$,

$$\sum_{x,\widetilde{x}\in\mathbb{Z}^d} P[\overline{\eta}_{t,x}\overline{\eta}_{t,\widetilde{x}}]f(x-\widetilde{x}) = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} P^{x-\widetilde{x}}_{S}\left[\exp\left(\frac{\kappa_2}{2}\int_0^{2t} \delta_0(S_u)du\right)f(S_{2t})\right]. \qquad (2.10)$$

Proof: It follows from Lemma 2.1.1 and (2.7) that

(1) $$\text{LHS of (2.9)} = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} P^{x,\widetilde{x}}_{X,\widetilde{X}}\left[\exp\left(\kappa_2\int_0^t \delta_0(X_s-\widetilde{X}_s)ds\right)\eta_{0,X_t}\eta_{0,\widetilde{X}_t}\right]g(x,\widetilde{x}).$$

We now observe that the operators

$$\begin{aligned}
f(x,\widetilde{x}) &\mapsto P^{x,\widetilde{x}}_{X,\widetilde{X}}\left[\exp\left(\kappa_2\int_0^t \delta_0(X_s-\widetilde{X}_s)ds\right)f(X_t,\widetilde{X}_t)\right], \\
f(x,\widetilde{x}) &\mapsto P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[\exp\left(\kappa_2\int_0^t \delta_0(Y_s-\widetilde{Y}_s)ds\right)f(Y_t,\widetilde{Y}_t)\right]
\end{aligned}$$

are dual to each other with respect to the counting measure on $\mathbb{Z}^d \times \mathbb{Z}^d$. Therefore,

$$\text{RHS of (1)} = \text{RHS of (2.9)}.$$

Taking $g(x,\widetilde{x}) = f(x-\widetilde{x})$ in particular, we have by (2.9) and Lemma 2.1.3 that

$$\begin{aligned}
\text{LHS of (2.10)} &= \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[\exp\left(\kappa_2\int_0^t \delta_0(Y_s-\widetilde{Y}_s)ds\right)f(Y_t-\widetilde{Y}_t)\right] \\
&= \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} P^{x-\widetilde{x}}_{S}\left[\exp\left(\kappa_2\int_0^t \delta_0(S_{2u})du\right)f(S_{2t})\right] = \text{RHS of (2.10)}.
\end{aligned}$$

□

Remark: In the case of BCPP, D. Griffeath obtained a Feynman-Kac formula for

$$\sum_{y\in\mathbb{Z}^d} P[\eta_{t,x}\eta_{t,\widetilde{x}+y}]$$

[4, proof of Theorem 1]. However, this does not seem to be enough for our purpose. Note that the Feynman-Kac formulae in the present paper (Lemma 2.1.1 and Lemma 2.1.4) are stronger, since they give the expression for each summand of the above summation.

2.2 Central limit theorems for Markov chains

We prepare central limit theorems for Markov chains, which is obtained by perturbation of random walks.

Lemma 2.2.1. *Let $((Z_t)_{t\geq 0}, P^x)$ be a continuous-time random walk on $\mathbb{Z}^d$ starting from x, with the generator*

$$L_Z f(x) = \sum_{y\in\mathbb{Z}^d} a_{y-x}(f(y)-f(x)),$$

where we assume that

$$\sum_{x\in\mathbb{Z}^d} |x|^2 a_x < \infty.$$

Then, for any $B \in \sigma[Z_u \ ; \ u \in [0,\infty)]$, $x \in \mathbb{Z}^d$, and $f \in C_{\mathrm{b}}(\mathbb{R}^d)$,

$$\lim_{t\to\infty} P^x[f((Z_t - mt)/\sqrt{t}) : B] = P^x(B)\int_{\mathbb{R}^d} f\,d\nu,$$

where $m = \sum_{x\in\mathbb{Z}^d} x a_x$ and ν is the Gaussian measure with

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \sum_{x\in\mathbb{Z}^d} x_i x_j a_x, \quad i,j = 1,..,d. \tag{2.11}$$

Proof: By subtracting a constant, we may assume that $\int_{\mathbb{R}^d} f\,d\nu = 0$. We first consider the case that $B \in \mathscr{F}_s \stackrel{\text{def}}{=} \sigma[Z_u \ ; \ u \in [0,s]]$ for some $s \in (0,\infty)$. It is easy to see from the central limit theorem for (Z_t) that for any $x \in \mathbb{Z}^d$,

$$\lim_{t\to\infty} P^x[f((Z_{t-s} - mt)/\sqrt{t})] = 0.$$

With this and the bounded convergence theorem, we have

$$P^x[f((Z_t - mt)/\sqrt{t}) : B] = P^x[P^{Z_s}[f((Z_{t-s} - mt)/\sqrt{t})] : B] \longrightarrow 0 \ \text{ as } t \nearrow \infty.$$

Next, we take $B \in \sigma[Z_u \ ; \ u \in [0,\infty)]$. For any $\varepsilon > 0$, there exist $s \in (0,\infty)$ and $\widetilde{B} \in \mathscr{F}_s$ such that $P^x[|\mathbf{1}_B - \mathbf{1}_{\widetilde{B}}|] < \varepsilon$. Then, by what we already have seen,

$$\overline{\lim_{t\to\infty}} P^x[f((Z_t - mt)/\sqrt{t}) : B] \leq \overline{\lim_{t\to\infty}} P^x[f((Z_t - mt)/\sqrt{t}) : \widetilde{B}] + \|f\|\varepsilon = \|f\|\varepsilon,$$

where $\|f\|$ is the sup norm of f. Similarly,

$$\underline{\lim_{t\to\infty}} P^x[f((Z_t - mt)/\sqrt{t}) : B] \geq -\|f\|\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we are done. $\square$

Lemma 2.2.2. *Let $Z = ((Z_t)_{t\geq 0}, P^x)$ be as in Lemma 2.2.1 and and $D \subset \mathbb{Z}^d$ be transient for Z. On the other hand, let $\widetilde{Z} = ((\widetilde{Z}_t)_{t\geq 0}, \widetilde{P}^x)$ be the continuous-time Markov chain on $\mathbb{Z}^d$ starting from x, with the generator*

$$L_{\widetilde{Z}} f(x) = \sum_{y\in\mathbb{Z}^d} \widetilde{a}_{x,y}(f(y)-f(x)),$$

where we assume that $\widetilde{a}_{x,y} = a_{y-x}$ if $x \notin D \cup \{y\}$ and that D is also transient for $\widetilde{Z}$. Furthermore, we assume that a function $v : \mathbb{Z}^d \to \mathbb{R}$ satisfies

$$v \equiv 0 \text{ outside } D,$$
$$\widetilde{P}^z \left[\exp\left(\int_0^\infty |v(\widetilde{Z}_t)| dt \right) \right] < \infty \text{ for some } z \in \mathbb{Z}^d.$$

Then, for $f \in C_b(\mathbb{R}^d)$,

$$\lim_{t\to\infty} \widetilde{P}^z \left[\exp\left(\int_0^t v(\widetilde{Z}_u) du \right) f((\widetilde{Z}_t - mt)/\sqrt{t}) \right] = \widetilde{P}^z \left[\exp\left(\int_0^\infty v(\widetilde{Z}_t) dt \right) \right] \int_{\mathbb{R}^d} f \, d\nu,$$

where ν is the Gaussian measure such that (2.11) holds.

Proof: Define

$$\begin{aligned} H_D(\widetilde{Z}) &= \inf\{t \geq 0 \,;\, \widetilde{Z}_t \in D\}, \;\; T_D(\widetilde{Z}) = \sup\{t \geq 0 \,;\, \widetilde{Z}_t \in D\}, \\ e_t &= \exp\left(\int_0^t v(\widetilde{Z}_s) ds \right). \end{aligned}$$

Then, for $s < t$,

$$\begin{aligned} & \widetilde{P}^z \left[e_t f((\widetilde{Z}_t - mt)/\sqrt{t}) \right] \\ &= \widetilde{P}^z \left[e_t f((\widetilde{Z}_t - mt)/\sqrt{t}) : T_D(\widetilde{Z}) < s \right] + \varepsilon_{s,t} \\ &= \widetilde{P}^z \left[e_s f((\widetilde{Z}_t - mt)/\sqrt{t}) : T_D(\widetilde{Z}) < s \right] + \varepsilon_{s,t} \\ &= \widetilde{P}^z \left[e_s \mathbf{1}_{\widetilde{Z}_s \notin D} \widetilde{P}^{\widetilde{Z}_s} \left[f((\widetilde{Z}_{t-s} - mt)/\sqrt{t}) : H_D(\widetilde{Z}) = \infty \right] \right] + \varepsilon_{s,t}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} |\varepsilon_{s,t}| &= \left| \widetilde{P}^z \left[e_t f((\widetilde{Z}_t - mt)/\sqrt{t}) : T_D(\widetilde{Z}) \geq s \right] \right| \\ &\leq \|f\| \widetilde{P}^z \left[\exp\left(\int_0^\infty |v(\widetilde{Z}_t)| dt \right) : T_D(\widetilde{Z}) \geq s \right] \to 0 \;\text{ as } s \to \infty. \end{aligned}$$

We now observe that

$$\widetilde{P}^x(\,\cdot\,| H_D(\widetilde{Z}) = \infty) = P^x(\,\cdot\,| H_D(Z) = \infty) \;\; \text{for } x \notin D,$$

where $H_D(Z)$ is defined similarly as $H_D(\widetilde{Z})$. Hence, for $x \notin D$ and fixed $s > 0$, we have by Lemma 2.2.1 that

$$\lim_{t\to\infty} \widetilde{P}^x \left[f((\widetilde{Z}_{t-s} - mt)/\sqrt{t}) : H_D(\widetilde{Z}) = \infty \right] = \widetilde{P}^x [H_D(\widetilde{Z}) = \infty] \int_{\mathbb{R}^d} f \, d\nu.$$

Therefore,

$$\begin{aligned} & \lim_{t\to\infty} \widetilde{P}^z \left[e_s \mathbf{1}_{\widetilde{Z}_s \notin D} \widetilde{P}^{\widetilde{Z}_s} \left[f((\widetilde{Z}_{t-s} - mt)/\sqrt{t}) : H_D(\widetilde{Z}) = \infty \right] \right] \\ &= \widetilde{P}^z \left[e_s \mathbf{1}_{\widetilde{Z}_s \notin D} \widetilde{P}^{\widetilde{Z}_s} [H_D(\widetilde{Z}) = \infty] \right] \int_{\mathbb{R}^d} f \, d\nu \\ &= \widetilde{P}^z \left[e_s : T_D(\widetilde{Z}) < s \right] \int_{\mathbb{R}^d} f \, d\nu. \end{aligned}$$

Thus, letting $t \to \infty$ first, and then $s \to \infty$, in (2.12), we get the lemma. $\square$

2.3 A Nash type upper bound for the Schrödinger semi-group

We will use the following lemma to prove (1.16). The lemma can be generalized to symmetric Markov chains on more general graphs. However, we restrict ourselves to random walks on $\mathbb{Z}^d$, since it is enough for our purpose.

Lemma 2.3.1. *Let $((Z_t)_{t\ge 0}, P^x)$ be continuous-time random walk on $\mathbb{Z}^d$ with the generator:*

$$L_Z f(x) = \sum_{y\in\mathbb{Z}^d} a_{y-x}(f(y)-f(x)),$$

where we assume that

the set $\{x \in \mathbb{Z}^d\ ;\ a_x \neq 0\}$ is bounded and contains a linear basis of $\mathbb{R}^d$,
$a_x = a_{-x}$ for all $x \in \mathbb{Z}^d$,

Let $v : \mathbb{Z}^d \to \mathbb{R}$ be a function such that

$$C_v \stackrel{\text{def}}{=} \sup_{x\in\mathbb{Z}^d} P^x\left[\exp\left(\int_0^\infty |v(Z_t)|dt\right)\right] < \infty.$$

Then, there exists $C \in (0,\infty)$ such that

$$\sup_{x\in\mathbb{Z}^d} P^x\left[\exp\left(\int_0^t v(Z_u)du\right) f(Z_t)\right] \le C t^{-d/2} \sum_{x\in\mathbb{Z}^d} f(x) \tag{2.13}$$

for all $t > 0$ and $f : \mathbb{Z}^d \to [0,\infty)$ with $\sum_{x\in\mathbb{Z}^d} f(x) < \infty$.

Proof: We adapt the argument in [1, Lemma 3.1.3]. For a bounded function $f : \mathbb{Z}^d \to \mathbb{R}$, we introduce

$$\begin{aligned}
(T_t f)(x) &= P^x\left[\exp\left(\int_0^t v(Z_u)du\right) f(Z_t)\right], \quad x \in \mathbb{Z}^d, \\
T_t^h f &= \frac{1}{h} T_t[fh], \text{ where } h(x) = P^x\left[\exp\left(\int_0^\infty v(Z_t)dt\right)\right].
\end{aligned}$$

Then, $(T_t)_{t\ge 0}$ extends to a symmetric, strongly continuous semi-group on $\ell^2(\mathbb{Z}^d)$. We now consider the measure $\sum_{x\in\mathbb{Z}^d} h(x)^2\delta_x$ on $\mathbb{Z}^d$, and denote by $(\ell^{p,h}(\mathbb{Z}^d), \|\cdot\|_{p,h})$ the associated $\mathbb{L}^p$-space. Then, it is standard (e.g., proofs of [2, page 74, Theorem 3.10] and [8, page 16, Proposition 3.3]) to see that $(T_t^h)_{t\ge 0}$ defines a symmetric strongly continuous semi-group on $\ell^{2,h}(\mathbb{Z}^d)$ and that for $f \in \ell^{2,h}(\mathbb{Z}^d)$,

$$\begin{aligned}
\mathscr{E}^h(f,f) &\stackrel{\text{def.}}{=} \lim_{t\searrow 0} \frac{1}{t} \sum_{x\in\mathbb{Z}^d} f(x)(f - T_t^h f)(x) h(x)^2 \\
&= \tfrac{1}{2} \sum_{x,y\in\mathbb{Z}^d} a_{y-x}|f(y)-f(x)|^2 h(x)h(y).
\end{aligned}$$

By the assumptions on (a_x), we have the Sobolev inequality:

$$\textbf{(1)} \qquad \sum_{x\in\mathbb{Z}^d} |f(x)|^{\frac{2d}{d-2}} \le c_1 \left(\frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} a_{y-x} |f(y)-f(x)|^2 \right)^{\frac{d}{d-2}} \quad \text{for all } f \in \ell^2(\mathbb{Z}^d),$$

where $c_1 \in (0,\infty)$ is independent of f. This can be seen via an isoperimetric inequality [9, page 40, (4.3)]. We have on the other hand that

$$\textbf{(2)} \qquad 1/C_\nu \le h(x) \le C_\nu.$$

We see from (1) and (2) that

$$\sum_{x\in\mathbb{Z}^d} |f(x)|^{\frac{2d}{d-2}} h(x)^2 \le c_2 \mathscr{E}^h(f,f)^{\frac{d}{d-2}} \quad \text{for all } f \in \ell^{2,h}(\mathbb{Z}^d),$$

where $c_2 \in (0,\infty)$ is independent of f. This implies that there is a constant C such that

$$\|T_t^h\|_{2\to\infty,h} \le Ct^{-d/4} \quad \text{for all } t>0,$$

e.g.,[3, page 75, Theorem 2.4.2], where $\|\cdot\|_{p\to q,h}$ denotes the operator norm from $\ell^{p,h}(\mathbb{Z}^d)$ to $\ell^{q,h}(\mathbb{Z}^d)$. Note that $\|T_t^h\|_{1\to 2,h} = \|T_t^h\|_{2\to\infty,h}$ by duality. We therefore have via semi-group property that

$$\textbf{(3)} \qquad \|T_t^h\|_{1\to\infty,h} \le \|T_{t/2}^h\|^2_{2\to\infty,h} \le C^2 t^{-d/2} \text{ for all } t>0.$$

Since $T_t f = hT_t^h[f/h]$, the desired bound (2.13) follows from (2) and (3). □

3 Proof of Theorem 1.2.1 and Theorem 1.2.3

3.1 Proof of Theorem 1.2.1

(a) $\Longleftrightarrow$ (b): Define

$$h(x) = P_S^x \left[\exp\left(\frac{\kappa_2}{2} \int_0^\infty \delta_0(S_t)dt \right) \right].$$

Since $\max_{x\in\mathbb{Z}^d} h(x) = h(0)$, we have that

$$\sup_{t\ge 0} P[|\overline{\eta}_t|^2] \stackrel{(2.10)}{=} \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} h(x-\widetilde{x}) \begin{cases} \le h(0)|\eta_0|^2, \\ \ge h(0)\sum_{x\in\mathbb{Z}^d} \eta_{0,x}^2. \end{cases}$$

Therefore, it is enough to show that (a) is equivalent to $h(0)<\infty$. In fact, Khas'minskii's lemma (e.g., [2, page 71] or [8, page 8]) says that $\sup_{x\in\mathbb{Z}^d} h(x) < \infty$ if $\frac{\kappa_2}{2}\sup_{x\in\mathbb{Z}^d} G(x) < 1$. Since $\max_{x\in\mathbb{Z}^d} G(x) = G(0)$, (a) implies that $h(0)<\infty$. On the other hand, we have that

$$\exp\left(\frac{\kappa_2}{2} \int_0^t \delta_0(S_s)ds \right) = 1 + \frac{\kappa_2}{2} \int_0^t \delta_{0,S_s} \exp\left(\frac{\kappa_2}{2} \int_s^t \delta_0(S_u)du \right) ds,$$

and hence that $h(x)=1+\frac{\kappa_2}{2}h(0)G(x)$. Thus, $h(0)<\infty$ implies (a) and that

$$h(x)=1+\frac{\kappa_2 G(x)}{2-\kappa_2 G(0)}. \tag{3.1}$$

(a),(b) $\Rightarrow$ (c): Since (b) implies that $\lim_{t\to\infty}|\overline{\eta}_t|=|\overline{\eta}_\infty|$ in $\mathbb{L}^2(P)$, it is enough to prove that

$$U_t \stackrel{\text{def.}}{=} \sum_{x\in\mathbb{Z}^d} \overline{\eta}_{t,x} f\left((x-mt)/\sqrt{t}\right) \longrightarrow 0 \quad \text{in } \mathbb{L}^2(P) \text{ as } t\nearrow\infty$$

for $f\in C_{\mathrm{b}}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f\,d\nu=0$.
We set $f_t(x,\widetilde{x})=f((x-m)/\sqrt{t})f((\widetilde{x}-m)/\sqrt{t})$. Then, by Lemma 2.1.4,

$$P[U_t^2]=\sum_{x,\widetilde{x}\in\mathbb{Z}^d} P[\overline{\eta}_{t,x}\overline{\eta}_{t,\widetilde{x}}]f_t(x,\widetilde{x})=\sum_{x,\widetilde{x}\in\mathbb{Z}^d}\eta_{0,x}\eta_{0,\widetilde{x}}P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_t f_t(Y_t,\widetilde{Y}_t)\right],$$

where $e_t=\exp\left(\kappa_2\int_0^t\delta_0(Y_s-\widetilde{Y}_s)ds\right)$. Note that by Lemma 2.1.3 and (a),

(1) $\quad P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_\infty\right]=h(x-\widetilde{x})\le h(0)<\infty.$

Since $|\eta_0|<\infty$, it is enough to prove that for each $x,\widetilde{x}\in\mathbb{Z}^d$

$$\lim_{t\to\infty}P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_t f_t(Y_t,\widetilde{Y}_t)\right]=0.$$

To prove this, we apply Lemma 2.2.2 to the Markov chain $\widetilde{Z}_t \stackrel{\text{def.}}{=} (Y_t,\widetilde{Y}_t)$ and the random walk (Z_t) on $\mathbb{Z}^d\times\mathbb{Z}^d$ with the generator

$$L_Z f(x,\widetilde{x})=\sum_{y,\widetilde{y}\in\mathbb{Z}^d} a_{x,\widetilde{x},y,\widetilde{y}}\left(f(y,\widetilde{y})-f(x,\widetilde{x})\right) \quad\text{with}\quad a_{x,\widetilde{x},y,\widetilde{y}}=\begin{cases} P[K_{\widetilde{y}-\widetilde{x}}] & \text{if } x=y \text{ and } \widetilde{x}\neq\widetilde{y},\\ P[K_{y-x}] & \text{if } x\neq y \text{ and } \widetilde{x}=\widetilde{y},\\ 0 & \text{if otherwise.}\end{cases}$$

Let $D=\{(x,\widetilde{x})\in\mathbb{Z}^d\times\mathbb{Z}^d\ ;\ x=\widetilde{x}\}$. Then,

(2) $a_{x,\widetilde{x},y,\widetilde{y}}=\Gamma_{y,\widetilde{y},x,\widetilde{x}}$ if $(x,\widetilde{x})\notin D\cup\{(y,\widetilde{y})\}$,

since

$$\begin{aligned}&\Gamma_{y,\widetilde{y},x,\widetilde{x}}\\ &\quad= P[(K_{y-x}-\delta_{y,x})\delta_{\widetilde{y},\widetilde{x}}+(K_{\widetilde{y}-\widetilde{x}}-\delta_{\widetilde{y},\widetilde{x}})\delta_{y,x}+(K_{y-x}-\delta_{y,x})(K_{\widetilde{y}-x}-\delta_{\widetilde{y},x})\delta_{x,\widetilde{x}}].\end{aligned}$$

Moreover, by (1.13),

(3) D is transient both for (Z_t) and for $(\widetilde{Z}_t)$.

Finally, the Gaussian measure $\nu \otimes \nu$ is the limit law in the central limit theorem for the random walk (Z_t). Therefore, by (1)–(3) and Lemma 2.2.2,

$$\lim_{t\to\infty} P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_t f_t(Y_t,\widetilde{Y}_t)\right] = P^{x,\widetilde{x}}_{Y,\widetilde{Y}}\left[e_\infty\right]\left(\int_{\mathbb{R}^d} f\,d\nu\right)^2 = 0.$$

(c) $\Rightarrow$ (b): This can be seen by taking $f \equiv 1$.
(1.16):By (2.10),

$$\sum_{x,\widetilde{x}\in\mathbb{Z}^d} P[\overline{\eta}_{t,x}\overline{\eta}_{t,\widetilde{x}}]f(x-\widetilde{x}) = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} \eta_{0,x}\eta_{0,\widetilde{x}} P^{x-\widetilde{x}}_S\left[\exp\left(\frac{\kappa_2}{2}\int_0^{2t}\delta_0(S_u)du\right)f(S_{2t})\right].$$

We apply Lemma 2.3.1 to the right-hand-side to get (1.16). □

3.2 Proof of Theorem 1.2.3

By the shift-invariance, we may assume that $b = 0$. We have by Lemma 2.1.4 that

$$P[\overline{\eta}^a_{t,x}\overline{\eta}^0_{t,\widetilde{x}}] = P^{a,0}_{Y,\widetilde{Y}}\left[\exp\left(\kappa_2\int_0^t \delta_0(Y_u-\widetilde{Y}_u)du\right) : (Y_t,\widetilde{Y}_t) = (x,\widetilde{x})\right],$$

and hence by Lemma 2.1.3 that

$$P[|\overline{\eta}^a_t||\overline{\eta}^0_t|] = P^{a,0}_{Y,\widetilde{Y}}\left[\exp\left(\kappa_2\int_0^{2t}\delta_0(Y_u-\widetilde{Y}_u)du\right)\right] = P^a_S\left[\exp\left(\frac{\kappa_2}{2}\int_0^{2t}\delta_0(S_u)du\right)\right].$$

By Theorem 1.2.1, both $|\overline{\eta}^a_t|$ and $|\overline{\eta}^0_t|$ are convergent in $\mathbb{L}^2(P)$ if $\frac{\kappa_2}{2}G(0) < 1$. Therefore, letting $t \nearrow \infty$, we conclude that

$$P[|\overline{\eta}^a_\infty||\overline{\eta}^0_\infty|] = P^a_S\left[\exp\left(\frac{\kappa_2}{2}\int_0^\infty \delta_0(S_u)du\right)\right] \stackrel{(3.1)}{=} 1 + \frac{\kappa_2 G(a)}{2-\kappa_2 G(0)}.$$

□

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