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**TRANSIENCE AND NON-EXPLOSION OF CERTAIN  
STOCHASTIC NEWTONIAN SYSTEMS**

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**Abstract:** We give sufficient conditions for non-explosion and transience of the solution  $(x_t, p_t)$  (in dimensions  $\geq 3$ ) to a stochastic Newtonian system of the form

$$\begin{cases} dx_t = p_t dt \\ dp_t = -\frac{\partial V(x_t)}{\partial x} dt - \frac{\partial c(x_t)}{\partial x} d\xi_t \end{cases},$$

where  $\{\xi_t\}_{t \geq 0}$  is a  $d$ -dimensional Lévy process,  $d\xi_t$  is an Itô differential and  $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ ,  $V \in C^2(\mathbb{R}^d, \mathbb{R})$  such that  $V \geq 0$ .

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# 1 Introduction

This work contributes to the series of papers [13, 15], [3, 4], [6], [20] and [19] which are devoted to the qualitative study of the Newton equations driven by random noise. For related results see also [5], [23], [26, 27], [1], [22] and the references given there. Newton equations of this type are interesting in their own right: as models for the dynamics of particles moving in random media (cf. [25]), in the theory of interacting particles (cf. [28], [29]) or in the theory of random matrices (cf. [24]), to mention but a few. On the other hand, the study of these equations fits nicely into the the larger context of (stochastic) partial differential equations, in particular Hamilton-Jacobi, heat and Schrödinger equations, driven by random noise (see [32, 33] and [14, 16, 17, 18]).

In most papers on this subject the driving stochastic process is a diffusion process with continuous sample paths, usually a standard Wiener process. Motivated by the recent growth of interest in Lévy processes, which can be observed both in mathematics literature and in applications, the present authors started in [20] and [19] the analysis of Newton systems driven by jump processes, in particular symmetric stable Lévy processes. In [20] we studied the rate of escape of a “free” particle driven by a stable Lévy process and its applications to the scattering theory of a system describing a particle driven by a stable noise and a (deterministic) external force.

In this paper we study non-explosion and transience of Newton systems of the form

$$\begin{cases} dx_t = p_t dt \\ dp_t = -\frac{\partial V(x_t)}{\partial x} dt - \frac{\partial c(x_t)}{\partial x} d\xi_t \end{cases}, \quad (1)$$

where  $\xi_t = (\xi_t^1, \dots, \xi_t^d)$  is a  $d$ -dimensional Lévy process,  $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ ,  $V \in C^2(\mathbb{R}^d)$ ,  $V \geq 0$  and  $\left(\frac{\partial c(x_t)}{\partial x} d\xi_t\right)_i := \sum_{j=1}^d \frac{\partial c_i(x_t)}{\partial x_j} d\xi_t^j$  is an Itô stochastic differential.

In Section 3 we give conditions under which the solutions do not explode in finite time. For symmetric  $\alpha$ -stable driving processes  $\xi_t = \xi_t^{(\alpha)}$  we show in Section 4 that the solution process of the system (1) is always transient in dimensions  $d \geq 3$ . We consider it as an interesting open problem to find necessary and sufficient conditions for transience and recurrence for the system (1) in dimensions  $d < 3$ . Even in the case of a driving Wiener process (white noise) only some partial results are available for  $d = 1$ , see [4, 3].

## 2 Lévy Processes

The driving processes for our Newtonian system will be Lévy processes. Recall that a  $d$ -dimensional *Lévy process*  $\{\xi_t\}_{t \geq 0}$  is a stochastic process with state space  $\mathbb{R}^d$  and independent and stationary increments; its paths  $t \mapsto \xi_t$  are continuous in probability which amounts to saying that there are almost surely no fixed discontinuities. We can (and will) always choose a modification with *càdlàg* (i.e., right-continuous with finite left limits) paths and  $\xi_0 = 0$ . Unless otherwise stated, we will always consider the augmented natural filtration of  $\{\xi_t\}_{t \geq 0}$  which satisfies the “usual conditions”. Because of the independent increment property the Fourier

transform of the distribution of  $\xi_t$  is of the form

$$\mathbb{E}(e^{i\eta\xi_t}) = e^{-t\psi(\eta)}, \quad t > 0, \eta \in \mathbb{R}^d,$$

with the *characteristic exponent*  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$  which is given by the *Lévy-Khinchine formula*

$$\psi(\eta) = -i\beta\eta + \eta Q \eta + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{iy\eta} + iy\eta \mathbf{1}_{\{|y|<1\}}) \nu(dy). \quad (2)$$

Here  $\beta \in \mathbb{R}^d$ ,  $Q = (q_{ij}) \in \mathbb{R}^{d \times d}$  is a positive semidefinite matrix and  $\nu$  is a Lévy measure, i.e., a Radon measure on  $\mathbb{R}^d \setminus \{0\}$  with  $\int_{y \neq 0} |y|^2 \wedge 1 \nu(dy) < \infty$ . The Lévy-triple  $(\beta, Q, \nu)$  can also be used to obtain the *Lévy decomposition* of  $\xi_t$ ,

$$\xi_t = W_t^Q + \iint_{[0,t] \times \{0 < |y| < 1\}} y \tilde{N}(dy, ds) + \iint_{[0,t] \times \{|y| \geq 1\}} y N(dy, ds) + \beta t \quad (3)$$

where  $\Delta\xi_t := \xi_t - \xi_{t-}$ ,  $\xi_{0-} := \xi_0$ ,  $N(dy, ds) = \sum_{0 \leq t \leq s} \mathbf{1}_{\{\Delta\xi_t \neq 0\}} \delta_{(\Delta\xi_t, t)}(dy, ds)$ , is the canonical jump measure,  $\tilde{N}(dy, ds) = N(dy, ds) - \nu(dy) ds$  is the compensated jump measure,  $W_t^Q$  is a Brownian motion with covariance matrix  $Q$  and  $\beta t$  is a deterministic drift with  $\beta = \mathbb{E}(\xi_1 - \sum_{s \leq 1} \Delta\xi_s \mathbf{1}_{\{|\Delta\xi_s| \geq 1\}})$ . Notice that the first two terms in the above decomposition (3) are martingales.

**Lemma 1.** *Let  $\{\xi_t\}_{t \geq 0}$  be a  $d$ -dimensional Lévy process whose jumps are bounded by  $2R$ . Then*

$$\mathbb{E}([\xi^i, \xi^j]_t) \leq t \max_{1 \leq i, j \leq d} |q_{ij}| + t \int_{0 < |y| < 2R} |y|^2 \nu(dy), \quad t > 0,$$

where  $[\xi^i, \xi^j]_\bullet$  denotes the quadratic (co)variation process.

This Lemma is a simple consequence of the well-known formula

$$\mathbb{E}([\xi^i, \xi^j]_t) = \mathbb{E}\left([W^i, W^j]_t + \sum_{s \leq t} \Delta\xi_s^i \Delta\xi_s^j\right) = t \left(q_{ij} + \int_{|y| < 2R} y^i y^j \nu(dy)\right).$$

It is well-known that Lévy processes are Feller processes. The infinitesimal generator  $(A, \mathfrak{D}(A))$  of the process (more precisely: of the associated Feller semigroup) is a *pseudo-differential operator*  $A|_{C_c^\infty(\mathbb{R}^d)} = -\psi(D)$  with *symbol*  $-\psi$ , i.e.,

$$-\psi(D)u(x) := -(2\pi)^{-d/2} \int_{\mathbb{R}^d} \psi(\eta) \widehat{u}(\eta) e^{iy\eta} d\eta, \quad u \in C_c^\infty(\mathbb{R}^d), \quad (4)$$

where  $\widehat{u}(\eta)$  denotes the Fourier transform of  $u$ . The test functions  $C_c^\infty(\mathbb{R}^d)$  are an operator core. Later on, we will also use the following simple fact.

**Lemma 2.** Let  $u \in C_c^\infty(\mathbb{R}^d)$  and  $u_R(x) := Ru(\frac{x}{R})$ ,  $R \geq 1$ . Then

$$|\psi(D)u_R(x)| \leq C_\psi R \int_{\mathbb{R}^d} (1 + |\eta|^2) |\widehat{u}(\eta)| d\eta = C_{\psi,u} R$$

uniformly for all  $x \in \mathbb{R}^d$  with an absolute constant  $C_{\psi,u}$ .

*Proof.* Observe that  $\widehat{u}_R(\eta) = R^{d+1} \widehat{u}(R\eta)$ . Therefore,

$$\begin{aligned} |\psi(D)u_R(x)| &= (2\pi)^{-d/2} \left| \int_{\mathbb{R}^d} e^{ix\eta} \psi(\eta) \widehat{u}_R(\eta) d\eta \right| \\ &\leq (2\pi)^{-d/2} R \int_{\mathbb{R}^d} R^d |\psi(\eta) \widehat{u}(R\eta)| d\eta \\ &= (2\pi)^{-d/2} R \int_{\mathbb{R}^d} \left| \psi\left(\frac{\eta}{R}\right) \widehat{u}(\eta) \right| d\eta \\ &\leq (2\pi)^{-d/2} C_\psi R \int_{\mathbb{R}^d} \left(1 + \left|\frac{\eta}{R}\right|^2\right) |\widehat{u}(\eta)| d\eta \\ &\leq (2\pi)^{-d/2} C_\psi R \int_{\mathbb{R}^d} (1 + |\eta|^2) |\widehat{u}(\eta)| d\eta, \end{aligned}$$

where we used that  $|\psi(\eta)| \leq C_\psi(1 + |\eta|^2)$  for all  $\eta \in \mathbb{R}^d$  with some absolute constant  $C_\psi > 0$ . Since  $u \in C_c^\infty(\mathbb{R}^d)$ ,  $\widehat{u}$  is a rapidly decreasing function which means that the integral in the last line is finite.  $\square$

Our standard references for the analytic theory of Lévy and Feller processes is the book [10] by Jacob, see also [11]; for stochastic calculus of semimartingales and stochastic differential equations we use Protter [30].

### 3 Non-explosion

Let  $(X_t, P_t) = (X(t, x_0, p_0), P(t, x_0, p_0))$  be a solution of the system (1) with initial condition  $(x_0, p_0) \in \mathbb{R}^{2d}$  at  $t = 0$ , where  $\xi_t = (\xi_t^1, \dots, \xi_t^d)$  is a  $d$ -dimensional Lévy process,  $d \geq 1$ ,  $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ ,  $V \in C^2(\mathbb{R}^d)$ ,  $V \geq 0$  and  $\partial c/\partial x$  is uniformly bounded. Clearly, these conditions ensure local (i.e., for small times) existence and uniqueness of the solution, see e.g., [30].

The random times

$$T_m := \inf\{s \geq 0 : |X_s| \vee |P_s| \geq m\} \tag{5}$$

are stopping times w.r.t. the (augmented) natural filtration of the Lévy process  $\{\xi_t\}_{t \geq 0}$  and so is the *explosion time*  $T_\infty := \sup_m T_m$  of the system (1).

**Theorem 3.** Under the assumptions stated above, the explosion time  $T_\infty$  of the system (1) is almost surely infinite, i.e.,  $\mathbb{P}(T_\infty = \infty) = 1$ .

*Proof. Step 1.* Let  $\tau_m := \inf\{s \geq 0 : |P_s| \geq m\}$  and  $\tau_\infty := \sup_m \tau_m$ . It is clear that  $T_m \leq \tau_m$  and so  $T_\infty \leq \tau_\infty$ . Suppose that  $T_\infty(\omega) < t < \tau_m(\omega) \leq \tau_\infty(\omega)$  for some  $t > 0$  and  $m \in \mathbb{N}$ . From the first equation in (1) we deduce that for every  $k \in \mathbb{N}$

$$\sup_{s \in [0, T_k(\omega)]} |X_s(\omega)| \leq |x_0| + t \sup_{s \in [0, t]} |P_s(\omega)| \leq |x_0| + tm.$$

On the other hand, since  $T_k(\omega) < T_\infty(\omega) < t < \tau_\infty(\omega)$ , we find that  $\sup_{k \in \mathbb{N}} \sup_{s \in [0, T_k]} |X_s(\omega)| = \infty$ . This, however, leads to a contradiction, and so  $\tau_\infty = T_\infty$ .

*Step 2.* We will show that  $\mathbb{P}(\tau_\infty = \infty) = 1$ . Set  $H(x, p) := \frac{1}{2}p^2 + V(x)$  and  $H_t = H(X_t, P_t)$ . Since  $H(x, p)$  is twice continuously differentiable, we can use Itô's formula (for jump processes and in the slightly unusual form of Protter [30, p. 71, (\*\*\*)]). For this observe that only the quadratic variation of the Lévy process  $[\xi, \xi] := ([\xi^i, \xi^j])_{ij} \in \mathbb{R}^{d \times d}$  contributes to the quadratic variation of  $\{(X_t, P_t)\}_{t \geq 0}$ :

$$[(X, P), (X, P)] = \begin{pmatrix} 0 & 0 \\ 0 & \left[ \frac{\partial c}{\partial x} \xi, \frac{\partial c}{\partial x} \xi \right] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \left( \frac{\partial c}{\partial x} \right) [\xi, \xi] \left( \frac{\partial c}{\partial x} \right)^T \end{pmatrix} \in \mathbb{R}^{2d \times 2d}.$$

Therefore,

$$dH_t = P_{t-} dP_t + \frac{1}{2} \operatorname{tr} \left( \frac{\partial c(X_{t-})}{\partial x} d[\xi, \xi]_t \left( \frac{\partial c(X_{t-})}{\partial x} \right)^T \right) + \frac{\partial V(X_t)}{\partial x} P_t dt + \Sigma_t,$$

where

$$\Sigma_t = \frac{1}{2} \sum_{0 \leq s \leq t} (P_s^2 - P_{s-}^2 - 2P_{s-}(P_s - P_{s-}) - (P_s - P_{s-})^2) = 0.$$

The first equation in (1),  $dX_t = P_t dt$ , implies that  $X_t$  is a continuous function; the second equation,  $dP_t = -\partial V(X_t)/\partial x dt - \partial c(X_t)/\partial x d\xi_t$ , gives

$$dH_t = -P_{t-} \frac{\partial c(X_t)}{\partial x} d\xi_t + \frac{1}{2} \operatorname{tr} \left( \frac{\partial c(X_t)}{\partial x} d[\xi, \xi]_t \left( \frac{\partial c(X_t)}{\partial x} \right)^T \right). \quad (6)$$

Let  $\sigma_R := \inf\{t > 0 : |\xi_t| \geq R\}$  be the first exit time of the process  $\{\xi_t\}_{t \geq 0}$  from the ball  $B_R(0)$ . Then

$$\sigma = \sigma_{\ell, m, R} := \ell \wedge \sigma_R \wedge \tau_m, \quad \ell, m \in \mathbb{N},$$

is again a stopping time and we calculate from (6) that

$$\begin{aligned} H_{\sigma-} - H_0 &= - \int_0^{\sigma-} P_{t-} \frac{\partial c(X_t)}{\partial x} d\xi_t + \frac{1}{2} \int_0^{\sigma-} \operatorname{tr} \left( \frac{\partial c(X_t)}{\partial x} d[\xi, \xi]_t \left( \frac{\partial c(X_t)}{\partial x} \right)^T \right) \\ &= \mathbf{I} + \mathbf{II}. \end{aligned} \quad (7)$$

*Step 3.* Recall that  $-\psi(D)$  is the generator of the Lévy process  $\xi_t$ . We want to estimate  $|\mathbb{E}(\mathbf{I})|$ . For this purpose choose a function  $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$  such that  $\phi(x) = x$  if  $|x| \leq 1$ ,  $\operatorname{supp} \phi \subset \{x : |x| \leq 2\}$  and define  $\phi_R(x) = R\phi\left(\frac{x}{R}\right)$ . Clearly,

$$\phi_R(\xi_t) = \xi_t, \quad t < \sigma_R, \quad (8)$$

and, since  $\phi_R \in C_c^\infty(\mathbb{R}^d) \subset \mathfrak{D}(A)$  is in the domain of the generator of  $\xi_t$ , we find that

$$M_t^{\phi_R} := \phi_R(\xi_t) + \int_0^t \psi(D)\phi_R(\xi_s) ds \quad (9)$$

is an  $L^2$ -martingale (w.r.t. the natural filtration of  $\{\xi_t\}_{t \geq 0}$ ). The stopped process  $(M_{t \wedge \tau_m \wedge \ell}^{\phi_R})_{t \geq 0}$  is again an  $L^2$ -martingale for fixed  $m, \ell \in \mathbb{N}$ . We can now use (8) and (9) to get

$$\mathbf{I} = - \int_0^{\sigma^-} P_{t-} \frac{\partial c(X_t)}{\partial x} dM_{t \wedge \tau_m \wedge \ell}^{\phi_R} + \int_0^{\sigma^-} P_{t-} \frac{\partial c(X_t)}{\partial x} \psi(D)\phi_R(\xi_t) dt = \mathbf{I}' + \mathbf{I}''.$$

Clearly,  $\int_0^\bullet P_{t-} (\partial c(X_t)/\partial x) dM_{t \wedge \tau_m \wedge \ell}^{\phi_R}$  is a local martingale. Since

$$\begin{aligned} & \left[ \int_0^\bullet P_{s-} \frac{\partial c(X_s)}{\partial x} dM_{s \wedge \tau_m \wedge \ell}^{\phi_R}, \int_0^\bullet P_{s-} \frac{\partial c(X_s)}{\partial x} dM_{s \wedge \tau_m \wedge \ell}^{\phi_R} \right]_t \\ &= \int_0^t P_{s-}^2 \left( \frac{\partial c(X_s)}{\partial x} \right)^2 d[M_{\bullet}^{\phi_R}, M_{\bullet}^{\phi_R}]_{s \wedge \tau_m \wedge \ell} \\ &= \int_0^{t \wedge \tau_m \wedge \ell} P_{s-}^2 \left( \frac{\partial c(X_s)}{\partial x} \right)^2 d[M_{\bullet}^{\phi_R}, M_{\bullet}^{\phi_R}]_{s \wedge \tau_m \wedge \ell} \end{aligned}$$

we find for every  $t > 0$

$$\begin{aligned} & \left| \mathbb{E} \left[ \int_0^\bullet P_{s-} \frac{\partial c(X_s)}{\partial x} dM_{s \wedge \tau_m \wedge \ell}^{\phi_R}, \int_0^\bullet P_{s-} \frac{\partial c(X_s)}{\partial x} dM_{s \wedge \tau_m \wedge \ell}^{\phi_R} \right]_t \right| \\ & \leq m^2 \left\| \frac{\partial c}{\partial x} \right\|_\infty^2 \mathbb{E} [M_{\bullet}^{\phi_R}, M_{\bullet}^{\phi_R}]_t < \infty, \end{aligned}$$

where we used that  $|P_{s-}| \leq m$  if  $s \leq \ell \wedge \tau_m$  and that  $M_t^{\phi_R}$  is an  $L^2$ -martingale. This shows that  $\int_0^\bullet P_{t-} (\partial c(X_t)/\partial x) dM_t^{\phi_R}$  is a martingale (cf. [30], p.66 Corollary 3) and we may apply optional stopping to the bounded stopping time  $\sigma$  to get

$$\begin{aligned} \mathbb{E}(\mathbf{I}') &= -\mathbb{E} \left( \int_0^\sigma P_{t-} \frac{\partial c(X_t)}{\partial x} dM_t^{\phi_R} \right) + \mathbb{E} \left( P_{\sigma-} \frac{\partial c(X_\sigma)}{\partial x} \Delta M_\sigma^{\phi_R} \right) \\ &= \mathbb{E} \left( P_{\sigma-} \frac{\partial c(X_\sigma)}{\partial x} \Delta M_\sigma^{\phi_R} \right). \end{aligned}$$

Therefore

$$|\mathbb{E}(\mathbf{I}')| \leq md^2 \left\| \frac{\partial c}{\partial x} \right\|_\infty \mathbb{E} |\Delta M_\sigma^{\phi_R}| \leq 2mRd^2 \left\| \frac{\partial c}{\partial x} \right\|_\infty \|\phi\|_\infty, \quad (10)$$

where we used

$$\left| \Delta M_\sigma^{\phi_R} \right| = |\phi_R(\xi_\sigma) - \phi_R(\xi_{\sigma-})| \leq 2R \|\phi\|_\infty$$

and the notation

$$\left\| \frac{\partial c}{\partial x} \right\|_\infty := \max_{i,j=1,\dots,d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial c_i(x)}{\partial x_j} \right|.$$

*Step 4.* For the estimate of  $\mathbb{E}(\mathbf{I}'')$ , we use Lemma 2 with  $u = \phi$  to get  $\|\psi(D)\phi_R\|_\infty \leq C_{\psi,\phi}$ , and also  $\sigma \leq \ell$ , so

$$|\mathbb{E}(\mathbf{I}'')| \leq C_{\psi,\phi} R \mathbb{E} \left( \sup_{t < \sigma} \left| P_{t-} \frac{\partial c(X_t)}{\partial x} \right| \right) \ell \leq C_2 \left\| \frac{\partial c}{\partial x} \right\|_\infty R m \ell. \quad (11)$$

Put together, the estimates (10), (11) give

$$|\mathbb{E}(\mathbf{I})| \leq C_3 R m \ell. \quad (12)$$

*Step 5.* We proceed with  $|\mathbb{E}(\mathbf{II})|$ . From

$$\|AB\|_\infty \leq d \|A\|_\infty \|B\|_\infty, \quad A, B \in \mathbb{R}^{d \times d},$$

where  $\|A\|_\infty = \max_{i,j=1,\dots,d} |A_{ij}|$ , we get

$$\int_0^t \text{tr} \left[ \frac{\partial c(X_s)}{\partial x} d[\xi, \xi]_s \left( \frac{\partial c(X_s)}{\partial x} \right)^T \right] \leq d^3 \left\| \frac{\partial c}{\partial x} \right\|_\infty^2 \|\xi, \xi\|_t \infty.$$

Since we have  $\sup_{s \leq t} |\xi_s| \leq R$  for  $t < \sigma_R$ , the jumps  $|\Delta \xi_s|$ ,  $s \leq t$ , cannot exceed  $2R$ . Lemma 1 then shows

$$\mathbb{E} \left( [\xi^i, \xi^j]_{\ell \wedge \sigma_{R-}} \right) \leq \ell \int_{0 < |y| \leq 2R} |y|^2 \nu(dy) + \ell \|Q\|_\infty$$

and so

$$|\mathbb{E}(\mathbf{II})| \leq C_4 \ell \left( \int_{0 < |y| \leq 2R} |y|^2 \nu(dy) + \|Q\|_\infty \right). \quad (13)$$

*Step 6.* Combining (7), (12), (13) we obtain

$$\mathbb{E}(H_{\sigma-}) \leq H_0 + C_3 R m \ell + C_4 \ell \left( \int_{0 < |y| \leq 2R} |y|^2 \nu(dy) + \|Q\|_\infty \right). \quad (14)$$

On the other hand, by Jensen's inequality,

$$\begin{aligned} \mathbb{E}(H_{\sigma-}) &= \frac{1}{2} \mathbb{E}(P_{\sigma-}^2) + \mathbb{E}(V(X_{\sigma-})) \geq \frac{1}{2} \mathbb{E}(P_{\sigma-}^2) \\ &\geq \frac{1}{2} [\mathbb{E}(|P_{\sigma-}|)]^2 \\ &\geq \frac{1}{2} [\mathbb{E}(|P_{\ell \wedge \tau_m \wedge \sigma_{R-}}| \mathbf{1}_{\{\tau_m < \ell \wedge \sigma_R\}})]^2 \\ &= \frac{1}{2} [\mathbb{E}(|P_{\tau_m} - \Delta P_{\tau_m}| \mathbf{1}_{\{\tau_m < \ell \wedge \sigma_R\}})]^2. \end{aligned}$$

Clearly,  $|P_{\tau_m}| \geq m$  and, since on  $\{s < \sigma_R\}$  the driving Lévy process has jumps of size  $|\Delta\xi_s| \leq 2R$ , we find from (1) that

$$|\Delta P_{\tau_m}| \mathbf{1}_{\{\tau_m < \ell \wedge \sigma_R\}} \leq 2R \left\| \frac{\partial c}{\partial x} \right\|_{\infty} \mathbf{1}_{\{\tau_m < \ell \wedge \sigma_R\}}.$$

Choosing  $m$  sufficiently large, say  $m > 2R\|(\partial c/\partial x)\|_{\infty}$ , we arrive at

$$\begin{aligned} \mathbb{E}(H_{\sigma^-}) &\geq \frac{1}{2} [\mathbb{E}(m - |\Delta P_{\tau_m}|) \mathbf{1}_{\{\tau_m < \ell \wedge \sigma_R\}}]^2 \\ &\geq \frac{1}{2} \left( m - 2R \left\| \frac{\partial c}{\partial x} \right\|_{\infty} \right)^2 \{\mathbb{P}(\tau_m < \ell \wedge \sigma_R)\}^2. \end{aligned} \quad (15)$$

We can now combine (14) and (15) to find

$$\begin{aligned} \{\mathbb{P}(\tau_m < \ell \wedge \sigma_R)\}^2 &\leq \frac{2(H_0 + C_3 R m \ell)}{(m - 2R\|(\partial c/\partial x)\|_{\infty})^2} \\ &\quad + \frac{2C_4 \ell}{(m - 2R\|(\partial c/\partial x)\|_{\infty})^2} \left( \int_{0 < |y| \leq 2R} |y|^2 \nu(dy) + \|Q\|_{\infty} \right). \end{aligned}$$

Letting first  $m \rightarrow \infty$  and then  $R \rightarrow \infty$  shows  $\mathbb{P}(\tau_{\infty} \leq \ell) = 0$  for all  $\ell \in \mathbb{N}$ , so  $\mathbb{P}(\tau_{\infty} = \infty) = 1$ , and the claim follows.  $\square$

## 4 Transience

We will now prove that the solution  $\{(X_t, P_t)\}_{t \geq 0}$  of the Newton system (1) is transient, at least if the driving noise is a symmetric stable Lévy process  $\xi_t = \xi_t^{(\alpha)}$  with index  $\alpha \in (0, 2)$ . Symmetric  $\alpha$ -stable Lévy processes have no drift, no Brownian part and their Lévy measures are  $\nu(dy) = c_{\alpha} |y|^{-d-\alpha} dy$ , where

$$c_{\alpha} = \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+d}{2}\right)}{\pi^{\alpha/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}. \quad (16)$$

We restrict ourselves to presenting this particular case, but it is clear that, with minor alterations, the proof of Theorem 6 below remains valid for *any* driving Lévy process *with rotationally symmetric Lévy measure*.

Our proof is based on the following result which extends a well-known transience criterion for diffusion processes to jump processes, see for instance [8] or [21].

Denote by  $\{T_t\}_{t \geq 0}$  the operator semigroup associated with a stochastic process and let  $(A, \mathfrak{D}(A))$  be its generator. The *full generator* is the set

$$\widehat{A} := \left\{ (f, g) \in C_b \times C_b : T_t f - f = \int_0^t T_s g ds \right\},$$

see Ethier, Kurtz [7] p. 24. It is clear that  $(u, Au) \in \widehat{A}$  for all  $u \in \mathfrak{D}(A)$ .

**Lemma 4.** Let  $\{\eta_t\}_{t \geq 0}$  be an  $\mathbb{R}^n$ -valued, càdlàg strong Markov process with generator  $(A, \mathfrak{D}(A))$  and full generator  $\tilde{A}$ . Let  $D \subset \mathbb{R}^n$  be a bounded Borel set and assume that there exists a sequence  $\{u_k\}_{k \in \mathbb{N}} \subset C_b(\mathbb{R}^n)$  and some function  $u \in C(\mathbb{R}^n)$ , such that the following conditions are satisfied:

- (i)  $A$  has an extension  $\tilde{A}$  such that  $\tilde{A}u_k$  is pointwise defined,  $(u_k, \tilde{A}u_k) \in \hat{A}$  and  $\lim_{k \rightarrow \infty} (u_k, \tilde{A}u_k) = (u, \tilde{A}u)$  exists locally uniformly.
- (ii)  $u \geq 0$  and  $\inf_D u > a > 0$  for some  $a > 0$ .
- (iii)  $u(y_0) < a$  for some  $y_0 \notin \bar{D}$ .
- (iv)  $\tilde{A}u \leq 0$  in  $D^c$ .

Then  $\{\eta_t\}_{t \geq 0}$  is transient.

*Proof.* Since  $(u_k, \tilde{A}u_k) \in \hat{A}$ , we know that

$$M_t^k = u_k(\eta_t) - \int_0^t \tilde{A}u_k(\eta_s) ds, \quad k \in \mathbb{N},$$

are martingales, see Ethier, Kurtz [7, p. 162, Prop. 4.1.7]. We set

$$\tau_D = \inf\{t > 0 : \eta_t \in D\} \quad \text{and} \quad \sigma_R = \inf\{t > 0 : |\eta_t - \eta_0| > R\}$$

and from an optional stopping argument we find for any fixed  $T > 0$

$$\mathbb{E}^{y_0} \left( M_{\tau_D \wedge \sigma_R \wedge T}^k \right) = \mathbb{E}^{y_0} (M_0^k) = \mathbb{E}^{y_0} (u_k(\eta_0)).$$

On the other hand,

$$\mathbb{E}^{y_0} \left( M_{\tau_D \wedge \sigma_R \wedge T}^k \right) = \mathbb{E}^{y_0} \left( u_k(\eta_{\tau_D \wedge \sigma_R \wedge T}) - \int_0^{\tau_D \wedge \sigma_R \wedge T} \tilde{A}u_k(\eta_s) ds \right),$$

and because of assumption (i) we can pass to the limit  $k \rightarrow \infty$  to get

$$\begin{aligned} a > u(y_0) &= \lim_{k \rightarrow \infty} u_k(y_0) \\ &= \lim_{k \rightarrow \infty} \mathbb{E}^{y_0} \left( u_k(\eta_{\tau_D \wedge \sigma_R \wedge T}) - \int_0^{\tau_D \wedge \sigma_R \wedge T} \tilde{A}u_k(\eta_s) ds \right) \\ &= \mathbb{E}^{y_0} \left( u(\eta_{\tau_D \wedge \sigma_R \wedge T}) - \int_0^{\tau_D \wedge \sigma_R \wedge T} \tilde{A}u(\eta_s) ds \right) \\ &\geq \mathbb{E}^{y_0} (u(\eta_{\tau_D \wedge \sigma_R \wedge T})) \\ &\geq \mathbb{E}^{y_0} (u(\eta_{\tau_D \wedge \sigma_R \wedge T}) \mathbf{1}_{\{\tau_D < \infty\}}), \end{aligned}$$

where we used in the penultimate step that  $\tilde{A}u|_{D^c} \leq 0$ .

As  $u \in C^+(\mathbb{R}^n)$ , we may use dominated convergence and let  $T \rightarrow \infty$  and Fatou's Lemma to let  $R \rightarrow \infty$ . Thus,

$$\begin{aligned} a > u(y_0) &\geq \liminf_{R \rightarrow \infty} \mathbb{E}^{y_0} (u(\eta_{\tau_D \wedge \sigma_R}) \mathbf{1}_{\{\tau_D < \infty\}}) \geq \mathbb{E}^{y_0} (u(\eta_{\tau_D}) \mathbf{1}_{\{\tau_D < \infty\}}) \\ &\geq (\inf_D u) \mathbb{P}^{y_0}(\tau_D < \infty) > a \mathbb{P}^{y_0}(\tau_D < \infty). \end{aligned}$$

Therefore,  $\mathbb{P}^{y_0}(\tau_D < \infty) < 1$ , and, see e.g [2],  $\{\eta_t\}_{t \geq 0}$  is transient.  $\square$

We will now turn to the task to determine the infinitesimal generator of the solution process  $\{(X_t, P_t)\}_{t \geq 0}$ . The following result is, in various settings, common knowledge. We could not find a precise reference in our situation, though. Since we need some technical details of the proof, we include the standard argument.

**Lemma 5.** *Let  $\{\xi_t\}_{t \geq 0}$  be a  $d$ -dimensional Lévy process with characteristic exponent  $\psi$  and Lévy triple  $(\alpha, Q, \nu)$ . The (pointwise) infinitesimal generator of the process  $(X_t, P_t) = (X(t, x_0, p_0), P(t, x_0, p_0))$  solving (1) is of the form*

$$\begin{aligned} Au(x, p) &= \frac{\partial u(x, p)}{\partial x} p - \frac{\partial u(x, p)}{\partial p} \left( \frac{\partial V(x)}{\partial x} + \frac{\partial c(x)}{\partial x} \beta \right) \\ &+ \frac{1}{2} \text{tr} \left( \frac{\partial^2 u(x, p)}{\partial p^2} \left( \frac{\partial c(x)}{\partial x} \right) Q \left( \frac{\partial c(x)}{\partial x} \right)^T \right) \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x, p - \frac{\partial c(x)}{\partial x} y) - u(x, p) + \frac{\partial u(x, p)}{\partial p} \frac{\partial c(x)}{\partial x} y \mathbf{1}_{\{|y| < 1\}} \right) \nu(dy). \end{aligned}$$

for all  $u \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$  and with  $\beta = \mathbb{E}^0 (\xi_1 - \sum_{0 \leq s \leq 1} \Delta \xi_s \mathbf{1}_{\{|\Delta \xi_s| \geq 1\}})$ . In particular, the pairs  $(u, Au)$ ,  $u \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$ , are in the full generator  $\widehat{A}$  of the process.

*Proof.* For  $u = u(x, p) \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$  we can use Itô's formula (for jump processes, now in the usual form [30, p. 70, Theorem II.32]) and get with a similar calculation to the one made in the proof of Theorem 3

$$\begin{aligned} u(X_t, P_t) - u(x_0, p_0) &= \int_0^t \frac{\partial u}{\partial x} P_s ds - \int_0^t \frac{\partial u}{\partial p} \frac{\partial V}{\partial x} ds - \int_0^t \frac{\partial u}{\partial p} \frac{\partial c}{\partial x} d\xi_s \\ &+ \frac{1}{2} \int_0^t \text{tr} \left( \frac{\partial^2 u}{\partial p^2} \left( \frac{\partial c}{\partial x} \right) Q \left( \frac{\partial c}{\partial x} \right)^T \right) ds \\ &+ \sum_{0 \leq s \leq t} \left( u(X_s, P_s) - u(X_s, P_{s-}) + \frac{\partial u}{\partial p}(X_s, P_{s-}) \frac{\partial c}{\partial x} \Delta \xi_s \right). \end{aligned}$$

Here we used the fact that the continuous martingale part of  $\xi_t$  is  $W_t^Q$ , and so  $[\xi, \xi]_t^c = [W^Q, W^Q]_t = Qt$ . Note that we suppressed arguments in those places where no ambiguity

is possible. Since  $P_s = P_{s-} + \Delta P_s = P_{s-} - \frac{\partial c}{\partial x} \Delta \xi_s$  we find, using the Lévy decomposition (3),

$$\begin{aligned}
& u(X_t, P_t) - u(x_0, p_0) \\
&= \int_0^t \frac{\partial u}{\partial x} P_s ds - \int_0^t \frac{\partial u}{\partial p} \frac{\partial V}{\partial x} ds - \int_0^t \frac{\partial u}{\partial p} \frac{\partial c}{\partial x} \beta ds - \int_0^t \frac{\partial u}{\partial p} \frac{\partial c}{\partial x} dW_s^Q \\
&\quad - \int_0^t \frac{\partial u}{\partial p} \frac{\partial c}{\partial x} \int_{0 < |y| < 1} y \tilde{N}(dy, ds) + \frac{1}{2} \int_0^t \text{tr} \left( \frac{\partial^2 u}{\partial p^2} \left( \frac{\partial c}{\partial x} \right) Q \left( \frac{\partial c}{\partial x} \right)^T \right) ds \\
&\quad + \iint \left( u(X_s, P_{s-} - \frac{\partial c}{\partial x} y) - u(X_s, P_{s-}) + \frac{\partial u(X_s, P_{s-})}{\partial p} \frac{\partial c}{\partial x} y \mathbf{1}_{\{|y| < 1\}} \right) \tilde{N}(dy, ds) \\
&\quad + \iint \left( u(X_s, P_{s-} - \frac{\partial c}{\partial x} y) - u(X_s, P_{s-}) + \frac{\partial u(X_s, P_{s-})}{\partial p} \frac{\partial c}{\partial x} y \mathbf{1}_{\{|y| < 1\}} \right) \nu(dy) ds
\end{aligned}$$

with the double integrals ranging over  $[0, t] \times \mathbb{R}^d \setminus \{0\}$ . The function  $u$  has compact support, and we may take expectations on both sides of the above relation and differentiate in  $t$ . Since the terms driven by  $\tilde{N}(dy, ds)$  or  $dW_s^Q$  are martingales, we find

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E}(u(X_t, P_t)) \Big|_{t=0} \\
&= \frac{\partial u(x_0, p_0)}{\partial x} p_0 - \frac{\partial u(x_0, p_0)}{\partial p} \frac{\partial V(x_0)}{\partial x} - \frac{\partial u(x_0, p_0)}{\partial p} \frac{\partial c(x_0)}{\partial x} \beta \\
&\quad + \frac{1}{2} \text{tr} \left( \frac{\partial^2 u(x_0, p_0)}{\partial p^2} \left( \frac{\partial c(x_0)}{\partial x} \right) Q \left( \frac{\partial c(x_0)}{\partial x} \right)^T \right) \\
&\quad + \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x_0, p_0 - \frac{\partial c(x_0)}{\partial x} y) - u(x_0, p_0) + \frac{\partial u(x_0, p_0)}{\partial p} \frac{\partial c(x_0)}{\partial x} y \mathbf{1}_{\{|y| < 1\}} \right) \nu(dy),
\end{aligned}$$

which is what we claimed. Notice, that the convergence is pointwise, so that it is not clear that  $C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$  is in the domain of the generator. However, our calculation shows that  $Au \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$  and

$$\mathbb{E} u(X_t, P_t) - u(x_0, p_0) = \int_0^t \mathbb{E} (Au)(X_s, P_s) ds$$

which means that  $(u, Au)$  is in the full generator  $\widehat{A}$ .  $\square$

If the driving Lévy process has no drift, no Brownian part and a rotationally symmetric Lévy measure, the form of the infinitesimal generator becomes much simpler. In this case we have for all  $u \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$

$$\begin{aligned}
Au(x, p) &= \frac{\partial u(x, p)}{\partial x} p - \frac{\partial u(x, p)}{\partial p} \frac{\partial V(x)}{\partial x} \\
&\quad + \text{v.p.} \int_{\mathbb{R}^d} \left( u(x, p - \frac{\partial c(x)}{\partial x} y) - u(x, p) \right) \nu(dy),
\end{aligned} \tag{17}$$

where  $\text{v.p.} \int_{\mathbb{R}^d} f(y) \nu(dy) := \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} f(y) \nu(dy)$  stands for the principal value integral. It is not hard to see that

$$\begin{aligned} & \text{v.p.} \int_{\mathbb{R}^d} \left( u(x, p - \frac{\partial c(x)}{\partial x} y) - u(x, p) \right) \nu(dy) \\ &= \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x, p - \frac{\partial c(x)}{\partial x} y) - u(x, p) + \frac{\partial u(x, y)}{\partial x} \frac{\partial c(x)}{\partial x} y \mathbf{1}_{\{|y| < 1\}} \right) \nu(dy) \end{aligned}$$

or also

$$= \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x, p - \frac{\partial c(x)}{\partial x} y) + u(x, p + \frac{\partial c(x)}{\partial x} y) - 2u(x, p) \right) \nu(dy)$$

holds. The latter two representations do exist in the sense of ordinary integrals (just use a simple Taylor expansion for  $u$  up to order two) and are frequently used in the literature. For our purposes, formula (17) is better suited. Notice that all three representations extend  $A$  onto  $C^2$ .

**Theorem 6.** *Let  $d \geq 3$ ,  $V \in C^2(\mathbb{R}^d)$ ,  $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$  and  $\{\xi_t\}_{t \geq 0}$  be a symmetric  $\alpha$ -stable Lévy process,  $0 < \alpha < 2$ . Then the process  $\{(X_t, P_t)\}_{t \geq 0}$  solving (1) is transient.*

*Proof.* We want to apply Lemma 4. Take the function

$$u_\gamma(x, p) = (H(x, p) - V_0)^{-\gamma} = \left( \frac{1}{2} p^2 + V(x) - V_0 \right)^{-\gamma}$$

with  $V_0 = \inf V - 1$  and with a parameter  $\gamma > 0$  which we will choose later. It is not hard to see that for this  $u = u_\gamma(x, p)$  and

$$D := \left\{ (x, p) \in \mathbb{R}^{2d} : |x| + |p| \leq 1 \right\}, \quad a := \frac{1}{2} \min_{(x, p) \in D} u_\gamma(x, p)$$

conditions (ii), (iii) of Lemma 4 are satisfied.

Moreover, we have

$$\frac{\partial u_\gamma}{\partial x} p - \frac{\partial u_\gamma}{\partial p} \frac{\partial V}{\partial x} = 0.$$

Since  $\{\xi_t\}_{t \geq 0}$  is a symmetric  $\alpha$ -stable process, its Lévy measure is of the form  $\nu(dy) = c_\alpha |y|^{-d-\alpha} dy$  with  $c_\alpha$  given by (16), and (17) shows that

$$\tilde{A}u_\gamma(x, p) = c_\alpha \text{v.p.} \int_{\mathbb{R}^d} \left( u_\gamma \left( x, p + \frac{\partial c}{\partial x} y \right) - u_\gamma(x, p) \right) \frac{dy}{|y|^{d+\alpha}}.$$

We will see in Corollary 9 below (with  $B = \partial c / \partial x$  and  $b = 2(V(x) - V_0)$ ) that we can choose  $\gamma > 0$  in such a way that  $\tilde{A}u_\gamma(x, p) \leq 0$ . This, however, means that also condition (iv) of Lemma 4 is met.

Let  $\chi_k \in C_c^\infty(\mathbb{R}^d)$  be a cut-off function with  $\mathbf{1}_{B_k(0)} \leq \chi_k \leq \mathbf{1}_{B_{2k}(0)}$  and set  $u_k(x, p) := u_\gamma(x, p) \chi_k(x) \chi_k(p)$ . Clearly,  $u_k \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$  and we know from Lemma 5 that the pair

$(u_k, Au_k)$  is in the full generator  $\widehat{A}$ . The following considerations are close to those in [31]. Write  $\|g\|_A = \|g\mathbf{1}_A\|_\infty$ . Using a Taylor expansion we find for some  $0 < \theta < 1$  and all  $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$

$$\begin{aligned} & f(x, p + \frac{\partial c}{\partial x} y) - f(x, p) \\ &= \frac{\partial f(x, p)}{\partial p} \frac{\partial c(x)}{\partial x} y + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f(x, p + \theta \frac{\partial c}{\partial x} y)}{\partial p_i \partial p_j} \left( \frac{\partial c}{\partial x} y \right)_i \left( \frac{\partial c}{\partial x} y \right)_j \end{aligned}$$

and, therefore, for all compact sets  $K \subset \mathbb{R}^d$  and  $(x, p) \in K \times K$ ,

$$\begin{aligned} & \left| \text{v.p.} \int_{\mathbb{R}^d} (f(x, p + \frac{\partial c}{\partial x} y) - f(x, p)) \nu(dy) \right| \\ & \leq \left| \text{v.p.} \int_{|y| < 1} (f(x, p + \frac{\partial c}{\partial x} y) - f(x, p)) \nu(dy) \right| + 2 \int_{|y| \geq 1} \nu(dy) \|f\|_{K \times \mathbb{R}^d} \\ & \leq \frac{d^4}{2} \left\| \frac{\partial c}{\partial x} \right\|_K^2 \int_{0 < |y| < 1} |y|^2 \nu(dy) \left\| \frac{\partial^2 f}{\partial p^2} \right\|_{K \times \tilde{K}} + 2 \int_{|y| \geq 1} \nu(dy) \|f\|_{K \times \mathbb{R}^d}, \end{aligned}$$

where  $\tilde{K} = K + \{p \in \mathbb{R}^d : |p| \leq \|\partial c / \partial x\|_K\}$ . Since the estimate of the local part in (17) is obvious, we find

$$\|\tilde{A}f\|_{K \times K} \leq C \left( \|f\|_{K \times \mathbb{R}^d} + \left\| \frac{\partial f}{\partial x} \right\|_{K \times K} + \left\| \frac{\partial f}{\partial p} \right\|_{K \times K} + \left\| \frac{\partial^2 f}{\partial p^2} \right\|_{K \times \tilde{K}} \right),$$

for any  $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$  with  $\|f\|_{K \times \mathbb{R}^d} < \infty$  and with an absolute constant  $C = C(K, c, V)$  depending only on  $K$ ,  $\|\partial c / \partial x\|_K$  and  $\|\partial V / \partial x\|_K$ . Since  $p \mapsto u_\gamma(x, p)$  vanishes at infinity, condition (i) of Lemma 4 is satisfied for the sequence  $(u_k, Au_k) \rightarrow (u_\gamma, \tilde{A}u_\gamma)$ .

The theorem follows now directly from Lemma 4.  $\square$

## Appendix

We will now give the somewhat technical proof that for some  $\gamma > 0$  the function  $u_\gamma(x, p) = (\frac{1}{2}p^2 + V(x) - V_0)^{-\gamma}$  which we used in the proof of Theorem 6 satisfies condition (iv) of Lemma 4. We begin with a few elementary lemmas.

Recall that Euler's Beta function  $B(x, y)$  is given by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \quad (18)$$

and satisfies the relations

$$B(x, y) = B(y, x) \quad \text{and} \quad B(x, y) = \frac{x+y}{y} B(x, y+1), \quad (19)$$

cf. Gradshteyn and Ryzhik [9, §8.38]. A change of variable in (18) according to  $t = s^2$  yields

$$B(x, y) = \int_{-1}^1 (s^2)^{x-\frac{1}{2}} (1-s^2)^{y-1} ds, \quad x, y > 0.$$

**Lemma 7.** *For any  $v \in \mathbb{R} \setminus \{0\}$ ,  $a \geq 1$ ,  $d \geq 3$  we have*

$$J(v) = \int_{-1}^1 (1-s^2)^{\frac{d-3}{2}} \ln(v^2 + 2vs + a) ds > \ln(a) I_{\frac{d-3}{2}}. \quad (20)$$

*Proof.* We observe that  $J(v) = J(-v)$  and

$$\ln(v^2 + 2vs + a) - \ln(a) = \ln\left(\frac{v^2}{a} + 2\frac{v}{a}s + 1\right) \geq \ln\left(\frac{v^2}{a^2} + 2\frac{v}{a}s + 1\right).$$

Therefore, we may assume that  $a = 1$  and  $v \geq 0$ . Since  $J(0) = \ln(a) = 0$ , it is enough to show that  $J(v)$  is increasing. This is clear for  $v \geq 1$  since  $v \mapsto v^2 + 2vs + 1$  increases for all parameter values  $|s| \leq 1$ . For  $0 < v < 1$  we calculate the derivative

$$J'(v) = 2 \int_{-1}^1 \frac{v+s}{v^2 + 2vs + 1} (1-s^2)^{\frac{d-3}{2}} ds.$$

In the case  $d = 3$  a few lines of simple calculations give

$$J'(v) = \left(1 - \frac{1}{v^2}\right) \ln\left(\frac{1+v}{1-v}\right) + \frac{2}{v}$$

which is clearly positive. If  $d > 3$ , we use the symmetry of the measure  $(1-s^2)^{\frac{d-3}{2}} ds$  and find

$$\begin{aligned} J'(v) &= \int_{-1}^1 \left( \frac{v+s}{v^2 + 2vs + 1} + \frac{v-s}{v^2 - 2vs + 1} \right) (1-s^2)^{\frac{d-3}{2}} ds \\ &= 2v \int_{-1}^1 \frac{v^2 + 1 - 2s^2}{(v^2 + 1)^2 - 4v^2 s^2} (1-s^2)^{\frac{d-3}{2}} ds \\ &= \frac{2v}{(v^2 + 1)^2} \int_{-1}^1 (v^2 + 1 - 2s^2) \sum_{j=0}^{\infty} \left( \frac{2v}{v^2 + 1} \right)^{2j} s^{2j} (1-s^2)^{\frac{d-3}{2}} ds, \end{aligned}$$

since  $2v(v^2 + 1)^{-1} \leq 1$ . The integrand can be written as

$$\begin{aligned} &(v^2 + 1 - 2s^2) \sum_{j=0}^{\infty} \left( \frac{2v}{v^2 + 1} \right)^{2j} s^{2j} \\ &= (v^2 + 1) \sum_{j=0}^{\infty} \left( \frac{2v}{v^2 + 1} \right)^{2j} s^{2j} - 2 \sum_{j=0}^{\infty} \left( \frac{2v}{v^2 + 1} \right)^{2j} s^{2j+2} \end{aligned}$$

$$\begin{aligned}
&= (v^2 + 1) + \sum_{j=1}^{\infty} \left\{ (v^2 + 1) \left( \frac{2v}{v^2 + 1} \right)^{2j} - 2 \left( \frac{2v}{v^2 + 1} \right)^{2j-2} \right\} s^{2j} \\
&= (v^2 + 1) + \frac{2(v^2 - 1)}{v^2 + 1} \sum_{j=1}^{\infty} \left( \frac{2v}{v^2 + 1} \right)^{2j-2} s^{2j} \\
&\geq (v^2 + 1) + \frac{2(v^2 - 1)}{v^2 + 1} s^2 + \frac{2(v^2 - 1)}{v^2 + 1} \left( \frac{2v}{v^2 + 1} \right)^2 \frac{s^4}{1 - s^2}
\end{aligned}$$

since  $v^2 - 1 \leq 0$ . This gives

$$\begin{aligned}
J'(v) &\geq \frac{2v}{v^2 + 1} \left( \int_{-1}^1 (1 - s^2)^{\frac{d-3}{2}} ds + \frac{2(v^2 - 1)}{(v^2 + 1)^2} \int_{-1}^1 s^2 (1 - s^2)^{\frac{d-3}{2}} ds \right. \\
&\quad \left. + \frac{2(v^2 - 1)}{(v^2 + 1)^2} \left( \frac{2v}{v^2 + 1} \right)^2 \int_{-1}^1 s^4 (1 - s^2)^{\frac{d-5}{2}} ds \right) \\
&= \frac{2v}{v^2 + 1} \left( B\left(\frac{1}{2}, \frac{d-1}{2}\right) + \frac{2(v^2 - 1)}{(v^2 + 1)^2} B\left(\frac{3}{2}, \frac{d-1}{2}\right) + \frac{v^2 - 1}{v^2 + 1} \frac{8v^2}{(v^2 + 1)^3} B\left(\frac{5}{2}, \frac{d-3}{2}\right) \right).
\end{aligned}$$

Using (19) we find for all dimensions  $d \geq 4$

$$B\left(\frac{1}{2}, \frac{d-1}{2}\right) = dB\left(\frac{3}{2}, \frac{d-1}{2}\right) \quad \text{and} \quad B\left(\frac{5}{2}, \frac{d-3}{2}\right) = \frac{3}{d-3} B\left(\frac{3}{2}, \frac{d-1}{2}\right),$$

and so

$$\begin{aligned}
J'(v) &\geq \frac{2v}{v^2 + 1} B\left(\frac{3}{2}, \frac{d-1}{2}\right) \left( d + \frac{2(v^2 - 1)}{(v^2 + 1)^2} + \frac{3}{d-3} \frac{8v^2(v^2 - 1)}{(v^2 + 1)^4} \right) \\
&\geq \frac{2v}{v^2 + 1} B\left(\frac{3}{2}, \frac{d-1}{2}\right) \left( 4 + \frac{2(v^2 - 1)}{(v^2 + 1)^2} + \frac{24v^2(v^2 - 1)}{(v^2 + 1)^4} \right).
\end{aligned}$$

It is now straightforward to check that

$$4 + \frac{2(v^2 - 1)}{(v^2 + 1)^2} + \frac{24v^2(v^2 - 1)}{(v^2 + 1)^4} \geq 0$$

for all  $v \in \mathbb{R}$ . □

**Lemma 8.** *Let  $d \geq 3$ ,  $0 < \alpha < 2$ . There exists some  $\gamma = \gamma(\alpha, d) > 0$  such that*

$$\text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + \lambda y|^2 + 1)^\gamma} - \frac{1}{(|p|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} < 0 \quad (21)$$

holds for all  $p \in \mathbb{R}^d$ ,  $\lambda \in \mathbb{R}$ .

*Proof.* With the reasoning following Lemma 5 it is clear that the integral (21) exists. Without loss of generality we may assume that  $\lambda = 1$ . Denote the left-hand side of (21) by  $I(\gamma)$ . Changing to polar coordinates we get

$$I(\gamma) = \iint_{S^{d-2} \times (0+, \infty)} Z(r) r^{-1-\alpha} dr d\theta = |S^{d-2}| \int_{0+}^{\infty} Z(r) r^{-1-\alpha} dr,$$

(in the sense of an improper integral at the lower limit  $0+$ ) where

$$Z(r) = \int_{-1}^1 \left( \frac{1}{(r^2 + |p|^2 + 2r|p|s + 1)^\gamma} - \frac{1}{(|p|^2 + 1)^\gamma} \right) (1 - s^2)^{\frac{d-3}{2}} ds.$$

Write  $Z(r) = |p|^{-2\gamma} \tilde{Z}(r)$  and observe that with  $v = r/|p|$

$$\tilde{Z}(r) = \int_{-1}^1 \left( \frac{1}{(v^2 + 1 + 2vs + |p|^{-2})^\gamma} - \frac{1}{(1 + |p|^{-2})^\gamma} \right) (1 - s^2)^{\frac{d-3}{2}} ds.$$

An application of Lemma 7 with  $a = 1 + |p|^{-2}$  implies

$$\begin{aligned} \left. \frac{\partial \tilde{Z}(r)}{\partial \gamma} \right|_{\gamma=0} &= - \int_{-1}^1 (\ln(v^2 + 2vs + a) - \ln(a)) (1 - s^2)^{\frac{d-3}{2}} ds \\ &= - \left( J(v) - \ln(a) I_{\frac{d-3}{2}} \right) < 0, \end{aligned}$$

and therefore

$$I'(0) = -|p|^{-\alpha-2\gamma} \int_{0+}^{\infty} \left( J(v) - \ln(a) I_{\frac{d-3}{2}} \right) v^{-1-\alpha} dv < 0.$$

Since  $I(0) = 0$ , the claim follows.  $\square$

Assertion (iv) of Lemma 4 follows finally from

**Corollary 9.** *Let  $d \geq 3$  and  $0 < \alpha < 2$ . Then there exists some  $\gamma = \gamma(\alpha, d) > 0$  such that for all  $B \in \mathbb{R}^{d \times d}$ ,  $b > 0$ ,  $p \in \mathbb{R}^d$*

$$\text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + By|^2 + b)^\gamma} - \frac{1}{(|p|^2 + b)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \leq 0. \quad (22)$$

*Proof.* An argument similar to the one used in the proof of Lemma 8 shows that the integral (22) is well-defined for every  $\gamma > 0$ . Since

$$\begin{aligned} &\text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + By|^2 + b)^\gamma} - \frac{1}{(|p|^2 + b)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\ &= \frac{1}{b^\gamma} \text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|b^{-1/2}p + b^{-1/2}By|^2 + 1)^\gamma} - \frac{1}{(|b^{-1/2}p|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}}, \end{aligned}$$

we may assume that  $b = 1$ . Depending on the rank of the matrix  $B$  we distinguish between three cases.

*Case 1:*  $\text{rank } B = 0$ . Nothing is to prove in this case.

*Case 2:*  $\text{rank } B = d$ . We have

$$\begin{aligned}\mathcal{J}(\lambda) &= \text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + By|^2 + 1)^\gamma} - \frac{1}{(|p + \lambda y|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\ &= \lambda^\alpha \text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + \lambda^{-1}By|^2 + 1)^\gamma} - \frac{1}{(|p + y|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}}\end{aligned}$$

and, therefore,

$$\lim_{\lambda \rightarrow 0} \lambda^{-\alpha} \mathcal{J}(\lambda) < 0 \quad \text{and, by Lemma 8,} \quad \lim_{\lambda \rightarrow \infty} \lambda^{-\alpha} \mathcal{J}(\lambda) > 0.$$

Since  $\mathcal{J}(\lambda)$  is a continuous function, there exists some  $\lambda^* = \lambda^*(p, B)$  such that  $\mathcal{J}(\lambda^*) = 0$ . Thus,

$$\begin{aligned}\text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + By|^2 + 1)^\gamma} - \frac{1}{(|p + \lambda y|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\ = \mathcal{J}(\lambda^*) + \text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + \lambda^* y|^2 + 1)^\gamma} - \frac{1}{(|p|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \leq 0,\end{aligned}$$

where we used Lemma 8 again.

*Case 3:*  $\text{rank } B = k$ ,  $1 < k < d$ . In this case we can find an orthogonal matrix  $S \in \mathbb{R}^{d \times d}$  such that

$$B = S \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix} S^T$$

where  $\tilde{B} \in \mathbb{R}^{k \times k}$  has full rank. Since the measure  $|y|^{-d-\alpha} dy$  is invariant under orthogonal transformations we can assume that  $B$  is already of the form  $\begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}$ ; otherwise we would make a change of variables in (22) with  $p' = Sp$  in place of  $p$ . Write  $y = (y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ ,  $p = (p_1, p_2) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$  and set  $b = 1 + |p_2|^2$ . Then

$$\begin{aligned}\text{v.p.} \int_{\mathbb{R}^d} \left( \frac{1}{(|p + By|^2 + 1)^\gamma} - \frac{1}{(|p + \lambda y|^2 + 1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\ = \text{v.p.} \iint_{\mathbb{R}^d} \left( \frac{1}{(|p_1 + B'y_1|^2 + b)^\gamma} - \frac{1}{(|p_1|^2 + b)^\gamma} \right) \frac{dy_1 dy_2}{(|y_1|^2 + |y_2|^2)^{\frac{d+\alpha}{2}}} \\ = \text{v.p.} \int_{\mathbb{R}^k} \left( \frac{1}{(|p_1 + B'y_1|^2 + b)^\gamma} - \frac{1}{(|p_1|^2 + b)^\gamma} \right) \int_{\mathbb{R}^{d-k}} \frac{dy_2}{(|y_1|^2 + |y_2|^2)^{\frac{d+\alpha}{2}}} dy_1 \\ = \int_{\mathbb{R}^{d-k}} \frac{d\eta_2}{(1 + |\eta_2|^2)^{\frac{d+\alpha}{2}}} \text{v.p.} \int_{\mathbb{R}^k} \left( \frac{1}{(|p_1 + B'y_1|^2 + b)^\gamma} - \frac{1}{(|p_1|^2 + b)^\gamma} \right) \frac{dy_1}{|y_1|^{k+\alpha}}\end{aligned}$$

where we used the change of variables  $|y_1|\eta_2 = y_2$  in the last step. Since  $B'$  has full rank, the claim follows from case 2.  $\square$

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