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THE LAW OF THE MAXIMUM OF A BESSEL BRIDGE

Jim Pitman
Department of Statistics, University of California
367 Evans Hall # 3860, Berkeley, CA 94720-3860, U. S. A.
pitman@stat.berkeley.edu

Marc Yor
Laboratoire de Probabilités, Tour 56, 4 Place Jussieu
75252 Paris Cedex 05, France

Abstract Let M_δ be the maximum of a standard Bessel bridge of dimension δ. A series formula for $P(M_\delta \le a)$ due to Gikhman and Kiefer for $\delta = 1, 2, \ldots$ is shown to be valid for all real $\delta > 0$. Various other characterizations of the distribution of M_δ are given, including formulae for its Mellin transform, which is an entire function. The asymptotic distribution of M_δ is described both as $\delta \to \infty$ and as $\delta \downarrow 0$.

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1 Introduction

For real $\delta > 0$ let $(R_\delta(t), t \geq 0)$ denote a $\mathrm{BES}_0(\delta)$ process, that is a Bessel process of dimension δ started at 0, and let $(r_\delta(u), 0 \leq u \leq 1)$ denote a standard $\mathrm{BES}(\delta)$ bridge defined by conditioning $(R_\delta(u), 0 \leq u \leq 1)$ on $R_\delta(1) = 0$. The law of $(R_\delta(t), t \geq 0)$ is that of the pathwise unique non-negative solution of the stochastic differential equation

$$R_0 = 0; \quad dR_t^2 = \delta\, dt + 2R_t d\beta_t \tag{1}$$

where $(\beta_t, t \geq 0)$ is a standard Brownian motion, while $(r_\delta(u), 0 \leq u \leq 1)$ can be conveniently constructed from $(R_\delta(t), t \geq 0)$ by the space-time transformation

$$r_\delta(u) := (1-u) R_\delta\left(\frac{u}{1-u}\right). \tag{2}$$

See [39, 49] for background and motivation for the study of these processes. Let M_δ denote the maximum of the standard $\mathrm{BES}(\delta)$ bridge, and $M_{\delta *}$ the maximum up to time 1 of the unconditioned $\mathrm{BES}_0(\delta)$ process:

$$M_\delta := \sup_{0 \leq u \leq 1} r_\delta(u); \qquad M_{\delta *} := \sup_{0 \leq u \leq 1} R_\delta(u). \tag{3}$$

The distribution of M_δ for $\delta = 1, 2, \ldots$ arises as a limit in the theory of empirical distributions [50]. As shown by Doob [15], asymptotic results of Kolmogorov [30] and Smirnov [51] give expressions for $P(M_1 \leq a)$ for $a > 0$, in particular

$$P(M_1 \leq a) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(-2n^2 a^2). \tag{4}$$

Gikhman [19] and Kiefer [28] found the following formula for $\delta = 1, 2, \ldots$:

$$P(M_\delta \leq a) = \frac{2}{C_\delta a^\delta} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} \exp\left(-\frac{j_{\nu,n}^2}{2a^2}\right) \tag{5}$$

where $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ is the sequence of positive zeros of J_ν, the Bessel function of index

$$\nu := (\delta - 2)/2, \quad \text{and} \quad C_\delta := 2^{(\delta-2)/2}\Gamma(\delta/2) = 2^\nu \Gamma(\nu+1). \tag{6}$$

Appendix A recalls the definition of J_ν, basic properties of the $j_{\nu,n}$, and some other formulae for Bessel functions which are used in this paper. The equality of the right side of (4) and the right side of (5) for $\delta = 1$ is an instance of the functional equation for Jacobi's theta function.

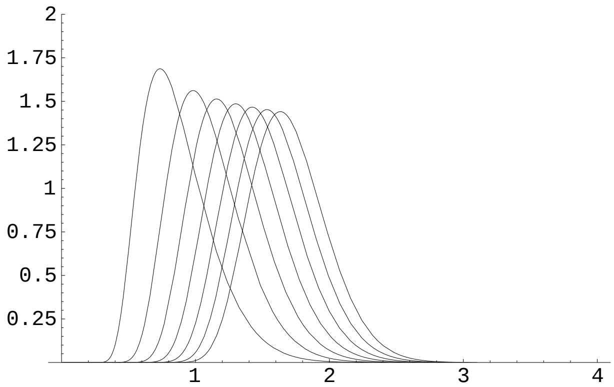

Figure 1. *Graphs of the densities of M_δ for $\delta = 1, 2, \ldots, 7$.*

One purpose of this paper is to establish the following result, which was claimed without proof in [42]:

Theorem 1 *The Gikhman-Kiefer formula* (5) *is valid for all real* $\delta > 0$.

In particular, for each real $\delta > 0$ the right side of formula (5) defines an increasing function of $a \in (0, \infty)$ with limit 0 at 0 and limit 1 at ∞. These facts are not obvious from the formula, even for integer δ.

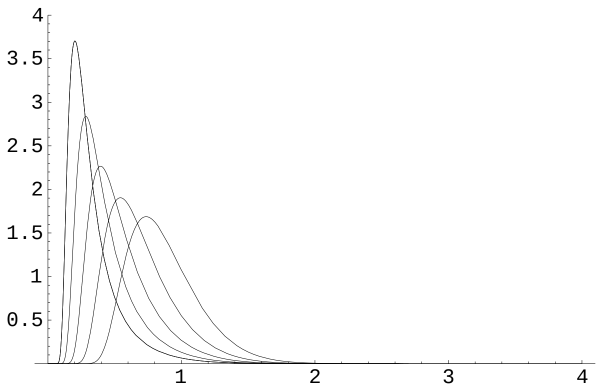

Figure 2. *Graphs of the densities of* M_δ *for* $\delta = 2^{-n}$ and $n = 0, 1, 2, 3, 4$.

As pointed out by Williams [57], the law of the standard Brownian excursion described by Lévy [32] and Itô-McKean [24] is identical to the law of $(r_3(u), 0 \le u \le 1)$. Thus the distribution of M_3 found by Gikhman and Kiefer is identical to the distribution of the maximum of the standard Brownian excursion, found by Chung [12] and Kennedy [26]. Due to Vervaat's [54] transformation of a standard Brownian bridge into a standard Brownian excursion, the distribution of M_3 appears again as the distribution of the maximum minus the minimum of a standard Brownian bridge.

As observed by Watson [56] and Chung [13], the formula (4) implies easily that for $\lambda > 0$

$$E \exp(-\tfrac{1}{2}\lambda^2 M_1^2) = \frac{\frac{\pi}{2}\lambda}{\sinh(\frac{\pi}{2}\lambda)}. \tag{7}$$

That is to say, there is the equality in distribution

$$M_1^2 \stackrel{d}{=} T_{3,\pi/2} \tag{8}$$

where $T_{\delta,x}$ denotes the hitting time of $x > 0$ by the $\mathrm{BES}_0(\delta)$ process $(R_\delta(t), t \ge 0)$. Note that by Brownian scaling, $T_{\delta,x} \stackrel{d}{=} x^2 T_\delta$ for all $x, \delta > 0$, where $T_\delta := T_{\delta,1}$. Chung discovered the identity

$$E \exp(-\tfrac{1}{2}\lambda^2 M_3^2) = \left(\frac{\frac{\pi}{2}\lambda}{\sinh(\frac{\pi}{2}\lambda)} \right)^2 \tag{9}$$

which combined with (7) leads to the remarkable conclusion that

$$M_3^2 \stackrel{d}{=} M_1^2 + \widehat{M}_1^2 \stackrel{d}{=} \frac{\pi^2}{4}(T_3 + \widehat{T}_3) \tag{10}$$

where $\widehat{M}_1$ is an independent copy of M_1 and $\widehat{T}_3$ an independent copy of $T_3 := T_{3,1}$. As far as we know, no proof of these identities (8) and (10) has ever been given in probabilistic terms, without involving some analysis related to the functional equation for Jacobi's theta function. See [5, 42, 60, 4] for further discussion of this circle of results.

Stimulated by these remarkable results involving M_δ and T_δ for $\delta = 1$ and 3, we were led to study the distributions of M_δ and T_δ for a general $\delta > 0$ and to look for relations involving the distributions of these random variables. Though the distribution of M_δ is our main concern in this paper, we often mention corresponding results for

$$M_{\delta *} := \sup_{0 \le u \le 1} R_\delta(u).$$

Results for $M_{\delta *}$ are typically simpler, due to the identity in law

$$M_{\delta *}^{-2} \stackrel{d}{=} T_\delta \tag{11}$$

which is an immediate consequence of the inverse relation

$$P(T_\delta > t) = P(\sup_{0 \le u \le t} R_\delta(u) < 1)$$

and Brownian scaling.

An analog of (11) for M_δ is provided by the following absolute continuity relation between the law of M_δ^{-2} and the law of

$$\widetilde{T}_\delta := T_\delta + \widehat{T}_\delta$$

for $\widehat{T}_\delta$ an independent copy of T_δ:

Theorem 2 (The agreement formula.) [5, 41, 42] *For each $\delta > 0$ there is the identity*

$$E\left[g(M_\delta^{-2})\right] = C_\delta E\left[\widetilde{T}_\delta^\nu g(\widetilde{T}_\delta)\right] \tag{12}$$

for every non-negative measurable function g, where as in (6)

$$C_\delta := 2^{(\delta-2)/2}\Gamma(\delta/2), \qquad \nu := (\delta - 2)/2.$$

In particular, $M_\delta^{-2} \stackrel{d}{=} \widetilde{T}_\delta$ if and only if $\delta = 2$. In [42] we presented the agreement formula as the specialization to Bessel processes of a general result for one-dimensional diffusions. As explained in [5, 60, 42], the agreement formula follows from the fact that a certain σ-finite measure on the space of continuous non-negative paths with finite lifetimes can be explicitly disintegrated in two different ways, according to the lifetime, or according to the value of the maximum. For $\delta = 3$ this σ-finite measure is Itô's excursion law for excursions away from 0 of the reflecting Brownian motion BES(1). The agreement formula is then an expression of Williams' [57, 59] results that on the one hand a Brownian excursion of length t is a BES(3) bridge of length t, and on the other hand, a Brownian excursion of height h can be constructed by back-to-back splicing of two independent copies of BES(3) run until its first hitting time of h. For $\delta \in (2,4)$ there is a similar interpretation in terms of Itô's law of excursions of the recurrent BES$(4-\delta)$ process [39, 5].

This paper presents some variations and applications of the Gikhman-Kiefer and agreement formulae. In Section 2 we recall from [42] a general formula which determines the distribution of the maximum of a one-dimensional diffusion bridge. Section 3 shows how this formula implies the Gikhman-Kiefer formula (5) for arbitrary real $\delta > 0$. In Section 4 we review some known results regarding the distribution of T_δ and relate these results to the Gikhman-Kiefer and agreement formulae. Section 5 shows how the general results of Section 2 yield another characterization of the law of M_δ by a Laplace transform which is convenient for some purposes. Section 6 gives applications of the results of previous sections to the computation of moments of M_δ. In particular, we deduce from the agreement formula some simple evaluations of particular moments of M_δ for integer δ which are not at all obvious from the Gikhman-Kiefer formula. For instance

$$E(M_4^2) = 2; \quad E(M_6^2) = \frac{8}{3}; \quad E(M_6^4) = 8; \tag{13}$$

$$E(M_1) = \sqrt{\frac{\pi}{2}}\log 2; \quad E(M_3) = \sqrt{\frac{\pi}{2}}; \quad E(M_5) = \frac{3}{5}\sqrt{2\pi}; \quad E(M_5^3) = \frac{3}{2}\sqrt{2\pi}. \tag{14}$$

Section 7 shows how the general results of previous sections yield a number of special results in dimensions one and three, including formulae (7) and (9), and expressions for the Mellin transforms of M_1 and M_3 in terms of the Riemann zeta function. In Section 8 we obtain the following result regarding the asymptotic behaviour of the distributions of M_δ and $M_{\delta *}$ as $\delta \to \infty$.

Theorem 3 *As $\delta \to \infty$*

$$M_\delta - \tfrac{1}{2}\sqrt{\delta} \xrightarrow{d} N(0, \tfrac{1}{8}); \qquad M_{\delta *} - \sqrt{\delta} \xrightarrow{d} N(0, \tfrac{1}{2}), \tag{15}$$

where $N(0,v)$ denotes a normal variable with mean zero and variance v.

This theorem is related to the classical result, due to Poincaré and others, that for a vector picked uniformly at random from the surface of the sphere of radius $\sqrt{\delta}$ in $\mathbb{R}^\delta$, as $\delta \to \infty$ through the integers, for each fixed k the distribution of the first k coordinates approaches that of k independent standard normal variables. See [33, 62] for other results in this vein and further references.

Section 9 establishes the following description of the asymptotic distribution of M_δ as $\delta \downarrow 0$:

Theorem 4 *As $\delta \downarrow 0$*

$$\frac{M_\delta}{\sqrt{\delta}} \xrightarrow{d} \frac{1}{\sqrt{\mathcal{E}}} \tag{16}$$

where $\mathcal{E}$ denotes a standard exponential variable. The same holds with $M_{\delta *}$ instead of M_δ.

This result is related to the asymptotic behaviour of $j_{\nu,1}$, the smallest positive zero of J_ν, as $\nu \downarrow -1$. From a more probabilistic standpoint, it is related to the Ray-Knight description of

Brownian local times, and to the Poisson structure of the jumps of the path-valued processes $(R_\delta^2, \delta \geq 0)$ and $(r_\delta^2, \delta \geq 0)$ when these processes are constructed with stationary independent increments as in [39].

We record in Section 10 some relations between the distributions of M_δ and T_δ and the distribution of last exit times derived from the $\mathrm{BES}_0(\delta)$ process. Finally, in Section 11 we record the evaluation of a series involving the zeros of J_ν which we obtain by comparison of the Gikhman-Kiefer and agreement formulae for the distribution of M_δ.

2 The maximum of a diffusion bridge

Let $p(t; x, y)$ denote the symmetric transition density of a regular one-dimensional diffusion R on $[0, \infty)$ relative to the speed measure of the diffusion [24]. Let $M(t) := \sup_{0 \leq s \leq t} R(s)$, and for $x, y \geq 0$ and $t > 0$ let $P_{t;x,y}$ denote the probability law governing an R-bridge of length t from x to y. So

$$P_{t;x,y}(M(t) > a) = P(M(t) > a \,|\, R(0) = x, R(t) = y).$$

By a first passage argument, for $0 \leq x, y < a$ there is the formula [42, (2.9)]

$$P_{t;x,y}(M(t) > a) p(t; x, y) = \int_0^t f_{xa}(u) p(t - u; a, y)\, du \tag{17}$$

where $f_{xa}(u)$ denotes the density at u of the distribution of the first passage time to a for the diffusion started at x. Let $\phi_\lambda^\uparrow$ and $\phi_\lambda^\downarrow$ denote the increasing and decreasing solutions of $Au = \lambda u$ for A the infinitesimal generator of the diffusion [24], normalized so that

$$\int_0^\infty e^{-\lambda t} p(t; x, y)\, dt = \phi_\lambda^\uparrow(x) \phi_\lambda^\downarrow(y) \qquad (0 \leq x \leq y, \lambda > 0). \tag{18}$$

It is well known that for $0 \leq x < a$ the first passage density f_{xa} is determined by the Laplace transform

$$\int_0^\infty e^{-\lambda t} f_{xa}(t) dt = \frac{\phi_\lambda^\uparrow(x)}{\phi_\lambda^\uparrow(a)}. \tag{19}$$

So the Laplace transform equivalent of (17) is the following identity [42, (2.10)]: for $0 \leq x, y < a$ and $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} P_{t;x,y}(M(t) > a) p(t; x, y)\, dt = \phi_\lambda^\uparrow(y) \phi_\lambda^\downarrow(a) \frac{\phi_\lambda^\uparrow(x)}{\phi_\lambda^\uparrow(a)}. \tag{20}$$

We call (20) the *first passage transform*. For each choice of x and y, this transform determines the distribution of $M(t)$ for the diffusion bridge from x to y of length t for all $t > 0$. Assuming now that $0 \leq x \leq y < a$, we can subtract (20) from (18) to deduce the formula

$$\int_0^\infty e^{-\lambda t} P_{t;x,y}(M(t) \leq a) p(t; x, y)\, dt = \phi_\lambda^\uparrow(x) \left(\frac{\phi_\lambda^\downarrow(y) \phi_\lambda^\uparrow(a) - \phi_\lambda^\downarrow(a) \phi_\lambda^\uparrow(y)}{\phi_\lambda^\uparrow(a)} \right). \tag{21}$$

This transform also determines the $P_{t;x,y}$ distribution of $M(t)$ for all $t > 0$, except if $x = y = 0$ and 0 is an entrance boundary point, as is the case for $\mathrm{BES}(\delta)$ for $\delta \geq 2$; then $\phi_\lambda^\downarrow(0) = \infty$ and (21) holds only in the trivial sense of $\infty = \infty$.

There is also the following companion of (17), which reflects the Williams decomposition at the time the maximum is attained: for $0 \leq x, y < a$

$$\frac{P_{t;x,y}(M(t) \in da)}{s(da)} p(t;x,y) = \int_0^t f_{xa}(u) f_{ya}(t-u)\, du \tag{22}$$

where $s(da)$ is the scale measure associated with the diffusion. See [14],[42] and [6, §II.3] for further discussion of this formula and its relation to (17). As shown in [42], application of (22) to BES(δ) yields the agreement formula (12) for the standard Bessel bridge.

3 The Gikhman-Kiefer Formula

For a BES(δ) process, the functions $\phi_\lambda^\uparrow$ and $\phi_\lambda^\downarrow$ are known [27] to be

$$\phi_\lambda^\uparrow(x) = I_\nu(\sqrt{2\lambda}x)x^{-\nu}, \quad \phi_\lambda^\downarrow(y) = K_\nu(\sqrt{2\lambda}y)y^{-\nu}, \tag{23}$$

where I_ν and K_ν are the usual modified Bessel functions, whose definition is recalled in (133) and (134). From (151)

$$\phi_\lambda^\uparrow(0) = \lim_{x \downarrow 0} I_\nu(\sqrt{2\lambda}x)x^{-\nu} = \frac{1}{\Gamma(\nu+1)}\left(\frac{\lambda}{2}\right)^{\nu/2}. \tag{24}$$

Let R_δ denote the BES(δ) process with $R_\delta(0) = 0$. The density of $R_\delta(t)$ relative to the speed measure $2y^{\delta-1}\,dy$ is well known to be

$$p_\delta(t;0,y) = \frac{P(R_\delta(t) \in dy)}{2y^{\delta-1}\,dy} = \Gamma(\delta/2)^{-1}(2t)^{-\delta/2}e^{-y^2/(2t)}, \tag{25}$$

as can be verified using (18) for $x = 0$, with (23) and (24). So (21) yields for $0 < y < a$

$$\int_0^\infty e^{-\lambda t} P\left(\sup_{0 \leq s \leq t} R_\delta(s) \leq a \,|\, R_\delta(t) = y\right)(2t)^{-\nu-1}e^{-y^2/(2t)}\,dt = \tag{26}$$

$$\frac{(\lambda/2)^{\nu/2}}{y^\nu}\left(\frac{K_\nu(\sqrt{2\lambda}y)I_\nu(\sqrt{2\lambda}a) - K_\nu(\sqrt{2\lambda}a)I_\nu(\sqrt{2\lambda}y)}{I_\nu(\sqrt{2\lambda}a)}\right). \tag{27}$$

The Laplace transform (27) appears for positive integer δ in the work of both Gikhman [19] and Kiefer [28]. Kiefer derived this transform by a passage to the limit after an application of the Feynman-Kac formula. Gikhman derived the same transform more simply by consideration of solutions of the heat equation in $\mathbb{R}^\delta$. The interpretation (26) of the transform is easily shown to be consistent with these alternative approaches for positive integer δ. Following the method used by Gikhman and Kiefer for $\delta = 1, 2, \ldots$, the derivation of the Gikhman-Kiefer formula (5) for arbitrary $\delta > 0$ is now completed by inversion of the transform (27) using the Fourier-Bessel expansion (163). Thus

$$P(M_{\delta *} \leq a \,|\, R_\delta(1) = y) = 2e^{y^2/2}\sum_{n=1}^\infty \left(\frac{j_{\nu,n}}{ay}\right)^\nu \frac{J_\nu(yj_{\nu,n}/a)}{a^2 J_{\nu+1}^2(j_{\nu,n})}\exp\left(-\frac{j_{\nu,n}^2}{2a^2}\right). \tag{28}$$

Finally, (5) follows from (28) by passage to the limit as $y \downarrow 0$, using (150).

If $\bar{F}(u)$ is a decreasing function of $u > 0$, of the form

$$\bar{F}(u) = \int_u^\infty da\, f(a),$$

then by application of Fubini's theorem and the Gikhman-Kiefer formula (5) we deduce that

$$E(\bar{F}(M_\delta)) = \frac{2^{1-\nu}}{\Gamma(1+\nu)} \sum_{n=1}^\infty \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} \int_0^\infty da\, f(a) a^{-\delta} \exp\left(-\frac{j_{\nu,n}^2}{2a^2}\right). \tag{29}$$

For instance, if we take $f(a) = qa^{-q-1}$ for $q > 0$ we recover a formula for negative moments of M_δ which we record later as (70). Similarly, if we take $f(a) = \lambda^2 a \exp(-\frac{1}{2}\lambda^2 a^2)$ and apply (136) then we obtain the formula

$$E \exp\left(-\tfrac{1}{2}\lambda^2 M_\delta^2\right) = \frac{2^{1-\nu}\lambda^{2+\nu}}{\Gamma(1+\nu)} \sum_{n=1}^\infty \frac{j_{\nu,n}^\nu K_\nu(\lambda j_{\nu,n})}{J_{\nu+1}^2(j_{\nu,n})}. \tag{30}$$

For $\nu = \frac{1}{2}$, we find from (143) and (142) that

$$j_{\frac{1}{2},n} = n\pi, \quad J_{\frac{1}{2}+1}^2(n\pi) = \frac{2}{\pi^2 n}; \quad K_{\frac{1}{2}}(\lambda\pi n) = \sqrt{\frac{1}{2\lambda n}}\, e^{-\lambda\pi n}$$

so (30) simplifies to yield Chung's formula (9):

$$E \exp\left(-\tfrac{1}{2}\lambda^2 M_3^2\right) = (\lambda\pi)^2 \sum_{n=1}^\infty n e^{-\lambda\pi n} = (\lambda\pi)^2 \frac{e^{-\lambda\pi}}{(1-e^{-\lambda\pi})^2} = \left(\frac{\frac{\pi}{2}\lambda}{\sinh(\frac{\pi}{2}\lambda)}\right)^2. \tag{31}$$

Formula (30) simplifies similarly to give (7) for $\delta = 1$. But it appears that these are the only dimensions for which it is reasonable to expect any substantial simplification, as there is no simple formula for the $j_{\nu,n}$ except for $\nu = \pm\frac{1}{2}$. So far as the Bessel functions are concerned, the next simplest case is $\delta = 5$ corresponding to $\nu = \frac{3}{2}$. The $j_{\frac{3}{2},n}$ are then the positive roots of $x \cos x = \sin x$, for which there seems to be no simple expression.

4 The law of T_δ and the agreement formula

Let $f_X(x) := P(X \in dx)/dx$ denote the density at x of a random variable X. It is known [27] that f_{T_δ} is determined for all $\delta > 0$ by the Laplace transform

$$E \exp(-\lambda T_\delta) = \int_0^\infty e^{-\lambda t} f_{T_\delta}(t) dt = \frac{(\sqrt{2\lambda})^\nu}{C_\delta I_\nu(\sqrt{2\lambda})} \tag{32}$$

for C_δ as in (6). As indicated by Ismail-Kelker ([22], Theorems 1.10 and 4.10) this Laplace transform (32) can be inverted to give

$$f_{T_\delta}(t) = \frac{1}{C_\delta} \sum_{n=1}^\infty \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} e^{-\frac{1}{2} j_{\nu,n}^2 t} \qquad (t > 0). \tag{33}$$

We note that Ismail-Kelker state this result only for $\nu > 0$, but that their argument can be adapted also to the case $\nu > -1$. The agreement formula (12) can be restated in terms of densities as

$$f_{M_\delta^{-2}}(t) = C_\delta t^\nu \int_0^t dx\, f_{T_\delta}(x) f_{T_\delta}(t-x) \qquad (t > 0). \tag{34}$$

On the other hand, the Gikhman-Kiefer formula (5) amounts to

$$f_{M_\delta^{-2}}(t) = \frac{t^\nu}{C_\delta} \sum_{n=1}^{\infty} (j_{\nu,n}^2 t - 2(\nu+1)) \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} e^{-\frac{1}{2} j_{\nu,n}^2 t} \qquad (t > 0). \tag{35}$$

The similarity between (33) and (35) suggests that it should be possible to pass between (35) and (33) via the agreement formula (34). In Section 11 we indicate how this can be done for $\delta \leq 1$, but we do not see how to do this for general δ.

In view of (32), by equating the right sides of (34) and (35), then multiplying by $C_\delta t^{-\nu} e^{-\frac{1}{2}\lambda^2 t}$ and integrating from 0 to ∞, we obtain the following identity: for all $\lambda > 0$

$$\frac{\lambda^{2\nu}}{I_\nu^2(\lambda)} = \int_0^\infty dt \sum_{n=1}^{\infty} \left(j_{\nu,n}^2 t - 2(\nu+1) \right) \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} e^{-\frac{1}{2}(j_{\nu,n}^2 + \lambda^2) t}. \tag{36}$$

By uniqueness of Laplace transforms, any one of these three formulae (34), (35) and (36) follows easily from the other two. It does not seem at all easy to verify (36) directly for general $\nu > -1$.

Confirmation of (36) for $\nu \in (-1, 0)$. For $\nu \in (-1,0)$ a switch of summation and integration in (36) can be justified by (138) and (139). Then (36) becomes

$$\frac{\lambda^{2\nu}}{I_\nu^2(\lambda)} = \sum_{n=1}^{\infty} \left(\frac{4 j_{\nu,n}^2}{(j_{\nu,n}^2 + \lambda^2)^2} - \frac{4(\nu+1)}{j_{\nu,n}^2 + \lambda^2} \right) \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} \qquad (-1 < \nu < 0). \tag{37}$$

In view of the Wronskian relation (156), this identity (37) can be verified by differentiation of the identity

$$\lambda^{2\nu} \frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} = \frac{-4}{\Gamma(\nu)\Gamma(1-\nu)} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} \frac{1}{(j_{\nu,n}^2 + \lambda^2)} \qquad (-1 < \nu < 0) \tag{38}$$

which is a simplification of the particular case $\mu = -\nu$ of the following identity due to Ismail-Kelker [22, (4.10)]:

$$\lambda^{\nu-\mu} \frac{I_\mu(\lambda)}{I_\nu(\lambda)} = -2 \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{\nu-\mu+1} J_\mu(j_{\nu,n})}{J_\nu'(j_{\nu,n})} \frac{1}{(j_{\nu,n}^2 + \lambda^2)} \qquad (-1 < \nu < \mu). \tag{39}$$

The passage from the case $\mu = -\nu$ of (39) to (38) involves the identity

$$\frac{j_{\nu,n} J_{-\nu}(j_{\nu,n})}{J_\nu'(j_{\nu,n})} = \frac{2}{\Gamma(\nu)\Gamma(1-\nu) J_{\nu+1}^2(j_{\nu,n})} \tag{40}$$

which is valid for all $\nu \in (-1, 0)$ and all $n = 1, 2, \ldots$ by application of the Wronskian identity (154) and $J_\nu(j_{\nu,n}) = 0$.

For instance, for $\nu = -1/2$ we find using (144) that after the substitution $\lambda = \frac{\pi}{2}x$ formula (37) reduces to

$$\frac{\frac{\pi}{2}}{\cosh^2 \frac{\pi}{2}x} = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{2(2n-1)^2}{((2n-1)^2+x^2)^2} - \frac{1}{(2n-1)^2+x^2}\right) \tag{41}$$

which is the differentiated form of the standard identity [20, 1.421.2]

$$\tanh \tfrac{\pi}{2}x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{x}{(2n-1)^2+x^2}. \tag{42}$$

5 The first passage transform and its derivatives

Since a BES(δ) bridge of length t from 0 to 0 can be constructed by Brownian scaling from the standard BES(δ) bridge, the general first passage transform (20) and (25) imply that for each $\delta > 0$ the distribution of M_δ is determined by the following formula: for $x > 0$

$$\int_0^\infty dt\, t^{-\nu-1}\, e^{-t} P\left(M_\delta \geq \frac{x}{\sqrt{2t}}\right) = Q_{\delta,0}(x) := \frac{2}{\Gamma(\nu+1)}\frac{K_\nu(x)}{I_\nu(x)}. \tag{43}$$

For $a > 0$ let Γ_a denote a random variable with the gamma(a) distribution

$$P(\Gamma_a \in dt)/dt = \Gamma(a)^{-1}t^{a-1}e^{-t} \qquad (t > 0). \tag{44}$$

In the recurrent case with $\delta < 2$, that is $\nu < 0$, formula (43) amounts to the result of [10, §4] that

$$P(\sqrt{2\Gamma_\alpha}\, M_\delta > x) = \frac{Q_{\delta,0}(x)}{\Gamma(\alpha)} \tag{45}$$

where $\alpha := -\nu \in (0,1)$ and Γ_α is assumed independent of M_δ. In this case, we can construct M_δ as the maximum of the particular standard Bessel bridge

$$r_\delta(u) := R_\delta(ug_T)/\sqrt{g_T} \qquad (0 \leq u \leq 1)$$

where g_T is the time of the last zero of the $\mathrm{BES}_0(\delta)$ process R_δ before an independent exponential time $T := 2\Gamma_1$. Then g_T is independent of r_δ with $g_T \overset{d}{=} 2\Gamma_\alpha$, so there is the interpretation

$$\sup_{0\leq t\leq g_T} R_\delta(t) \overset{d}{=} \sqrt{2\Gamma_\alpha}\, M_\delta \qquad (0 < \delta < 2,\ \alpha := 1 - \delta/2). \tag{46}$$

Formula (45) can then be understood in terms of the excursions away from 0 of the recurrent Bessel process R_δ. For $\delta \geq 2$ this probabilistic interpretation of the first passage transform (43) breaks down. Still, it is possible to characterize the distribution of M_δ for arbitrary $\delta > 0$ via the distribution of $\sqrt{2\Gamma_a}M_\delta$ for suitable $a > 0$, as shown by the following theorem. Section 6 gives applications of this characterization to the computation of moments of M_δ.

Theorem 5 *For each $\delta := 2 + 2\nu > 0$, and each $n = 0, 1, 2, \ldots$, the distribution of M_δ is uniquely determined by the following formula: for $x > 0$*

$$\int_0^\infty dt\, t^{n-\nu-1}\, e^{-t} P\left(M_\delta \geq \frac{x}{\sqrt{2t}}\right) = Q_{\delta,n}(x) \tag{47}$$

where $Q_{\delta,0}(x)$ *is defined by* (43) *and* $Q_{\delta,n+1}(x)$ *is determined for* $n = 0, 1, 2, \ldots$ *by the recursion*

$$Q_{\delta,n+1}(x) := (n-\nu) Q_{\delta,n}(x) - \frac{x}{2} \frac{d}{dx} Q_{\delta,n}(x). \tag{48}$$

Moreover, if $n > \nu$ *then*

$$Q_{\delta,n}(x) = \Gamma(n-\nu) - \frac{4\, n!\, x^{2n-2\nu}}{\Gamma(\nu+1)} \sum_{m=1}^{\infty} \frac{j_{\nu,m}^{2\nu}}{J_{\nu+1}^2(j_{\nu,m})(j_{\nu,m}^2 + x^2)^{n+1}} \tag{49}$$

and the identity (47) *can be recast as*

$$P(\sqrt{2\Gamma_{n-\nu}}\, M_\delta > x) = \frac{Q_{\delta,n}(x)}{\Gamma(n-\nu)} \tag{50}$$

where $\Gamma_{n-\nu}$ *with* gamma$(n-\nu)$ *distribution is assumed to be independent of* M_δ*.*

Proof. Formula (47) for $n = 0$ is just (43). Formula (47) for $n = 1, 2, \ldots$ and the recursion (48) are consequences of the following lemma. Formula (49) follows easily from (47) by applying the Gikhman-Kiefer formula (5) and integrating term by term. The interpretation (50) of (47) is evident by conditioning on $\Gamma_{n-\nu}$. □

Lemma 6 *Let* g *be a non-negative Borel function. For real* r *and* $x > 0$ *let*

$$\Psi_r(x) := \int_0^\infty t^{r-1} e^{-t} g\left(\frac{x}{\sqrt{2t}}\right) dt \tag{51}$$

and suppose that $\Psi_r(x_0) < \infty$ *for some* r *and* x_0*. Then* $\Psi_r(x)$ *is differentiable at each* $x \in (x_0, \infty)$*, with*

$$\Psi_{r+1}(x) = r\Psi_r(x) - \frac{x}{2}\frac{d}{dx}\Psi_r(x). \tag{52}$$

Proof. Make the change of variable $t = x^2 u$ and then differentiate with respect to x. □

The first few functions $Q_{\delta,n}(x)$ and their derivatives can be computed as follows, using the recurrence (48), which implies easily the following recurrence for the derivatives $Q'_{\delta,n}(x)$:

$$Q'_{\delta,n+1}(x) := (n - \nu - \tfrac{1}{2}) Q'_{\delta,n}(x) - \frac{x}{2}\frac{d}{dx} Q'_{\delta,n}(x). \tag{53}$$

The evaluation of $Q'_{\delta,0}$ uses the Wronskian formula (155), while the evaluation of $Q'_{\delta,1}$ via (53) involves the recurrence formula (158) for Bessel functions.

$$Q_{\delta,0}(x) := \frac{2}{\Gamma(\nu+1)} \frac{K_\nu(x)}{I_\nu(x)}; \quad -Q'_{\delta,0}(x) = \frac{2}{\Gamma(\nu+1) x I_\nu^2(x)} \tag{54}$$

$$Q_{\delta,1}(x) = \frac{1 - 2\nu I_\nu(x) K_\nu(x)}{\Gamma(\nu+1) I_\nu^2(x)}; \quad -Q'_{\delta,1}(x) = \frac{2 I_{\nu+1}(x)}{\Gamma(\nu+1) I_\nu^3(x)} \tag{55}$$

$$Q_{\delta,2}(x) = \frac{(1-\nu) I_\nu(x)(1 - 2\nu I_\nu(x) K_\nu(x)) + x I_{\nu+1}(x)}{\Gamma(\nu+1) I_\nu^3(x)} \tag{56}$$

$$-Q'_{\delta,2}(x)=\frac{-xI_\nu^2+3xI_{\nu-1}I_{\nu+1}+3xI_{\nu+1}^2+(2-4\nu)I_\nu I_{\nu+1}-xI_\nu I_{\nu+2}}{2\Gamma(\nu+1)I_\nu^4} \tag{57}$$

where in the last formula we abbreviate $I_\mu(x)$ to I_μ. The interpretation (47) implies that $Q_{\delta,n}(x)$ is a decreasing function of x, so $-Q'_{\delta,n}(x)\geq 0$ for all $x>0$. For $n=0$ and $n=1$ this is obvious from the above formulae and the non-negativity of I_μ. But the fact that $-Q'_{\delta,2}(x)\geq 0$ does not seem at all evident from (57).

An alternative expression for the derivatives $Q'_{\delta,n}(x)$ can be obtained as follows. Differentiate formula (47) with respect to x, then make the change of variable $u=x/\sqrt{2t}$ to obtain the following interpretation of the functions $-Q'_{\delta,n}(x)$ for $n=0,1,2\dots$, where the second equality is read from the agreement formula (12):

$$-Q'_{\delta,n}(x)=\frac{x^{2n-2\nu-1}}{2^{n-\nu-1}}E\left[M_\delta^{2\nu-2n}e^{-\frac{x^2}{2M_\delta^2}}\right]=\frac{x^{2n-2\nu-1}}{2^{n-\nu-1}}C_\delta\,E\left[\widetilde{T}_\delta^n e^{-\frac{x^2}{2}\widetilde{T}_\delta}\right]. \tag{58}$$

It will be seen in the next section that $E(M_\delta^r)<\infty$ for all real r. It follows that for all real n formula (47) serves to define a function $Q_{\delta,n}(x)$ which is finite for all $x>0$ and $\delta>0$, with derivative $Q'_{\delta,n}(x)$ such that (58) holds. Moreover $-Q'_{\delta,n}(x)$ defined by (58) is non-negative and continuous, with integral over $(0,\infty)$ equal to ∞ or $\Gamma(n-\nu)$ according to whether $n\leq\nu$ or $n>\nu$. In the latter case, $-Q'_{\delta,n}(x)/\Gamma(n-\nu)$ is the probability density of $\sqrt{2\Gamma_{n-\nu}}M_\delta$ for $\Gamma_{n-\nu}$ with gamma$(n-\nu)$ distribution independent of M_δ.

We now point out some particular instances of the above formulae in dimensions 1 and 3, obtained by using (143) and (142) to evaluate the Bessel functions involved. Note that $\sqrt{2\Gamma_{1/2}}\overset{d}{=}|N|$ where N is a standard normal variable. For further discussion of these formulae and related results, see Section 7. Let $x>0$. Then

$$P(\sqrt{2\Gamma_{1/2}}M_1\leq x)=\tanh x \tag{59}$$

$$P(\sqrt{2\Gamma_{3/2}}M_1\leq x)=\tanh x-\frac{x}{\cosh^2 x} \tag{60}$$

$$P(\sqrt{2\Gamma_{5/2}}M_1\leq x)=\tanh x-x-\frac{2}{3}x^2\tanh x+x\tanh^2 x+\frac{2}{3}x^2\tanh^3 x \tag{61}$$

$$P(\sqrt{2\Gamma_{1/2}}M_3\leq x)=\coth x-\frac{x}{\sinh^2 x} \tag{62}$$

$$P(\sqrt{2\Gamma_{3/2}}M_3\leq x)=\coth x+\frac{x}{\sinh^2 x}-\frac{2x^2\cosh x}{\sinh^3 x} \tag{63}$$

Formula (62) was obtained independently by Jansons [25] for M_3 identified as the maximum of a standard Brownian excursion. This particular interpretation of formula (50) is extended by the following construction of a random variable with the distribution of $\sqrt{2\Gamma_{n-\nu}}M_\delta$ described in (50) for all $\delta\in(2,4)$ and $n=1,2,\dots$. It is known [39] that for $\delta\in(2,4)$ the standard Bessel bridge r_δ is identical to the standard excursion derived from the recurrent Bessel process $R_{4-\delta}$. As shown in [36], if $T=2\Gamma_1$ is constructed as the first point of a Poisson process $(N(t),t\geq 0)$ of rate $1/2$ independent of $R_{4-\delta}$, and (g_T,d_T) denotes the excursion interval of $R_{4-\delta}$ away from

0 that straddles time T, then for each $n = 1, 2, \ldots$

$$(d_T - g_T \,|\, N(d_T) = n) \stackrel{d}{=} 2\Gamma_{n-\nu}$$

where $\nu = (\delta - 2)/2$. It then follows by Brownian scaling that there is the following variation of the identity (46):

$$\left(\sup_{g_T \le t \le d_T} R_{4-\delta}(t) \,\middle|\, N(d_T) = n \right) \stackrel{d}{=} \sqrt{2\Gamma_{n-\nu}}\, M_\delta \quad (2 < \delta < 4, \nu := (\delta - 2)/2). \tag{64}$$

We do not know of any analog of (46) and (64) for $\delta = 2$ or $\delta \ge 4$. But see [45, 46, 10, 37] for other characterizations of the distribution of some Brownian or Bessel functional F via the distribution of $\sqrt{2\Gamma_a}F$ for a suitable independent gamma variable Γ_a.

Formula (50) for $n = 2$ and $\delta = 5$, with (56), with (146) and (148), gives a rather complicated expression in terms of hyperbolic functions for $P(\sqrt{2\Gamma_{1/2}}M_5 \le x)$.

Remark. In connection with the consistency of formula (50) as n varies, let X be a non-negative random variable and for $r, x > 0$ let

$$F_r(x) := P(\sqrt{2\Gamma_r}X > x)$$

where Γ_r is a gamma(r) variable independent of X. Observe that if $g(x) := P(X > x)$ then $\Psi_r(x)$ defined by (51) equals $\Gamma(r)F_r(x)$. So (52) in this case, after division of both sides by $\Gamma(r+1) = r\Gamma(r)$, yields

$$F_{r+1}(x) = F_r(x) - \frac{x}{2r}\frac{d}{dx}F_r(x). \tag{65}$$

6 Moments

For each $\delta > 0$, each of the functions

$$s \mapsto E(M_\delta^s) \text{ and } s \mapsto E(M_{\delta*}^s)$$

is a well defined entire function of a complex variable $s := \Re s + i\Im s$. This follows immediately from asymptotic estimates on the tails of the distributions of M_δ and $M_{\delta*}$ given by Gruet and Shi [21]. Alternatively, it can be verified by elementary arguments that each of the four random variables $M_\delta^2, M_\delta^{-2}, M_{\delta*}^2$ and $M_{\delta*}^{-2}$ has a moment generating function which is defined in some neighbourhood of the origin. Using the agreement formula (12) and (11), for arbitrary complex s and q there are the formulae

$$E\left[M_\delta^s\right] = C_\delta E\left[\widetilde{T}_\delta^{(\delta-s-2)/2}\right]; \qquad E\left[\widetilde{T}_\delta^q\right] = C_\delta^{-1} E\left[M_\delta^{\delta-2-2q}\right] \tag{66}$$

$$E\left[M_{\delta*}^s\right] = E\left[T_\delta^{-s/2}\right]; \qquad E\left[T_\delta^q\right] = E\left[M_{\delta*}^{-2q}\right]. \tag{67}$$

We now deduce from results of the previous sections various formulae for $E\left[M_\delta^s\right]$ and $E\left[M_{\delta*}^s\right]$ which translate via (66) and (67) into formulae for $E\left[\widetilde{T}_\delta^q\right]$ and $E\left[T_\delta^q\right]$.

Corollary 7 *For each $\delta = 2 + 2\nu > 0$ and each $n = 0, 1, 2, \ldots$ the distribution of M_δ is the unique probability distribution on $(0, \infty)$ determined by the Mellin transform*

$$E(M_\delta^s) = \frac{2^{-\frac{s}{2}}}{\Gamma(n - \nu + \frac{s}{2})} \int_0^\infty s\, x^{s-1} Q_{\delta,n}(x)\, dx \qquad (\Re s > 0 \vee \delta - 2 - 2n) \tag{68}$$

where $Q_{\delta,0}(x) := (2/\Gamma(1+\nu))K_\nu(x)/I_\nu(x)$ and $Q_{\delta,n+1}(x)$ is determined for $n = 0, 1, 2, \ldots$ by the recursion (48)*. Also,*

$$E(M_\delta^s) = \frac{2^{-\frac{s}{2}}}{\Gamma(n - \nu + \frac{s}{2})} \int_0^\infty x^s(-Q'_{\delta,n}(x))\, dx \qquad (\Re s > \delta - 2 - 2n) \tag{69}$$

where $-Q'_{\delta,0}(x) = 2/(\Gamma(\nu+1)xI_\nu^2(x))$ and $Q'_{\delta,n+1}(x)$ is determined for $n = 0, 1, 2, \ldots$ by the recursion (53)*; and*

$$E(M_\delta^s) = \frac{-s\, 2^{\nu+1-\frac{s}{2}}\, \Gamma(\nu + 1 - \frac{s}{2})}{C_\delta} \sum_{m=1}^\infty \frac{j_{\nu,m}^{s-2}}{J_{\nu+1}^2(j_{\nu,m})} \qquad (\Re s < 0). \tag{70}$$

Proof. Consider first positive real s. Multiply (47) by sx^{s-1}, integrate x from 0 to ∞, and then switch the order of integration and expectation. This gives (68) for $s > 0 \vee \delta - 2 - 2n$, hence the result for all s with $\Re s > 0 \vee \delta - 2 - 2n$. The alternative form (69) follows from (68) by integration by parts. Formula (70) is obtained as indicated below (29). □

If $\nu < n$ then both (68) and (70) are valid for suitable s. The consistency of the two formulae then amounts to the expression (49) for $Q_{\delta,n}(x)$, which is in turn an expression of the Gikhman-Kiefer formula (5). As far as we know, the only way to enter this circle of identities for general δ is by the method of Section 3.

Analogous formulae for $M_{\delta*}$. By combination of (11) with (32), (166) and (33) we obtain

$$E(M_{\delta*}^s) = \frac{2^{1-s/2}}{C_\delta \Gamma(\delta/2)} \int_0^\infty dx\, \frac{x^{s+\nu-1}}{I_\nu(x)} \qquad (\Re s > 0). \tag{71}$$

$$E(M_{\delta*}^s) = \frac{2^{1-s/2}}{C_\delta} \sum_{n=1}^\infty \frac{j_{\nu,n}^{s-1+\nu}}{J_{\nu+1}(j_{\nu,n})} \qquad (\Re s < -\nu - 1/2) \tag{72}$$

where the condition on $\Re s$ ensures absolute convergence of the series, by (138) and (139). These asymptotic relations also show that the series in (72) is divergent if $\Re s \geq -\nu + 1/2$.

Some simple moments. From (32) and the series expansion for I_ν given in (133)

$$E[\exp(-\lambda \widetilde{T}_\delta)] = \left(1 + \sum_{k=1}^\infty a_k(\delta)\lambda^k\right)^{-2} = 1 + \sum_{m=1}^\infty b_m(\delta)\lambda^m \tag{73}$$

where $(a_k(\delta))^{-1} = k!\delta(\delta+2)\cdots(\delta+2k)$ and the $b_m = b_m(\delta)$ are obtained from the $a_k = a_k(\delta)$ by formal power series manipulation. Thus

$$b_1 = -2a_1, b_2 = 3a_1^2 - 2a_2, b_3 = -4a_1^3 + 6a_1a_2 - 2a_3$$

and so on. In view of (66), this implies that

$$C_\delta^{-1}\, E(M_\delta^{\delta-2-2n}) = E(\widetilde{T}_\delta^n) = (-1)^n n! b_n(\delta) \tag{74}$$

is a rational function of δ for every $n = 0, 1, \ldots$. The first few of these functions are displayed in the following table:

n	0	1	2	3	4
$E(\widetilde{T}_\delta^n)$	1	$\frac{2}{\delta}$	$\frac{2^2(\delta+3)}{\delta^2(\delta+2)}$	$\frac{2^3(\delta^2+9\delta+24)}{\delta^3(\delta+2)(\delta+4)}$	$\frac{2^4(\delta^4+20\delta^3+159\delta^2+588\delta+720)}{\delta^4(\delta+2)^2(\delta+4)(\delta+6)}$

Assuming $q < \delta$, the form of C_δ implies $E(M_\delta^q)$ is a rational number if both δ and q are even positive integers, and a rational multiple of $\sqrt{2\pi}$ if both δ and q are odd positive integers. For instance, (74) gives the formulae (13) and (14) presented in the introduction. As a check, these exact results agree with the approximate values of these moments found by numerical integration using (68).

A generalization of Riemann's zeta function. According to formula (70), the series

$$\Sigma(\nu, q) := \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{-2-q}}{J_{\nu+1}^2(j_{\nu,n})} \tag{75}$$

is absolutely convergent for all q with $\Re q > 0$. This can also be checked using (138) and (139). In the cases $\nu = \pm\frac{1}{2}$, discussed further in Section 7, we find from (144) and (145) that

$$\Sigma(\tfrac{1}{2}, q) = \tfrac{1}{2}\pi^{-q}\zeta(q+1); \quad \Sigma(-\tfrac{1}{2}, q) = (2^{q+1}-1)\Sigma(\tfrac{1}{2}, q)$$

where ζ is the Riemann zeta function. Formula (70) implies that for each $\nu > -1$ the function $q \mapsto q\Gamma(\nu + 1 + \frac{q}{2})\Sigma(\nu, q)$ is the restriction to $(\Re q > 0)$ of an entire function. This is a generalization of Riemann's well known result on the analytic continuation of his zeta function. See Section 7 for references and further discussion. A natural question, left open here, is whether this generalization of Riemann's zeta function satisfies a functional equation. See also [55, §15.51], [2] and [62, §11.6] regarding the generalization of the Riemann zeta function obtained by consideration of the sum (75) with the factor $J_{\nu+1}^2(j_{\nu,n})$ replaced by 1.

By comparison of (70) and (74) we obtain a sequence of evaluations of $\Sigma(\nu, q)$ for $q = 2n - 2\nu > 0$ and $n = 0, 1, \ldots$. For instance, for $n = 0$ and $n = 1$ these evaluations read

$$\Sigma(\nu, -2\nu) = (-\nu)^{-1} 2^{2\nu-2}\Gamma(1+\nu)^2 \qquad (\nu \in (-1, 0)); \tag{76}$$

$$\Sigma(\nu, 2-2\nu) = (1-\nu^2)^{-1} 2^{2\nu-3}\Gamma(1+\nu)^2 \qquad (\nu \in (-1, 1)). \tag{77}$$

For $m = 1, 2, \ldots$ the $(m-1)$th identity for $\nu = -\frac{1}{2}$ and the mth identity for $\nu = +\frac{1}{2}$ reduce to the classical formula $\zeta(2m) = 2^{2m-1}\pi^{2m}|B_{2m}|/(2m)!$ where B_n is the nth Bernoulli number.

Other evaluations of $\Sigma(\nu, q)$ can be obtained by application of the formula (69). For instance, after a simplification using the reflection formula (165) for the gamma function, we find that

$$\Sigma(\nu, q) = \frac{\sin(-\nu - q/2)\pi}{q\pi} \int_0^\infty dx\, \frac{x^{-q-1}}{I_\nu^2(x)} \qquad (-1 < \nu < 0 < q < -2\nu). \tag{78}$$

Good numerical approximations to $\Sigma(\nu, q)$ are obtained by evaluating the sum of the first N terms in the series and using the estimates (138) and (139) to approximate the remainder with Hurwitz's zeta function $\zeta(s,a) := \sum_{n=0}^{\infty}(n+a)^{-s}$, which is available in *Mathematica*. Thus

$$\Sigma(\nu, q) = \sum_{n=1}^{N} \frac{j_{\nu,n}^{-2-q}}{J_{\nu+1}^2(j_{\nu,n})} + \frac{\pi^{-q}}{2}\zeta(1+q, N+1+\nu/2-1/4) + \epsilon_{\nu,N} \tag{79}$$

where the error term $\epsilon_{\nu,N}$ converges to zero as $N \to \infty$ much more rapidly than if the series were simply truncated at N.

Some logarithmic moments. As another application of formula (74) we obtain the following result, indicated already in [42, (3.19)] in the special case $n = 0$.

Proposition 8 *For each $n = 0, 1, 2, \ldots$,*

$$E(\log M_{2n+2}) = \tfrac{1}{2}(\log 2 - \gamma) + \theta_n \tag{80}$$

where $\gamma \approx 0.577216$ is Euler's constant, and

$$\theta_n := \tfrac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k}\right) + 2^n n! \left.\frac{d}{d\delta}E(\widetilde{T}_\delta^n)\right|_{\delta=2n+2} \tag{81}$$

is a rational number for every n.

Proof. Differentiate formula (66) with respect to δ at $\delta = 2n+2$ for fixed non-negative integer $q = n$ to obtain

$$E(\log M_{2n+2}) = \frac{d}{d\delta}\left[C_\delta E(\widetilde{T}_\delta^n)\right]\Big|_{\delta=2n+2} = C_\delta^{-1}\left.\frac{d}{d\delta}C_\delta\right|_{\delta=2n+2} + C_{2n+2}\left.\frac{d}{d\delta}E(\widetilde{T}_\delta^n)\right|_{\delta=2n+2}$$

where the first equality is justified by dominated convergence, using the fact that M_δ can be constructed to be increasing in δ, and the second equality appeals again to (66). The conclusion now follows from (6), using the standard evaluation [1, 6.3.2] of the digamma function $\psi(z) := d(\log\Gamma(z))/dz$ for $z = 1, 2, \ldots$ involving Euler's constant $\gamma := -\psi(1)$. $\square$

With some help from *Mathematica*, the first few values of θ_n and $E(\log(M_{2n+2}))$ were found to be as follows:

n	0	1	2	3	4
$\delta = 2n+2$	2	4	6	8	10
θ_n	0	$\frac{1}{4}$	$\frac{29}{72}$	$\frac{247}{480}$	$\frac{126{,}697}{210{,}000}$
$E(\log(M_{2n+2}))$	0.058	0.308	0.461	0.573	0.661

7 Dimensions one and three

For these dimensions, corresponding to $\nu = -\frac{1}{2}$ and $\frac{1}{2}$, the functions $I_{\pm\frac{1}{2}}(x)$ and $K_{\pm\frac{1}{2}}(x)$ can be expressed in terms of hyperbolic functions as in (143), so that

$$\frac{K_{-\frac{1}{2}}(x)}{I_{-\frac{1}{2}}(x)} = \frac{\pi}{e^{2x}+1}; \qquad \frac{K_{\frac{1}{2}}(x)}{I_{\frac{1}{2}}(x)} = \frac{\pi}{e^{2x}-1}.$$

Let $\zeta(s)$ denote the Riemann zeta function defined for complex $s = \Re s + i\Im s$ with $\Re s > 1$ by the standard series and integral representations [1, §23.2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad = \quad \frac{1}{\Gamma(s)} \int_0^{\infty} dx \, \frac{x^{s-1}}{e^x - 1} \tag{82}$$

$$= \quad \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} dx \, \frac{x^{s-1}}{e^x + 1} \tag{83}$$

As shown by Riemann, the formula

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) \qquad (\Re s > 1) \tag{84}$$

defines the restriction to $(\Re s > 1)$ of a unique entire analytic function ξ which satisfies the functional equation $\xi(s) = \xi(1-s)$ for all complex s. See any of the texts [11, 16, 48, 53] for details. If we apply (68) for $\delta = 3$ and then appeal to (82), the duplication formula for the gamma function (164), and the facts that both $E(M_3^s)$ and $\xi(s)$ are entire functions of s, we recover the remarkable result of Biane-Yor [5] that

$$E(M_3^s) = \left(\frac{\pi}{2}\right)^{s/2} 2\xi(s) \tag{85}$$

for all complex s. See also [5, 60, 4] regarding alternative proofs of (85) and the relation of this result to Chung's identity (9) and the agreement formula. A similar application of (68) for $\delta = 1$, using (83), yields

$$E(M_1^s) = \frac{(1-2^{1-s})}{(s-1)} \left(\frac{\pi}{2}\right)^{s/2} 2\xi(s) \qquad (s \neq 1) \tag{86}$$

and hence $E(M_1) = \sqrt{\pi/2}\,\log 2$ by passage to the limit as $s \to 1$. Let N be a standard normal variable independent of M_1. Since $\sqrt{2\Gamma_{1/2}} \overset{d}{=} |N|$, formula (59) can be rewritten

$$P(|N|M_1 > x) = 1 - \tanh(x) = \frac{2}{1+e^{2x}}. \tag{87}$$

Equivalently, the symmetric random variable NM_1 has the *logistic distribution* on $\mathbb{R}$

$$P(NM_1 > y) = \frac{1}{1+e^{2y}} \qquad (y \in \mathbb{R}). \tag{88}$$

According to (88), a logistic distribution is obtained as a scale mixture of centered normal distributions when the scale parameter is given the Kolmogorov-Smirnov distribution (4). This was observed by Stefansky [52], who gave an analytic proof. Formula (87) can also be derived by an analysis of one-dimensional Brownian motion stopped at an independent exponential time, using excursion theory [49, Ex. 4.24 of Ch. XII].

Differentiate (88) to obtain

$$P(NM_1 \in dy)/dy = \frac{1}{2\cosh^2(y)} \qquad (y \in \mathbb{R}) \tag{89}$$

and thence the Fourier transform

$$E\exp(i\lambda NM_1) = \int_{-\infty}^{\infty} \frac{e^{i\lambda y}\,dy}{2\cosh^2(y)} = \frac{\frac{\pi}{2}\lambda}{\sinh(\frac{\pi}{2}\lambda)}. \tag{90}$$

On the other hand, the expectation in (90) can be computed by conditioning on M_1 to recover the identity (7).

As a check on (62), let us show that it agrees with Chung's formula (9). Indeed, the left side of (9) equals $E(\exp(i\lambda NM_3))$ where N is standard normal and independent of M_3. Since $\sqrt{2\Gamma_{1/2}} \stackrel{d}{=} |N|$, for $x > 0$ the density of NM_3 at either x or $-x$ is half the derivative of the right side of (62). So (62) amounts to

$$P(NM_3 \in dx)/dx = \frac{x\coth(x) - 1}{\sinh^2(x)} \qquad (x \in \mathbb{R}). \tag{91}$$

According to the table of Fourier-Laplace transforms in Biane-Yor [5], the Fourier transform in λ of the function in (91) is the right side of (9). Thus if we consider Chung's formula (9), formula (62) for the distribution of $\sqrt{2\Gamma_{1/2}}M_3$, and the abovementioned formula for the Fourier transform of the density (91), any two of these three formulae imply the third.

8 Limits as $\delta \to \infty$

The asymptotic normality of the distribution of M_δ as $\delta \to \infty$, described in Theorem 3, is a consequence of the following description of the asymptotic distribution of the bridge $(r_\delta(u), 0 \le u \le 1)$ as $\delta \to \infty$:

Theorem 9 *As $\delta \to \infty$*

$$\left(r_\delta(u) - \sqrt{\delta}\sqrt{u(1-u)},\, 0 \le u \le 1\right) \stackrel{d}{\to} (Y_u, 0 \le u \le 1) \tag{92}$$

in the sense of weak convergence of distributions on $C[0,1]$, where $(Y_u, 0 \le u \le 1)$ is a centered Gaussian process with covariance

$$E(Y_u Y_v) = \frac{1}{2}\frac{u^{3/2}}{(1-u)^{1/2}}\frac{(1-v)^{3/2}}{v^{1/2}} \qquad (0 < u \le v < 1). \tag{93}$$

Proof. By writing $a^2-b^2=(a-b)(a+b)$ the conclusion (92) is easily seen to be equivalent to

$$\left(\sqrt{\delta}\left(\frac{r_\delta^2(u)}{\delta}-u(1-u)\right),\ 0\le u\le 1\right) \xrightarrow{d} (X_u, 0\le u\le 1) \tag{94}$$

where $(X_u := 2\sqrt{u(1-u)}Y_u,\ 0\le u\le 1)$ is a centered Gaussian process with covariance

$$E(X_uX_v)=2u^2(1-v)^2 \quad (0\le u\le v\le 1). \tag{95}$$

The result (94) for $(r_\delta(u), 0\le u\le 1)$ follows from the representation (2) and the corresponding result for $(R_\delta(t), t\ge 0)$, which is

$$\left(\sqrt{\delta}\left(\frac{R_\delta^2(t)}{\delta}-t\right), t\ge 0\right) \xrightarrow{d} (Z_t, t\ge 0) \tag{96}$$

in the sense of weak convergence of distributions on $C[0,\infty)$ endowed with the topology of uniform convergence on compacts, where

$$(Z_t := 2\int_0^t \sqrt{u}\, d\beta_u, t\ge 0) \stackrel{d}{=} (2\beta_{t^2/2}, t\ge 0)$$

for β a standard Brownian motion. To obtain (96), rewrite the SDE (1)

$$R_\delta^2(t) = 2\int_0^t R_\delta(s)d\beta_s + \delta t$$

as

$$\sqrt{\delta}\left(\frac{R_\delta^2(t)}{\delta}-t\right) = 2\int_0^t \frac{R_\delta(s)}{\sqrt{\delta}}\, d\beta_s. \tag{97}$$

By application of the Burkholder-Davis-Gundy and Doob inequalities (see e.g. [49]), we see that for each $T>0$ and $p\ge 1$,

$$\left\|\ \sup_{0\le t\le T}\left|\frac{R_\delta^2(t)}{\delta}-t\right|\ \right\|_{L^p} = O\left(\frac{1}{\sqrt{\delta}}\right) \quad \text{as } \delta\to\infty.$$

Application of this estimate in (97) now yields (96). □

Proof of Theorem 3. Let $(Y_\delta(u), 0\le u\le 1)$ denote the process on the left side of (92), so $r_\delta(u) = Y_\delta(u)+\sqrt{\delta}\sqrt{u(1-u)}$. Then

$$M_\delta - \tfrac{1}{2}\sqrt{\delta} = \sup_{0\le u\le 1} Y_\delta(u) - \sqrt{\delta}(\tfrac{1}{2}-\sqrt{u(1-u)}).$$

Since the function $u\to\frac{1}{2}-\sqrt{u(1-u)}$ is non-negative and continuous on $[0,1]$ with unique zero at $u=\frac{1}{2}$, it follows from the convergence in distribution (92) of $(Y_\delta(u), 0\le u\le 1)$ to $(Y_u, 0\le u\le 1)$ that there is the convergence in probability

$$M_\delta - \tfrac{1}{2}\sqrt{\delta} - Y_\delta(1/2) \xrightarrow{P} 0 \text{ as } \delta\to\infty.$$

Therefore, $M_\delta-\frac{1}{2}\sqrt{\delta}$ has the same limit in distribution as $Y_\delta(1/2)$ as $\delta\to\infty$, that is $N(0,1/8)$ by (93). The proof of the result for $M_{\delta *}$ is similar but easier, so we leave the details of this to the reader. □

9 Limits as $\delta \downarrow 0$

This section offers several different approaches to the asymptotic distribution of M_δ as $\delta \downarrow 0$, as presented in Theorem 4.

Proof of Theorem 4. As shown by [10, §4], formula (50) for $n = 0$ and $\delta \in (0, 2)$ can be recast as the following identity in distribution:

$$B_{\alpha,1-\alpha} M_\delta^2 \stackrel{d}{=} (T_\delta + L_{4-\delta})^{-1} \qquad (0 < \alpha := (2 - \delta)/2 < 1) \tag{98}$$

where on the left side $B_{\alpha,1-\alpha}$ is a beta$(\alpha, 1 - \alpha)$ variable which is independent of M_δ, and on the right side T_δ and $L_{4-\delta}$ are independent with $L_{4-\delta}$ distributed like the last hit of 1 by a $\text{BES}_0(4 - \delta)$ process, that is $L_{4-\delta} \stackrel{d}{=} (2\Gamma_\alpha)^{-1}$. Formula (98) can be combined with the agreement formula to obtain a property of the distribution of T_δ for $\delta \in (0, 2)$. Like (113), this can be expressed as an integral equation for $\phi(\lambda)$ as in (114). But the resulting equation is fairly complicated, and we did not pursue the question of uniqueness in this case.

Since $B_{\alpha,1-\alpha}$ converges in probability to 1 as $\alpha \uparrow 1$, we see from (98) that (16) amounts to

$$\delta T_\delta \stackrel{d}{\to} \mathcal{E} \text{ as } \delta \downarrow 0. \tag{99}$$

In terms of Laplace transforms, (99) is equivalent to

$$E(e^{-\lambda \delta T_\delta}) \to (1 + \lambda)^{-1} \text{ as } \delta \downarrow 0 \tag{100}$$

which is easily verified, either by (73) or by (32) and (133). In view of (11), the result (16) with $M_{\delta *}$ instead of M_δ follows also from (99). □

In terms of the density $f_\delta(t)$ of T_δ, the density of δT_δ at $x > 0$ is $\delta^{-1} f_\delta(x/\delta)$. As an alternate approach to (99), it can be seen from the series expansion (33) that for $x > 0$ this density $\delta^{-1} f_\delta(x/\delta)$ converges pointwise to the standard exponential density e^{-x} due to the asymptotics for the zeros $j_{\nu,n}$ of J_ν as $\nu \downarrow -1$ described by formulae (140) and (141). The same asymptotics for the $j_{\nu,n}$ allow (16) to be derived from the Gikhman-Kiefer formula (5) for $P(M_\delta \leq a)$.

We now provide a more complete picture of the structure of the $\text{BES}_0(\delta)$ bridge for small δ. It might be imagined that underlying the convergence in distribution of $M_\delta/\sqrt{\delta}$ to $1/\mathcal{E}$ as $\delta \downarrow 0$ there might be convergence in distribution of $(r_\delta(u)/\sqrt{\delta}, 0 \leq u \leq 1)$ to some limiting process in $C[0, 1]$ or $D[0, 1]$ whose maximum was distributed like $M_\delta/\sqrt{\delta}$ to $1/\sqrt{\mathcal{E}}$. But it is easily seen that this is not the case. It is known [43] that as $\delta \downarrow 0$, the length of the longest zero-free interval of $(r_\delta(u), 0 \leq u \leq 1)$ converges in probability to zero. It follows easily that $(r_\delta(u)/\sqrt{\delta}, 0 \leq u \leq 1)$ does not have a weak limit in either $C[0, 1]$ or $D[0, 1]$. Still, some particular functionals of $(r_\delta(u)/\sqrt{\delta}, 0 \leq u \leq 1)$ have weak limits which can be described in terms of Poisson point processes, as we now explain. For instance, if U_δ is the a.s. unique time that $(r_\delta(u), 0 \leq u \leq 1)$ attains its maximum value M_δ, then

$$(U_\delta, M_\delta/\sqrt{\delta}) \stackrel{d}{\to} (U, 1/\sqrt{\mathcal{E}}) \text{ as } \delta \downarrow 0 \tag{101}$$

where U is independent of $\mathcal{E}$ and U has uniform distribution on $(0, 1)$. This is a consequence of [42, Theorem 3.1] (which is recalled after Construction 13 in Section 10), using (99) for both T_δ and $\widehat{T}_\delta$.

Results such as (101) can also be explained in terms of the Poisson structure of the jumps of the $C[0,1]$-valued process $(r_\delta^2, \delta \geq 0)$ where $r_\delta := (r_\delta(u), 0 \leq u \leq 1)$ and it is assumed that $(r_\delta^2, \delta \geq 0)$ has been constructed with stationary independent increments, as in [39]. As shown in [35, Proposition 14], results of [39] imply that the path of r_δ^2 may be constructed as an infinite sum of random functions $\omega_{\delta,i} := (\omega_{\delta,i}(u), 0 \leq u \leq 1)$ which we call *blips*. Each blip is a path that vanishes except on an open subinterval of $[0,1]$ during which interval the blip is strictly positive. The blips $\omega_{\delta,i}$ contributing to r_δ^2 are the second coordinates of those points (d_j, ω_j) of a Poisson point process (PPP) on $(0,\infty) \times C[0,1]$ with $d_j \leq \delta$. The intensity measure of the PPP is the product of Lebesgue measure on $(0,\infty)$ and a σ-finite measure N_0 on $C[0,1]$. By combination of the description of N_0 in [35, Proposition 14] and [42, Theorem 1.1], there is the formula

$$N_0(\omega : \sup_{0\leq u\leq 1} \sqrt{\omega(u)} > x) = \int_x^\infty 2m^{-3}k(m)\,dm \tag{102}$$

for some continuous decreasing function $k(m)$ with $k(0+) = 1$ and $k(\infty-) = 0$. Consider now the PPP on $(0,\infty)$ whose points are $m_{\delta,i} := \sup_{0\leq u\leq 1} \sqrt{\omega_{\delta,i}(u)}$ as $\omega_{\delta,i}$ ranges over the collection of all blips whose sum is r_δ^2. Let S_δ be the maximal point of this PPP. By the construction of r_δ^2, it is obvious that

$$S_\delta \leq M_\delta \tag{103}$$

and it will be seen that

$$S_\delta / M_\delta \xrightarrow{P} 1 \text{ as } \delta \downarrow 0. \tag{104}$$

Now from (102) we find that

$$\begin{aligned} P(S_\delta/\sqrt{\delta} \leq y) &= \exp\left(-2\delta \int_{y\sqrt{\delta}}^\infty k(m)m^{-3}\,dm\right) && (105)\\ &= \exp\left(-2\int_y^\infty k(z\sqrt{\delta})z^{-3}\,dz\right) \qquad (m = z\sqrt{\delta}) && (106)\\ &\to \exp\left(-2\int_y^\infty z^{-3}\,dz\right) = \exp(-1/y^2) \text{ as } \delta \downarrow 0. && (107) \end{aligned}$$

by dominated convergence, using $k(0+) = 1$. Compare this calculation with (16) to see that $S_\delta/\sqrt{\delta}$ and $M_\delta/\sqrt{\delta}$ have the same limit in distribution as $\delta \downarrow 0$. Now (104) follows from (103).

In [45, 46] we gave a characterization of the joint law of the sequence

$$M_{1,\delta} \geq M_{2,\delta} \geq \cdots$$

of ranked heights of the excursions of $(r_\delta(u), 0 \leq u \leq 1)$ away from 0. So $M_{1,\delta} := M_\delta$, $M_{2,\delta}$ is the maximum of $(r_\delta(u), 0 \leq u \leq 1)$ on $[0,1] - I$ where I is the excursion interval during which $M_{1,\delta}$ is attained, and so on. The above considerations, combined with a simple variation of [40, Proposition 6.2], and the fact that a suitably normalized local time process for $(r_\delta(u), 0 \leq u \leq 1)$ at 0 converges uniformly in probabilty to the identity function as $\downarrow 0$, yield easily the following proposition:

Proposition 10 *Let $U_{i,\delta}$ denote the time at which the excursion maximum at level $M_{i,\delta}$ is attained. Then as $\delta \downarrow 0$*

$$(M_{i,\delta}/\sqrt{\delta}, U_{i,\delta}; i \geq 1) \xrightarrow{d} (H_i, U_i; i \geq 1)$$

where $H_1 > H_2 > \cdots$ are the ranked points of a PPP on $(0,\infty)$ with intensity $2h^{-3}dh$, that is $H_i = 1/\sqrt{\Gamma_i}$ where $\Gamma_i = \sum_{j=1}^{i} \mathcal{E}_j$ for independent standard exponential variables $\mathcal{E}_j$, and the $U_i, i \geq 1$ are independent uniform $(0,1)$ random variables, which are also independent of the H_i.

As another amplification of (99), we now consider the asymptotics of $\int_0^{T_\delta} ds\, h(R_\delta(s))$ as $\delta \downarrow 0$ for a general class of Borel functions h. Recall that $(R_1(t), t \geq 0)$ denotes a reflecting Brownian motion started at 0. Let $(L_t^x(R_1), t \geq 0, x \geq 0)$ denote the process of local times of R_1, normalized as occupation densities relative to Lebesgue measure, and recall the Ray-Knight theorem that

$$(L_{T_1}^x(R_1), 0 \leq x \leq 1) \stackrel{d}{=} (R_2^2(1-x), 0 \leq x \leq 1)$$

where T_1 is the hitting time of 1 by R_1.

Proposition 11 *Let $f : [0,1] \to \mathbb{R}$ be continuous, and $g : [0,1] \to \mathbb{R}$ be Borel measurable with $\int_0^1 \frac{da}{a}|g(a)| < \infty$. Then as $\delta \downarrow 0$*

$$\left(\delta \int_0^{T_\delta} ds\, f(R_\delta(s)), \int_0^{T_\delta} ds\, g(R_\delta(s))\right) \tag{108}$$

$$\stackrel{d}{\to} \left(\tfrac{1}{2} f(0) L_{T_1}^0(R_1), \frac{1}{4}\int_0^{T_1} \frac{ds}{R_1(s)} g\left(\sqrt{R_1(s)}\right)\right) \tag{109}$$

Proof. Apply [49, Prop. (1.11) of Ch. 11] to obtain the representation

$$\frac{-1}{\nu} R_\delta(t) = \left[R_1\left(\int_0^t \frac{ds}{(R_\delta(s))^{2\delta-2}}\right)\right]^{\frac{-1}{2\nu}}. \tag{110}$$

The result (109) follows easily from this representation as $\delta \downarrow 0$. Then $\nu \downarrow -1$ and $-1/(2\nu) \downarrow 1/2$, hence the appearance of $\sqrt{R_1(s)}$ on the right side of (109). □

10 Relation to last exit times

Let $L_\delta := \sup\{t : R_\delta(t) = 1\}$ denote the last hitting time of 1 by $\mathrm{BES}_0(\delta)$. It is well known that $P(L_\delta < \infty) = 0$ if $\delta \leq 2$ (the *recurrent case*) whereas $P(L_\delta < \infty) = 1$ if $\delta > 2$ (the *transient case*), when [18, 38]

$$L_\delta \stackrel{d}{=} (2\Gamma_\nu)^{-1} \qquad (\delta = 2 + 2\nu > 2). \tag{111}$$

The analog of (11) for L_δ is

$$1/L_\delta \stackrel{d}{=} \left(\inf_{t \geq 1} R_\delta(t)\right)^2. \tag{112}$$

See Yor [61] for an elementary approach to these results and their relation to the laws of the functionals $\int_0^\infty ds\, \exp[2(B_s - \nu s)]$ derived from a standard one-dimensional Brownian motion (B_s), for $\nu > 0$.

Theorem 12 *For each $\delta = 2 + 2\nu > 2$ the law of T_δ is uniquely characterized by the identity in distribution*

$$T_\delta + (2\Gamma_\nu)^{-1} \stackrel{d}{=} U^{-1/\nu} \widetilde{T}_\delta \tag{113}$$

where on the left side Γ_ν is a random variable with gamma(ν) *distribution, independent of T_δ, and on the right side U has uniform distribution on $[0,1]$, $\widetilde{T}_\delta = T_\delta + \widehat{T}_\delta$ where $\widehat{T}_\delta$ is an independent copy of T_δ, and U and $\widetilde{T}_\delta$ are assumed to be independent.*

Proof. Fix $\delta > 2$. Set $V_\delta := U^{-1/2\nu}$, so $P(V_\delta \in dv) = 2\nu v^{1-\delta} dv \ \ (v > 1)$, and let $L_\delta := 1/(2\Gamma_\nu)$. Given some distribution of T_δ on $(0, \infty)$, for $\lambda > 0$ let

$$\phi(\lambda) := E(e^{-\frac{1}{2}\lambda^2 T_\delta}); \quad \theta(\lambda) := E(e^{-\frac{1}{2}\lambda^2 L_\delta}) = \frac{2^{1-\nu}}{\Gamma(\nu)} \lambda^\nu K_\nu(\lambda), \tag{114}$$

where the last equality is read from (136). The claim is that the distribution of T_δ satisfies the identity (113) if and only if

$$\phi(\lambda) = \frac{\lambda^\nu}{2^\nu \Gamma(\nu+1) I_\nu(\lambda)}. \tag{115}$$

By taking Laplace transforms, the identity (113) is equivalent to

$$\phi(\lambda)\theta(\lambda) = 2\nu \int_1^\infty dv\, v^{1-\delta} \phi^2(\lambda v)$$

which is in turn equivalent to

$$\lambda^{-2\nu}\phi(\lambda)\theta(\lambda) = 2\nu \int_\lambda^\infty dw\, w^{1-\delta} \phi^2(w)$$

by the change of variable $v = w/\lambda$. Since $\phi(\lambda)$ and $\theta(\lambda)$ are bounded between 0 and 1, both sides of this identity vanish as $\lambda \to \infty$, so this identity is equivalent to

$$\frac{d}{d\lambda}\left(\lambda^{-2\nu}\phi(\lambda)\theta(\lambda)\right) = -2\nu\lambda^{1-\delta}\phi^2(\lambda).$$

The Wronskian identity (155) shows that this differential equation is solved by the $\phi(\lambda)$ defined by (115). As it is easily seen that the equation has a unique solution, the conclusion follows. □

Theorem 12 is related to the agreement formula (12) by consideration of the following construction, which we introduced in [42], following [58, 59, 39, 5].

Construction 13 Given two continuous path processes with random finite lifetimes, each with initial value 0 and final value 1, say $R := (R(t), 0 \le t \le \eta)$ and $(\widehat{R} := (\widehat{R}(t), 0 \le t \le \widehat{\eta})$ with $R(\eta) = \widehat{R}(\widehat{\eta}) = 1$, construct a random element $\widetilde{r}$ of $C[0,1]$ with $\widetilde{r}(0) = \widetilde{r}(1) = 0$ by first pasting R and $\widehat{R}$ back to back and then transforming the resulting path by Brownian scaling to have lifetime 1; that is

$$\widetilde{r}(u) := \begin{cases} \zeta^{-1/2} R(u\zeta) & \text{if } 0 \le u \le V \\ \zeta^{-1/2} \widehat{R}((1-u)\zeta) & \text{if } V \le u \le 1 \end{cases} \tag{116}$$

where $\zeta := \eta + \widehat{\eta}$ and $V := \eta/\zeta$.

Suppose now that $R = R_\delta$ and $\widehat{R} = \widehat{R}_\delta$ are two independent copies of $\mathrm{BES}_0(\delta)$. We showed in [42, Theorem 3.1] that if we take $\eta = T_\delta$ to be the hitting time of 1 by R and $\widehat{\eta} = \widehat{T}_\delta$ the hitting time of 1 by $\widehat{R}$, then the law of the resulting process $\widetilde{r}$ is absolutely continuous with respect to that of the standard $\mathrm{BES}_0(\delta)$ bridge r_δ, with density $C_\delta^{-1} M_\delta^{\delta-2}$. The agreement formula (12) expresses the consequence of this relation regarding the distributions of $M_\delta := \sup_{0 \le u \le 1} r_\delta(u)$ and of $\sup_{0 \le u \le 1} \widetilde{r}_\delta(u) = (T_\delta + \widehat{T}_\delta)^{-1/2}$. For $\delta > 2$ it can be shown that exactly the same law of $\widetilde{r}$ is obtained if we repeat the above construction with R and $\widehat{R}$ two independent copies of $\mathrm{BES}_0(\delta)$, with $\eta = T_\delta$ as before, but with $\widehat{\eta} = \widehat{L}_\delta$, the time of the last hit of 1 by $\widehat{R}$. This is due to Williams' path decomposition at the maximum for the process $(\widehat{R}_\delta(t), 0 \le t \le \widehat{L}_\delta)$, according to which the distribution of $V_\delta := \sup_{0 \le t \le \widehat{L}_\delta} \widehat{R}_\delta(t)$ is given by

$$P(V_\delta > v) = v^{2-\delta} \qquad (v \ge 1) \tag{117}$$

and the process $(\widehat{R}_\delta(t), 0 \le t \le \widehat{L}_\delta)$ given $V_\delta = x$ can be constructed by pasting back to back two independent $\mathrm{BES}_0(\delta)$ processes run till their hitting times of x, where the first process starts at 0 and the second starts at 1. The identity in distribution (113) follows from this decomposition by Brownian scaling, along with (111) and the consequence of (117) that $V_\delta^2 \stackrel{d}{=} U^{-1/\nu}$. See also [44] where we discuss the variation of the above construction with $\eta = L_\delta, \widehat{\eta} = \widehat{L}_\delta$, which does not yield the same law of $\widetilde{r}$, but one with a different density relative to the law of r_δ.

A last exit result for $\delta \in (0,2)$. For $\delta \in (0,2)$ formula (98) has an interpretation explained in [10] in terms of the last time R_δ visits zero before an independent exponential time. Another result in this vein for $\delta \in (0,2)$ can be given as follows. Let G_δ denote the time of the last visit of the $\mathrm{BES}_0(\delta)$ process R_δ to zero before the time T_δ that it first reaches 1. It is known [39] that the process $(R_\delta(G_\delta + t), 0 \le t \le T_\delta - G_\delta)$ has the same distribution as $(R_{4-\delta}(t), 0 \le t \le T_{4-\delta})$. So the last exit decomposition of R_δ at time G_δ represents T_δ as the sum of G_δ and an independent copy of $T_{4-\delta}$. It then follows from (32) that

$$E\left(e^{-\frac{1}{2}\lambda^2 G_\delta}\right) = 2^{-2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \lambda^{2\nu} \frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} \qquad (0 < \delta = 2 + 2\nu < 2). \tag{118}$$

The identity (38) allows us to invert this Laplace transform to obtain the following expression for the density of G_δ at $t > 0$:

$$f_{G_\delta}(t) = \frac{(-\nu) 2^{1-2\nu}}{\Gamma(1+\nu)^2} \sum_{n=1}^\infty \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^2(j_{\nu,n})} e^{-\frac{1}{2} j_{\nu,n}^2 t} \qquad (0 < \delta = 2 + 2\nu < 2) \tag{119}$$

Thus the distribution of G_δ is a mixture of exponential distributions with rates $\frac{1}{2} j_{\nu,n}^2$, $n = 1, 2, \ldots$. To be precise, $G_\delta \stackrel{d}{=} 2\mathcal{E}/j_{\nu,N_\nu}^2$ where $\mathcal{E}$ has standard exponential distribution and N_ν is a positive integer valued random variable independent of $\mathcal{E}$ with distribution

$$P(N_\nu = n) = \frac{(-\nu) 2^{2-2\nu}}{\Gamma(1+\nu)^2} \frac{j_{\nu,n}^{2\nu-2}}{J_{\nu+1}^2(j_{\nu,n})} \qquad (n = 1, 2, \ldots). \tag{120}$$

The particular cases of (118) and (119) for $\delta = 1, \nu = -1/2$ were found by Knight [29]. See also Pitman-Yor [47] for an expression for the Laplace transform of the time of last zero before the first hit of 1 for a general diffusion on $[0,1]$ started at 0, with 0 a reflecting boundary point. As a check on (120), the fact that this formula defines a probability distribution on the positive integers amounts to the previous identity (76). As a final remark, we note that the last exit decomposition at time G_δ gives the formula

$$f_{G_\delta}(t) = p_\delta(t;0,0)\, P(M_\delta \le 1/\sqrt{t}) q_\delta \tag{121}$$

where the second factor is the probability that a Bessel bridge of length t from 0 to 0 has a maximum less than 1, and q_δ is the rate per unit local time at 0 of excursions that reach level 1, for a local time process $(\ell_\delta(t), t \ge 0)$ of R_δ at 0 normalized so that $E[\ell_\delta(t)] = \int_0^t ds\, p_\delta(s;0,0)$. It is known [5] that

$$p_\delta(t;0,0)q_\delta = (-\nu)2^{-\nu}\Gamma(1+\nu)^{-1}t^{-\nu-1}$$

so (121) becomes

$$f_{G_\delta}(t) = (-\nu)2^{-\nu}\Gamma(1+\nu)^{-1}t^{-\nu-1}P(M_\delta \le 1/\sqrt{t}). \tag{122}$$

Thus for $\nu \in (-1,0)$, formula (122) allows the formula (119) for the density of G_δ to be derived directly from the Gikhman-Kiefer formula (5), and vice versa.

11 A series involving the zeros of J_ν

For $\nu > -1$ let

$$g_{\nu,n} := \frac{j_{\nu,n}^{\nu+1}}{J_{\nu+1}(j_{\nu,n})} \sim (-1)^{n-1}\sqrt{\frac{\pi}{2}}\, j_{\nu,n}^{\nu+\frac{3}{2}} \sim (-1)^{n-1}\sqrt{\frac{\pi}{2}}(n\pi)^{\nu+\frac{3}{2}} \tag{123}$$

where $\sim$ denotes asymptotic equivalence as $n \to \infty$ for fixed ν and the asymptotic equivalences are read from (138) and (139). Note that for $\nu = \pm\frac{1}{2}$ the first $\sim$ above is actually an equality, and so is the second if $\nu = \frac{1}{2}$.

Consider the equality of right sides of (34) and (35), with the expression (33) substituted for $f_{T_\delta}(t)$. Provided that a switch in the order of summation and integration can be justified, we find that the right side of (34) equals

$$\frac{t^\nu}{C_\delta}\sum_{n,m} g_{\nu,n}g_{\nu,m}e^{-\frac{1}{2}j_{\nu,m}^2 t}\int_0^t dx\, e^{-\frac{1}{2}(j_{\nu,n}^2 - j_{\nu,m}^2)x}$$

$$= \frac{t^\nu}{C_\delta}\sum_{n\ne m}\frac{2g_{\nu,n}g_{\nu,m}}{(j_{\nu,n}^2 - j_{\nu,m}^2)}\left(e^{-\frac{1}{2}j_{\nu,m}^2 t} - e^{-\frac{1}{2}j_{\nu,n}^2 t}\right) + \frac{t^\nu}{C_\delta}\sum_n g_{\nu,n}^2 t e^{-\frac{1}{2}j_{\nu,n}^2 t}.$$

The second sum from the diagonal terms matches the part of the sum in (35) involving $j_{\nu,n}^2 t$. So the equality of right sides of (34) and (35) reduces to the identity

$$\sum_{n\ne m}\frac{2g_{\nu,n}g_{\nu,m}}{(j_{\nu,n}^2 - j_{\nu,m}^2)}\left(e^{-\frac{1}{2}j_{\nu,n}^2 t} - e^{-\frac{1}{2}j_{\nu,m}^2 t}\right) = \sum_{n=1}^\infty \frac{g_{\nu,n}^2}{j_{\nu,n}^2}e^{-\frac{1}{2}j_{\nu,n}^2 t}. \tag{124}$$

On the left side, the coefficient of $e^{-\frac{1}{2}j_{\nu,k}^2 t}$ is $2\sum_{m\ne k} g_{\nu,k}g_{\nu,m}/(j_{\nu,k}^2 - j_{\nu,m}^2)$. Thus if we equate coefficients of $e^{-\frac{1}{2}j_{\nu,k}^2 t}$, then replace k by n, we obtain the identity

$$\sum_{m=1}^{\infty}{}' \frac{g_{\nu,m}}{j_{\nu,m}^2 - j_{\nu,n}^2} = -\frac{(\nu+1)}{2}\frac{g_{\nu,n}}{j_{\nu,n}^2} \qquad (n = 1,2,\ldots) \tag{125}$$

where the notation $\sum'$ denotes a sum where the term with denominator zero is to be excluded. By (123) the series in (125) is absolutely convergent iff $\nu < -\frac{1}{2}$. It appears that the above calculation can be justified by absolute convergence for $-1 < \nu < \frac{1}{2}$. As a check, we show at the end of this section how formula (125) can also be derived by letting $x \to j_{\nu,n}$ in the following Mittag-Leffler expansion of Ismail and Kelker [22, (4.10)]:

$$\frac{x^\nu}{J_\nu(x)} = \frac{(ix)^\nu}{I_\nu(ix)} = \sum_{m=1}^{\infty} \frac{2g_{\nu,m}}{j_{\nu,m}^2 - x^2}. \tag{126}$$

Ismail and Kelker deduced from (126) the formula (33) for the density of $T_{2+2\nu}$. While it appears that formula (33) can be justified for all $\nu > -1$ by passage to the limit as $\mu \to \infty$ from [22, Theorem 1.9 and Corollary 4.8], formulae (125) and (126) can only be valid for $\nu \in (-1, \frac{1}{2})$. Indeed (123) shows that the terms in (125) and (126) converge to zero iff $\nu < \frac{1}{2}$. For $\nu \in (-1, -\frac{1}{2})$ the series in (126) is absolutely convergent, and (126) can then be derived as indicated in [22]. Formula (126) for $\nu = -\frac{1}{2}$ reduces to a classical expansion of $1/\cos x$ [20, 1.422.1]. This yields (125) for $\nu = -\frac{1}{2}$, which reduces to

$$\sum_{m=1}^{\infty}{}' \frac{(-1)^m(2m-1)}{(2m-1)^2 - (2n-1)^2} = \frac{(-1)^{n-1}}{4(2n-1)} \qquad (n = 1, 2, \ldots). \tag{127}$$

We believe that both (126) and (125) are valid also for $\nu \in (-\frac{1}{2}, \frac{1}{2})$, but we will not attempt to prove that here. We note that (126) for $\nu = \frac{1}{2}$ would amount to

$$\frac{\pi y}{\sin \pi y} = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^2}{m^2 - y^2}. \tag{128}$$

While false with regard to convergence of partial sums, this formula is true for any method of summation of divergent series which makes $\sum_{m=1}^{\infty} (-1)^{m-1} = 1/2$, by virtue of the correct formula [20, 1.422.5]

$$\frac{\pi y}{\sin \pi y} = 1 + 2y^2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^2 - y^2}. \tag{129}$$

For $\nu = \frac{1}{2}$ formula (125) would assign the value $(-1)^{n-1}\frac{3}{4}$ to the divergent series $\sum_{m=1}^{\infty}{}' (-1)^m m^2/(m^2 - n^2)$ for every $n = 1, 2, \ldots$. In fact

$$\sum_{m=1}^{N}{}' \frac{(-1)^m m^2}{m^2 - n^2} \sim (-1)^{n-1}\frac{3}{4} + (-1)^N \frac{1}{2} \qquad \text{as } N \to \infty,$$

as can be seen from the formula

$$\sum_{m=1}^{\infty}{}' \frac{(-1)^m}{m^2 - n^2} = \frac{2 + (-1)^n}{4n^2} \qquad (n = 1, 2, \ldots) \tag{130}$$

which follows from (129) by passage to the limit as $y \to n$. The case of (130) with n even appears as [20, 0.237.4].

Formula (125) is a close relative of the formula

$$\sum_{m=1}^{\infty}{}' \frac{1}{j_{\nu,m}^2 - j_{\nu,n}^2} = -\frac{(\nu+1)}{2j_{\nu,n}^2} \qquad (n = 1, 2, \ldots) \tag{131}$$

due to Calogero [8]. See also [9, 3] for variations and extensions of this formula. The following derivation of (125) from (126) parallels an argument of Ismail-Muldoon [23], who indicated how (131) can be derived from the Mittag-Leffler expansion [55, p. 498]

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} = \sum_{m=1}^{\infty} \frac{2x}{j_{\nu,m}^2 - x^2}. \tag{132}$$

Derivation of (125) from (126). Observe first that from the differential equation satisfied by J_ν, that is

$$z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) = 0,$$

it is evident that

$$J_\nu''(j_{\nu,n}) = \frac{-1}{j_{\nu,n}} J_\nu'(j_{\nu,n}) \qquad (n = 1, 2, \ldots)$$

and hence as $\varepsilon \to 0$

$$J_\nu(j_{\nu,n} + \varepsilon) = J_\nu'(j_{\nu,n}) \varepsilon \left(1 - \frac{\varepsilon}{2j_{\nu,n}} + o(\varepsilon)\right) \qquad (n = 1, 2, \ldots).$$

Thus we can apply (126) to compute as follows, where we use the above and the consequence of (157) that $J_\nu'(j_{\nu,n}) = -J_{\nu+1}(j_{\nu,n})$, and the first equality is justified by absolute convergence for $\nu \in (-1, -\frac{1}{2})$:

$$\begin{aligned}
\sum_{m=1}^{\infty}{}' \frac{g_{\nu,m}}{j_{\nu,m}^2 - j_{\nu,n}^2} &= \lim_{x \to j_{\nu,n}} \left\{ \frac{x^\nu}{2J_\nu(x)} - \frac{g_{\nu,n}}{j_{\nu,n}^2 - x^2} \right\} \\
&= g_{\nu,n} \lim_{\varepsilon \to 0} \left\{ -\frac{(1 + \varepsilon/j_{\nu,n})^\nu J_\nu'(j_{\nu,n})}{2j_{\nu,n} \varepsilon J_\nu'(j_{\nu,n}) \left(1 - \frac{\varepsilon}{2j_{\nu,n}} + o(\varepsilon)\right)} + \frac{1}{\varepsilon(2j_{\nu,n} + \varepsilon)} \right\} \\
&= g_{\nu,n} \lim_{\varepsilon \to 0} \left\{ \frac{-\left((1 + \frac{\nu\varepsilon}{j_{\nu,n}} + o(\varepsilon)\right)(2j_{\nu,n} + \varepsilon) + 2j_{\nu,n}\left(1 - \frac{\varepsilon}{2j_{\nu,n}} + o(\varepsilon)\right)}{2j_{\nu,n}\varepsilon \left(1 - \frac{\varepsilon}{2j_{\nu,n}} + o(\varepsilon)\right)(2j_{\nu,n} + \varepsilon)} \right\} \\
&= \frac{g_{\nu,n}}{4j_{\nu,n}^2} \lim_{\varepsilon \to 0} \left\{ \frac{-1 - 2\nu - 1 + o(\varepsilon)}{1 + o(\varepsilon)} \right\} = \frac{-g_{\nu,n}(\nu + 1)}{2j_{\nu,n}^2}.
\end{aligned}$$

□

A Appendix: Some Useful Formulae

We record here for the reader's convenience some basic formulae which are used throughout the paper.

A.1 Bessel Functions

Except where otherwise indicated, the following formulae can all be found in [1, Chapter 9].

Series Expansions. For all real ν

$$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}; \qquad I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)} \tag{133}$$

For ν not an integer

$$K_\nu(z) = \frac{\pi(I_{-\nu}(z) - I_\nu(z))}{2\sin(\nu\pi)} = \frac{\Gamma(\nu)\Gamma(1-\nu)}{2}(I_{-\nu}(z) - I_\nu(z)). \tag{134}$$

Negative indices. For integer n and real ν

$$I_{-n}(z) = I_n(z); \qquad K_{-\nu}(z) = K_\nu(z). \tag{135}$$

An integral representation. [31]

$$\int_0^\infty t^{\nu-1} \exp\left(-pt - \frac{q}{t}\right) dt = 2\left(\frac{q}{p}\right)^{\nu/2} K_\nu\left(2\sqrt{pq}\right) \tag{136}$$

The zeros of J_ν. Let $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ be the sequence of positive zeros of J_ν.

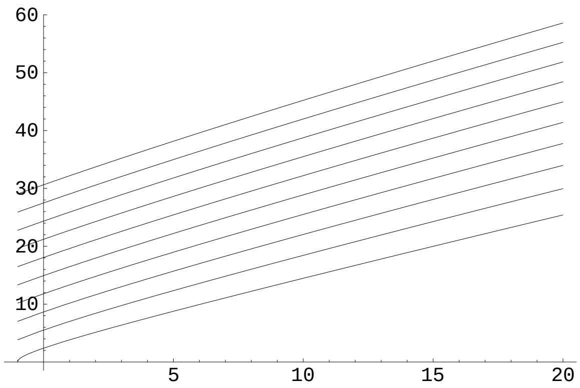

Figure 3. *Graphs of $\nu \to j_{\nu,n}$ for $\nu \in [-1, 20]$ and $n = 1, 2, \ldots, 10$.*

There is the asymptotic expansion [55, p. 506]

$$j_{\nu,n} = (n + \nu/2 + 1/4)\pi - \frac{(4\nu^2 - 1)}{8(n + \nu/2 + 1/4)\pi} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384(n + \nu/2 + 1/4)^3\pi^3} + \cdots. \tag{137}$$

In particular,

$$j_{\nu,n} = (n + \nu/2 + 1/4)\pi + o(1) \text{ as } n \to \infty \text{ for fixed } \nu. \tag{138}$$

More crudely, $j_{\nu,n} \sim n\pi$, where we write $a_{n,\nu} \sim b_{n,\nu}$ to indicate that $a_{n,\nu}/b_{n,\nu} \to 1$ as $n \to \infty$ for each fixed ν. From [55, p. 505] we find that

$$J_{\nu+1}(j_{\nu,n}) \sim (-1)^{n-1}\sqrt{\frac{2}{\pi j_{\nu,n}}} \sim (-1)^{n-1}\frac{1}{\pi}\sqrt{\frac{2}{n}}. \tag{139}$$

According to Piessens [34]

$$j_{\nu,1} = 2(\nu+1)^{1/2}(1+o(\nu+1)) \text{ as } \nu \downarrow -1 \tag{140}$$

and

$$\lim_{\nu\downarrow -1} j_{\nu,n} = j_{1,n-1} > 0 \text{ for } n > 1. \tag{141}$$

Indices $\pm\frac{1}{2}$.

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x; \quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \tag{142}$$

$$I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cosh x; \quad I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh x; \quad K_{\pm\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}. \tag{143}$$

$$j_{-\frac{1}{2},n} = (2n-1)\frac{\pi}{2}; \qquad j_{\frac{1}{2},n} = n\pi. \tag{144}$$

Index $\frac{3}{2}$.

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \tag{145}$$

$$I_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right); \quad K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2x}}\left(1+\frac{1}{x}\right)\, e^{-x} \tag{146}$$

Index $\frac{5}{2}$.

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-3\cos x}{x} - \left(1-\frac{3}{x^2}\right)\sin x\right) \tag{147}$$

$$I_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-3\cosh x}{x} + \left(1+\frac{3}{x^2}\right)\sinh x\right) \tag{148}$$

$$K_{\frac{5}{2}}(x) = \sqrt{\frac{\pi}{2x}}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\, e^{-x} \tag{149}$$

Asymptotics as $z \to 0$

$$J_\nu(z) \sim (\tfrac{1}{2}z)^\nu/\Gamma(\nu+1), \qquad (\nu \neq -1,-2,\ldots) \tag{150}$$

$$I_\nu(z) \sim (\tfrac{1}{2}z)^\nu/\Gamma(\nu+1), \qquad (\nu \neq -1,-2,\ldots) \tag{151}$$

$$K_\nu(z) \sim \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu} \qquad (\nu > 0) \tag{152}$$

$$K_0(z) \sim -\log z \tag{153}$$

Wronskians. $W(f,g) := fg' - f'g$

$$W(J_\nu(z), J_{-\nu}(z)) = J_{\nu+1}(z)J_{-\nu}(z)) + J_\nu(z)J_{-(\nu+1)}(z) = \frac{-2}{\Gamma(\nu)\Gamma(1-\nu)z} \tag{154}$$

$$W(K_\nu(z), I_\nu(z)) = I_\nu(z)K_{\nu+1}(z) - I_{\nu+1}(z)K_\nu(z) = 1/z \tag{155}$$

$$W(I_\nu(z), I_{-\nu}(z)) = I_\nu(z)I_{-(\nu+1)}(z) - I_{\nu+1}(z)I_{-\nu}(z) = \frac{-2}{\Gamma(\nu)\Gamma(1-\nu)z} \tag{156}$$

Derivatives.

$$\frac{d}{dz}J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z) \tag{157}$$

$$\frac{d}{dz}I_\nu(z) = I_{\nu-1}(z) - \frac{\nu}{z}I_\nu(z) = I_{\nu+1}(z) + \frac{\nu}{z}I_\nu(z) \tag{158}$$

$$\frac{d}{dz}K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z) = K_{\nu+1}(z) + \frac{\nu}{z}K_\nu(z) \tag{159}$$

Fourier-Bessel expansions. Following Kiefer [28, p. 428], from [7, (3.3)] there is the following expansion, obtained by calculus of residues, which appears along with similar expansions in [17, 7.15 (58)]. For ν not a negative integer and $0 \le x \le X \le 1$

$$\frac{J_\nu(xz)}{J_\nu(z)}\left[J_\nu(z)Y_\nu(Xz) - J_\nu(Xz)Y_\nu(z)\right] = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{J_\nu(xj_{\nu,n})J_\nu(Xj_{\nu,n})}{J^2_{\nu+1}(j_{\nu,n})(z^2 - j^2_{\nu,n})} \tag{160}$$

where Y_ν is related to K_ν by [17, 7.2 (5) and (17)]

$$J_\nu(z) + iY_\nu(z) = -\frac{2i}{\pi}e^{-i\nu\pi/2}K_\nu(-iz).$$

Take $z = iw$ and use $J_\nu(iw) = e^{i\nu\pi/2}I_\nu(w)$ to deduce

$$\frac{I_\nu(xw)}{I_\nu(w)}\left[I_\nu(w)K_\nu(Xw) - I_\nu(Xw)K_\nu(w)\right] = 2\sum_{n=1}^{\infty}\frac{J_\nu(xj_{\nu,n})J_\nu(Xj_{\nu,n})}{J^2_{\nu+1}(j_{\nu,n})(w^2 + j^2_{\nu,n})}. \tag{161}$$

Let $x \to 0$ and use (150), (151) to obtain for $0 \le X \le 1$

$$\frac{w^\nu}{I_\nu(w)}\left[I_\nu(w)K_\nu(Xw) - I_\nu(Xw)K_\nu(w)\right] = 2\sum_{n=1}^{\infty}\frac{j^\nu_{\nu,n}J_\nu(Xj_{\nu,n})}{J^2_{\nu+1}(j_{\nu,n})(w^2 + j^2_{\nu,n})}. \tag{162}$$

For $w = \sqrt{2\lambda}a, X = y/a$ with $\lambda > 0$ and $0 < y < a$ this yields the Laplace transform

$$\frac{(2\lambda)^{\nu/2}a^\nu}{I_\nu(\sqrt{2\lambda}a)}\left[I_\nu(\sqrt{2\lambda}a)K_\nu(\sqrt{2\lambda}y) - I_\nu(\sqrt{2\lambda}y)K_\nu(\sqrt{2\lambda}a)\right]$$

$$= \int_0^\infty dt\, e^{-\lambda t}\sum_{n=1}^{\infty}\frac{j^\nu_{\nu,n}J_\nu(yj_{\nu,n}/a)}{a^2J^2_{\nu+1}(j_{\nu,n})}\exp\left(-\frac{j^2_{\nu,n}\,t}{2a^2}\right). \tag{163}$$

A.2 The Gamma Function

Duplication.

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\Gamma(\tfrac{1}{2})\Gamma(2z) \tag{164}$$

Reflection.

$$\Gamma(\nu)\Gamma(1-\nu) = -\nu\Gamma(-\nu)\Gamma(\nu) = \frac{\pi}{\sin \pi\nu} \quad (\nu \text{ not an integer}) \tag{165}$$

A.3 Negative Moments

For any non-negative random variable X, and $\Re p > 0$, by application of Fubini's theorem,

$$E(X^{-p}) = \frac{1}{\Gamma(p)} \int_0^\infty d\lambda\, \lambda^{p-1} E(e^{-\lambda X}) = \frac{2^{1-p}}{\Gamma(p)} \int_0^\infty d\xi\, \xi^{2p-1} E \exp(-\tfrac{1}{2}\xi^2 X). \tag{166}$$

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