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Some Examples of Dynamics for Gelfand-Tsetlin Patterns

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Abstract

We give three examples of stochastic processes in the Gelfand-Tsetlin cone in which each component evolves independently apart from a blocking and pushing interaction. These processes give rise to couplings between certain conditioned Markov processes, last passage times and exclusion processes. In the first two examples, we deduce known identities in distribution between such processes whilst in the third example, the components of the process cannot escape past a wall at the origin and we obtain a new relation.

Key words: Gelfand-Tsetlin cone; conditioned Markov process; exclusion process; last passage percolation; random matrices.

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1 Introduction

In [1], the authors Baik, Deift and Johansson show that suitably rescaled, the law of the longest increasing subsequence of a uniformly chosen random permutation of $\{1, 2, \dots, n\}$ converges, as n tends to infinity, to that of the *Tracy-Widom distribution*. The latter, first identified in [28], describes the typical fluctuations of the largest eigenvalue of a large random Hermitian matrix from the *Gaussian unitary ensemble* (see [11] for a definition). This somewhat surprising discovery has been followed by much research which has shown that the Tracy-Widom distribution also occurs as a limiting law in various other models such as last passage percolation [12; 13], exclusion processes [25], random tilings [14; 13] and polynuclear growth [15; 22]. See also the survey [16].

Eigenvalues of random matrices are closely related to multi-dimensional random walks whose components are conditioned not to collide. In particular, both fall into a class of processes with determinantal correlation structure and exhibit pairwise repulsion at a distance. On the other hand, models such as the exclusion process are defined by local "hard edged" interactions rather than particles repelling each other remotely. This paper is concerned with showing how it is possible to connect these two types of model by coupling processes of one class with processes from the other.

In common with previous works in this area, we realise these couplings via the construction of a stochastic process in the *Gelfand-Tsetlin* cone

$$\mathbb{K}_n = \{(x^1, x^2, \dots, x^n) \in \mathbb{R}^1 \times \mathbb{R}^2 \times \dots \times \mathbb{R}^n : x_i^{k+1} \le x_i^k \le x_{i+1}^{k+1}\}.$$

A configuration $(x^1, \dots, x^n) \in \mathbb{K}_n$ is called a *Gelfand-Tsetlin pattern* and we may represent the *interlacing* conditions $x_i^{k+1} \le x_i^k \le x_{i+1}^{k+1}$ diagrammatically as follows.

$$\begin{array}{ccccccccc}
 & & & & x_1^1 & & & & \\
 & & & x_1^2 & & x_2^2 & & & \\
 & & x_1^3 & & x_3^2 & & x_3^3 & & \\
 & \iddots & & & \vdots & & & \ddots & \\
x_1^n & x_2^n & x_3^n & & \dots & & & x_{n-1}^n & x_n^n
\end{array}$$

Suitable processes in the Gelfand-Tsetlin cone appear naturally in several settings, for example the particle process associated with a random domino tiling of the Aztec diamond [19] and the eigenvalues of a GUE matrix and its minors [2]. In other cases, the process in $\mathbb{K}_n$ is not evident at first sight and must be constructed, for example see the recent studies of asymmetric simple exclusion processes [4; 6].

Most frequently, dynamics for the process in $\mathbb{K}_n$ are constructed using a combinatorial procedure known as the Robinson-Schensted-Knuth algorithm (see O'Connell [20]). With RSK dynamics, the $n(n+1)/2$ components of the process are driven by a noise with only n degrees of freedom, leading to strong correlations between components.

In this paper we consider some alternative dynamics in which every component of the process evolves independently except for certain blocking and pushing interactions that ensures the process stays in $\mathbb{K}_n$. This approach yields a new relation between an exclusion type process constrained by an impenetrable wall and a multi-dimensional random walk with components conditioned to neither become disordered nor jump over the wall. Dynamics of this type have previously been considered by Warren [29] for Brownian particles (see also Toth and Veto [27]), by Nordemstam in the context of shuffling domino tilings of the Aztec diamond [19], by Borodin and Ferrari in the context of surface growth models [3]. Analogous dynamics have also previously been studied in the context of growth models where they are known as Gates and Westcott dynamics, see Prähofer and Spohn [22] for example.

2 Description of dynamics and results

From here on, we work exclusively with Gelfand-Tsetlin patterns with integer valued components and hence modify our definition of $\mathbb{K}_n$ to

$$\mathbb{K}_n = \{(x^1, x^2, \ldots, x^n) \in \mathbb{Z}^1 \times \mathbb{Z}^2 \times \ldots \times \mathbb{Z}^n : x_i^{k+1} \le x_i^k \le x_{i+1}^{k+1}\}.$$

2.1 Poisson case

Our first example consists of a continuous time $\mathbb{K}_n$ valued Markov process $(\mathfrak{X}(t); t \ge 0)$ that determines the positions of $n(n+1)/2$ interlaced particles on the integer lattice $\mathbb{Z}$ at time t. The stochastic evolution of the pattern $\mathfrak{X}$ is as follows.

Fix a vector of rates $q \in (0,\infty)^n$ and identify each particle with its corresponding component in $\mathfrak{X}$. The particle $\mathfrak{X}_1^1$ jumps rightwards at rate $q_1 > 0$, i.e. after an exponentially distributed waiting time of mean q_1^{-1}. The two particles, $\mathfrak{X}_1^2, \mathfrak{X}_2^2$ corresponding to the second row of the pattern each jump rightwards at rate q_2 independently of $\mathfrak{X}_1^1$ and each other unless either

- $\mathfrak{X}_1^2(t) = \mathfrak{X}_1^1(t)$, in which case any rightward jump of $\mathfrak{X}_1^2$ is suppressed (*blocked*), or
- $\mathfrak{X}_2^2(t) = \mathfrak{X}_1^1(t)$, in which case $\mathfrak{X}_2^2$ will be forced to jump (*pushed*) if $\mathfrak{X}_1^1$ jumps.

In general, for $k > 1$ and $1 \le j < k$, each particle $\mathfrak{X}_j^k$ attempts to jump rightwards at rate q_k, and will succeed in doing so unless it is blocked by particle $\mathfrak{X}_j^{k-1}$. Particle $\mathfrak{X}_k^k$ can always jump rightwards at rate q_k without impediment. In addition, if $\mathfrak{X}_j^{k-1} = \mathfrak{X}_{j+1}^k$, particle $\mathfrak{X}_{j+1}^k$ is pushed to the right when $\mathfrak{X}_j^{k-1}$ jumps. This blocking and pushing ensures that $\mathfrak{X}(t)$ remains in $\mathbb{K}_n$ for every $t \ge 0$. We will show that for certain initial conditions on $\mathfrak{X}(0)$, the bottom layer of the pattern, $(\mathfrak{X}^n(t); t \ge 0)$, is distributed as a multi-dimensional random walk with components conditioned not to become disordered (Theorem 2.1).

To describe the result more precisely, recall that for $z \in \mathscr{W}^n = \{z \in \mathbb{Z}^n : z_1 \le z_2 \le \ldots \le z_n\}$, the *Schur function* $S_z : \mathbb{R}^n \to \mathbb{R}$ can be defined (see for example [10]) as a sum of geometrically weighted patterns,

$$S_z(q_1, \ldots, q_n) = \sum_{x \in \mathbb{K}_n(z)} w^q(x). \tag{2.1}$$

The sum is over $\mathbb{K}_n(z) = \{x \in \mathbb{K}_n : x^n = z\}$, the set of all Gelfand-Tsetlin patterns $x = (x^1, \ldots, x^n) \in \mathbb{K}_n$ with bottom row x^n equal to z and the geometric weight function is

$$w^q(x) = \prod_{i=1}^n q_i^{|x^i| - |x^{i-1}|},$$

where $|z|=\sum_{i=1}^{d} z_i$ for $z\in\mathbb{R}^d$ and $|x^0|=0$.

This definition gives a natural probability mass function on patterns $x\in\mathbb{K}_n(z)$,

$$M_z(x)=\frac{w^q(x)}{S_z(q)}. \tag{2.2}$$

Suppose that $(Z(t);\ t\geq 0)$ is an n-dimensional random walk in which component i is, independently of the other components, a Poisson counting process of rate q_i. The function $h:\mathscr{W}^n\to\mathbb{R}$ defined by

$$h(x)=q_1^{-x_1}\dots q_n^{-x_n}S_x(q). \tag{2.3}$$

is harmonic for Z killed at the first instant it leaves $\mathscr{W}^n$ (see [17] for example). Hence, h may be used to define a new process, $Z^\dagger$, with conservative Q-matrix on $\mathscr{W}^n$ defined by

$$Q(x,x+e_i)=q_i\frac{h(x+e_i)}{h(x)}=\frac{S_{x+e_i}(q)}{S_x(q)},\quad 1\leq i\leq n,\ x\in\mathscr{W}^n,$$

where e_i is the standard basis vector, and the other off diagonal rates in Q are zero.

This Doob h-transform, $Z^\dagger$, may be interpretted as a version of Z conditioned not to leave $\mathscr{W}^n$ and is closely related to the Charlier orthogonal polynomial ensemble (again see [17]).

In section 3 we prove the following result, obtained independently by Borodin and Ferrari by another method in [3].

Theorem 2.1. *If $(\mathfrak{X}(t);\ t\geq 0)$ has initial distribution $M_z(\cdot)$ for some $z\in\mathscr{W}^n$ then $(\mathfrak{X}^n(t);\ t\geq 0)$ is distributed as an n dimensional Markov process with conservative Q-matrix*

$$Q(x,x+e_i)=\frac{S_{x+e_i}(q)}{S_x(q)}\mathbb{1}_{[x+e_i\in\mathscr{W}^n]},\quad 1\leq i\leq n,\ x\in\mathscr{W}^n.$$

and all other off diagonal entries set to zero, started from z.

Note that from structure of the initial distribution and the construction of $\mathfrak{X}$, this theorem implies that in fact *every* row of the pattern is distributed as a conditioned Markov process of appropriate dimension and rates.

Theorem 2.1 readily yields a coupling of the type discussed in the introduction – the (shifted) left hand edge $(\mathfrak{X}_1^1(t),\mathfrak{X}_1^2(t)-1,\dots,\mathfrak{X}_1^n(t)-n+1;t\geq 0)$ of $\mathfrak{X}$ has the same "hard edged" interactions as an asymmetric exclusion process (the particle with position $\mathfrak{X}_1^k(t)-k+1$, $1\leq k\leq n$ takes unit jumps rightwards at rate q_k but is barred from occupying the same site as any particle to its right). However, Theorem 2.1 implies that $(\mathfrak{X}_1^n(t);t\geq 0)$ has the same law as $(Z_1^\dagger(t);t\geq 0)$, the first component of the random walk Z conditioned to stay in $\mathscr{W}^n$, when started from $Z^\dagger(0)=z$. Further we observe that when $z=(0,\dots,0)$, M_z is concentrated on the origin and a version of the left hand edge can be constructed from the paths of Z via $\mathfrak{X}_1^1(t)=Z_1(t)$ and

$$\mathfrak{X}_1^{k+1}(t)=Z_{k+1}(t)+\inf_{0\leq s\leq t}\left(\mathfrak{X}_1^k(s)-Z_{k+1}(s)\right),\quad 1\leq k<n.$$

Iterating this expression and appealing to Theorem 2.1,

$$\left(Z_1^\dagger(t);t\geq 0\right)\overset{\text{dist}}{=}\left(\inf_{0=t_0\leq t_1\leq\ldots\leq t_n=t}\sum_{i=1}^{n}\left(Z_i(t_i)-Z_i(t_{i-1})\right);t\geq 0\right). \tag{2.4}$$

This identity was previously derived by O'Connell and Yor in [21] using a construction based on the Robinson-Schensted-Knuth correspondence.

2.2 Geometric jumps

For our second example we consider a discrete time process $(\mathfrak{X}(t);\ t \in \mathbb{Z}_+)$ (where $\mathbb{Z}_+$ is the set of non-negative integers) in $\mathbb{K}_n$ in which components make independent geometrically distributed jumps perturbed by interactions that maintain the interlacing constraints.

Let q be a fixed vector in $(0,1)^n$ and update the pattern at time t beginning with the top particle by setting $\mathfrak{X}_1^1(t+1) = \mathfrak{X}_1^1(t)+\xi$, where ξ is a geometric random variable with mean $(1-q_1)/q_1$. That is, the top most particle always takes geometrically distributed jumps rightwards without experiencing pushing or blocking.

Suppose rows 1 through $k-1$ have been updated for some $k > 1$ and we wish to update the position of the particle corresponding to the j^{th} component of the k^{th} row in the pattern, $\mathfrak{X}_j^k$. If $\mathfrak{X}_{j-1}^{k-1}(t+1) > \mathfrak{X}_j^k(t)$, then $\mathfrak{X}_j^k(t)$ is pushed to an intermediate position $\tilde{\mathfrak{X}}_j^k(t) = \mathfrak{X}_{j-1}^{k-1}(t+1)$, while if $\mathfrak{X}_{j-1}^{k-1}(t+1) \leq \mathfrak{X}_j^k(t)$, no pushing occurs and $\tilde{\mathfrak{X}}_j^k(t) = \mathfrak{X}_j^k(t)$.

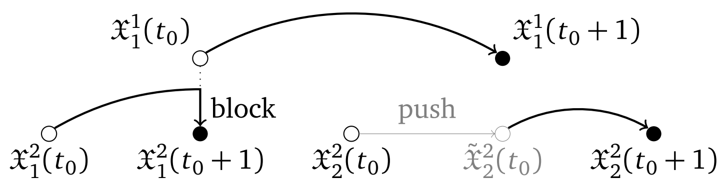

Figure 1: Example of blocking and pushing

The particle $\mathfrak{X}_j^k$ then attempts to make a rightward jump of size that is geometrically distributed with mean $(1-q_k)/q_k$ from its intermediate position $\tilde{\mathfrak{X}}_j^k(t)$ (so the particle is pushed *before* it attempts to jump). It always succeeds if $j = k$ (i.e. it is the right most particle) while if $j < k$, it cannot jump past $\mathfrak{X}_j^{k-1}(t)$, the position of particle to the right of it on the row above *before* the update. The leftmost particle $\mathfrak{X}_1^k$, $k > 1$ is not subject to pushing by any particle, but is still blocked by the "ghost" of the particle $\mathfrak{X}_1^{k-1}$.

To state the result, let us write $x \prec x'$ when the inequality $x_1 \leq x_1' \leq x_2 \leq \ldots \leq x_{n-1}' \leq x_n \leq x_n'$ holds for $x, x' \in \mathbb{R}^n$ and suppose M_z is as defined in (2.2). Then,

Theorem 2.2. *If $\mathfrak{X}(0)$ has initial distribution $M_z(\cdot)$ for some $z \in \mathscr{W}^n$ then $(\mathfrak{X}^n(t); t \in \mathbb{Z}_+)$ is distributed as an n dimensional Markov process in $\mathscr{W}^n$ with transition kernel*

$$p(x,x') = \prod_{i=1}^{n}(1-q_i)\frac{S_{x'}(q)}{S_x(q)}\,\mathbb{1}_{[x \prec x']},\ x, x' \in \mathscr{W}^n,$$

beginning at z.

The Markov process with transition kernel p can be described by a Doob h-transform - suppose Z is now a discrete time random walk beginning at $z \in \mathscr{W}^n$ in which the k^{th} component makes a geometric(q_k) rightward jump at each time step, independently of the other components. Then the function h defined in (2.3) is harmonic for Z killed at the instant that the interlacing condition $Z(t) \prec Z(t+1)$ fails to hold (see [20]). The corresponding h-transform $Z^\dagger$ is the discrete analogue of a process that arises from eigenvalues of Wishart matrices [8].

The right hand edge of the pattern, $(\mathfrak{X}_1^1(t), \mathfrak{X}_2^2(t), \ldots, \mathfrak{X}_n^n(t); t \in \mathbb{Z}_+)$ has a simple connection to the last passage percolation model with geometric weights that may be formulated as follows. Suppose that $\eta_k(t)$ are independent geometric(q_k) random variables attached to sites in the lattice $1 \le k \le n$, $t \ge 1$. An increasing path π from $(1,1)$ to (t,k) is a collection of sites $\{(t_1,k_1), \ldots, (t_N,k_N)\}$, $N = t + k - 2$, such that the step $(t_{m+1},k_{m+1}) - (t_m,k_m) \in \{(1,0),(0,1)\}$, and we denote the set of such paths by $\Pi(t,k)$. The quantity of interest is the k-dimensional process of last passage times

$$G_k(t) = \max_{\pi \in \Pi(t,k)} \sum_{(i,j)\in\pi} \eta_j(i), \ t \in \mathbb{Z}_+.$$

It is not difficult to confirm that $(G_1(t), \ldots, G_n(t); t \in \mathbb{Z}_+)$ has the same law as the right hand edge $(\mathfrak{X}_1^1(t), \mathfrak{X}_2^2(t), \ldots, \mathfrak{X}_n^n(t); t \in \mathbb{Z}_+)$ when $\mathfrak{X}$ has initial distribution M_z, $z = (0, \ldots, 0)$ (i.e. $\mathfrak{X}_k^j(0) = 0$, $1 \le k \le j \le n$). But, a version of the right hand edge may be constructed from paths of Z begun at the origin so that Theorem 2.2 gives

$$\left(Z_n^\dagger(t); t \ge 0\right) \stackrel{\text{dist}}{=} \left(\max_{\pi\in\Pi(t,n)} \sum_{(i,j)\in\pi} (Z_j(i) - Z_j(i-1)); t \ge 0 \right). \tag{2.5}$$

As a consequence, Theorem 2.2 provides a new proof that such last passage percolation times have the same distribution as the rightmost particle in the conditioned process $Z^\dagger$ (the distribution of which, at a fixed time, is given by the Meixner ensemble – see Johansson [13] or [12]). This is a key step in obtaining the Tracy-Widom distribution in this setting.

Note that the dynamics discussed above are different to those exhibited in [3] for geometric jumps. In particular, the particles in the process we described above are blocked by the position of the particle immediately above and to the right of them at the *previous* time step.

2.3 With wall at the origin

The final example of the paper uses the ideas introduced above to construct a continuous time process $(\mathfrak{X}(t);\ t \ge 0)$ on a *symplectic* Gelfand-Tsetlin cone. The latter are so termed because they are in direct correspondence with the symplectic tableau arising from the representations of the symplectic group [26].

The space $\mathbb{K}_n^0$ of integer valued symplectic Gelfand-Tsetlin patterns may be defined (see for example [7] or [23]) as the set of point configurations $(x^1, x^2, \ldots, x^{n-1}, x^n)$ such that

- $x^{2i-1}, x^{2i} \in \mathbb{Z}_+^i$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$ and $x^n \in \mathbb{Z}_+^{(n+1)/2}$ if n is odd,
- $x^{2i-1} \prec x^{2i}$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$,
- $x^{2i} \preceq x^{2i+1}$ for $1 \le i \le \lfloor \frac{n-1}{2} \rfloor$.

So the all the points in a symplectic pattern lie to the right of an impenetrable wall at the origin, represented diagrammatically below.

$$
\left|\begin{array}{ccccccccc}
x_1^1 & & & & & & & & \\
& x_1^2 & & & & & & & \\
x_1^3 & & x_3^2 & & & & & & \\
& x_1^4 & & x_2^4 & & & & & \\
x_1^5 & & x_2^5 & & x_3^5 & & & & \\
& \vdots & & \vdots & & \ddots & & & \\
x_1^n & & x_2^n & & x_3^n & \dots & x_n^n & &
\end{array}\right.
$$

In the vein of previous sections, we construct a process $(\mathfrak{X}(t), t \geq 0)$ in $\mathbb{K}_n^0$ in which only one particle jumps of its own volition at each instant and a blocking and pushing interaction maintains the interlacing constraints.

Fix $q \in (0,1)^n$. The top particle $\mathfrak{X}_1^1$ jumps right at rate q_1 and left at rate q_1^{-1}, *apart* from at origin where its left jumps are suppressed. The second row also only has one particle, $\mathfrak{X}_2^1$, which jumps rightwards at rate q_1^{-1} and leftwards at rate q_1 (notice rates are reversed), *except* at instances when $\mathfrak{X}_1^1(t) = \mathfrak{X}_1^2(t)$. In the latter case, it is pushed rightwards if $\mathfrak{X}_1^1$ jumps to the right and any leftward jumps are suppressed.

The remaining particles evolve in a similar fashion – on row $2k-1$, particles take steps to the right at rate q_k and left at rate q_k^{-1} when they are not subject to the blocking or pushing required to keep the process in the state space, in particular $\mathfrak{X}_1^{2k-1}$ has any leftward jump from the origin suppressed. On row $2k$, the rates are reversed but the same blocking and pushing mantra applies.

We will deduce that for appropriate initial conditions, the marginal distribution of each row $(\mathfrak{X}^k(t); t \geq 0)$ is a Markov process. The Q-matrices for the marginal processes can be written in terms of *symplectic* Schur functions, the definition of which is similar to that of the classic Schur function (2.1) – they are sums over geometrically weighted symplectic Gelfand-Tsetlin patterns.

Fix $k > 0$ and suppose that either $n = 2k-1$ or $n = 2k$. Now let $\mathscr{W}_0^k = \{z \in \mathbb{Z}^k : 0 \leq z_1 \leq z_2 \ldots \leq z_k\}$ and define $\mathbb{K}_n^0(z)$ to be the set of symplectic patterns x in $\mathbb{K}_n^0$ with bottom row x^n equal to $z \in \mathscr{W}_0^k$. The geometric weight w_n^q on $\mathbb{K}_n^0$ is

$$
w_{2k-1}^{(q_1,\ldots,q_k)}(x) = q_k^{|x^{2k-1}|-|x^{2k-2}|} \prod_{i=1}^{k-1} q_i^{|x^{2i-1}|-|x^{2i-2}|+|x^{2i-1}|-|x^{2i}|}
$$

and

$$
w_{2k}^{(q_1,\ldots,q_k)}(x) = \prod_{i=1}^{k} q_i^{|x^{2i-1}|-|x^{2i-2}|+|x^{2i-1}|-|x^{2i}|},
$$

using the convention that $|x^0| = 0$ and empty products are equal to 1 (so $w_1^{(q_1)}(x) = q_1^{|x^1|}$).

Then, the symplectic Schur function $Sp_z^n : \mathbb{R}^k \to \mathbb{R}$, $z \in \mathscr{W}_0^k$, $k \geq 1$ is defined (see [9]) by

$$
Sp_z^n(q_1, \ldots, q_k) = \sum_{x \in \mathbb{K}_n^0(z)} w_n^q(x). \tag{2.6}
$$

For even n, Sp^n gives the characters of irreducible representations of the symplectic group $Sp(n)$ [26]. For odd n, Sp^n was introduced by Proctor [23] and can interpretted as the character of the irreducible representations of a group that interpolates between the classical groups $Sp(n)$ and $Sp(n+1)$ [18].

Define the Q-matrix $Q_n : \mathscr{W}_0^k \times \mathscr{W}_0^k \to \mathbb{R}$ as follows. For $x \in \mathscr{W}_0^k$ and $x \pm e_i \in \mathscr{W}_0^k$, some $1 \le i \le k$,

$$Q_n(x, x \pm e_i) = \frac{Sp^n_{x\pm e_i}(q)}{Sp^n_x(q)}. \tag{2.7}$$

All other off diagonal entries vanish and the diagonals are given by

$$-Q_{2k-1}(x,x) = \sum_{i=1}^{k-1}\left(q_i + q_i^{-1}\right) + q_k^{-1}\,\mathbb{1}_{[x_1>0]} + q_k, \tag{2.8}$$

and

$$-Q_{2k}(x,x) = \sum_{i=1}^{k}\left(q_i + q_i^{-1}\right). \tag{2.9}$$

A corollary of the intertwinings we prove in sections 5.1 and 5.2 is that Q_n is conservative.

For $z \in \mathscr{W}_0^k$ define $M_z^n : \mathbb{K}_n^0 \to [0,1]$ by

$$M_z^n(x) = \frac{w_n^q(x)}{Sp_z^n(q)}.$$

Then the definitions of the symplectic Schur functions imply that $M_z^n(\cdot)$ gives a probability distribution on patterns in $\mathbb{K}_n^0(z)$.

From these ingredients we obtain

Theorem 2.3. *Suppose $\mathfrak{X}$ has initial distribution given by $M_z^n(\cdot)$, then $(\mathfrak{X}^n(t); t \ge 0)$ is distributed as a Markov process with Q-matrix Q_n, started from z.*

The relevance of this theorem to the discussion in the introduction may again be seen by examining the evolution of the right hand edge of $\mathfrak{X}$. Suppose we have a system of n particles with positions $(\mathfrak{X}_1^1(t), \mathfrak{X}_1^2(t)+1, \mathfrak{X}_2^3(t)+2, \ldots, \mathfrak{X}^n_{\lfloor (n+1)/2\rfloor}(t)+n-1; t \ge 0)$.

Particle $i > 1$ attempts to jump rightwards at rate $\gamma_i = q_{(i+1)/2}$ if i is odd or $\gamma_i = q_{i/2}^{-1}$ if i is even and leftwards at rate γ_i^{-1}. An attempted left jump succeeds only if the destination site is vacant, otherwise it is suppressed. A rightward jump always succeeds, and, any particle occupying the destination site is pushed rightwards. A particle being pushed rightwards also pushes any particle standing in its way, so a rightward jump by a particle could cause many particles to be pushed. So far we have essentially described the dynamics of the "PushASEP" process introduced in [4]. Our process differs by the presence of a wall: the leftmost particle (identified with $\mathfrak{X}_1^1$) is modified so that any leftward jump at the origin suppressed. Also, the particle rates are restricted in that for odd i, the jump rates of particle i and $i+1$ are inverses of each other (which is not the case in [4]).

As in the previous examples, the bottom row $(\mathfrak{X}^n(t); t \ge 0)$ may be realised as a Doob h-transform and we deduce identities analogous to (2.4) and (2.5). For simplicity, we shall only consider the case that $n = 2k$. The case of odd n can be treated with similar arguments but it is complicated slightly due to the non-standard behaviour of $\mathfrak{X}_1^n$ at the wall.

Let Z be a k-dimensional random walk in which the i^{th} component jumps rightwards at rate q_i^{-1} and leftwards at rate q_i. It is readily seen that Q_{2k} is the Q-matrix of $Z^\dagger$, the h-transform of Z killed on leaving $\mathscr{W}_0^k$ under harmonic functions

$$h_{2k}(x) = q_1^{-x_1} q_2^{-x_2} \ldots q_k^{-x_k} Sp_x^{2k}(q),\ x \in \mathscr{W}_0^k.$$

Theorem 2.3 shows that $\left(Z_k^\dagger(t); t \geq 0\right)$ has the same law as $\left(\mathfrak{X}_k^{2k}(t); t \geq 0\right)$ when $\mathfrak{X}$ is initially distributed according to M_z^{2k} and $Z(0) = z \in \mathscr{W}_0^k$.

But if $z = (0, \ldots, 0)$, a process with the same law as the right hand edge of $\mathfrak{X}$ can be constructed from the paths of Z and a random walk $\tilde{Z}$ that is independent of, but identically distributed to $-Z$. The resulting identity in distribution can be stated succinctly in terms of the $2k$-dimensional random walk $\bar{Z}(t) = (Z_1(t), \tilde{Z}_1(t), Z_2(t), \tilde{Z}_2(t), \ldots, Z_k(t), \tilde{Z}_k(t))$ as follows

$$\left(Z_k^\dagger(t); t \geq 0\right) \stackrel{dist}{=} \left(\sup_{0 \leq t_1 \leq \ldots \leq t_{2k+1} = t} \sum_{i=1}^{2k} \left(\bar{Z}_i(t_{i+1}) - \bar{Z}_i(t_i)\right); t \geq 0 \right).$$

The Brownian analogue of this result will be considered in [5].

3 Proof of Theorem 2.1

Let $(\mathfrak{X}(t);\ t \geq 0)$ be the process on $\mathbb{K}_n$ satisfying the dynamics described in section 2.1. It is clear from this description that the law of $\{\mathfrak{X}^n(t);\ t \geq 0\}$ is conditionally independent of $\{\mathfrak{X}^j(t);\ t \geq 0, j < n-1\}$ given $\{\mathfrak{X}^{n-1}(t);\ t \geq 0\}$. That is, the dynamics of the particle in row n depend on the evolution of particles in the rows above only through the particles in row $n-1$. Hence the theorem may be proven inductively by studying only the bottom and penultimate layers of the pattern.

To this end, we assume for induction that the conclusion of 2.1 holds. Then, when $\mathfrak{X}(0)$ is distributed according to $M_z(\cdot)$, the bottom layer $(\mathfrak{X}^n(t); t \geq 0)$ is Markovian and evolves according to the conservative Q-matrix Q_X defined via

$$Q_X(x, x+e_i) = \frac{S_{x+e_i}(q)}{S_x(q)} \mathbb{1}_{[x+e_i \in \mathscr{W}^n]}, \quad 1 \leq i \leq n,\ x \in \mathscr{W}^n,$$

and all other off diagonal entries set to zero.

We will define a Markov process $(X(t), Y(t); t \geq 0)$ on $\mathscr{W}^{n,n+1} = \{(x,y) \in \mathscr{W}^n \times \mathscr{W}^{n+1} : x \preceq y\}$ (recall $x \preceq y$ means that $y_i \leq x_i \leq y_{i+1}$, $1 \leq i \leq n$) in which X evolves according to Q_X while Y evolves independently of X apart from the blocking and pushing interaction. One should think of (X, Y) as the penultimate and bottom layer of our construction in $\mathbb{K}_{n+1}$. So, to complete the induction step it is sufficient to show that marginally Y is Markovian and evolves according to

$$Q_Y(y, y+e_i) = q_i \frac{\tilde{h}(y+e_i)}{\tilde{h}(y)},$$

for $y \in \mathscr{W}^{n+1}$, where $\tilde{h}$ is given by

$$\tilde{h}(y) = q_1^{-x_1} \ldots q_{n+1}^{-x_{n+1}} S_y(q_1, \ldots, q_{n+1}),\ y \in \mathscr{W}^{n+1}.$$

for some $q_{n+1} > 0$, and all other off diagonal entries vanish. The diagonal entries are given by

$$Q_Y(y,y) = -\sum_{i=1}^{n+1} q_i.$$

Appropriate dynamics for (X,Y) are specified by the conservative Q-matrix $\mathscr{A}$ with off diagonal entries given by

$$\mathscr{A}((x,y),(x',y')) = \begin{cases} Q_X(x,x+e_i), & (x',y') = (x+e_i,y),\ x_i < y_{i+1}, \\ Q_X(x,x+e_i), & (x',y') = (x+e_i,y+e_{i+1}),\ x_i = y_{i+1}, \\ q_{n+1}, & (x',y') = (x,y+e_j), \\ 0 & \text{otherwise} \end{cases}$$

for $(x,y),(x',y') \in \mathscr{W}^{n,n+1}$ and $1 \le i \le n$, $1 \le j \le n+1$. The diagonal entry $-\mathscr{A}((x',y'),(x',y'))$ is given by

$$\sum_{i=1}^{n} q_i + q_{n+1} \sum_{i=1}^{n} \mathbb{1}_{[y'_i < x'_i]} + q_{n+1} = \sum_{i=1}^{n+1} q_i + q_{n+1} \sum_{i=1}^{n} \mathbb{1}_{[y'_i < x'_i]}. \tag{3.1}$$

Now, as an immediate consequence of the definition of the Schur function in (2.1), we have

$$\begin{aligned} S_z(q_1,q_2,\dots,q_n,q_{n+1}) &= \sum_{x \in \mathbb{K}_n(z)} w^q(x) \\ &= \sum_{z' \in \mathscr{W}^n : z' \preceq z} q_{n+1}^{|z|-|z'|} \sum_{x \in \mathbb{K}_n(z')} \prod_{j=1}^{n} q_j^{|x^j|-|x^{j-1}|} \\ &= \sum_{z' \preceq z} q_{n+1}^{|z|-|z'|} S_{z'}(q_1,\dots,q_n). \end{aligned}$$

So the marginal distribution of the penultimate row of particles under the initial distribution defined in (2.2) is given by $m(\cdot,y)$ where $y \in \mathscr{W}^{n+1}$ is fixed and $m : \mathscr{W}^{n,n+1} \to [0,1]$ is defined by

$$m(x,y) = q_{n+1}^{|y|-|x|} \frac{S_x(q)}{S_y(\tilde{q})},$$

where $\tilde{q} = (q_1,q_2,\dots,q_n,q_{n+1})$.

Furthermore,

$$\Lambda(y,(x',y')) = m(x',y')\, \mathbb{1}_{[y'=y]}. \tag{3.2}$$

defines a Markov kernel from $\mathscr{W}^{n+1}$ to $\mathscr{W}^{n,n+1}$. That is, for each $y \in \mathscr{W}^{n+1}$, $\Lambda(y,\cdot)$ defines a probability distribution on $\mathscr{W}^{n,n+1}$.

The heart of our proof is showing that the conservative Q_Y is *intertwined* with $\mathscr{A}$ via Λ,

$$Q_Y \Lambda = \Lambda \mathscr{A}. \tag{3.3}$$

From here, lemma A.1 shows that Λ intertwines the corresponding transition kernels. That is, if $(p_t; t \ge 0)$ are the transition kernels corresponding to Q_Y and $(q_t; t \ge 0)$ those to $\mathscr{A}$, then for $y \in \mathscr{W}^{n+1}$, $(x',y') \in \mathscr{W}^{n,n+1}$ and $t \ge 0$,

$$p_t(y,y') m(x',y') = \sum_{x \prec y} m(x,y) q_t((x,y),(x',y')),$$

An immediate consequence of this relationship is that for bounded $f : \mathscr{W}^{n+1} \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}^y[f(Y(t))] &= \sum_{(x',y')} \sum_{x \prec y} m(x,y) q_t((x,y),(x',y')) f(y') \\
&= \sum_{(x',y')} p_t(y,y') m(x',y') f(y') \\
&= \sum_{y'} p_t(y,y') f(y') \sum_{x' \prec y'} m(x',y') \\
&= \sum_{y'} p_t(y,y') f(y'),
\end{aligned}$$

where $\mathbb{E}^y$ is the expectation operator corresponding to the measure under which (X,Y) has initial distribution $\Lambda(y,\cdot)$.

When $0 \le t_1 \le \ldots \le t_N$ and $f_1,\ldots,f_N : \mathscr{W}^{n+1} \to \mathbb{R}$ are bounded, the preceeding argument generalises and the intertwining shows that

$$\mathbb{E}^y[f_1(Y(t_1))\ldots f_N(Y(t_N))] = \sum_{y^1,\ldots,y^N} p_t(y,y^1) p_t(y^1,y^2)\ldots p_t(y^{N-1},y^N) f_1(y^1)\ldots f_N(y^N).'$$

This is essentially the argument of Rogers and Pitman [24] and establishes

Theorem 3.1. *Suppose $(X(t),Y(t); t \ge 0)$ is a Markov process with Q-matrix $\mathscr{A}$ and initial distribution $\Lambda(y,\cdot)$, for some $y \in \mathscr{W}^{n+1}$. Then Q_Y and $\mathscr{A}$ are interwined via Λ and as a consequence, $(Y(t); t \ge 0)$ is distributed as a Markov process with Q-matrix Q_Y, started from y.*

The intertwining (3.3) is equivalent to

$$Q_Y(y,y') = \sum_{x \preceq y} \frac{m(x,y)}{m(x',y')} \mathscr{A}((x,y),(x',y')), \;\; y \in \mathscr{W}^{n+1}, \; (x',y') \in \mathscr{W}^{n,n+1}, \tag{3.4}$$

where the summation is over the points x in $\mathscr{W}^n$ that interlace with y. As the particles can only make unit jumps rightwards, both sides of the expression vanish unless either $y' = y$ or $y' = y + e_j$, for some $1 \le j \le n+1$.

We first consider the case when $y = y'$, corresponding to the diagonal entries of Q_Y. The right hand side of the expression is

$$\sum_{x \preceq y'} \frac{m(x,y')}{m(x',y')} \mathscr{A}((x,y'),(x',y')).$$

Using the definition of m, this becomes

$$\sum_{x \preceq y'} q_{n+1}^{|x'|-|x|} \frac{S_x(q)}{S_{x'}(q)} \mathscr{A}((x,y'),(x',y')). \tag{3.5}$$

Now, $\mathscr{A}((x,y'),(x',y'))$ is non zero for $x \preceq y'$ only if $x = x'$ or $x = x' - e_i$ for some $1 \le i \le n$. When $x = x'$, $-\mathscr{A}((x,y'),(x',y'))$ is the rate of leaving at (x',y'), given in (3.1). On the other hand if $x = x' - e_i$, $\mathscr{A}((x,y'),(x',y'))$ is the rate at which the i^{th} X particle jumps rightwards (without pushing a Y particle). But, such values of x are included in the summation only if $x = x' - e_i \preceq y = y'$, i.e. $x'_i > y'_i$.

Combining this with (3.5) and (3.1) and the fact that $q_{n+1}^{|x'|-|x'-e_i|} = q_{n+1}$, we see that if $y = y'$ the right hand side of (3.4) is

$$\sum_{i=1}^{n} q_{n+1} \frac{S_{x'-e_i}(q)}{S_{x'}(q)} Q_X(x'-e_i, x')\, \mathbb{1}_{[x'_i > y'_i]} - \sum_{i=1}^{n+1} q_i - \sum_{i=1}^{n} q_{n+1}\, \mathbb{1}_{[y'_i < x'_i]}.$$

The first summand above is

$$q_{n+1} \frac{S_{x'-e_i}(q)}{S_{x'}(q)} Q_X(x'-e_i, x')\, \mathbb{1}_{[x'_i > y'_i]} = q_{n+1}\, \mathbb{1}_{[y'_i < x'_i]},$$

so the first and last summations above disappear and we are left with $-\sum_{i=1}^{n+1} q_i$, which is exactly $Q_Y(y', y')$.

If $y \neq y'$, the only other possibility is that $y' = y + e_i$ for some $1 \leq i \leq n+1$. Let us first deal with the simplest case, where $i = 1$, that is, $y' = y + e_1$. The only value of x for which $\mathscr{A}((x, y'-e_1), (x', y'))$ is non zero is $x = x'$ as the first Y particle is never pushed by an X particle. Furthermore, $y'_1 - 1 < y'_1 \leq x'_1$ and so the jump of Y_1 is certainly not blocked. Hence,

$$\begin{aligned}
\sum_{x : x \preceq y} \frac{m(x,y)}{m(x',y')} \mathscr{A}((x,y),(x',y')) &= \frac{m(x', y'-e_1)}{m(x',y')} \mathscr{A}((x', y'-e_1), (x', y')) \\
&= \frac{q_{n+1}^{|y'-e_1|-|x'|}}{q_{n+1}^{|y'|-|x'|}} \frac{S_{x'}(q)}{S_{x'}(q)} \frac{S_{y'}(\tilde{q})}{S_{y'-e_1}(\tilde{q})} q_{n+1} \\
&= Q_Y(y'-e_1, y').
\end{aligned}$$

So in this case, (3.4) is satisfied.

For $i > 1$, consider the dichotomy $x'_{i-1} < y'_i$ or $x'_{i-1} = y'_i$. Suppose we are in the former case, i.e. $y' = y + e_i$ and $x'_{i-1} < y'_i$. It is not possible that the movement in the i^{th} component of Y could have been instigated due to pushing by the $(i-1)^{th}$ X particle (a push could only have occurred if $x'_{i-1} - 1 = y'_i - 1$). Thus, as in the $i = 1$ case above, $\mathscr{A}((x, y'-e_i), (x', y'))$ is non zero only for $x = x'$ and almost identical calculations verify (3.4).

The second $i > 1$ subcase is that $x'_{i-1} = y'_i$ and $y = y' - e_i$. Here the only possibility is that the i^{th} Y particle "did not jump but was pushed", which one may confirm by noting that x' does not interlace with $y' - e_i$ when $x'_{i-1} = y'_i$. So, the right hand side of (3.4) is given by

$$\frac{m(x'-e_{i-1}, y'-e_i)}{m(x', y')} \mathscr{A}((x'-e_{i-1}, y'-e_i), (x', y')).$$

Using the definitions of m and $\mathscr{A}$, this becomes

$$\frac{q_{n+1}^{|y'-e_i|-|x'-e_{i-1}|}}{q_{n+1}^{|y'|-|x'|}} \frac{S_{x'-e_{i-1}}(q)}{S_{y'-e_i}(\tilde{q})} \frac{S_{y'}(\tilde{q})}{S_{x'}(q)} \frac{S_{x'}(q)}{S_{x'-e_{i-1}}(q)},$$

a quantity which is easily seen to equal $Q_Y(y'-e_i, y')$.

This concludes the proof that Q_Y and $\mathscr{A}$ are intertwined via Λ.

4 Proof of Theorem 2.2

It is again sufficient to consider any pair of consecutive rows (X,Y) and construct the process iteratively.

Let $(X(t); t \in \mathbb{Z}_+)$ be an n dimensional Markov chain in $\mathscr{W}^n$ with one step transition kernel

$$p_X(x,x') = a(q)\frac{S_{x'}(q)}{S_x(q)}\mathbb{1}_{[x \prec x']}\,.$$

where $q \in (0,1)^n$, $a(q) = \prod_{i=1}^n (1-q_i)$ and for $x, x' \in \mathbb{R}^n$, $x \prec x'$ indicates that the inequality $x_1 \le x'_1 \le x_2 \le \ldots \le x'_{n-1} \le x_n \le x'_n$ holds.

Let $\xi_i(t)$ $(t \in \mathbb{Z}_+, 1 \le i \le n+1)$ be geometric(q_{n+1}) random variables that are independent of each other and of X,

$$\mathbb{P}(\xi_i(k) = j) = (1-q_{n+1})q_{n+1}^j, \quad j = 0,1,2,\ldots.$$

Define a process $(Y(t); t \in \mathbb{Z}_+)$ in $\mathscr{W}^{n+1}$ in terms of X using the recursion

$$Y_1(t+1) = \min(Y_1(t) + \xi_1(t+1), X_1(t)),$$

$$Y_{n+1}(t+1) = \max(Y_{n+1}(t), X_n(t+1)) + \xi_{n+1}(t+1),$$

$$Y_j(t+1) = \min(\max(Y_j(t), X_{j-1}(t+1)) + \xi_j(t+1), X_j(t)),$$

for $2 \le j \le n$.

The recursion encodes the blocking and pushing mechanism, maintaining the initial interlacing relationship, so $X(t) \prec Y(t)$ for each t.

We will prove that if Λ is as defined in (3.2) then

Theorem 4.1. *If (X,Y) is initially distributed according to $\Lambda(y,\cdot)$, $y \in \mathscr{W}^{n+1}$, and then evolves according to the recursion above, the marginal process $(Y(t); t \ge 0)$ is distributed as an $n+1$ dimensional Markov process with transition kernel*

$$p_Y(y,y') = a(q_1,\ldots,q_{n+1})\frac{S_{y'}(q_1,\ldots,q_{n+1})}{S_y(q_1,\ldots,q_{n+1})}\mathbb{1}_{[y \prec y']}\,.$$

started from y.

Our strategy, again, is to prove that Λ interwines the corresponding transition probabilities. Suppose $(x,y),(x',y') \in \mathscr{W}^{n,n+1}$, $x \prec x'$ and $y \prec y'$. Let us write down $q((x,y),(x',y'))$, the one step transition probabilities for (X,Y). Firstly note that

$$q((x,y),(x',y')) = r(y',x',x,y)p_X(x,x').$$

where

$$r(y',x',x,y) = \mathbb{P}(Y(1) = y' | X(1) = x', X(0) = x, Y(0) = y).$$

Using the definition of Y, r can be conveniently expressed in terms of the "blocking" and "pushing" factors $b,c : \mathbb{Z}^2 \to \mathbb{R}_+$

$$b(u,v) = (1-q_{n+1})\,\mathbb{1}_{[v<u]} + \mathbb{1}_{[u=v]},$$

$$c(u,v) = q_{n+1}^{-v}\,\mathbb{1}_{[u\le v]} + q_{n+1}^{-u}\,\mathbb{1}_{[u>v]}\,.$$

Then $r(y',x',x,y)$ is equal to

$$q_{n+1}^{y_1'-y_1}\,b(x_1,y_1')\left(\prod_{i=2}^{n} q_{n+1}^{y_i'}\,b(x_i,y_i')c(x_{i-1}',y_i)\right)q_{n+1}^{y_{n+1}'}(1-q_{n+1})c(x_n',y_{n+1}).$$

To prove the theorem we will need the following "integrating out" lemma.

Lemma 4.2. *Suppose $v_2, v_1', u' \in \mathbb{Z}$ satisfy $v_1' \le v_2$, $v_1' \le u'$ and $q_{n+1} \ne 1$. Then we have*

$$\sum_{u=v_1'}^{v_2\wedge u'} q_{n+1}^{-u}\,b(u,v_1')c(u',v_2) = q_{n+1}^{-u'-v_2} \tag{4.1}$$

The lemma may be understood more readily by imagining that we are considering the $n=1$ case, so that there is one "X" particle nestled between two "Y" particles. We may fix the initial and final positions of the "Y" particles (v and v' in the lemma above) and also the final position of the "X" particle (u in the lemma) – it is the starting location of the X particle that we are integrating out. The summation is over the possible values that the X particle *may* have started from. It must be at least equal to the final position of the left most Y particle v_1', as this particle cannot overtake the X particle (see recursion equations above). Also, it cannot exceed either the initial position of the second Y particle v_2 (due to the interlacing constraint) or the final position of the X particle u' (as the particles may only jump rightwards).

Proof. After using the definitions of b and c, the sum becomes

$$\sum_{u=v_1'}^{v_2\wedge u'} q_{n+1}^{-u}\left((1-q_{n+1})\,\mathbb{1}_{[v_1'<u]} + \mathbb{1}_{[u=v_1']}\right)\left(q_{n+1}^{-v_2}\,\mathbb{1}_{[u'\le v_2]} + q_{n+1}^{-u'}\,\mathbb{1}_{[u'>v_2]}\right)$$

Now expand the brackets in the summand and sum the terms individually. We find

$$\sum_{u=v_1'}^{v_2\wedge u'}(1-q_{n+1})\,\mathbb{1}_{[v_1'<u]}\,q_{n+1}^{-v_2-u}\,\mathbb{1}_{[u'\le v_2]} = q_{n+1}^{-v_2}\left(q_{n+1}^{-u'} - q_{n+1}^{-v_1'}\right)\mathbb{1}_{[u'\le v_2]},$$

$$\sum_{u=v_1'}^{v_2\wedge u'}(1-q_{n+1})\,\mathbb{1}_{[v_1'<u]}\,q_{n+1}^{-u'-u}\,\mathbb{1}_{[u'>v_2]} = q_{n+1}^{-u'}\left(q_{n+1}^{-v_2} - q_{n+1}^{-v_1'}\right)\mathbb{1}_{[u'>v_2]},$$

$$\sum_{u=v_1'}^{v_2\wedge u'}\mathbb{1}_{[u=v_1']}\,q_{n+1}^{-v_2-u}\,\mathbb{1}_{[u'\le v_2]} = q_{n+1}^{-v_2-v_1'}\,\mathbb{1}_{[u'\le v_2]}$$

and

$$\sum_{u=v_1'}^{v_2\wedge u'}\mathbb{1}_{[u=v_1']}\,q_{n+1}^{-u'-u}\,\mathbb{1}_{[u'>v_2]} = q_{n+1}^{-u'-v_1'}\,\mathbb{1}_{[u'>v_2]}\,.$$

Summing the above expressions gives the result. □

The interesting thing about this scheme, as we will see in a moment, is that we may apply it successively from left to right when there are n particles so that the leftmost particles get heavier and heavier until we have reduced the problem to the $n = 1$ case.

When the initial distribution is $\Lambda(y,\cdot)$, the joint distribution after one time step is given by

$$\pi(x',y') = \sum_{x\prec y} m(x,y)q((x,y),(x',y')).$$

Expanding the sum and incorporating the conditions $y'_i \le x_i$ and $x \prec x'$ into the summation indices yields

$$\pi(x',y') = \sum_{x_n=y'_n}^{y_{n+1}\wedge x'_n} \cdots \sum_{x_1=y'_1}^{y_2\wedge x'_1} m(x,y)q((x,y),(x',y')). \tag{4.2}$$

The summand in (4.2) equals

$$\begin{aligned}
&q_{n+1}^{y'_1+y_2+\ldots+y_{n+1}+y'_{n+1}-x_1-\ldots-x_n} b(x_1,y'_1)\left(\prod_{i=2}^{n} q_{n+1}^{y'_i} b(x_i,y'_i)c(x'_{i-1},y_i)\right)\\
&\times c(x'_n,y_{n+1})a(\tilde q)\frac{S_{x'}(q)}{S_y(\tilde q)},
\end{aligned}$$

for $x \preceq y, x' \preceq y', x \prec x', y \prec y'$ and vanishes elsewhere.

Now, one notices that we may use lemma 4.2 to iteratively evaluate the summation over $x_1, x_2, \ldots, x_n$ (in that order). More concretely, first apply the lemma with $u' = x'_1, v = (y_1, y_2), v' = (y'_1, y'_2)$ to reveal that the sum $\sum_{x_1=y'_1}^{y_2\wedge x'_1} m(x,y)Q((x,y),(x',y'))$ is equal to

$$\begin{aligned}
&q_{n+1}^{y'_1+y'_2+y_3+\ldots+y_{n+1}+y'_{n+1}-x'_1-x_2-\ldots-x_n} b(x_2,y'_2)\\
&\quad\times\left(\prod_{i=3}^{n} q_{n+1}^{y'_i} b(x_i,y'_i)c(x'_{i-1},y_i)\right) c(x'_n,y_{n+1})a(\tilde q)\frac{S_{x'}(q)}{S_y(\tilde q)}.
\end{aligned}$$

This expression is again in a suitable form to apply lemma 4.2, but this time with $u' = x'_2, v = (y_2, y_3), v' = (y'_2, y'_3)$ and summing over x_2. Continuing in this fashion shows that (4.2) is equal to

$$q_{n+1}^{y'_1+y'_2\ldots+y'_{n+1}-x'_1-\ldots-x'_n} a(\tilde q)\frac{S_{x'}(q)}{S_y(\tilde q)} = m(x',y')p_Y(y,y'). \tag{4.3}$$

Hence we have verified the intertwining

$$m(x',y')p_Y(y,y') = \sum_{x\preceq y} m(x,y)q((x,y),(x',y')),$$

and Theorem 4.1 follows from the argument of [24] discussed in the previous section.

5 Proof of Theorem 2.3

As in the previous two examples, we give a row by row construction. This time the asymmetry between odd rows and even rows means we have to specify how to iterate from even rows to odd rows and odd rows to even rows separately (presented below in 5.1 and 5.2 respectively).

En route to proving Theorem 2.3, we need to conclude that Q_n is a conservative Q-matrix for each n.

This will be achieved by an inductive argument. Let H(n) denote the hypothesis that Q_n is a conservative Q-matrix. It is easy to establish H(1), that Q_1 is conservative – recall that for $x_1 \geq 0$, $Sp^1_{(x_1)} = q_1^{x_1}$ so

$$Q_1(x, x+e_1) + Q_1(x, x-e_1) + Q_1(x,x) = \frac{q_1^{x_1+1}}{q_1^{x_1}} + \frac{q_1^{x_1-1}}{q_1^{x_1}} \mathbb{1}_{[x_1>0]} - q_1 - q_1^{-1} \mathbb{1}_{[x_1>0]},$$

a quantity equal to zero, and the off diagonal entries are clearly positive.

Under the assumption that H($2n-1$) holds we will define a conservative Q-matrix $\mathscr{A}_0$ on $\mathscr{W}_0^{n,n} = \{(x,y) \in \mathscr{W}_0^n \times \mathscr{W}_0^n : x \prec y\}$ in terms of Q_{2n-1} and prove the intertwining relationship

$$Q_{2n}\Lambda = \Lambda \mathscr{A}_0.$$

where Λ is a Markov kernel. Expanding the intertwining and summing both sides shows that $\sum_{x'} Q_{2n}(x,x') = 0$, so we conclude that H($2n$) holds as well. The step from H($2n$) to H($2n+1$) follows a similar argument.

5.1 Part I: Iterating from an odd row to an even row

Suppose H($2n-1$) holds and identify $Q_X \equiv Q_{2n-1}$. Introduce a Q-matrix $\mathscr{A}_0$ on $\mathscr{W}_0^{n,n}$ with off diagonal entries defined by

$$\mathscr{A}_0((x,y),(x',y')) = \begin{cases} Q_X(x, x \pm e_j), & (x',y') = (x \pm e_j, y) \\ Q_X(x, x - e_{i+1}), & (x',y') = (x - e_{i+1}, y - e_i),\ x_{i+1} = y_i \\ Q_X(x, x + e_j), & (x',y') = (x+e_j, y+e_j),\ x_j = y_j \\ q_n^{\mp 1}, & (x',y') = (x, y \pm e_j) \\ 0 & \text{otherwise} \end{cases},$$

for $(x,y),(x',y') \in \mathscr{W}_0^{n,n}$, $1 \leq i < n$, $1 \leq j \leq n$. The diagonal entry $-\mathscr{A}_0((x,y),(x,y))$ is given by

$$\sum_{i=1}^{n-1} \left(q_i + q_i^{-1} \right) + q_n + q_n^{-1} \mathbb{1}_{[x_1>0]} + \sum_{i=1}^{n-1} \left(q_n^{-1} \mathbb{1}_{[y_i < x_{i+1}]} + q_n \mathbb{1}_{[y_i > x_i]} \right) + q_n \mathbb{1}_{[y_n > x_n]} + q_n^{-1}, \tag{5.1}$$

so under the assumption that Q_X is conservative, $\mathscr{A}_0$ is also conservative.

Define $m : \mathscr{W}_0^{n,n} \to [0,1]$ by

$$m(x,y) = q_n^{|x|-|y|} \frac{Sp_x^{2n-1}(q)}{Sp_y^{2n}(q)}.$$

Note that the geometric factor is now $q_n^{|x|-|y|}$ instead of the usual $q_n^{|y|-|x|}$. By definition (2.6),

$$
\begin{aligned}
Sp_z^{2n}(q) &= \sum_{x\in\mathbb{K}_{2n}^0(z)} w_{2n}^q(q) \\
&= \sum_{x\in\mathbb{K}_{2n}^0(z)} q_n^{|x^{2n-1}|-|x^{2n}|} q_n^{|x^{2n-1}|-|x^{2n-2}|} \prod_{i=1}^{n-1} q_i^{2|x^{2i-1}|-|x^{2i}|-|x^{2i-2}|} \\
&= \sum_{z'\in\mathscr{W}_0^n : z'\prec z} q_n^{|z'|-|z|} \sum_{x\in\mathbb{K}_{2n-1}^0(z')} q_n^{|x^{2n-1}|-|x^{2n-2}|} \prod_{i=1}^{n-1} q_i^{2|x^{2i-1}|-|x^{2i}|-|x^{2i-2}|} \\
&= \sum_{z'\in\mathscr{W}_0^n : z'\prec z} q_n^{|z'|-|z|} Sp_{z'}^{2n-1}(q).
\end{aligned}
$$

Hence, m gives a Markov kernel Λ from $\mathscr{W}_0^n$ to $\mathscr{W}_0^{n,n}$ defined by

$$\Lambda(y,(x',y')) = m(x',y')\,\mathbb{1}_{[y'=y]}\,.$$

We then have

Theorem 5.1. *Assume Q_{2n-1} is a conservative Q-matrix and $(X(t),Y(t);t\geq 0)$ is a Markov process with Q-matrix $\mathscr{A}_0$ and initial distribution $\Lambda(y,\cdot)$ for some $y\in\mathscr{W}_0^n$. Then Q_{2n} is a conservative Q-matrix and $(Y(t);t\geq 0)$ is distributed as a Markov process with Q-matrix Q_{2n}, started from y.*

Suppose $Q_Y\equiv Q_{2n}$, then as usual we prove an intertwining relationship

$$Q_Y\Lambda = \Lambda\mathscr{A}_0.$$

This is equivalent to

$$Q_Y(y,y') = \sum_{x\prec y} \frac{m(x,y)}{m(x',y')}\mathscr{A}_0((x,y),(x',y')),\;\; y\in\mathscr{W}_0^n,\; (x',y')\in\mathscr{W}_0^{n,n}. \tag{5.2}$$

where the sum is over $x\in\mathscr{W}_0^n$ such that $(x,y)\in\mathscr{W}_0^{n,n}$.

Particles may take unit steps in either direction so we need to check the equality (5.2) holds for $y=y'$, $y=y'+e_j$ and $y=y'-e_j$ for some $1\leq j\leq n$.

Let us first consider the case $y=y'$. When $x=x'$, $-\mathscr{A}_0((x,y),(x',y'))$ is the rate of leaving (x',y') and is given by (5.1). The only other possible values of x in the summation for which the summand is non-zero are $x=x'\pm e_i$, $1\leq i\leq n$. For such values (i.e. if $(x'\pm e_i,y)\in\mathscr{W}_0^{n,n}$), the summand is

$$\frac{m(x'\pm e_i,y')}{m(x',y')}Q_X(x'\pm e_i,x') = \frac{q_n^{|x'\pm e_i|-|y'|}}{q_n^{|x'|-|y'|}}\frac{Sp_{x'\pm e_i}^{2n-1}(q)}{Sp_{y'}^{2n}(q)}\frac{Sp_{y'}^{2n}(q)}{Sp_{x'}^{2n-1}(q)}\frac{Sp_{x'}^{2n-1}(q)}{Sp_{x'\pm e_i}^{2n-1}(q)},$$

which is a rather fancy way of writing $q_n^{\pm 1}$. But, for $(x',y')\in\mathscr{W}_0^{n,n}$,

- $(x'+e_i,y')\in\mathscr{W}_0^{n,n}$ only if $x_i'<y_i'$
- $(x'-e_i,y')\in\mathscr{W}_0^{n,n}$, $i>1$ only if $x_i'>y_{i-1}'$

- $(x'-e_1,y')\in\mathscr{W}_0^{n,n}$, only if $x'_1>0$.

So

$$\sum_{x\prec y, x\neq x'}\frac{m(x,y)}{m(x',y')}\mathscr{A}_0((x,y),(x',y')) = \sum_{i=1}^{n-1}\left(q_n\,\mathbb{1}_{[x'_i<y_i]}+q_n^{-1}\,\mathbb{1}_{[x'_{i+1}>y_i]}\right)+ q_n\,\mathbb{1}_{[x'_n<y_n]}+q_n^{-1}\,\mathbb{1}_{[x'_1>0]}$$

On subtracting the rate of leaving $-\mathscr{A}_0((x',y'),(x',y'))$ defined in (5.1) we find that the indicator functions all cancel and the right hand side of (5.2) is

$$\sum_{i=1}^{n}\left(q_i+q_i^{-1}\right),$$

which is equal to the left hand side.

Next we consider the case that $y'=y-e_i\in\mathscr{W}_0^n$. If $i=n$, the only possibility is that the Y particle jumped by itself. When $i<n$, the only possibilities are that the i^{th} component of Y was pushed by the $(i+1)^{th}$ component of X (i.e. $x=x'+e_{i+1}$) or it jumped by its own volition (i.e. $x=x'$). The former only occurs if $y'_i=x'_{i+1}$, while the latter can only occur if $y'_i<x'_{i+1}$, inducing a natural partition on the values we have to check the intertwining on. When $y'=y-e_i$, $y'_i<x'_{i+1}$, $i<n$, or $i=n$, the right hand side of (5.2) is

$$\frac{m(x',y'+e_i)}{m(x',y')}\mathscr{A}_0((x',y'+e_i),(x',y'))=\frac{q_n^{|x'|-|y'+e_i|}}{q_n^{|x'|-|y'|}}\frac{Sp^{2n-1}_{x'}(q)}{Sp^{2n}_{y'+e_i}(q)}\frac{Sp^{2n}_{y'}(q)}{Sp^{2n-1}_{x'}(q)}q_n=\frac{Sp^{2n}_{y'}(q)}{Sp^{2n}_{y'+e_i}(q)}.$$

When $y'=y-e_i$, $x'_{i+1}=y'_i$, $i<n$, the sum on the right hand side of the intertwining involves a single term,

$$\frac{m(x'+e_{i+1},y'+e_i)}{m(x',y')}Q_X(x'+e_{i+1},x').$$

Using the definitions of m and Q_X shows this summand is

$$\frac{q_n^{|x'+e_{i+1}|-|y'+e_i|}}{q_n^{|x'|-|y'|}}\frac{Sp^{2n-1}_{x'+e_{i+1}}(q)}{Sp^{2n}_{y'+e_i}(q)}\frac{Sp^{2n}_{y'}(q)}{Sp^{2n-1}_{x'}(q)}\frac{Sp^{2n-1}_{x'}(q)}{Sp^{2n-1}_{x'+e_{i+1}}(q)}.$$

Both of these quantities are equal to $Q_Y(y'+e_i,y)$.

Finally we consider the case $y'=y+e_i$, $1\le i\le n$. As in the previous case, the dichotomy $x'_i=y'_i$ and $x'_i<y'_i$ divides the possible values of x in the summation into two cases, each of which having only one term contributing to the sum. When $x'_i=y'_i$, the i^{th} Y particle must have been pushed, and

$$\frac{m(x'-e_i,y'-e_i)}{m(x',y')}Q_X(x'-e_i,x')=\frac{q_n^{|x'-e_i|-|y'-e_i|}}{q_n^{|x'|-|y'|}}\frac{Sp^{2n-1}_{x'-e_i}(q)}{Sp^{2n}_{y'-e_i}(q)}\frac{Sp^{2n}_{y'}(q)}{Sp^{2n-1}_{x'}(q)}\frac{Sp^{2n-1}_{x'}(q)}{Sp^{2n-1}_{x'-e_i}(q)}.$$

Simplifying the expression on the right hand side by cancelling common factors in the numerator and denominator reveal it to be simply $Q_Y(y'-e_i,y)$.

On the other hand, when $x'_i < y'_i$ the i^{th} Y particle cannot have been pushed so the right hand side of the intertwining (5.2) is

$$\frac{m(x', y' - e_i)}{m(x', y')} \mathscr{A}_0((x', y' - e_i), (x', y')) = \frac{q_n^{|x'|-|y'-e_i|}}{q_n^{|x'|-|y'|}} \frac{Sp_{x'}^{2n-1}(q)}{Sp_{y'-e_i}^{2n}(q)} \frac{Sp_{y'}^{2n}(q)}{Sp_{x'}^{2n-1}(q)} q_n^{-1} = \frac{Sp_{y'}^{2n}(q)}{Sp_{y'-e_i}^{2n}(q)}.$$

The proof of the intertwining relationship is concluded by noting that this is $Q_Y(y' - e_i, y)$ as required.

Now, summing both sides of the intertwining

$$\sum_{(x',y')} Q_Y(y, y') m(x', y') = \sum_{(x',y')} \sum_{x \prec y} m(x, y) \mathscr{A}_0((x, y), (x', y'))$$

over all pairs in (x', y') in $\mathscr{W}_0^{n,n}$ shows that Q_Y is conservative as $\sum_{x'} m(x', y') = 1$ and $\sum_{(x',y')} \mathscr{A}_0((x, y), (x', y')) = 0$.

We then apply lemma A.1 to recover the rest of the theorem.

5.2 Part II: Iterating from an even row to an odd

Suppose $Q_X \equiv Q_{2n}$, $Q_Y \equiv Q_{2n+1}$ and $\mathscr{A}_0$ is a conservative Q-matrix $\mathscr{A}_0$ on $\mathscr{W}_0^{n,n+1} = \{(x, y) \in \mathscr{W}_0^n \times \mathscr{W}_0^{n+1} : x \preceq y\}$ with off diagonal entries given by

$$\mathscr{A}_0((x, y), (x', y')) = \begin{cases} Q_X(x, x \pm e_i), & (x', y') = (x \pm e_i, y) \\ Q_X(x, x - e_i), & (x', y') = (x - e_i, y - e_i),\ x_i = y_i \\ Q_X(x, x + e_i), & (x', y') = (x + e_i, y + e_{i+1}),\ x_i = y_{i+1} \\ q_{n+1}^{\pm 1}, & (x', y') = (x, y \pm e_j) \\ 0 & \text{otherwise} \end{cases},$$

for $(x, y), (x', y') \in \mathscr{W}_0^{n,n+1}$, $1 \le i \le n$, $1 \le j \le n + 1$. The diagonal $-\mathscr{A}_0((x, y), (x, y))$ is given by

$$\sum_{i=1}^{n} (q_i + q_i^{-1}) + \sum_{i=1}^{n} (q_{n+1} \mathbb{1}_{[y_i < x_i]} + q_{n+1}^{-1} \mathbb{1}_{[y_{i+1} > x_i]}) + q_{n+1} + q_{n+1}^{-1} \mathbb{1}_{[y_1 > 0]}. \tag{5.3}$$

Hence $\mathscr{A}_0$ is conservative if Q_X is.

From definition (2.6) we calculate

$$\begin{aligned} Sp_z^{2n+1}(\tilde{q}) &= \sum_{x \in \mathbb{K}_{2n+1}^0(z)} w_{2n+1}^q(\tilde{q}) \\ &= \sum_{x \in \mathbb{K}_{2n+1}^0(z)} q_{n+1}^{|x^{2n+1}| - |x^{2n}|} \prod_{i=1}^{n} q_i^{2|x^{2i-1}| - |x^{2i}| - |x^{2i-2}|} \\ &= \sum_{z' \in \mathscr{W}_0^n : z' \preceq z} q_{n+1}^{|z| - |z'|} \sum_{x \in \mathbb{K}_{2n}^0(z')} \prod_{i=1}^{n} q_i^{2|x^{2i-1}| - |x^{2i}| - |x^{2i-2}|} \\ &= \sum_{z' \in \mathscr{W}_0^n : z' \preceq z} q_{n+1}^{|z| - |z'|} Sp_{z'}^{2n}(q). \end{aligned}$$

So the function $m : \mathscr{W}_0^{n,n+1} \to [0,1]$ given by

$$m(x,y) = q_{n+1}^{|y|-|x|} \frac{Sp_x^{2n}(q)}{Sp_y^{2n+1}(\tilde{q})}.$$

induces a Markov kernel from $\mathscr{W}_0^{n+1}$ to $\mathscr{W}_0^{n,n+1}$,

$$\Lambda(y,(x',y')) = m(x',y')\, \mathbb{1}_{[y'=y]}\,.$$

Our final theorem is

Theorem 5.2. *Assume Q_{2n} is a conservative Q-matrix and suppose $(X(t), Y(t); t \geq 0)$ is a Markov process with Q-matrix $\mathscr{A}_0$ and initial distribution $\Lambda(y,\cdot)$ for some $y \in \mathscr{W}_0^{n+1}$. Then Q_{2n+1} is a conservative Q-matrix and $(Y(t); t \geq 0)$ is distributed as a Markov process with Q-matrix Q_{2n+1}, started from y.*

The intertwining via Λ is equivalent to

$$Q_Y(y,y') = \sum_{x \preceq y} \frac{m(x,y)}{m(x',y')} \mathscr{A}_0((x,y),(x',y')),\;\; y \in \mathscr{W}_0^{n+1},\; (x',y') \in \mathscr{W}_0^{n,n+1}, \tag{5.4}$$

where we sum over x such that $(x,y) \in \mathscr{W}_0^{n,n+1}$.

We only need to check (5.4) holds for y of the form $y = y'$, $y = y' \pm e_j$ for $1 \leq j \leq n+1$ as both sides vanish otherwise.

Again we start with the case $y' = y$. When $x = x'$, the rate of leaving $-\mathscr{A}_0((x',y'),(x',y'))$ is given by (5.3). The only other possible values of x for which the summand is non-zero are $x = x' \pm e_i$ for $1 \leq i \leq n$. For such x values satisfying $(x,y) \in \mathscr{W}_0^{n,n+1}$, the definitions of m and Q_X give

$$\frac{m(x' \pm e_i, y')}{m(x',y')} Q_X(x' \pm e_i, x') = \frac{q_{n+1}^{|y'|-|x' \pm e_i|}}{q_{n+1}^{|y'|-|x'|}} \frac{Sp_{x' \pm e_i}^{2n}(q)}{Sp_{y'}^{2n+1}(\tilde{q})} \frac{Sp_{y'}^{2n+1}(\tilde{q})}{Sp_{x'}^{2n}(q)} \frac{Sp_{x'}^{2n}(q)}{Sp_{x' \pm e_i}^{2n}(q)},$$

which is equal to $q_{n+1}^{\mp 1}$. But, for $(x',y') \in \mathscr{W}_0^{n,n+1}$,

- $(x' + e_i, y') \in \mathscr{W}_0^{n,n+1}$ only if $x'_i < y'_{i+1}$ and
- $(x' - e_i, y') \in \mathscr{W}_0^{n,n+1}$ only if $x'_i > y'_i$.

So,

$$\sum_{x \prec y, x \neq x'} \frac{m(x,y)}{m(x',y')} \mathscr{A}_0((x,y),(x',y')) = \sum_{i=1}^{n} \left(q_{n+1}^{-1}\, \mathbb{1}_{[x'_i < y_{i+1}]} + q_{n+1}\, \mathbb{1}_{[x'_i > y_i]} \right).$$

If we now subtract the rate of leaving (5.3) we find that at $y = y'$ the right hand side of (5.4) is equal to

$$-\left(\sum_{i=1}^{n} (q_i + q_i^{-1}) + q_{n+1} + q_{n+1}^{-1}\, \mathbb{1}_{[y'_1 > 0]} \right),$$

which is equal to $Q_Y(y', y')$.

The remaining cases are $y = y' \pm e_i$ for some $1 \le i \le n+1$. Let us deal with $y' = y - e_i$. If $i = n+1$, this case corresponds to a leftward jump in the rightmost Y particle, a situation that cannot arise through pushing by an X particle. If $i < n+1$, then the jump arose by pushing if $x'_i = y'_i$, while if $x'_i > y'_i$ then the Y particle jumped by its own volition. In the case of pushing ($i < n+1$, $x'_i = y'_i$), familiar calculations show

$$\frac{m(x'+e_i, y'+e_i)}{m(x', y')} Q_X(x'+e_i, x') = \frac{Sp^{2n+1}_{y'}(\tilde{q})}{Sp^{2n+1}_{y'+e_i}(\tilde{q})} = Q_Y(y, y').$$

In the case of no pushing, i.e. $i < n+1$ and $x'_i > y'_i$ or $i = n+1$, the summand is

$$\frac{m(x', y'+e_i)}{m(x', y')} \mathscr{A}_0((x', y'+e_i), (x', y')) = \frac{Sp^{2n+1}_{y'}(\tilde{q})}{Sp^{2n+1}_{y'+e_i}(\tilde{q})} = Q_Y(y, y').$$

Finally we consider the case $y' = y + e_i$, $1 \le i \le n+1$ corresponding to a rightward jump in the i^{th} Y particle. For $i > 1$, consider the dichotomy $x'_{i-1} = y'_i$ or $x'_{i-1} < y'_i$, corresponding to the i^{th} Y particle being pushed upwards by the $(i-1)^{th}$ X particle and a free jump respectively. The case $i = 1$ corresponds to the leftmost Y particle jumping rightwards, an event that cannot arise as a result of pushing. In the case of pushing, i.e. $i > 1$ and $x'_{i-1} = y'_i$, the summand is equal to

$$\frac{m(x'-e_{i-1}, y'-e_i)}{m(x', y')} \mathscr{A}_0((x'-e_{i-1}, y'-e_i), (x', y')).$$

Using the definitions of $\mathscr{A}_0$ and m, this is

$$\frac{q_{n+1}^{|y'-e_i|-|x'-e_{i-1}|}}{q_{n+1}^{|y'|-|x'|}} \frac{Sp^{2n}_{x'-e_{i-1}}(q)}{Sp^{2n+1}_{y'-e_i}(\tilde{q})} \frac{Sp^{2n+1}_{y'}(\tilde{q})}{Sp^{2n}_{x'}(q)} Q_X(x'-e_{i-1}, x') = \frac{Sp^{2n+1}_{y'}(\tilde{q})}{Sp^{2n+1}_{y'-e_i}(\tilde{q})} = Q_Y(y, y'),$$

as required.

If $y'_i > x'_{i-1}$ ($i > 1$) or $i = 1$, then the i^{th} Y particle jumped of its own accord and the only term in the summation is

$$\frac{m(x', y'-e_i)}{m(x', y')} \mathscr{A}_0((x', y'-e_i), (x', y')) = \frac{Sp^{2n+1}_{y'}(\tilde{q})}{Sp^{2n+1}_{y'-e_i}(\tilde{q})} = Q_Y(y, y').$$

This concludes the verification of the intertwining relationship and the theorem follows.

A A lemma on intertwinings of Q-matrices

Lemma A.1. *Suppose that L and L' are uniformly bounded conservative Q-matrices on discrete spaces U and V that are intertwined by a Markov kernel $\Lambda : U \times V \to [0,1]$ from U to V, i.e.*

$$L\Lambda = \Lambda L'.$$

Then the transition kernels for the Markov processes with Q-matrices L and L' are also intertwined.

Note that we always use the lemma with $U = W', V = W \times W'$ where $W \subset \mathbb{Z}^n$ and either $W' \subset \mathbb{Z}^n$ or $W' \subset \mathbb{Z}^{n+1}$. Our Markov kernel Λ is always such that $\Lambda(u,(v,v')) > 0$ only if $u = v'$, but this, of course, is not necessary for the lemma.

Proof. The intertwining relationship $L\Lambda = \Lambda L'$ may be written

$$\sum_{\tilde{v}\in V} \Lambda(u,\tilde{v})L'(\tilde{v},v) = \sum_{\tilde{u}\in U} L(u,\tilde{u})\Lambda(\tilde{u},v), \quad u \in U, v \in V.$$

Let $(p_t; t \geq 0)$ denote the transition kernels for the Markov process corresponding to Q-matrix L and fix $u_0 \in U$. Multiplying both sides of the expanded intertwining relationship above by $p_t(u_0,u)$ and summing over $u \in U$ gives

$$\sum_{u\in U} p_t(u_0,u) \sum_{\tilde{v}\in V} \Lambda(u,\tilde{v})L'(\tilde{v},v) = \sum_{u\in U} p_t(u_0,u) \sum_{\tilde{u}\in U} L(u,\tilde{u})\Lambda(\tilde{u},v), \quad v \in V. \tag{A.1}$$

Now, let $c \in \mathbb{R}$ be a uniform bound for the absolute values of the entries of L and L'. Then

$$|p_t(u_0,u)\Lambda(u,\tilde{v})L'(\tilde{v},v)| \leq c\Lambda(u,\tilde{v})p_t(u_0,u),$$

so the double sum on the left hand side is absolutely convergent. Also,

$$\sum_{\tilde{u}\in U} |p_t(u_0,u)\Lambda(\tilde{u},v)L(u,\tilde{u})| \leq \sum_{\tilde{u}\in U} |p_t(u_0,u)L(u,\tilde{u})| \leq 2cp_t(u_0,u)$$

and the same conclusion holds for the double sum on the right hand side. So, we may exchange the order of the sums on both sides to give

$$\sum_{\tilde{v}\in V} L'(\tilde{v},v) \sum_{u\in U} p_t(u_0,u)\Lambda(u,\tilde{v}) = \sum_{\tilde{u}\in U} \Lambda(\tilde{u},v) \sum_{u\in U} p_t(u_0,u)L(u,\tilde{u}), \quad v \in V.$$

Now, as $(p_t; t \geq 0)$ is the transition kernel corresponding to the Markov process with Q matrix L, it satisfies the Kolmogorov forward equation

$$\frac{d}{dt}p_t(u_0,\tilde{u}) = \sum_{u\in U} p_t(u_0,u)L(u,\tilde{u}).$$

Let us define $q_t(v) = \sum_{\tilde{u}\in U} p_t(u_0,\tilde{u})\Lambda(\tilde{u},v)$ for $v \in V$.
We may differentiate the summation term by term in t using Fubini's theorem and the absolute bounds on the summands discussed above. Hence, the right hand side of (A.1) is simply $\frac{d}{dt}q_t(v)$.
Then, using the definition of q_t in the left hand side of (A.1), we see that

$$\frac{d}{dt}q_t(v) = \sum_{\tilde{v}\in V} L'(\tilde{v},v)q_t(\tilde{v}), \quad t \geq 0. \tag{A.2}$$

Now let $(p'_t; t \geq 0)$ denote the transition kernels of the Markov process with Q-matrix L', and

$$p'_t(v) = \sum_{\tilde{v}} q_0(\tilde{v})p'_t(\tilde{v},v) = \sum_{\tilde{v}\in V} \Lambda(u_0,\tilde{v})p'_t(\tilde{v},v).$$

Then $p'_0(v) = q_0(v)$ for all $v \in V$ and p'_t also satisfies the forward equation (A.2) in L'.

But when the rates are uniformly bounded there is exactly one solution to the forward differential equation with the same boundary conditions as q_t so $q_t(v) = p'_t(v)$ for all $t \geq 0$ and $v \in V$.

By definition of $q_t(v)$, we then have

$$\sum_{\tilde{u} \in U} p_t(u_0, \tilde{u}) \Lambda(\tilde{u}, v) = \sum_{\tilde{v} \in V} \Lambda(u_0, \tilde{v}) p'_t(\tilde{v}, v),$$

and since the argument holds for arbitrary $u_0 \in U$ we're done.

□

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