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Path properties of a class of locally asymptotically self similar processes *

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Abstract

Various paths properties of a stochastic process are obtained under mild conditions which allow for the integrability of the characteristic function of its increments and for the dependence among them. The main assumption is closely related to the notion of local asymptotic self-similarity. New results are obtained for the class of multifractional random processes .

Key words: Hausdorff dimension, level sets, local asymptotic self-similarity, local nondeterminism, local times.

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1 Introduction

Let $X = (X(t),\ t \in \mathbb{R}^+)$ be a real valued separable random process with Borel sample functions. For any Borel set B, the occupation measure of X on B is defined as follows

$$\mu_B(A) \;=\; \lambda\{s \in B\,:\, X(s) \in A\} \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}),$$

where λ is the Lebesgue measure on $\mathbb{R}^+$. If μ_B is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that X has a local time on B and define its local time, $L(B,.)$, to be the Radon-Nikodym derivative of μ_B. Here x is the so-called space variable, and B is the time variable. Sometimes, we write $L(t,x)$ instead of $L([0,t],x)$.

The relation between the smoothness of the local time in its variables and the irregularity of the underlying process has been investigated by S. Berman for Gaussian processes; for instance, if $L(t,x)$ is jointly continuous, then X is nowhere differentiable and has uncountable level sets, see for example [7; 8; 9] for details. The computations required to establish the joint continuity of the local time in the Gaussian case were based on the concept of local nondeterminism [9]. In Berman's papers, the smoothness of $L(t,x)$ in time and space was important to prove the irregularity of the original process.

Geman and Horowitz have obtained in a series of papers weaker conclusions under more general conditions; for instance, the irregularity of the sample paths of X was based on the continuity of the local time only as a function of the time parameter, we refer to [20] for a survey in this area. Berman has extended in [10] his definition of local nondeterminism to a wide class of stochastic processes, and has refined his previous results under an assumption of higher order integrability of the local time instead of the smoothness in the space variable. Kôno and Shieh [23],[27] have refined the previous results for the class of self similar processes with stationary increments under additional assumptions on the joint density of $(X(t), X(s))$. Their conclusions have been given in terms of the exponent of self-similarity. Our results may be considered as a continuation in this direction. The assumptions on the process X considered in this paper are framed on the characteristic function of the one-dimensional increments of X (assumption $(\mathcal{H})$ in Section 2) and on the dependence among these increments (assumption (H_m) in Section 2). These assumptions are weak enough to include existing methods related to the Fourier analytic approach initiated by S. Berman, especially various notions of local nondeterminism. They are applicable to cases where the marginal laws and the dependence structure up to a certain order k are known, but where the law of the process itself is unknown, such as the weak Brownian motion of order k (cf. [18]).

Assumption $(\mathcal{H})$ is closely related to the notion of local asymptotic self-similarity. A real valued stochastic process $\{X(t),\ t \in \mathbb{R}^+\}$ is said to be locally asymptotically self similar (lass for brevity) at $t \in \mathbb{R}^+$ if there exists a non-degenerate process $\{Y(u),\ u \in \mathbb{R}^+\}$, such that

$$\lim_{\rho \to 0^+} \frac{X(t+\rho u) - X(t)}{\rho^H} = Y(u),\ u \in \mathbb{R}^+, \tag{1}$$

where the convergence is in the sense of the finite dimensional distributions and $0 < H < 1$ is the lass exponent. Y is called the tangent process at t and if Y is unique in law, it has stationary increments and it is self similar with exponent H (c.f. [17]). Conversely, note that any non degenerate H-self similar process X with stationary increments is H-lass and its tangent process is X itself.

The notion of lass processes was introduced in Benassi *et al.* [6] and Lévy-Véhel and Peltier [24] in order to relax the self-similarity property of fractional Brownian motion. Various examples of lass processes appear nowadays in the literature; for example, filtered white noises [4] and the multi-scale fractional Brownian motion [22]. An important class of lass processes are multifractional processes, for which the H-lass parameter is no more constant but a regular function of time, such as multifractional Brownian motions [6; 24] and the linear multifractional stable process [28].

The paper is organized as follows. In Section 2, we introduce our assumptions and explain how they are related to the study of lass processes. The continuity of the local time and related irregularity properties of the underlying process are obtained in Section 3 under assumption $(\mathcal{H})$. The joint continuity of the local time required initially in the Gaussian case and the higher order integrability in the space variable assumed in [10] are circumvented. Section 4 contains results on the Hausdorff measure and the Hausdorff dimension of the progressive level sets. In Section 5 the joint continuity and Hölder conditions of local time are shown under assumptions $(\mathcal{H})$ and (H_m), and the Hausdorff dimension of level sets at deterministic levels is deduced. The higher smoothness of the local time in the space variable is studied in Section 6. In section 7 we verify $(\mathcal{H})$ and (H_m) for some classes of processes, including sub-Gaussian processes and linear multifractional stable processes. We show how the lass property helps to verify the local nondeterminism of Gaussian multifractional processes. In this sense this paper may be considered as a continuation of previous work [13; 14; 21; 22] on local times of Gaussian multifractional processes. Let us finally mention that an incorrect result in [16] on the equivalence of versions of the multifractional Brownian motion has been cited in [13]. Proposition 7.2 of this paper proves the local nondeterminism for all versions and confirms therefore the correctness of the results in [13] not only for the moving average version, but also for the harmonisable version of the multifractional Brownian motion.

2 Assumptions

Let $\{X(t),\ t \in [0,T]\}$ be a real valued separable stochastic process with measurable sample paths. The first assumption concerns the integrability of the characteristic function of the increment $X(t)-X(s)$, for $0 < s < t \leq T < \infty$, s and t sufficiently close.

Assumption $(\mathcal{H})$ **:** There exist positive numbers $(\rho_0, H) \in (0,+\infty) \times (0,1)$ and a positive function $\psi \in L^1(\mathbb{R})$ such that for all $\lambda \in \mathbb{R}$, $t,s \in [0,T]$, $0 < |t-s| < \rho_0$ we have

$$\left| \mathbb{E} \exp\left(i\lambda \frac{X(t)-X(s)}{|t-s|^H} \right) \right| \leq \psi(\lambda).$$

Comments 1. *1. If X is H-self similar with stationary increments, the assumption $(\mathcal{H})$ is reduced to : $\psi(\lambda) = |E\left(e^{i\lambda X(1)}\right)|$ belongs to $L^1(\mathbb{R})$, which is a classical condition in the investigation of local times of H-self similar processes with stationary increments (see [23]).*

2. Assumption $(\mathcal{H})$ is closely related to the study of lass processes. Indeed, according to (1), we have

$$\lim_{\rho \to 0^+} \mathbb{E}\left\{ \exp\left(i\lambda \frac{X(t+\rho)-X(t)}{\rho^H} \right) \right\} = \mathbb{E}\left(\exp(i\lambda Y(1)) \right).$$

Then in view of the first comment and the fact that Y is H-self similar with stationary increments, $(\mathcal{H})$ seems to be natural for the study of local times of lass processes. We refer to [6] for a use of $(\mathcal{H})$ for computing the Hausdorff dimension of the graph of lass processes.

The second assumption characterizes the dependence among the finite-dimensional increments of X :

Assumption (H_m) **:** There exist $m \geq 2$ and positive constants δ and c, both may depend on m, such that for all $t_1, ..., t_m$ with $0 := t_0 < t_1 < t_2 < ... < t_m \leq T$, and $|t_m - t_1| \leq \delta$, we have

$$\left| \mathbb{E} \exp \left(i \sum_{j=1}^{m} u_j [X(t_j) - X(t_{j-1})] \right) \right| \leq \prod_{j=1}^{m} \left| \mathbb{E} \exp \left(icu_j [X(t_j) - X(t_{j-1})] \right) \right|,$$

for all $u_1, ..., u_m \in \mathbb{R}$.

Comments 2. 1. *If X has independent increments, then (H_m) holds for all $m \geq 2$ trivially. When (H_m) holds for all $m \geq 2$, we say that X has characteristic function locally approximately independent increments (see [25], Definition 2.5). This concept is called in the literature the* local nondeterminism *(LND), and classical examples of LND processes are the fractional Brownian motion (fBm) and the linear fractional stable motion (LFSM) (see [23]).*

2. *Observe that the assumption (H_m) is decreasing in m in the sense that if (H_m) is satisfied then (H_{m-1}) holds. Hence, (H_2) is the minimal condition. We need sometimes only (H_2), nevertheless we give in Section 7 examples of processes satisfying (H_m) for all $m \geq 2$.*

3. *According to Lemma 2.1 in Berman [12], a sufficient condition for (H_m) to hold for a centered Gaussian process is as follows: There exists $m \geq 2$ such that*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t_m - t_1 < \varepsilon} \max_{i \neq j} \frac{|Cov\,[X(t_i) - X(t_{i-1}), X(t_j) - X(t_{j-1})]\,|}{\sqrt{Var[X(t_i) - X(t_{i-1})] Var[X(t_j) - X(t_{j-1})]}} < \frac{1}{m-1}.$$

For $m = 2$, corresponding to (H_2), the preceding condition becomes

$$\lim_{\varepsilon \to 0} \sup_{0 < t-s \leq \varepsilon} \frac{|Cov[X(t) - X(s), X(s)]|}{\sqrt{Var[X(t) - X(s)] Var[X(s)]}} < 1.$$

According to Theorem 3.1 in [9], this is a necessary and sufficient condition for a Gaussian Markov process to be LND.

We end this section by recalling the notion of approximate moduli of continuity. First, t is said to be a point of dispersion (resp. a point of density) for a bounded Lebesgue measurable set F if

$$\lim_{\varepsilon \to 0} \frac{\lambda\{F \cap (t - \varepsilon, t + \varepsilon)\}}{\varepsilon} = 0, \quad (\text{ resp. } = 1)$$

where λ denotes Lebesgue measure. The approximate lim sup of a function $f(s)$ for $s \to t$ is at least y if t is not a point of dispersion for the set $\{s : f(s) \geq y\}$. We refer to Geman and Horowitz ([20], Appendix page 22) for details and to Example 1 in the same reference for geometric interpretation. We note that

$$-\infty \leq \liminf_{s\to t} f(s) \leq \mathrm{ap} - \limsup_{s\to t} f(s) \leq \limsup_{s\to t} f(s) \leq +\infty \tag{2}$$

We will use $C,\ C_1, ...$ to denote unspecified positive finite constants which may not necessarily be the same at each occurrence.

3 Continuity in time and applications

Our first result in this section is the following

Theorem 3.1. *Suppose that $\{X(t),\ t \in [0,T]\}$ satisfies $(\mathcal{H})$. Then, almost surely*

i) X has a local time $L(t,x)$, continuous in t for a.e. $x \in \mathbb{R}$ and $L(t,.) \in \mathbb{L}^2(dx \times \mathbb{P})$.

ii)
$$ap - \limsup_{s\to t} \frac{|X(t)-X(s)|}{|t-s|^{1+H}\phi(|t-s|)} = +\infty \quad \text{for all} \quad t \in [0,T],$$

where $\phi(r),\ r \geq 0$ is any right-continuous function decreasing to 0 as $r \searrow 0$.

Remark 3.1. *It is straightforward from (i) of the previous theorem and Theorem A of Geman [19] that, with probability one,*

$$ap - \lim_{s\to t} \frac{|X(t)-X(s)|}{|t-s|} = +\infty, \quad \text{for a.e. } t \in [0,T].$$

The following is an immediate consequence of Theorem 3.1.

Corollary 3.2. *With probability one, $X(.,w)$ is nowhere Hölder continuous of any order greater than $1+H$.*

Proof of Theorem 3.1. Let I be an interval of length smaller than ρ_0. For all $s,t \in I$, we have

$$\begin{aligned}
&\mathbb{P}\left(|X(t)-X(s)| \leq \varepsilon\right) \\
&= \mathbb{P}\left(\frac{|X(t)-X(s)|}{|t-s|^H} \leq \frac{\varepsilon}{|t-s|^H}\right) \\
&= \frac{1}{2\pi}\int_{-\varepsilon/|t-s|^H}^{\varepsilon/|t-s|^H}\int_{\mathbb{R}} \mathbb{E}\exp\left(i\lambda\frac{X(t)-X(s)}{|t-s|^H}\right)\exp\left(-i\lambda x\right)d\lambda dx \\
&\leq \frac{\varepsilon}{\pi|t-s|^H}\int_{\mathbb{R}}\psi(\lambda)d\lambda
\end{aligned}$$

Then,

$$\begin{aligned}
\int_I\int_I \sup_{\varepsilon>0}\frac{1}{\varepsilon}\mathbb{P}\left(|X(t)-X(s)| \leq \varepsilon\right)dsdt &\leq \frac{1}{\pi}\int_{\mathbb{R}}\psi(\lambda)d\lambda\int_I\int_I\frac{1}{|t-s|^H}dsdt \\
&< +\infty,
\end{aligned} \tag{3}$$

since $0 < H < 1$. Therefore, by using Theorem 21.15 in [20], $L(t,x)$ exists and it belongs to $\mathbb{L}^2(dx \times \mathbb{P})$. Furthermore, (3) implies that

$$\int_I \sup_{\varepsilon>0} \frac{1}{\varepsilon} \mathbb{P}\left(|X(t) - X(s)| \leq \varepsilon\right) ds < +\infty, \quad \text{for } \ a.e. \ t \in I.$$

Then, according to Theorem B in Geman [19], the result of the theorem holds for any interval in $[0,T]$ of length smaller than ρ_0. Moreover, since $[0,T]$ is a finite interval, we can obtain the local time on $[0,T]$ by a standard patch-up procedure, i.e. we partition $[0,T]$ into $\cup_{i=1}^n [T_{i-1}, T_i]$ and define $L([0,T],x) = \sum_{i=1}^n L([T_{i-1},T_i],x)$, where $T_0 = 0$ and $T_n = T$.

The proof of (ii) will be an application of Theorem 1 of [11] to the sample paths of X. Hence, we shall estimate the so called modulator of local time defined as follows

$$M(t) = \left(2 \sum_{j=1}^{2^n} \int_{\mathbb{R}} L^2 \left(\left[\frac{j-1}{2^n}, \frac{j}{2^n} \right], x \right) dx \right)^{1/2}, \tag{4}$$

whenever $2^{-n-1} \leq t \leq 2^{-n}$. Now, consider $[\alpha, \beta] \subset [0,T]$ with $\beta - \alpha < \rho_0$. It follows from the occupation density formula and Parseval equation that

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}} L^2([\alpha,\beta],x) dx &= E \int_{\mathbb{R}} \widehat{L}([\alpha,\beta],x) \overline{\widehat{L}([\alpha,\beta],x)} dx \\ &= \int_\alpha^\beta \int_\alpha^\beta \int_{\mathbb{R}} \mathbb{E} \exp\left(iu(X(t) - X(s))\right) du dt ds. \end{aligned} \tag{5}$$

Combining the change of variable $v = (t-s)^H u$ and $(\mathcal{H})$, we obtain that (5) is smaller than

$$\int_\alpha^\beta \int_\alpha^t \frac{1}{(t-s)^H} ds dt \int_{\mathbb{R}} \psi(x) dx \leq C(\beta - \alpha)^{2-H}. \tag{6}$$

Therefore,

$$\mathbb{E} M^2(2^{-n}) \leq C (2^{-n})^{1-H}. \tag{7}$$

Consequently

$$\begin{aligned} &\liminf_{n \to +\infty} 2^n \left[E M^2(2^{-n}) \right]^{1/2} \left[(2^{-n})^{1+H} \phi(2^{-n}) \right]^{1/2} \\ &\qquad \leq C \liminf_{n \to +\infty} (2^n)^{1+(1-(2-H))/2-(1+H)/2} \phi^{1/2}(2^{-n}) = 0. \end{aligned}$$

Since, by Lemma 2.1 in [11], the modulator is increasing in t, then

$$\liminf_{t \to 0} t^{-1} \left[E M^2(t) \right]^{1/2} \left[t^{1+H} \phi(t) \right]^{1/2} = 0.$$

Therefore, (1.6) in Theorem 1 of Berman [11] holds for almost all sample functions. This completes the proof of (ii) $\square$.

We can get a better result for the prescribed local regularity, at a fixed point t, as follows.

Proposition 3.3. *If $\{X(t),\ t \in [0,T]\}$ satisfies $(\mathcal{H})$, then for every $t \in [0,T]$, the process X is a $J_H(t)$ (Jarnik) function on t; i.e.*

$$ap - \limsup_{s \to t} \frac{|X(t) - X(s)|}{|t-s|^H \phi(|t-s|)} = +\infty, \quad \textit{almost surely}$$

where $\phi(r),\ r \geq 0$ is any right-continuous function decreasing to 0 as $r \searrow 0$.

Now for the pointwise Hölder exponent of a stochastic process X at t_0, defined by

$$\alpha_X(t_0, \omega) = \sup \left\{ \alpha > 0,\ \lim_{\rho \to 0} \frac{X(t_0 + \rho, \omega) - X(t_0, \omega)}{\rho^\alpha} = 0 \right\}, \tag{8}$$

the result of the previous proposition implies

Corollary 3.4. *The pointwise Hölder exponent of X at any point t is almost surely smaller or equal to H.*

Remark 3.2. *Proposition 3.3 in [2] gives an analogous result for lass processes with exponent H.*

We need the following modification of Lemma 2.2 in Berman [10] for the proof.

Lemma 3.1. *Let $f(t),\ 0 \leq t \leq T$ be a deterministic real valued measurable function which has a local time $L(t,x)$ and let $\delta(s),\ s > 0$, be a positive nondecreasing function such that for every $0 \leq t \leq T$*

$$\liminf_{s \to 0} \frac{(\delta(s))^{1/2}}{s} \left(\int_{\mathbb{R}} L^2([t-s, t+s], x) dx \right)^{1/2} = 0. \tag{9}$$

Then

$$ap - \limsup_{s \to t} \frac{|f(t) - f(s)|}{\delta(|t-s|)} = +\infty. \tag{10}$$

Proof. According to Lemma 2.1 in [10], for arbitrary $M > 0$, $0 \leq t \leq T$, and $s > 0$, we have

$$\begin{aligned} &(2s)^{-1} \lambda \{t' \ :\ |t - t'| \leq s,\ |f(t') - f(t)| < M\delta(|t-t'|)\} \\ &\qquad \leq \frac{(2M\delta(s))^{1/2}}{2s} \left(\int_{\mathbb{R}} L^2([t-s, t+s], x) dx \right)^{1/2}. \end{aligned} \tag{11}$$

By (9), the right hand side of (11) has lim inf equal to zero. Hence, t is not a density point for the set,

$$\left\{ s :\ \frac{|f(t) - f(s)|}{\delta(|t-s|)} < M \right\}.$$

Therefore, it is not a dispersion point for the complementary set,

$$\left\{ s :\ \frac{|f(t) - f(s)|}{\delta(|t-s|)} \geq M \right\}.$$

Consequently the approximate limsup of the ratio is at least equal to M. Since M is arbitrary, the conclusion follows. ∎

Proof of Proposition 3.3. It suffices to show that for every $0 \le t \le T$, the assumption (9) holds almost surely for $\delta(s) = s^H \phi(s)$. This holds if the random variable

$$\Lambda(s) = \frac{(\delta(s))^{1/2}}{s} \left(\int_{\mathbb{R}} L^2([t-s, t+s], x) dx \right)^{1/2}$$

converges to 0 in probability as s tends to 0. Moreover, using Markov's inequality, we have for arbitrary $\varepsilon > 0$

$$\begin{aligned} \mathbb{P}\left(\Lambda(s) > \varepsilon\right) &\le s^H \phi(s) s^{-2} \varepsilon^{-2} \mathbb{E} \int_{\mathbb{R}} L^2([t-s, t+s], x) dx \\ &\le C s^H \phi(s) s^{-2} \varepsilon^{-2} (2s)^{2-H}, \end{aligned}$$

where the last inequality follows from (5) and (6). The last term is equal to $2^{2-H} C \varepsilon^{-2} \phi(s)$, which tends to zero as s tends to 0 by the assumption on ϕ. This completes the proof.

4 Hausdorff measure and dimension

Firstly, we recall the definition of the ϕ-Hausdorff measure and dimension. Let Φ be the class of functions $\phi : (0,1) \to (0,1)$ which are right continuous and increasing with $\phi(0^+) = 0$.
The ϕ-Hausdorff measure $\mathcal{H}_\phi(A)$ of a Borel subset A of $\mathbb{R}$ is defined by

$$\mathcal{H}_\phi(A) = \liminf_{\varepsilon \searrow 0} \left\{ \sum_{n=1}^{\infty} \phi(|I_n|) : \ \{I_n\}_{n \in \mathbb{N}} \ \text{is a countable cover of A} \right. \\ \left. \text{by compact intervals with length } |I_n| \le \varepsilon \right\}.$$

And the Hausdorff dimension of A is defined by

$$\begin{aligned} \dim A &= \inf\{\alpha \ / \ \mathcal{H}_\phi(A) = 0, \ \phi(r) = r^\alpha\} \\ &= \sup\{\alpha \ / \ \mathcal{H}_\phi(A) = +\infty, \ \phi(r) = r^\alpha\}. \end{aligned}$$

Theorem 4.1. *Suppose that $\{X(t), \ t \in [0,T]\}$ satisfies $(\mathcal{H})$, and let $Z_t = \{s \in [0,T] / X(s) = X(t)\}$ be the progressive level set. Consider the measure function $\phi(r) = r^{(1-H)/2} |\log(r)|^\theta$, $\theta > 1/2$. Then*

$$\mathbb{P}\left(\mathcal{H}_\phi(Z_t) = +\infty \ \ \text{for a.e.} \ \ t \right) = 1. \tag{12}$$

Furthermore, if (H_2) holds and if, for any $\beta < H$, X satisfies a uniform Hölder condition of order β almost surely, then for almost every $t \ge 0$

$$\mathbb{P}(\dim Z_t = 1 - H) = 1. \tag{13}$$

Remark 4.1. *Benassi et al. [6] have computed the Hausdorff dimension of the graph of lass processes under similar assumptions.*

Proof. The inequality (7) implies

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}M^2(2^{-n})}{[\phi(2^{-n})]^2} \leq C \sum_{n=1}^{\infty} \frac{(2^{-n})^{1-H}}{(2^{-n})^{1-H} n^{2\theta}} < +\infty.$$

Then, according to Theorem 2 and (4.6) in Berman [11], (12) is proved.
We prove (13) in two steps, by finding upper and lower bounds which appear to be identical.
Upper bound : We first prove that for almost every t

$$\dim Z_t \leq 1 - H, \quad \text{almost surely.} \tag{14}$$

For any $t \in [0,T]$, consider $t_0 \in [0,T]$ such that $0 < |t - t_0| < \rho_0$. Then, according (H_2), we have

$$\begin{aligned} \int_{\mathbb{R}} \left| \mathbb{E} e^{ixX(t)} \right| dx &\leq \int_{\mathbb{R}} \left| \mathbb{E} e^{ixX(t_0)} \right| \left| \mathbb{E} e^{ix(X(t)-X(t_0))} \right| dx, \\ &\leq \int_{\mathbb{R}} \psi(x) dx \times \frac{1}{|t-t_0|^H}. \end{aligned}$$

It follows that $X(t)$ has a bounded continuous density function $p_t(y)$. Furthermore, for any t, take $t_0 = t(1 - \dfrac{T\rho_0}{(T+\rho_0)^2})$, hence $t_0 \in [0,T]$, with $|t - t_0| < \rho_0$ and by the inverse Fourier transform we get

$$\mathbb{P}\left(|X(t) - x| \leq u\right) \leq \frac{C}{\pi} \frac{(T+\rho_0)^{2H} u}{(T\rho_0)^H} \times \frac{1}{t^H}. \tag{15}$$

Even though the processes considered in this paper are not necessarily Gaussian, the main ingredients needed to prove (14) are the existence of a continuous density, the inequality (15) and the Hölder continuity of X. The proof follows now the same lines as the first part of the proof of Theorem 2.1 in Berman [7]. We omit the details here.
Lower bound : Let us first recall the following property of the Hausdorff dimension: For any countable sequence of sets $E_1, E_2, \ldots$ we have

$$\dim \left(\bigcup_{i=1}^{\infty} E_i \right) = \sup_{i \geq 1} \dim E_i. \tag{16}$$

Now since $\mathbb{R}^+$ is a countable union of finite intervals, it suffices to prove the result for any I of small length. Let us prove that for any $|I| < \rho_0$, we have

$$\dim\{s \in I / X(s) = X(t)\} \geq 1 - H$$

We adapt the argument of Berman [7]. Using Parseval's identity, we obtain

$$\begin{aligned} H(s,t) &:= \int_{\mathbb{R}} L(t,x) L(s,x) dx, \\ &= \int_{\mathbb{R}} \int_0^t \int_0^s \exp\left(iu(X(t') - X(s'))\right) ds' dt' du. \end{aligned}$$

Furthermore, from a standard approximation, we have for any Borel function $g(s,t)$

$$\int_I \int_I g(s,t) H(ds, dt) = \int_{-\infty}^{+\infty} \int_I \int_I g(s,t) L(dt,x) L(ds,x) dx. \tag{17}$$

By Fubini's theorem, for every $0 < \gamma < 1 - H$, we have

$$\begin{aligned}
\mathbb{E}\left(\int_I \int_I \frac{1}{|t-s|^\gamma} H(ds, dt)\right) &= \int_{\mathbb{R}} \int_I \int_I \frac{1}{|t-s|^\gamma} \mathbb{E} \exp\left(iu(X(t) - X(s))\right) dsdtdu \\
&\leq \int_{\mathbb{R}} \psi(u) du \int_I \int_I \frac{1}{|t-s|^{H+\gamma}} dsdt, \qquad (18)
\end{aligned}$$

where we have used the assumption $(\mathcal{H})$ to obtain the last inequality. Since $0 < \gamma < 1 - H$, the second integral in (18) is finite. By Fubini's theorem and (17) we have

$$\int_I \int_I \frac{1}{|t-s|^\gamma} L(ds, x) L(dt, x) < \infty,$$

for almost all $x \in \mathbb{R}$, almost surely. This implies that

$$\int_I \int_I \frac{1}{|u-v|^\gamma} L(du, X(t)) L(dv, X(t)) < \infty, \quad \text{for almost all } t \in I \text{ almost surely.} \qquad (19)$$

According to Theorem 6.3 in Geman and Horwitz [20], the measure $L(., X(t))$ is a positive measure on I for almost all $t \in I$, almost surely. It follows from Lemma 1.5 in Berman [7] that the random measure $L(., X(t))$ is supported on Z_t. Moreover Z_t is closed, since X is continuous almost surely. Hence, combining Frostman's theorem (see e.g. Adler [1] page 196) and (19), we have almost surely $\dim Z_t \geq 1 - H$ for almost all $t \in I$. □

5 Joint continuity of local times

Now, we turn to the problem of studying the existence of jointly continuous local times. Throughout this section, we assume the supplementary integrability condition (20) which is verified for a wide class of stochastic processes including Gaussian and stable processes.

Theorem 5.1. *Let $\{X(t),\ t \in [0,T]\}$ be a stochastic process starting from zero and satisfying assumptions $(\mathcal{H})$ and (H_m) for all $m \geq 2$, where ψ satisfies*

$$\int_{|u|\geq 1} |u|^{\frac{1-H}{H}} \psi(u) du < \infty. \qquad (20)$$

Then X has a jointly continuous local time $L(t,x)$, such that for any compact $K \subset \mathbb{R}$ and any interval I with length less than ρ_0 [the constant appearing in the assumption $(\mathcal{H})$], we have

(i) *If $0 < \xi < 1 \wedge \frac{1-H}{2H}$, then $|L(I,x) - L(I,y)| \leq \eta |x-y|^\xi$, for all $x, y \in K$*

(ii) *If $0 < \delta < 1 - H$, then $\sup_{x \in K} L(I,x) \leq \eta |I|^\delta$,*

where η is a random variable, almost surely positive and finite.

Proof. It is well known that proving the joint continuity of the local times and the Hölder conditions (i) and (ii) is straightforward, when estimating the moments of local times. According to

Geman and Horowitz [20], equation (25.5) and (25.7), the following expressions for the moments of local times hold : for any $x, y \in \mathbb{R}$, $t,\ t+h \in [0,T]$ and any $m \geq 2$,

$$\begin{aligned} &\mathbb{E}\left[L(t+h,x) - L(t,x)\right]^m \\ &= \frac{1}{(2\pi)^m} \int_{[t,t+h]^m} \int_{\mathbb{R}^m} e^{-ix\sum_{j=1}^m u_j} \mathbb{E}\left(e^{i\sum_{j=1}^m u_j X(t_j)}\right) du_1...du_m dt_1...dt_m, \qquad (21) \end{aligned}$$

and for any even integer $m \geq 2$,

$$\begin{aligned} &\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \\ &= \frac{1}{(2\pi)^m} \int_{[t,t+h]^m} \int_{\mathbb{R}^m} \prod_{j=1}^m \left[e^{-iyu_j} - e^{-ixu_j}\right] \mathbb{E}\left(e^{i\sum_{j=1}^m u_j X(t_j)}\right) du_1...du_m dt_1...dt_m. \qquad (22) \end{aligned}$$

We estimate only (22), since (21) is treated in a same manner. By using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\xi}|\theta|^\xi$ for all $0 < \xi < 1$ and any $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned} &\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \qquad (23) \\ &\leq |y-x|^{m\xi}\pi^{-m} \int_{[t,t+h]^m} \int_{\mathbb{R}^m} \prod_{j=1}^m |u_j|^\xi \mathbb{E}\left[\exp\left(i\sum_{j=1}^m u_j X(t_j)\right)\right] du_1...du_m dt_1...dt_m. \end{aligned}$$

Furthermore, in order to use (H_m), we replace the integration over the domain $[t, t+h]$ by the integration over the subset $t < t_1 < ... < t_m < t+h$. We deal now with the inner multiple integral over the u's. Change the variables of integration by means of the transformation

$$u_j = v_j - v_{j+1},\ j = 1, ..., m-1;\ u_m = v_m.$$

Then the linear combination in the exponent in (23) is transformed according to

$$\sum_{j=1}^m u_j X(t_j) = \sum_{j=1}^m v_j (X(t_j) - X(t_{j-1})),$$

where $t_0 = 0$. Since $|a-b|^\xi \leq |a|^\xi + |b|^\xi$, for all $0 < \xi < 1$, it follows that

$$\begin{aligned} \prod_{j=1}^m |u_j|^\xi &= \prod_{j=1}^{m-1} |v_j - v_{j+1}|^\xi |v_m|^\xi \\ &\leq \prod_{j=1}^{m-1} (|v_j|^\xi + |v_{j+1}|^\xi)|v_m|^\xi. \qquad (24) \end{aligned}$$

Moreover, the last product is at most equal to a finite sum of terms each of the form $\prod_{j=1}^m |v_j|^{\xi\varepsilon_j}$, where $\varepsilon_j = 0, 1$, or 2 and $\sum_{j=1}^m \varepsilon_j = m$. Combining $(\mathcal{H})$, (H_m) for all $m \geq 2$, and the change of variable $v_j(t_j - t_{j-1})^H = \theta_j$, (23) becomes

$$\begin{aligned} &\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \qquad (25) \\ &\leq C_m \prod_{j=1}^m \int_{\mathbb{R}} |\theta_j|^{\xi\varepsilon_j} \psi(\theta_j) d\theta_j |y-x|^{m\xi}\pi^{-m} \int_{t<t_1...<t_m<t+h} \prod_{j=1}^m (t_j - t_{j-1})^{-H(1+\xi\varepsilon_j)} dt_1...dt_m. \end{aligned}$$

Furthermore, for $|u| \geq 1$, we have $|u|^{\xi\varepsilon_j} \leq |u|^{2\xi}$, then

$$\begin{aligned}\int_{\mathbb{R}} |u|^{\xi\varepsilon_j}\psi(u)du &= \int_{-1}^{1} |u|^{\xi\varepsilon_j}\psi(u)du + \int_{|u|\geq 1} |u|^{\xi\varepsilon_j}\psi(u)du \\ &\leq \int_{-1}^{1} \psi(u)du + \int_{|u|\geq 1} |u|^{2\xi}\psi(u)du.\end{aligned}$$

Combining the fact that ψ belongs to $L^1(\mathbb{R})$ and (20), we obtain that the last integrals are finite. Moreover, by an elementary calculation, for all $m \geq 1$, $h > 0$ and $b_j < 1$,

$$\int_{t<s_1<\ldots<s_m<t+h} \prod_{j=1}^{m}(s_j - s_{j-1})^{-b_j} ds_1 \ldots ds_m = h^{m-\sum_{j=1}^{m} b_j} \frac{\prod_{j=1}^{m}\Gamma(1-b_j)}{\Gamma(1+m-\sum_{j=1}^{m} b_j)},$$

where $s_0 = t$. It follows that (25) is dominated by

$$C_m \frac{|y-x|^{m\xi}|h|^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi))},$$

where we have used $\sum_{j=1}^{m} \varepsilon_j = m$. The rest of the proof follows now the lines of the proofs of Theorems 26.1 and 27.1 of Geman and Horowitz [20]. Therefore it will be omitted here. □

We can now establish the following result on the uniform dimension of the level set which improves (13) under the assumption (H_m) for all $m \geq 2$.

Proposition 5.1. *Suppose that $\{X(t), t \in [0,T]\}$ is $\beta-$Hölder continuous for any $\beta < H$ and satisfies the assumptions of the previous theorem. Then, with probability one, for any interval I, we have*

$$\dim\{t \in I / X(t) = x\} = 1 - H,$$

for all x such that $L(I,x) > 0$.

Proof. The proof of the upper bound appears already in that of (13), and the proof of the lower bound follows from Theorem 5.1 (ii) and Theorem 8.7.4 in Adler [1]. □

6 Regularity of the local time in the space variable

It is known that the Brownian motion local time satisfies a Hölder condition of any order smaller than 1/2, but not of order 1/2. The situation seems to be quite different for a large class of non Markovian Gaussian processes considered in the works of Berman [8] and Geman and Horowitz [20], where the authors have given several conditions that imply the higher smoothness of the local time as a function of x. These results have been extended to a wide class of self similar stochastic processes with stationary increments by Kôno and Shieh [23]. These results extend as follows.

Theorem 6.1. *Assume that $\{X(t),\ t \in [0,T]\}$ satisfies $(\mathcal{H})$ with*

$$\int_{\mathbb{R}} |u|^{2r}\psi(u)du < \infty,$$

for some nonnegative integer r such that $H < \dfrac{1}{2r+1}$. Then, the k-th derivatives, $L^{(k)}(T,x)$, of $L(T,.)$ exist up to $k = r$ almost surely. Moreover $L^{(k)}(T,x) \in L^2(dx \times \mathbb{P})$.

Proof. Let $I \subset [0,T]$ with length at most ρ_0. Using the change of variables $\dfrac{v}{|t-s|^H} = u$ and $(\mathcal{H})$ we obtain

$$\mathbb{E}\int_{-\infty}^{+\infty}\int_I\int_I |u|^{2r}\exp\left(iu(X(t)-X(s))\right)dsdtdu \leq \int_I\int_I \frac{dsdt}{|t-s|^{H(2r+1)}}\int_{\mathbb{R}}|v|^{2r}|\psi(v)|dv,$$

which is finite by the assumptions of the theorem. Then, the conclusion follows from the Fourier inversion formula (c.f. Berman [8]) and by a standard patch-up procedure. □

Remark that the function ψ plays in our proof the role of $\phi(x) = Ee^{ixX(1)}$ for H-self similar processes with stationary increments in the proof of Theorem 5.1 in [23]. By using this remark, we can also extend Theorem 5.2 in [23] to

Theorem 6.2. *Suppose that $\{X(t),\ t \in [0,T]\}$ satisfies $(\mathcal{H})$ and (H_m) for some $m \geq 2$ and*

$$\int_{\mathbb{R}} |u|^{2r+2/m+\varepsilon}|\psi(u)|du < \infty,$$

for some nonnegative integer r and some $\varepsilon > 0$ such that $H < \dfrac{1}{2r+2/m+1}$. Then, the local time $L(T,x)$ is of class $\mathcal{C}^r$ in x. Moreover $L^{(r)}(T,x)$ is Hölder continuous of a certain order.

7 Examples and extensions

The local time and some related sample paths properties of self similar processes with stationary increments have been studied by Kôno and Shieh [23]. This class is covered by the results of the present paper and now under weaker conditions.

We give in the sequel examples of lass processes for which our results hold and we explain how the self-similarity has been relaxed for these processes. We also generalize the results of the previous sections to multifractional processes.

7.1 Multifractional Brownian motions

The multifractional Brownian motion was introduced independently by Lévy-Véhel and Peltier [24] and Benassi et al. [6]. The definition due to Lévy-Véhel and Peltier is based on the moving average representation of fBm, where the constant Hurst parameter H is substituted by a functional $H(t)$ as follows :

$$\begin{aligned} B(t) &= \frac{1}{\Gamma(H(t)+1/2)}\left(\int_{-\infty}^{0}[(t-u)^{H(t)-1/2}-(-u)^{H(t)-1/2}]\,W(du)\right. \\ &+ \left.\int_0^t (t-u)^{H(t)-1/2}\,W(du)\right), \qquad t \geq 0, \end{aligned} \tag{26}$$

where $H(t) : [0, \infty) \longrightarrow [\mu, \nu] \subset (0, 1)$ is a Hölder continuous function with exponent $\beta > 0$, W is the standard Brownian motion defined on $(-\infty, +\infty)$. Benassi et al. [6] defined the multifractional Brownian motion by means of the harmonisable representation of fBm as follows :

$$\widehat{B}(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} \widehat{W}(d\xi), \tag{27}$$

where $\widehat{W}(\xi)$ is the Fourier transform of the series representation of white noise with respect to an orthonormal basis of $L^2(\mathbb{R})$.

However, it is proved by Lévy Véhel and Peltier ([24], Proposition 5) and by Benassi *et al.* ([6], Theorem 1.7) that if H is β-Hölder continuous and $\sup_{t\in\mathbb{R}^+} H(t) < \beta$, the multifractional Brownian motion is lass. That is

$$\lim_{\rho \longrightarrow 0^+} law \left\{ \frac{B(t+\rho u) - B(t)}{\rho^{H(t)}}, u \in \mathbb{R} \right\} = law\{B^{H(t)}(u), u \in \mathbb{R}\}, \tag{28}$$

where $B^{H(t)}$ is a fBm with Hurst parameter $H(t)$.

In addition, Boufoussi *et al.* [13] have proved that the mBm given by the moving average representation satisfies the assumptions $(\mathcal{H})$ and (H_m) for all $m \geq 2$. By using the lass property of the mBm, we give here a proof that both representations of mBm are LND .

Theorem 7.1. *If H is β-Hölder continuous and $\sup_{t\in\mathbb{R}^+} H(t) < \beta$, then for every $\varepsilon > 0$, and any $T > \varepsilon$, the mBms given by (26) and (27) are locally nondeterministic on $[\varepsilon, T]$.*

Proof. Let us use X to denote the two representations of the mBm. In a same way as in [13], proof of Theorem 3.3, we prove that there exists $\delta > 0$ such that

$$\begin{cases} E(X(t) - X(s))^2 > 0, & \text{whenever } 0 < |s - t| < \delta; \\ E(X(t))^2 > 0, & \text{for all } t \in [\varepsilon, T]. \end{cases}$$

It remains to prove that X satisfies assumption (8) in [13]. Fix $t > 0$, and using Lemma 3.1 in [13], we obtain

$$Var(X(s) - X(t)) \leq C_{H(t)} |s - t|^{2H(t)}, \quad \forall\ s \in [0, T]$$

Then, for all points $t < t_1 < ... < t_m < t + \delta t$, we have

$$\begin{aligned} Var(X(t_m) - X(t_{m-1})) &\leq 2Var(X(t_m) - X(t)) + 2Var(X(t_{m-1}) - X(t)) \\ &\leq \widetilde{C}_{H(t)} \delta^{2H(t)} \end{aligned}$$

Therefore

$$\lim_{\delta \to 0} \frac{Var(X(t_m)/X(t_1), ..., X(t_{m-1}))}{Var(X(t_m) - X(t_{m-1}))} \geq \lim_{\delta \to 0} \frac{Var(X(t_m)/X(t_1), ..., X(t_{m-1}))}{\widetilde{C}_{H(t)} \delta^{2H(t)}} \tag{29}$$

Moreover, if we add $X(t)$ to the conditional set we obtain

$$\begin{aligned}
&\frac{Var(X(t_m)/X(t_1),...,X(t_{m-1}))}{\delta^{2H(t)}} \\
&\geq \frac{Var(X(t_m)/X(t),X(t_1),...,X(t_{m-1}))}{\delta^{2H(t)}} \\
&= Var\left(\frac{X(t_m)-X(t)}{\delta^{H(t)}}/X(t),X(t_1)-X(t),...,X(t_{m-1})-X(t)\right) \\
&= Var\left(\frac{X(t_m)-X(t)}{\delta^{H(t)}}/X(t),\frac{X(t_1)-X(t)}{\delta^{H(t)}},...,\frac{X(t_{m-1})-X(t)}{\delta^{H(t)}}\right),
\end{aligned}$$

where the last equality follows from the fact that

$$\sigma\left(X(t),X(t_1)-X(t),...,X(t_{m-1})-X(t)\right)=\sigma\left(X(t),\frac{X(t_1)-X(t)}{\delta^{H(t)}},...,\frac{X(t_{m-1})-X(t)}{\delta^{H(t)}}\right)$$

Let $t_m-t=u_m\delta$ with $0<u_m<t$. Therefore, the fraction in (29) becomes

$$\begin{aligned}
&\lim_{\delta\to 0}Var\left(\frac{X(t+\delta u_m)-X(t)}{\delta^{H(t)}}/X(t),\frac{X(t+\delta u_1)-X(t)}{\delta^{H(t)}},...,\frac{X(t+\delta u_{m-1})-X(t)}{\delta^{H(t)}}\right) \\
&=\lim_{\delta\to 0}Var\left(Y_{t,\delta}(u_m)/X(t),Y_{t,\delta}(u_1),...,Y_{t,\delta}(u_{m-1})\right),
\end{aligned}$$

where we denote for simplicity

$$Y_{t,\delta}(u_m)=\frac{X(t+\delta u_m)-X(t)}{\delta^{H(t)}}$$

Moreover,

$$Var\left(Y_{t,\delta}(u_m)/X(t),Y_{t,\delta}(u_1),...,Y_{t,\delta}(u_{m-1})\right)=\frac{\det Cov(X(t),Y_{t,\delta}(u_1),..,Y_{t,\delta}(u_m))}{\det Cov(X(t),Y_{t,\delta}(u_1),...,Y_{t,\delta}(u_{m-1}))}.$$

Now, since the mBm is locally asymptotically self similar, $Y_{t,\rho}$ converges weakly to the fBm $B^{H(t)}$ of parameter $H(t)$ [t is fixed]. Consequently, the fraction above converges to

$$\frac{\det Cov(X(t),B^{H(t)}(u_1),...,B^{H(t)}(u_m))}{\det Cov(X(t),B^{H(t)}(u_1),...,B^{H(t)}(u_{m-1}))}.$$

Consequently,

$$\begin{aligned}
&\lim_{\delta\to 0}Var\left(Y_{t,\delta}(u_m)/X(t),Y_{t,\delta}(u_1),....,Y_{t,\delta}(u_{m-1})\right) \\
&=Var\left(B^{H(t)}(u_m)/B^{H(t)}(t),B^{H(t)}(u_1),...,B^{H(t)}(u_{m-1})\right) \\
&\geq C_{H(t)}[(u_m-u_{m-1})\wedge(t-u_m)]^{2H(t)},
\end{aligned}$$

where the last inequality follows from Lemma 7.1 in Pitt [26]. The last term is strictly positive since $0<u_1<u_2<...<u_m<t$. $\square$

7.2 Mixed Gaussian processes

Let $\{W(t),\ t \in [0,T]\}$ be a standard Brownian motion and $\{B^H(t),\ t \in [0,T]\}$ an independent fractional Brownian motion. Cheridito [15] has introduced the mixed fBm defined by $Y = \{W(t) + B^H(t),\ t \in [0,T]\}$ in order to model stock prices with long range dependence.

Since W and B^H are independent and both have smooth local time, it will be easy to see that Y has smooth local time. Moreover the moduli of continuity are obtained in terms of $(H \wedge \frac{1}{2})$. We omit the details.

In the sequel we consider $\{X_1(t),\ t \in [0,1]\}$ and $\{X_2(t),\ t \in [0,1]\}$ two mean zero Gaussian processes, where X_1 is LND and X_2, not necessarily independent of X_1, but only negligible in the sense that

$$E\left(X_2(t) - X_2(s)\right)^2 = o\left(E\left[X_1(t) - X_1(s)\right]^2\right), \quad as \quad t \to s. \tag{30}$$

Then, we prove that

Lemma 7.1. *The sum process $X = X_1 + X_2$ is LND on any interval $J \subset [0,1]$ of small length.*

Proof. The idea of the proof is inspired from that used in Guerbaz [21] to prove that the filtered white noise is LND. We present it here for the sake of completeness.

By using the elementary inequality $(x+y)^2 \geq \frac{x^2}{2} - y^2$ we obtain

$$\begin{aligned} &Var\left(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]\right) \\ &\geq \frac{1}{2} Var\left(\sum_{j=1}^{m} u_j[X_1(t_j) - X_1(t_{j-1})]\right) - Var\left(\sum_{j=1}^{m} u_j[X_2(t_j) - X_2(t_{j-1})]\right). \end{aligned}$$

Furthermore, since X_1 is LND, there exist δ_m and C_m such that for any $t_0 = 0 < t_1 < ... < t_m < 1$, with $t_m - t_1 < \delta_m$, we have

$$\begin{aligned} &Var(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]) \\ &\geq \frac{C_m}{2} \sum_{j=1}^{m} u_j^2 Var(X_1(t_j) - X_1(t_{j-1})) - m \sum_{j=1}^{m} u_j^2 Var(X_2(t_j) - X_2(t_{j-1})). \end{aligned} \tag{31}$$

Moreover, according to (30), for $0 < \varepsilon_m < \frac{C_m}{2m}$, there exists $\widehat{\delta}_m$ such that

$$\frac{Var(X_2(t_j) - X_2(t_{j-1}))}{Var(X_1(t_j) - X_1(t_{j-1}))} \leq \varepsilon_m, \quad \text{for all} \quad t_j - t_{j-1} \leq \widehat{\delta}_m.$$

Therefore

$$Var(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]) \geq \left(\frac{C_m}{2} - m\varepsilon_m\right) \sum_{j=1}^{m} u_j^2 Var(X_1(t_j) - X_1(t_{j-1})).$$

Furthermore, (30) implies that there exists a constant $C > 0$ such that

$$Var(X_1(t_j) - X_1(t_{j-1})) \geq C Var(X(t_j) - X(t_{j-1})).$$

Therefore, it suffices now to choose

$$\widetilde{\delta}_m < \widehat{\delta}_m \wedge \delta_m,$$

and to consider

$$\widetilde{C}_m = \left(\frac{C_m}{2} - m\varepsilon_m\right),$$

and the lemma is proved. □

7.3 Multifractional Gaussian processes

The multifractional Gaussian process (MGP) has been introduced in [3] as follows :

$$X(t) = \int_{\mathbb{R}} \frac{a(t,\lambda)(e^{it\lambda} - 1)}{|\lambda|^{1/2+H(t)}} W(d\lambda),$$

where $W(d\lambda)$ is the random Brownian measure on $L^2(\mathbb{R})$.

When H is constant, this process is a Filtered White Noise ([4], in short FWN). Moreover, if $a(t,\lambda) = 1$, a MGP is a mBm.

Assume that $a(t,\lambda)$ is $C^2(\mathbb{R}^2;\mathbb{R})$, and that there exists a function $a_\infty(t) \neq 0$ such that $\lim_{|\lambda|\to\infty} a(t,\lambda) = a_\infty(t)$ and that $\sigma(t,\lambda) = a(t,\lambda) - a_\infty(t)$ satisfies :

$$\left|\frac{\partial^{i+j}\sigma(t,\lambda)}{\partial^i t \partial^j \lambda}\right| \leq \frac{C}{|\lambda|^{j+\eta}}, \tag{32}$$

for $i,j = 0,1,2$ and $\eta > 0$ such that $0 < H + \eta < 1$.

The particular case of FWN has been studied in Guerbaz [21]. We now prove that the MGP satisfies the assumptions $(\mathcal{H})$ and (H_m) for all $m \geq 2$. The assumption $(\mathcal{H})$ may be deduced from Proposition 1 in [3]. To prove that X is LND, we first write

$$X(t) = a_\infty(t)\widehat{B}(t) + \int_{\mathbb{R}} \frac{\sigma(t,\lambda)(e^{it\lambda} - 1)}{|\lambda|^{1/2+H(t)}} W(d\lambda),$$

where $\widehat{B}$ is the mBm given by (27). Since $a_\infty(t)$ belongs to $C^2(\mathbb{R})$ and Theorem 7.1 implies that $\widehat{B}$ is LND, we conclude easily that the Gaussian process $X_1(t) = a_\infty(t)\widehat{B}(t)$ is LND. Moreover, by using (32), we obtain that the process

$$X_2(t) = \int_{\mathbb{R}} \frac{\sigma(t,\lambda)(e^{it\lambda} - 1)}{|\lambda|^{1/2+H(t)}} dW(\lambda),$$

satisfies

$$E\left(X_2(t) - X_2(s)\right)^2 = o\left(E\left(X_1(t) - X_1(s)\right)^2\right), \quad as \quad t \to s.$$

Then Lemma 7.1 achieves the proof.

7.4 Sub-Gaussian processes

Let $X = \{X(t),\ t \in [0,T]\}$ be a mean zero Gaussian process and let Z a nonnegative $\alpha/2$-stable random variable, where $1 < \alpha < 2$, i.e., for $\lambda > 0$,

$$E \exp(-\lambda Z) = \exp(-\lambda^{\alpha/2}).$$

Assume that the random variable Z is independent of X. The α-stable process $Y = \{Y(t) = Z^{1/2} X(t),\ t \in [0,T]\}$ is called a *sub-Gaussian process* with underlying Gaussian process X.

Proposition 7.2. *1. Suppose that $Var(X(t) - X(s)) \geq C|t-s|^{2H}$ for some $0 < H < 1$, $C > 0$ and t, s sufficiently close. Then Y satisfies the assumption $(\mathcal{H})$.*

2. If X satisfies (H_m) for some $m \geq 1$, then Y satisfies the same.

Proof. 1. Since X is a centered Gaussian process

$$\begin{aligned} E \exp\left(i\lambda \frac{Y(t) - Y(s)}{|t-s|^H} \right) &= E\left[E\left(\exp\left(i Z^{1/2} \lambda \frac{X(t) - X(s)}{|t-s|^H} \right) /Z \right) \right] \\ &= E \exp\left(-Z \frac{\lambda^2}{2} \frac{Var(X(t) - X(s))}{|t-s|^{2H}} \right). \end{aligned}$$

And since Z is $\alpha/2$-stable random variable, the last expression becomes

$$\exp\left(\frac{-|\lambda|^\alpha Var(X(t) - X(s))^{\alpha/2}}{2^{\alpha/2}|t-s|^{\alpha H}} \right).$$

Finally, the assumption on X in the first point implies that the last expression is dominated by

$$\psi(\lambda) = \exp\left(-\frac{C^{\alpha/2}}{2^{\alpha/2}} |\lambda|^\alpha \right),$$

which belongs to $L^1(\mathbb{R})$. Consequently, Y satisfies the assumption $(\mathcal{H})$.
2. Assume that X satisfies (H_m) for some $m \geq 2$. Then, there exist two positive constants c_m and δ_m, such that for all $0 = t_0 < t_1 < t_2 < ... < t_m \leq T$, with $|t_m - t_1| \leq \delta_m$, we have

$$\left| \mathbb{E} \exp\left(i \sum_{j=1}^m u_j [X(t_j) - X(t_{j-1})] \right) \right| \leq \prod_{j=1}^m \left| \mathbb{E} \exp\left(i c_m u_j [X(t_j) - X(t_{j-1})] \right) \right|,$$

for all $u_1, ..., u_m \in \mathbb{R}$. Since X is a Gaussian process, the previous expression reads

$$\exp\left(-\frac{1}{2} Var\left(\sum_{j=1}^m u_j [X(t_j) - X(t_{j-1})] \right) \right) \leq \exp\left(-\frac{1}{2} c_m \sum_{j=1}^m u_j^2 Var[X(t_j) - X(t_{j-1})] \right).$$

Moreover, using the fact that the function $u^{\alpha/2}$ is concave for $0<\alpha<2$, and conditioning in the same manner as above we obtain

$$\begin{aligned}
&\left|\mathbb{E}\exp\left(i\sum_{j=1}^{m}u_j[Y(t_j)-Y(t_{j-1})]\right)\right| \\
&= \exp\left(-\frac{1}{2^{\alpha/2}}\left[Var\left(\sum_{j=1}^{m}u_j[X(t_j)-X(t_{j-1})]\right)\right]^{\alpha/2}\right) \\
&\leq \exp\left(-\frac{1}{2^{\alpha/2}}\frac{c_m^{\alpha/2}}{m^{1-\alpha/2}}\sum_{j=1}^{m}u_j^{\alpha}\left(Var[X(t_j)-X(t_{j-1})]\right)^{\alpha/2}\right) \\
&= \prod_{j=1}^{m}\mathbb{E}\exp\left[iC_m u_j(Y(t_j)-Y(t_{j-1}))\right],
\end{aligned}$$

where $C_m=\frac{\sqrt{c_m}}{m^{\frac{2-\alpha}{2\alpha}}}$. □

7.5 Linear multifractional stable processes

The linear multifractional stable process (LMSP) is defined by the stochastic integral

$$\begin{aligned}
\Psi^{\alpha}_{H(t)}(t) &= \int_{\mathbb{R}} a\left[(t-u)_+^{H(t)-1/\alpha}-(-u)_+^{H(t)-1/\alpha}\right] \\
&+ b\left[(t-u)_-^{H(t)-1/\alpha}-(-u)_-^{H(t)-1/\alpha}\right]M_{\alpha,\beta}(du); \qquad t\in\mathbb{R}, \qquad (33)
\end{aligned}$$

where $M_{\alpha,\beta}(du)$ is a (strictly) α-stable, independently scattered random measure with control measure ds, and skewness intensity $\beta(.)\in[-1,1]$, $u\in\mathbb{R}$.

This process was introduced by Stoev and Taqqu in [28] as a natural generalization of the linear fractional stable process to the case where the self-similarity parameter H is no more constant, but a regular function of time.

According to Stoev and Taqqu ([29]), Theorem 5.1), the LMSP is a lass process. Its tangent process at each t_0 is a linear fractional stable process with parameter $H(t_0)$.

We are interested in this paragraph in deriving sufficient conditions for the existence and the regularity of the local time of LMSP. We restrict ourselves for simplicity to the case $a=1$ and $b=\beta(.)=0$; i.e. we consider the process

$$\Psi^{\alpha}_{H(t)}(t)=\int_{\mathbb{R}}(t-u)_+^{H(t)-1/\alpha}-(-u)_+^{H(t)-1/\alpha}dZ^{\alpha}(u), \quad t\geq 0, \qquad (34)$$

where Z^{α} is a Lévy $\alpha-$stable motion. Boufoussi *et al.* [13] have investigated the case $\alpha=2$, corresponding the mBm, i.e., $M_{\alpha,\beta}$ is the random Brownian measure on $L^2(\mathbb{R})$. The authors have assumed H to be κ-Hölder continuous with $\sup_{t\in\mathbb{R}^+}H(t)<\kappa$. This condition was needed to prove some estimates which imply the existence and the regularity of the local time of mBm. We extend their results here to the LMSP under weaker conditions. We can now prove the following existence result

Theorem 7.3. *The LMSP* $\{\Psi^\alpha_{H(t)}(t),\ t \in [0,T]\}$ *has almost surely a local time* $L(t,x)$*, continuous with respect to time and such that* $L(T,.) \in L^2(dx \times \mathbb{P})$*. Conversely, assume that* H *is continuous and denote by* α_H *its pointwise Hölder exponent, then, if* $H(t) \le \alpha_H(t)$*, the existence of square integrable local time on small intervals implies that* $0 < H(t) < 1$ *almost everywhere.*

Proof. The first part will be proved if we show that the LMSP satisfies $(\mathcal{H})$. On the other hand, denoting by $span(\Psi^\alpha_{H(u)}(u),\ u \le s)$ the subspace spanned by $(\Psi^\alpha_{H(u)}(u),\ u \le s)$ and with the notation of [25], we have for any $0 \le s \le t$ such that $|t-s| < 1$,

$$\begin{aligned} \|\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)\|^\alpha_\alpha &\ge \|\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s) - span(\Psi^\alpha_{H(u)}(u),\ u \le s)\|^\alpha_\alpha \\ &= \|\Psi^\alpha_{H(t)}(t) - span(\Psi^\alpha_{H(u)}(u),\ u \le s)\|^\alpha_\alpha \\ &= \int_s^t (t-u)^{\alpha(H(t)-1/\alpha)} du = \frac{1}{\alpha H(t)}(t-s)^{\alpha(H(t)-1/\alpha)+1} \\ &= \frac{1}{\alpha H(t)}(t-s)^{\alpha H(t)} \qquad (35) \\ &\ge \frac{1}{\alpha\mu}(t-s)^{\alpha\mu}, \qquad (36) \end{aligned}$$

where $\mu = \sup_{t\in\mathbb{R}^+} H(t)$. Therefore,

$$\begin{aligned} \mathbb{E}\exp\left(iu \frac{\left[\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)\right]}{|t-s|^\mu} \right) &= \exp\left(-|u|^\alpha \frac{\|\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)\|^\alpha_\alpha}{|t-s|^{\alpha\mu}} \right) \\ &\le \exp\left(-\frac{|u|^\alpha}{\alpha\mu} \right), \end{aligned}$$

which belongs to $L^1(\mathbb{R})$. Then, the LMSP satisfies $(\mathcal{H})$, and the first part is proved.
Let's now prove the second point : Let $[a,b]$ be an interval with small length. Since the local time exists on the interval $[a,b]$, then according to Geman and Horowitz [[20], Theorem 21.9] the following holds

$$\int_{\mathbb{R}} \int_a^b \int_a^b \mathbb{E}\exp\left(i\theta[\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)]\right) dsdtd\theta < \infty. \qquad (37)$$

Moreover

$$\mathbb{E}\exp\left(i\theta[\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)]\right) = \exp\left(-|\theta|^\alpha \|\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)\|^\alpha_\alpha \right).$$

On the other hand, by Theorem 2.1 in [28], for all $1 \le \alpha < 2$ we have

$$\|\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)\|^\alpha_\alpha \le C_\alpha \left(|t-s|^{\alpha H(t)} + |H(t)-H(s)|^\alpha \right). \qquad (38)$$

Since $H(t) \le \alpha_H(t)$, then for all t,s close, the expression (38) becomes

$$\|\Psi^\alpha_{H(t)}(t) - \Psi^\alpha_{H(s)}(s)\|^\alpha_\alpha \le C_{\alpha,H}|t-s|^{\alpha H(t)}. \qquad (39)$$

Combining (37) and (39) we obtain that

$$\int_a^b \int_a^b |t-s|^{-H(t)} dsdt < \infty.$$

Hence $H(t) < 1$ for almost every $t \in [a, b]$. Since $\mathbb{R}$ is a countable union of small intervals, the result is proved. $\square$

According to Theorem 2.6 in [25], proving that the LMSP satisfies (H_m) is equivalent to prove that

Proposition 7.4. *The LMSP is locally nondeterministic on every interval* $[\epsilon, T]$*, for any* $0 < \epsilon < T < \infty$.

Proof. To prove the LND for the LMSP we shall verify assumptions (a), (b) and (c) of Definition 2.4 in [25]. First, let us denote for simplicity

$$K(t,u) = (t-u)_+^{H(t)-1/\alpha} - (-u)_+^{H(t)-1/\alpha}.$$

Since $\|K(t,u)\|_\alpha = t^{H(t)}\|K(1,u)\|_\alpha$, then (a) holds away from the origin. The second condition in Definition 2.4 in [25] follows from (4.11) in [28]. It remains to show that the LMSP satisfies the last assumption, i.e.

$$\liminf_{\substack{c \searrow 0^+ \\ 0<t_m-t_1\leq c}} \frac{\|\Psi^\alpha_{H(t_m)}(t_m) - span(\Psi^\alpha_{H(t_i)}(t_i),\ i=1,...,m-1)\|^\alpha_\alpha}{\|\Psi^\alpha_{H(t_m)}(t_m) - \Psi^\alpha_{H(t_{m-1})}(t_{m-1})\|^\alpha_\alpha} > 0. \tag{40}$$

Since

$$\begin{aligned}
&\|\Psi^\alpha_{H(t_m)}(t_m) - span(\Psi^\alpha_{H(t_i)}(t_i),\ i=1,...,m-1)\|^\alpha_\alpha \\
&\geq \|\Psi^\alpha_{H(t_m)}(t_m) - span(\Psi^\alpha_{H(u)}(u),\ u \leq t_{m-1})\|^\alpha_\alpha \\
&\geq \frac{1}{\alpha\mu}(t_m - t_{m-1})^{\alpha H(t_m)},
\end{aligned} \tag{41}$$

where we have used (35) to obtain the last inequality. Combining (38) and (41), we obtain that the ratio in (40) is at least equal to

$$\begin{aligned}
&C_{\alpha,H(t_m)} \frac{(t_m - t_{m-1})^{\alpha H(t_m)}}{|t_m - t_{m-1}|^{\alpha H(t_m)} + |H(t_m) - H(t_{m-1})|^\alpha} \\
&= C_{\alpha,H(t_m)} \left[1 + \left(\frac{|H(t_m) - H(t_{m-1})|}{(t_m - t_{m-1})^{H(t_m)}}\right)^\alpha\right]^{-1}.
\end{aligned} \tag{42}$$

Since $H(t) \leq \alpha_H(t)$, $\lim_{t-s\to 0} \frac{|H(t)-H(s)|}{(t-s)^{H(t)}} = 0$. Then (40) holds and Ψ^α_H is LND. $\square$

Our main result in this paragraph reads

Theorem 7.5. *Assume* $1 \leq \alpha < 2$ *and* H *is continuous with* $H(t) < \alpha_H(t)$*. Then, the LMSP* Ψ^α_H *has jointly continuous local times* $L(t,x)$*. It satisfies for any compact* $U \subset \mathbb{R}$
(i)

$$\sup_{x\in U} \frac{|L(t+h,x) - L(t,x)|}{|h|^\gamma} < +\infty \quad a.s., \tag{43}$$

where $\gamma < 1 - H(t)$ and $|h| < \eta$, η being a small random variable almost surely positive and finite,
(ii) for any $I \subset [0,T]$,

$$\sup_{x,y\in U, x\neq y} \frac{|L(I,x) - L(I,y)|}{|x-y|^{\zeta}} < +\infty \quad a.s., \tag{44}$$

where $\zeta < \left(\dfrac{1}{2 \sup\limits_I H(t)} - \dfrac{1}{2} \right) \wedge 1.$

Proof. The proof follows the same lines as in ([13], Theorem 3.5), but here we use the LND in the sense of Nolan [25] instead of Berman [9]. □

Theorem 7.6. *Assume $1 \leq \alpha < 2$ and H is ρ-Hölder continuous with $1/\alpha \leq H(t) < \rho$ for all $t \geq 0$. Then, for any interval $[a,b] \subset \mathbb{R}^+$ and every $u \in \mathbb{R}$, the linear multifractional stable process Ψ_H^α satisfies*

$$dim\{t \in [a,b],\ \Psi_H^\alpha(t) = u\} = 1 - \min_{[a,b]} H(t), \tag{45}$$

with positive probability.

Proof. We omit the proof which uses a chaining argument similar to that of Theorem 4.2 in [13]. □

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