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Small Deviations of Gaussian Random Fields in L_q–Spaces

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Abstract

We investigate small deviation properties of Gaussian random fields in the space $L_q(\mathbb{R}^N, \mu)$ where μ is an arbitrary finite compactly supported Borel measure. Of special interest are hereby "thin" measures μ, i.e., those which are singular with respect to the N–dimensional Lebesgue measure; the so–called self–similar measures providing a class of typical examples. For a large class of random fields (including, among others, fractional Brownian motions), we describe the behavior of small deviation probabilities via numerical characteristics of μ, called mixed entropy, characterizing size and regularity of μ.

For the particularly interesting case of self–similar measures μ, the asymptotic behavior of the mixed entropy is evaluated explicitly. As a consequence, we get the asymptotic of the small deviation for N–parameter fractional Brownian motions with respect to $L_q(\mathbb{R}^N, \mu)$–norms.
While the upper estimates for the small deviation probabilities are proved by purely probabilistic methods, the lower bounds are established by analytic tools concerning Kolmogorov and entropy numbers of Hölder operators

Key words: random fields, Gaussian processes, fractional Brownian motion, fractal measures, self–similar measures, small deviations, Kolmogorov numbers, metric entropy, Hölder operators

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1 Introduction

The aim of the present paper is the investigation of the small deviation behavior of Gaussian random fields in the L_q–norm taken with respect to a rather arbitrary measure on $\mathbb{R}^N$. Namely, for a Gaussian random field $(X(t), t \in \mathbb{R}^N)$, for a measure μ on $\mathbb{R}^N$, and for any $q \in [1, \infty)$ we are interested in the behavior of the small deviation function

$$\varphi_{q,\mu}(\varepsilon) := -\log \mathbb{P}\left(\int_{\mathbb{R}^N} |X(t)|^q \, \mathrm{d}\mu(t) < \varepsilon^q\right) , \tag{1.1}$$

as $\varepsilon \to 0$ in terms of certain quantitative properties of the underlying measure μ. Let us illustrate this with an example. As a consequence of our estimates, we get the following corollary for the N–parameter fractional Brownian motion $W_H = (W_H(t), t \in \mathbb{R}^N)$ of Hurst index $H \in (0,1)$. For the exact meaning of the theorem, see Section 5.

Theorem 1.1. *Let $T \subset \mathbb{R}^N$ be a compact self–similar set of Hausdorff dimension $D > 0$ and let μ be the D-dimensional Hausdorff measure on T. Then for all $1 \le q < \infty$ and $0 < H < 1$ it follows that*

$$-\log \mathbb{P}\left(\int_T |W_H(t)|^q \, \mathrm{d}\mu(t) < \varepsilon^q\right) \approx \varepsilon^{-D/H} .$$

General small deviation problems attracted much attention during the last years due to their deep relations to various mathematical topics such as operator theory, quantization, strong limit laws in statistics, etc, see the surveys (12; 14). A more specific motivation for this work comes from (17), where the one–parameter case $N = 1$ was considered for fractional Brownian motions and Riemann–Liouville processes.

Before stating our main multi–parameter results, let us recall a basic theorem from (17), thus giving a clear idea of the entropy approach to small deviations in more general L_q–norms.

Recall that the (one–parameter) fractional Brownian motion (fBm) W_H with Hurst index $H \in (0,1)$ is a centered Gaussian process on $\mathbb{R}$ with a.s. continuous paths and covariance

$$\mathbb{E}\, W_H(t)\, W_H(s) = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\} , \quad t, s \in \mathbb{R}.$$

We write $f \sim g$ if $\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1$ while $f \preceq g$ (or $g \succeq f$) means that $\limsup_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty$. Finally, $f \approx g$ says that $f \succeq g$ as well as $g \succeq f$.

If $\mu = \lambda_1$, the restriction of the Lebesgue measure to $[0,1]$, then for W_H the behavior of $\varphi_{q,\mu}(\varepsilon)$ is well–known, namely, $\varphi_{q,\mu}(\varepsilon) \sim c_{q,H}\, \varepsilon^{-1/H}$, as $\varepsilon \to 0$. The exact value of the finite and positive constant $c_{q,H}$ is known only in few cases; sometimes a variational representation for $c_{q,H}$ is available. See more details in (12) and (18).

If μ is absolutely continuous with respect to λ_1, the behavior of $\varphi_{q,\mu}(\varepsilon)$ was investigated in (10), (15) and (16). Under mild assumptions, the order $\varepsilon^{-1/H}$ remains unchanged, only an extra factor depending on the density of μ (with respect to λ_1) appears. The situation is completely different for measures μ being singular to λ_1. This question was recently investigated in (20) for $q = \infty$ (here only the size of the support of μ is of importance) and in (24) for self–similar measures and $q = 2$. When passing from $q = \infty$ to a finite q, the problem becomes more involved because in the latter case the distribution of the mass of μ becomes important. Consequently, one has to introduce some kind of entropy of μ taking into account the size of its support as well as the distribution of the mass on $[0,1]$. This is done in the following way.

Let μ be a continuous measure on $[0,1]$, let $H>0$ and $q\in[1,\infty)$. We define a number $r>0$ by

$$1/r := H + 1/q\,. \tag{1.2}$$

Given an interval $\Delta\subseteq[0,1]$, we denote

$$J_\mu^{(H,q)}(\Delta) := |\Delta|^H\cdot\mu(\Delta)^{1/q} \tag{1.3}$$

and set

$$\sigma_\mu^{(H,q)}(n) := \inf\left\{\left(\sum_{j=1}^n J_\mu(\Delta_j)^r\right)^{1/r} \;:\; [0,1]\subseteq\bigcup_{j=1}^n\Delta_j\right\}\,, \tag{1.4}$$

where the Δ_j's are supposed to be intervals on the real line. The sequence $\sigma_\mu^{(H,q)}(n)$ may be viewed as some kind of outer *mixed* entropy of μ. Here "mixed" means that we take into account the measure as well as the length of an interval.

The main result of (17) shows a very tight relation between the behavior of $\sigma_\mu^{(H,q)}(n)$, as $n\to\infty$, and of the small deviation function (1.1). More precisely, the following is true.

Theorem 1.2. *Let μ be a finite continuous measure on $[0,1]$ and let W_H be a fBm of Hurst index $H\in(0,1)$. For $q\in[1,\infty)$, define $\sigma_\mu^{(H,q)}(n)$ as in (1.4).*

(a) *If*

$$\sigma_\mu^{(H,q)}(n)\succeq n^{-\nu}(\log n)^\beta$$

for certain $\nu\geq 0$ and $\beta\in\mathbb{R}$, then

$$-\log\mathbb{P}\{\|W_H\|_{L_q([0,1],\mu)}<\varepsilon\}\succeq\varepsilon^{-1/(H+\nu)}\cdot\log(1/\varepsilon)^{\beta/(H+\nu)}\,.$$

(b) *On the other hand, if*

$$\sigma_\mu^{(H,q)}(n)\preceq n^{-\nu}(\log n)^\beta$$

then

$$-\log\mathbb{P}\{\|W_H\|_{L_q([0,1],\mu)}<\varepsilon\}\preceq\varepsilon^{-1/(H+\nu)}\cdot\log(1/\varepsilon)^{\beta/(H+\nu)}\,. \tag{1.5}$$

Remarkably, there is another quantity, a kind of inner mixed entropy, equivalent to $\sigma_\mu^{(H,q)}(n)$ in the one–parameter case. This one is defined as follows. Given μ as before, for each $n\in\mathbb{N}$ we set

$$\delta_\mu^{(H,q)}(n) := \sup\left\{\delta>0 : \exists\,\Delta_1,\ldots,\Delta_n\subset[0,1],\; J_\mu^{(H,q)}(\Delta_i)\geq\delta\right\} \tag{1.6}$$

where the Δ_i are supposed to possess disjoint interiors.

It is shown in (17) that $\sigma_\mu^{(H,q)}(n)$ and $n^{1/r}\delta_\mu^{(H,q)}(n)$ are, in a sense, equivalent as $n\to\infty$, namely, it is proved that for each integer $n\geq1$, we have

$$\sigma_\mu^{(H,q)}(2n+1)\leq(2n+1)^{1/r}\,\delta_\mu^{(H,q)}(n)\quad\text{and}\quad n^{1/r}\,\delta_\mu^{(H,q)}(2n)\leq\sigma_\mu^{(H,q)}(n)\,. \tag{1.7}$$

Therefore, Theorem 1.2 can be immediately restated in terms of $\delta_\mu^{(H,q)}(n)$. For example, $\delta_\mu^{(H,q)}(n) \approx n^{-(1/q+1/a)}(\log n)^\beta$ with $a \le 1/H$ is equivalent to $\sigma_\mu^{(H,q)}(n) \approx n^{-(1/a-H)}(\log n)^\beta$ and thus to

$$-\log \mathbb{P}\{\|W_H\|_{L_q([0,1],\mu)} < \varepsilon\} \approx \varepsilon^{-a} \cdot (\log(1/\varepsilon))^{a\beta} .$$

Notice that in the case of measures on $[0,1]$ the restriction $a \le 1/H$ is natural; it is attained for the Lebesgue measure.

To our great deception, we did not find in the literature the notions of outer and inner mixed entropy as defined above, although some similar objects do exist: cf. the notion of weighted Hausdorff measures investigated in (22), pp. 117–120, or C-structures associated with metrics and measures in Pesin (27), p. 49, or multifractal generalizations of Hausdorff measures and packing measures in Olsen (26). Yet the quantitative properties of Hausdorff dimension and entropy of a set seem to have almost nothing in common (think of any countable set — its Hausdorff dimension is zero while the entropy properties can be quite non–trivial).

2 Main results in the multi–parameter case

Although in this article we are mostly interested in the behavior of the N–parameter fractional Brownian motion, an essential part of our estimates is valid for much more general processes. For example, to prove lower estimates for $\varphi_{q,\mu}(\varepsilon)$ we only need a certain non-degeneracy of interpolation errors (often called "non-determinism") while for upper estimates of $\varphi_{q,\mu}(\varepsilon)$ some Hölder type inequality suffices. Therefore, we start from the general setting. Let $X := (X(t), t \in T)$ be a centered measurable Gaussian process on a metric space (T, ρ). Here we endow T with the σ–algebra of Borel sets. For $t \in T$, any $A \subset T$ and $\tau > 0$ we set

$$v(t,\tau) := (\mathrm{Var}\,[X(t) - \mathbb{E}\,(X(t)\,|\,X(s),\ \rho(s,t) \ge \tau)])^{1/2} \tag{2.1}$$

and

$$v(A,\tau) := \inf_{t \in A} v(t,\tau) . \tag{2.2}$$

Note that $v(t,\tau)$ as well as $v(A,\tau)$ are obviously non–decreasing functions of $\tau > 0$.

We suppose that μ is a finite Borel measure on T and that $A_1, \ldots, A_n$ are disjoint measurable subsets of T. Let

$$v_1 := \inf_{t \in A_1} (\mathrm{Var} X(t))^{1/2} \cdot \mu(A_1)^{1/q},$$

and

$$v_i := v\big(A_i, \tau_i\big) \cdot \mu(A_i)^{1/q}, \qquad 2 \le i \le n\,, \tag{2.3}$$

where $\tau_i := \mathrm{dist}(A_i, \bigcup_{k=1}^{i-1} A_k)$. We set

$$V_\mu := V_\mu(A_1, \ldots, A_n) := \min_{1 \le i \le n} v_i. \tag{2.4}$$

Finally, given $n \in \mathbb{N}$ we define some kind of weighted inner entropy by

$$\delta_\mu(n) := \sup\left\{\delta > 0 : \exists \text{ disjoint } A_1, \ldots, A_n \subset T,\ V_\mu(A_1, \ldots, A_n) \ge \delta\right\} . \tag{2.5}$$

In this quite general setting we shall prove the following.

Theorem 2.1. *Let μ be a finite measure on T and let X be a measurable centered Gaussian random field on T. Let $q \in [1,\infty)$ and define $\delta_\mu(n)$ as in (2.5). If*

$$\delta_\mu(n) \succeq n^{-1/q-1/a} (\log n)^\beta$$

for certain $a > 0$ and $\beta \in \mathbb{R}$, then

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-a} \cdot \log(1/\varepsilon)^{a\beta} .$$

For the case $T \subset \mathbb{R}^N$ ($N \geq 1$) with the metric ρ generated by the Euclidean distance, i.e., $\rho(t,s) = |t-s|$, $t,s \in \mathbb{R}^N$, we give a slightly weaker upper bound for the small deviation probabilities. This bound, however, has the advantage of using simpler geometric characteristics. In particular, we do not need to care about the distances between the sets. If $A \subset T$ is measurable, we set

$$\overline{v}(A) := v\left(A, \tau_A\right)$$

where $\tau_A := \mathrm{diam}(A)/(2\sqrt{N})$, i.e.,

$$\overline{v}(A) = \inf_{t\in A} \left(\mathrm{Var}\left[X(t) - \mathbb{E}\left(X(t)\,|\,X(s),\ s\in T,\ 2\sqrt{N}\,|t-s| \geq \mathrm{diam}(A)\right)\right]\right)^{1/2} .$$

Given cubes $Q_1, \ldots, Q_n$ in T with disjoint interiors, similarly as in (2.4), we define the quantity

$$\overline{V}_\mu = \overline{V}_\mu(Q_1,\ldots,Q_n) := \inf_{1\leq i\leq n} \overline{v}(Q_i) \cdot \mu(Q_i)^{1/q}$$

and as in (2.5) we set

$$\overline{\delta}_\mu(n) := \sup\left\{\delta > 0 : \exists\, Q_1,\ldots,Q_n \subset T,\ \overline{V}_\mu(Q_1,\ldots,Q_n) \geq \delta\right\} \tag{2.6}$$

where the cubes Q_i are supposed to possess disjoint interiors.

We shall prove the following.

Theorem 2.2. *Let μ be a finite measure on $T \subset \mathbb{R}^N$ and let X be a centered Gaussian random field on T. For $q \in [1,\infty)$ and $\overline{\delta}_\mu(n)$ defined as in (2.6), if*

$$\overline{\delta}_\mu(n) \succeq n^{-1/q-1/a} (\log n)^\beta$$

for certain $a > 0$ and $\beta \in \mathbb{R}$, then

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-a} \cdot \log(1/\varepsilon)^{a\beta} .$$

Finally we apply our results to the N–parameter fractional Brownian motion, i.e., to the real–valued centered Gaussian random field $W_H := (W_H(t),\, t \in \mathbb{R}^N)$ with covariance

$$\mathbb{E}\left[W_H(s)W_H(t)\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \qquad (s,t) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $H \in (0,1)$ is the Hurst index.

It is known that W_H satisfies (see (29) and (33) for further information about processes satisfying similar conditions)

$$\mathrm{Var}\,[W_H(t) - \mathbb{E}\,(W_H(t)\,|\,W_H(s),\, |s-t| \geq \tau)] \geq c\,\tau^{2H}, \qquad t \in \mathbb{R}^N, \tag{2.7}$$

for all $0 \leq \tau \leq |t|$. Thus, in view of (2.7) it is rather natural to adjust the definition of $\overline{\delta}_\mu$ as follows. Namely, as in (1.3) for $N = 1$, we set

$$J_\mu^{(H,q)}(A) := (\mathrm{diam}(A))^H\,\mu(A)^{1/q} \tag{2.8}$$

for any measurable subset $A \subset \mathbb{R}^N$. Then the multi–parameter extension of (1.6) is

$$\delta_\mu^{(H,q)}(n) := \sup\left\{\delta > 0 : \exists\, Q_1, \ldots, Q_n \subset T,\ J_\mu^{(H,q)}(Q_i) \geq \delta\right\}$$

where the cubes Q_i are supposed to possess disjoint interiors.

Here we shall prove the following result.

Theorem 2.3. *Let μ be a measure on a bounded set $T \subset \mathbb{R}^N$ and let W_H be an N–parameter fractional Brownian motion with Hurst parameter H. Let $q \in [1,\infty)$. If*

$$\delta_\mu^{(H,q)}(n) \succeq n^{-1/q-1/a}\,(\log n)^\beta$$

for certain $a > 0$ and $\beta \in \mathbb{R}$, then

$$-\log \mathbb{P}\{\|W_H\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-a} \cdot \log(1/\varepsilon)^{a\beta}\ .$$

Note that this result does not follow from Theorem 2.2 directly, since inequality (2.7) only holds for $0 \leq \tau \leq |t|$. But we will show that the proof, based on Theorem 2.2, is almost immediate.

We now turn to lower estimates of $\varphi_{q,\mu}(\varepsilon)$. Let μ be a finite compactly supported Borel measure on $\mathbb{R}^N$. For a bounded measurable set $A \subset \mathbb{R}^N$ the quantity $J_\mu^{(H,q)}(A)$ was introduced in (2.8). Furthermore, for $H \in (0,1]$ and $q \in [1,\infty)$ the number r is now (compare with (1.2)) defined by

$$\frac{1}{r} := \frac{H}{N} + \frac{1}{q}\ .$$

Finally, for $n \in \mathbb{N}$, as in (1.4) we set

$$\sigma_\mu^{(H,q)}(n) := \inf\left\{\left(\sum_{j=1}^n J_\mu^{(H,q)}(A_j)^r\right)^{1/r} : \ T \subseteq \bigcup_{j=1}^n A_j\right\} \tag{2.9}$$

where the A_j's are compact subsets of $\mathbb{R}^N$ and T denotes the support of μ. With this notation we shall prove the following multi–parameter extension of (1.5).

Theorem 2.4. *Let $X := (X(t),\, t \in T)$ be a centered Gaussian random field indexed by a compact set $T \subset \mathbb{R}^N$ and satisfying*

$$\mathbb{E}\,|X(t) - X(s)|^2 \le c\,|t-s|^{2H}\,, \quad t,s \in T\,,$$

for some $0 < H \le 1$. If μ is a finite measure with support in T such that for certain $q \in [1,\infty)$, $\nu \ge 0$ and $\beta \in \mathbb{R}$

$$\sigma_\mu^{(H,q)}(n) \preceq n^{-\nu}\,(\log n)^\beta\,,$$

then

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} \le \varepsilon\} \preceq \varepsilon^{-a}\,\log(1/\varepsilon)^{a\beta}$$

where $1/a = \nu + H/N$.

Problem: Of course, Theorem 2.4 applies in particular to W_H. Recall that we have $\mathbb{E}(|W_H(t) - W_H(s)|^2) = |t-s|^{2H}$, $t,s \in \mathbb{R}^N$. Yet for general measures μ and $N > 1$ we do not know how the quantities $\sigma_\mu^{(H,q)}$ and $\delta_\mu^{(H,q)}$ are related (recall (1.7) for $N = 1$). Later on we shall prove a relation similar to (1.7) for a special class of measures on $\mathbb{R}^N$, the so–called self–similar measures. But in the general situation the following question remains open. Let $N > 1$. Does as in Theorem 1.2 for $N = 1$

$$\sigma_\mu^{(H,q)}(n) \succeq n^{-\nu}\,(\log n)^\beta$$

for certain $\nu \ge 0$ and $\beta \in \mathbb{R}$ always imply

$$-\log \mathbb{P}\{\|W_H\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-a} \cdot \log(1/\varepsilon)^{a\beta}$$

with a as in Theorem 2.4?

The rest of the paper is organized as follows. Section 3 is devoted to the study of upper estimates for small deviation probabilities, where Theorems 2.1, 2.2 and 2.3 are proved. In Section 4, we are interested in lower estimates for small deviation probabilities, and prove Theorem 2.4. Section 5 focuses on the case of self–similar measures. Finally, in Section 6 we discuss the L_∞–norm.

3 Upper estimates for small deviation probabilities

This section is divided into four distinct parts. The first three parts are devoted to the proof of Theorems 2.1, 2.2 and 2.3, respectively. The last part contains some concluding remarks.

3.1 Proof of Theorem 2.1

To prove Theorem 2.1, we shall verify the following quite general upper estimate for small deviation probabilities. As in the formulation of Theorem 2.1, let $X = (X(t), t \in T)$ be a measurable centered Gaussian process on a metric space (T,ρ), let μ be a finite Borel measure on T and for disjoint measurable subsets $A_1,\ldots,A_n$ in T the quantity $V_\mu = V_\mu(A_1,\ldots,A_n)$ is as in (2.4).

Proposition 3.1. *There exist a constant $c_1 \in (0,\infty)$ depending only on q and a numerical constant $c_2 \in (0,\infty)$ such that*

$$\mathbb{P}\left(\|X\|^q_{L_q(T,\mu)} \le c_1\, n\, V_\mu^q\right) \le \mathrm{e}^{-c_2\, n}.$$

Proof: For the sake of clarity, the proof is divided into three distinct steps.

Step 1. Reduction to independent processes. We define the predictions $\widehat{X}_1(t) := 0$, $t \in A_1$, and $\widehat{X}_i(t) := \mathbb{E}\{X(t)\,|\,X(s),\, s \in \cup_{k=1}^{i-1} A_k\}$, $t \in A_i$ (for $2 \le i \le n$). The prediction errors are

$$X_i(t) := X(t) - \widehat{X}_i(t), \qquad t \in A_i,\ 1 \le i \le n.$$

It is easy to see that $(X_i(t), t \in A_i)_{1\le i\le n}$ are n independent processes: for any $1 \le i \le n$, the random variable $X_i(t)$ is orthogonal to the span of $(X(s),\, s \in \cup_{k=1}^{i-1} A_k)$ for any $t \in A_i$, whereas all the random variables $X_j(u)$, $u \in A_j$, $j < i$, belong to this span.

The main ingredient in Step 1 is the following inequality.

Lemma 3.2. *For any $\varepsilon > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^n \int_{A_i} |X(t)|^q \,\mathrm{d}\mu(t) \le \varepsilon\right) \le \mathbb{P}\left(\sum_{i=1}^n \int_{A_i} |X_i(t)|^q \,\mathrm{d}\mu(t) \le \varepsilon\right).$$

Proof of Lemma 3.2: There is nothing to prove if $n = 1$. Assume $n > 1$. Let

$$\begin{aligned}
\mathcal{F}_{n-1} &:= \sigma\left(X(s),\, s \in \bigcup_{i=1}^{n-1} A_i\right),\\
S_{n-1} &:= \sum_{i=1}^{n-1} \int_{A_i} |X(t)|^q \,\mathrm{d}\mu(t),\\
U_n &:= \int_{A_n} |X(t)|^q \,\mathrm{d}\mu(t).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^n \int_{A_i} |X(t)|^q \,\mathrm{d}\mu(t) \le \varepsilon\right) &= \mathbb{P}\left(S_{n-1} + U_n \le \varepsilon\right)\\
&= \mathbb{E}\left\{\mathbb{P}\left(S_{n-1} + U_n \le \varepsilon \,\middle|\, \mathcal{F}_{n-1}\right)\right\}. \qquad (3.1)
\end{aligned}$$

By definition, $U_n = \int_{A_n} |X_n(t) + \widehat{X}_n(t)|^q \,\mathrm{d}\mu(t) = \|\widehat{X}_n + X_n\|^q_{L_q(A_n,\mu)}$. Observe that $(\widehat{X}_n(t),\, t \in A_n)$ and S_{n-1} are $\mathcal{F}_{n-1}$-measurable, whereas $(X_n(t),\, t \in A_n)$ is independent of $\mathcal{F}_{n-1}$. Therefore, by Anderson's inequality (see (1) or (13)),

$$\begin{aligned}
\mathbb{P}\left(S_{n-1} + U_n \le \varepsilon \,\middle|\, \mathcal{F}_{n-1}\right) &\le \mathbb{P}\left(\|X_n\|^q_{L_q(A_n,\mu)} \le (\varepsilon - S_{n-1})_+ \,\middle|\, \mathcal{F}_{n-1}\right)\\
&= \mathbb{P}\left(\|X_n\|^q_{L_q(A_n,\mu)} + S_{n-1} \le \varepsilon \,\middle|\, \mathcal{F}_{n-1}\right).
\end{aligned}$$

Plugging this into (3.1) yields that

$$\mathbb{P}\left(\sum_{i=1}^{n}\int_{A_i}|X(t)|^q\,\mathrm{d}\mu(t)\le\varepsilon\right)\le\mathbb{P}\left(\|X_n\|^q_{L_q(A_n,\mu)}+S_{n-1}\le\varepsilon\right).$$

Since the process $(X_n(t),\, t\in A_n)$ and the random variable S_{n-1} are independent, Lemma 3.2 follows by induction. □

Step 2. Evaluation of independent processes. In this step, we even do not use the specific definition of the processes $X_i(\cdot)$.

Lemma 3.3. *Let $(X_i(t),\, t\in A_i)_{1\le i\le n}$ be independent centered Gaussian processes defined on disjoint subsets $(A_i)_{1\le i\le n}$ of T. Then*

$$\mathbb{P}\left(\sum_{i=1}^{n}\int_{A_i}|X_i(t)|^q\,\mathrm{d}\mu(t)\le c_1\, n\widetilde{V}_\mu^q\right)\le \mathrm{e}^{-c_2\, n},$$

where c_1 depends only on q, c_2 is a numerical constant, and

$$\widetilde{V}_\mu := \min_{1\le i\le n}\inf_{t\in A_i}\{\mathrm{Var}(X_i(t))\}^{1/2}\,\mu(A_i)^{1/q}. \tag{3.2}$$

Proof of Lemma 3.3: Write

$$Y_i := \int_{A_i}|X_i(t)|^q\,\mathrm{d}\mu(t),\qquad 1\le i\le n,$$

which are independent random variables. We reduce $\sum_{i=1}^n Y_i$ to a sum of *Bernoulli* random variables. Let $S_i := Y_i^{1/q}$ and $m_i := \mathrm{median}(S_i)$, $1\le i\le n$. Consider random variables

$$B_i := \mathbf{1}_{\{Y_i\ge m_i^q\}},\qquad 1\le i\le n.$$

Since m_i^q is a median for Y_i, we have $\mathbb{P}(B_i=0)=\mathbb{P}(B_i=1)=1/2$. In other words, $(B_i,\, 1\le i\le n)$ is a collection of i.i.d. Bernoulli random variables.
Since $Y_i\ge m_i^q B_i$, we have, for any $x>0$,

$$\mathbb{P}\left(\sum_{i=1}^{n}Y_i\le x\right)\le\mathbb{P}\left(\sum_{i=1}^{n}m_i^qB_i\le x\right)\le\mathbb{P}\left(\sum_{i=1}^{n}B_i\le\frac{x}{\min_{1\le i\le n}m_i^q}\right).$$

In order to evaluate $\min_{1\le i\le n}m_i^q$, we use the following general result.

Fact 3.1. *Let $(X(t),\, t\in T)$ be a Gaussian random process. Assume that $S:=\sup_{t\in T}|X(t)|<\infty$ a.s. Let m be a median of the distribution of S. Then*

$$m\le\mathbb{E}(S)\le c\,m,$$

where $c:=1+\sqrt{2\pi}$.

The first inequality in Fact 3.1 is in Lifshits (13), p. 143, the second in Ledoux and Talagrand (9), p. 58.

Let us complete the proof of Lemma 3.3. By Fact 3.1, we have $m_i \geq c^{-1}\,\mathbb{E}(S_i) = c^{-1}\,\mathbb{E}(\|X_i\|_{L_q(A_i,\mu)})$. Recall (Ledoux and Talagrand (9), p. 60) that there exists a constant $c_q \in (0,\infty)$, depending only on q, such that $\mathbb{E}(\|X_i\|_{L_q(A_i,\mu)}) \geq c_q\{\mathbb{E}(\|X_i\|^q_{L_q(A_i,\mu)})\}^{1/q}$ and that $\mathbb{E}(|X_i(t)|^q) \geq c_q^q\,\{\mathrm{Var}(X_i(t))\}^{q/2}$, $t \in A_i$. Accordingly,

$$\begin{aligned} m_i^q &\geq c^{-q}c_q^q\,\mathbb{E}\left(\|X_i\|^q_{L_q(A_i,\mu)}\right) \\ &= c^{-q}c_q^q \int_{A_i} \mathbb{E}(|X_i(t)|^q)\;\mathrm{d}\mu(t) \\ &\geq c^{-q}c_q^{2q}\,\mu(A_i)\inf_{t\in A_i}\{\mathrm{Var}(X_i(t))\}^{q/2}. \end{aligned}$$

Thus $\min_{1\leq i\leq n} m_i^q \geq c^{-q}c_q^{2q}\,\widetilde{V}_\mu^q$. It follows that

$$\mathbb{P}\left(\sum_{i=1}^n Y_i \leq x\right) \leq \mathbb{P}\left(\sum_{i=1}^n B_i \leq \frac{x}{c^{-q}c_q^{2q}\,\widetilde{V}_\mu^q}\right).$$

Taking $x := \frac{c^{-q}c_q^{2q}}{3}\,\widetilde{V}_\mu^q n$, we obtain, by Chernoff's inequality,

$$\mathbb{P}\left(\sum_{i=1}^n Y_i \leq \frac{c^{-q}c_q^{2q}}{3}\,\widetilde{V}_\mu^q n\right) \leq \mathbb{P}\left(\sum_{i=1}^n B_i \leq \frac{n}{3}\right) \leq \mathrm{e}^{-c_2 n}.$$

Lemma 3.3 is proved, and Step 2 completed. $\square$

Step 3. Final calculations. We apply the result of Step 2 to the processes $(X_i(t),\, t \in A_i)$ constructed in Step 1. For any $2 \leq i \leq n$ and any $t \in A_i$, we have

$$\begin{aligned} \mathrm{Var}\,(X_i(t)) &= \mathrm{Var}\left(X(t) - \widehat{X}_i(t)\right) \\ &= \mathrm{Var}\left[X(t) - \mathbb{E}\left(X(t)\,\Big|\, X(s),\, s \in \bigcup_{k=1}^{i-1} A_k\right)\right] \\ &\geq \mathrm{Var}\left[X(t) - \mathbb{E}\left(X(t)\,\Big|\, X(s),\, \rho(s,t) \geq \mathrm{dist}(t, \bigcup_{k=1}^{i-1} A_k)\right)\right] \\ &\geq \mathrm{Var}\left[X(t) - \mathbb{E}\left(X(t)\,\Big|\, X(s),\, \rho(s,t) \geq \mathrm{dist}(A_i, \bigcup_{k=1}^{i-1} A_k)\right)\right] \\ &= v\,(t,\tau_i)^2, \end{aligned}$$

where $v(t,\cdot)$ is as in (2.1) and τ_i as in (2.3). Therefore, letting v_i be as in (2.3), we get

$$v_i \leq \inf_{t\in A_i}\{\mathrm{Var}(X_i(t))\}^{1/2}\,\mu(A_i)^{1/q}, \qquad 2 \leq i \leq n.$$

Moreover, from $X(t) = X_1(t)$, $t \in A_1$, it follows that

$$v_1 = \inf_{t \in A_1} \{\mathrm{Var}(X_1(t))\}^{1/2} \mu(A_1)^{1/q}$$

as well. Hence, by (3.2) we get $V_\mu = V_\mu(A_1, \ldots, A_n) \le \widetilde{V}_\mu$. It follows from Lemmas 3.2 and 3.3 that

$$\begin{aligned}
\mathbb{P}\left(\|X\|^q_{L_q(T,\mu)} \le c_1\, n V_\mu^q\right) &\le \mathbb{P}\left(\sum_{i=1}^n \int_{A_i} |X(t)|^q \, \mathrm{d}\mu(t) \le c_1\, n V_\mu^q\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^n \int_{A_i} |X_i(t)|^q \, \mathrm{d}\mu(t) \le c_1\, n V_\mu^q\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^n \int_{A_i} |X_i(t)|^q \, \mathrm{d}\mu(t) \le c_1\, n \widetilde{V}_\mu^q\right) \le \mathrm{e}^{-c_2\, n}.
\end{aligned}$$

This completes Step 3, and thus the proof of Proposition 3.1. □

Proof of Theorem 2.1: By assumption there is a constant $c > 0$ such that

$$\delta_n := c\, n^{-1/q-1/a} (\log n)^\beta < \delta_\mu(n) , \quad n \in \mathbb{N} .$$

Consequently, in view of the definition of $\delta_\mu(n)$ there exist disjoint measurable subsets $A_1, \ldots, A_n$ in T with $V_\mu = V_\mu(A_1, \ldots, A_n) \ge \delta_n$. From Proposition 3.1, we derive

$$\mathbb{P}\{\|X\|_{L_q(T,\mu)} \le c_1^{1/q}\, n^{1/q}\, \delta_n\} \le \mathbb{P}\{\|X\|_{L_q(T,\mu)} \le c_1^{1/q}\, n^{1/q}\, V_\mu\} \le \mathrm{e}^{-c_2 n} . \tag{3.3}$$

Letting $\varepsilon = c_1^{1/q}\, n^{1/q}\, \delta_n = c_1^{1/q}\, c\, n^{-1/a} (\log n)^\beta$, it follows that $c_2\, n \succeq \varepsilon^{-a} \log(1/\varepsilon)^{a\beta}$, hence (3.3) completes the proof of Theorem 2.1. □

3.2 Proof of Theorem 2.2

Proof of Theorem 2.2: This follows from Theorem 2.1 and the next proposition.

Proposition 3.4. *Let $T \subset \mathbb{R}^N$ and let μ be a finite measure on T. Then for $n \in \mathbb{N}$,*

$$\overline{\delta}_\mu(n) \le 2^{N/q}\, \delta_\mu\big([2^{-N} n]\big)$$

where, as usual, $[x]$ denotes the integer part of a real number x.

Proof: Let $Q_1, \ldots, Q_n$ be arbitrary cubes in T possessing disjoint interiors. Without loss of generality, we may assume that the diameters of the Q_i are non–increasing. Set $G := \{-1, 1\}^N$. We cut every cube Q_i into a union of 2^N smaller cubes (by splitting each side into two equal pieces):

$$Q_i = \bigcup_{g \in G} Q_i^g .$$

For any $i \le n$, let $g(i) \in G$ be such that

$$\mu(Q_i^{g(i)}) \geq \frac{\mu(Q_i)}{2^N}$$

(if the choice is not unique, we choose any one possible value). Let $g \in G$ be such that

$$\#\{i \in [1,n] \cap \mathbb{N} : \ g(i) = g\} \geq \frac{n}{2^N}. \tag{3.4}$$

We write $I_g := \{i \in [1,n] \cap \mathbb{N} : \ g(i) = g\}$, and consider the family of sets $A_i := Q_i^g$, $i \in I_g$. The following simple geometric lemma provides a lower bound for $\mathrm{dist}(A_i,\, A_j)$, $i \neq j$.

Lemma 3.5. *Let $Q^{\pm} := [-1,1]^N$ and $Q^+ := [0,1]^N$. Let $x_1,\, x_2 \in \mathbb{R}^N$ and $r_1,\, r_2 \in \mathbb{R}_+$ be such that the cubes $Q_i := x_i + r_i Q^{\pm}$, $i = 1$ and 2, are disjoint. Then*

$$\mathrm{dist}(x_1 + r_1 Q^+,\, x_2 + r_2 Q^+) \geq \min\{r_1,\, r_2\}.$$

Proof of Lemma 3.5: For any $x \in \mathbb{R}^N$, we write $x = (x^{(1)}, \ldots, x^{(N)})$. Since the cubes $x_1 + r_1 Q^{\pm}$ and $x_2 + r_2 Q^{\pm}$ are disjoint, there exists $\ell \in [1,N] \cap \mathbb{N}$ such that the intervals $[x_1^{(\ell)} - r_1,\, x_1^{(\ell)} + r_1]$ and $[x_2^{(\ell)} - r_2,\, x_2^{(\ell)} + r_2]$ are disjoint (otherwise, there would be a point belonging to both cubes Q_1 and Q_2). Without loss of generality, we assume that $x_1^{(\ell)} + r_1 < x_2^{(\ell)} - r_2$. Then, for any $y_1 \in x_1 + r_1 Q^+$ and $y_2 \in x_2 + r_2 Q^+$, we have

$$|y_2 - y_1| \geq |y_2^{(\ell)} - y_1^{(\ell)}| \geq y_2^{(\ell)} - y_1^{(\ell)} \geq x_2^{(\ell)} - (x_1^{(\ell)} + r_1) \geq r_2 \geq \min\{r_1,\, r_2\},$$

proving the lemma. □

We continue with the proof of Proposition 3.4. It follows from Lemma 3.5 that for any $i > k$ with $i \in I_g$ and $k \in I_g$,

$$\mathrm{dist}(A_i,\, A_k) \geq N^{-1/2} \min\{\mathrm{diam}(A_i),\, \mathrm{diam}(A_k)\} = N^{-1/2}\mathrm{diam}(A_i),$$

(by recalling that the diameters of Q_i are non–increasing). Let i_0 be the minimal element of I_g. Then, for $i \in I_g$, $i > i_0$,

$$\begin{aligned}
v\left(A_i, \mathrm{dist}(A_i, \bigcup_{k \in I_g, k<i} A_k)\right) \mu(A_i)^{1/q} \ &\geq \ v\left(A_i, \frac{\mathrm{diam}(A_i)}{\sqrt{N}}\right) \mu(A_i)^{1/q} \\
&\geq \ v\left(Q_i, \frac{\mathrm{diam}(Q_i)}{2\sqrt{N}}\right) \mu(Q_i)^{1/q}\, 2^{-N/q} \\
&= \ 2^{-N/q}\, \overline{v}(Q_i)\, \mu(Q_i)^{1/q} \\
&\geq \ 2^{-N/q}\, \overline{V}_\mu(Q_1, \ldots, Q_n)\,.
\end{aligned}$$

Similarly, using the inequality $\mathrm{Var}X \geq \mathrm{Var}[X - \mathbb{E}\,(X \mid \mathcal{F})]$ (for any random variable X and any σ-field $\mathcal{F}$), we obtain

$$\begin{aligned}
\inf_{t\in A_{i_0}} (\mathrm{Var}X(t))^{1/2} \cdot \mu(A_{i_0})^{1/q} &\geq v\left(A_{i_0}, \frac{\mathrm{diam}(A_{i_0})}{\sqrt{N}}\right) \mu(A_{i_0})^{1/q} \\
&\geq v\left(Q_{i_0}, \frac{\mathrm{diam}(Q_{i_0})}{2\sqrt{N}}\right) \mu(Q_{i_0})^{1/q}\, 2^{-N/q} \\
&= 2^{-N/q}\, \overline{v}(Q_{i_0})\, \mu(Q_{i_0})^{1/q} \geq 2^{-N/q}\, \overline{V}_\mu(Q_1, \ldots, Q_n) \,.
\end{aligned}$$

Note that the cardinality of I_g which takes the place of the parameter n in δ_μ is, according to (3.4), not smaller than $2^{-N}n$. Hence, since the cubes $Q_1, \ldots, Q_n$ were chosen arbitrarily in T, the proof of Proposition 3.4 follows by the definition of δ_μ and $\overline{\delta}_\mu$ in (2.5) and (2.6), respectively. □

3.3 Proof of Theorem 2.3

Proof of Theorem 2.3: Let $T \subset \mathbb{R}^N$ be a bounded set and let μ be a finite measure on T. We first suppose that T is "far away from zero", i.e., we assume

$$\mathrm{diam}(T) \leq \mathrm{dist}(\{0\}, T) \,. \tag{3.5}$$

By (2.7), for any $t \in T$,

$$v(t,\tau)^2 = \mathrm{Var}\left[W_H(t) - \mathbb{E}\,\left(W_H(t) \mid W_H(s),\, s \in T,\ |s-t| \geq \tau\right)\right] \geq c\, \tau^{2H}$$

for all $\tau \leq \mathrm{dist}(\{0\}, T)$. Consequently, for any cubes $Q_1, \ldots, Q_n$ in T with disjoint interiors, we obtain

$$\overline{V}_\mu(Q_1, \ldots, Q_n) \geq c' \min_{1\leq i\leq n} J_\mu^{(H,q)}(Q_i) \,,$$

hence $\overline{\delta}_\mu(n) \geq c'\, \delta_\mu^{(H,q)}(n)$. Theorem 2.3 follows now from Theorem 2.2.

Next let T be an arbitrary bounded subset of $\mathbb{R}^N$ and μ a finite measure on T. We choose an element $t_0 \in \mathbb{R}^N$ such that $T_0 := T + t_0$ satisfies (3.5). By what we have just proved,

$$-\log \mathbb{P}\{\|W_H\|_{L_q(T_0,\mu_0)} < \varepsilon\} \succeq \varepsilon^{-a} \log(1/\varepsilon)^{a\beta} \,,$$

where $\mu_0 := \mu * \delta_{\{t_0\}}$ ($\delta_{\{t_0\}}$ being the Dirac measure at t_0). Observe that $\|W_H\|_{L_q(T_0,\mu_0)} = \{\int_T |W_H(t+t_0)|^q\, \mathrm{d}\mu(t)\}^{1/q}$. Since $\widetilde{W}_H := (W_H(t+t_0) - W_H(t_0),\, t \in \mathbb{R}^N)$ is an N–parameter fBm as well, we finally arrive at:

$$-\log \mathbb{P}\left\{\int_T \left|\widetilde{W}_H(t) + W_H(t_0)\right|^q \mathrm{d}\mu(t) < \varepsilon^q\right\} \succeq \varepsilon^{-a} \log(1/\varepsilon)^{a\beta} \,.$$

Theorem 2.3 follows from the weak correlation inequality, see (12), proof of Theorem 3.7. □

3.4 Concluding remarks:

Suppose that $v(t,\tau) \geq c\,\tau^H$ and $\mu = \lambda_N$, the N–dimensional Lebesgue measure. Assuming that the interior of T is non–empty, we easily get $\delta_\mu(n) \succeq n^{-(1/q+H/N)}$, hence by Theorem 2.1,

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-N/H}.$$

Our estimates are suited rather well for stationary fields. For the non–stationary ones a logarithmic gap may appear. For example, let X be an N–parameter Brownian sheet with covariance

$$\mathbb{E}\, X(s)X(t) = \prod_{k=1}^{N} \min\{s_k, t_k\}.$$

Then we get $v(t,\tau) \geq c\,\tau^{N/2}$ for all $\tau < \min_{1\leq i\leq N} t_i$. By Theorem 2.2, for the Lebesgue measure and, say, the N–dimensional unit cube T,

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-2},$$

while it is known that in fact

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \approx \varepsilon^{-2}\, \log(1/\varepsilon)^{2N-2}\ .$$

We also note that cubes in (2.6) and in Theorem 2.2 can not be replaced by arbitrary closed convex sets. Indeed, disjoint "flat" sets are not helpful in this context, as the following example shows. Define a probability measure μ_0 on $[0,1]$ by

$$\mu_0 = (1-2^{-h})\ \sum_{k=0}^{\infty}\sum_{i=1}^{2^k} 2^{-k(1+h)}\delta_{\{i/2^k\}},$$

where $h > 0$ and $\delta_{\{x\}}$ stands for the Dirac mass at point x. Define a measure on the unit square T by $\mu = \mu_0 \otimes \lambda_1$. For a fixed k, by taking $Q_i = \{i/2^k\} \times [0,1]$, $1 \leq i \leq 2^k$, we get $n = 2^k$ disjoint sets with $\overline{V}_\mu(Q_1,\ldots,Q_n) \approx 2^{-k(1+h)/q}$, whatever the bound for the interpolation error is. If Theorem 2.2 were valid in this setting, we would get $\overline{\delta}_\mu(n) \succeq n^{-(1+h)/q}$, and

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \succeq \varepsilon^{-q/h},$$

while it is known, for example, for the 2–parameter Brownian motion, that in fact

$$-\log \mathbb{P}\{\|X\|_{L_q(T,\mu)} < \varepsilon\} \preceq \varepsilon^{-4}\ .$$

This would lead to a contradiction whenever $q/h > 4$.

4 Lower estimates for small deviation probabilities

This section is devoted to the study of lower estimates for small deviation probabilities, and is divided into three distinct parts. In the first part, we present some basic functional analytic tools, while in the second part, we establish a result for Kolmogorov numbers of operators with values in $L_q(T,\mu)$. In the third and last part, we prove Theorem 2.4.

4.1 Functional analytic tools

Let $[E, \|\cdot\|_E]$ and $[F, \|\cdot\|_F]$ be Banach spaces and let $u : E \to F$ be a compact operator. There exist several quantities to measure the degree of compactness of u. We shall need two of them, namely, the sequences $d_n(u)$ and $e_n(u)$ of Kolmogorov and (dyadic) entropy numbers, respectively. They are defined by

$$d_n(u) := \inf\{\|Q_{F_0} u\| : F_0 \subseteq F,\, \mathrm{dim} F_0 < n\}$$

where for a subspace $F_0 \subseteq F$ the operator $Q_{F_0} : F \to F/F_0$ denotes the canonical quotient map from F onto F/F_0 . The entropy numbers are given by

$$e_n(u) := \inf\{\varepsilon > 0 : \exists\, y_1, \ldots, y_{2^{n-1}} \in F \text{ with } u(B_E) \subseteq \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_F)\}$$

where B_E and B_F are the closed unit balls of E and F, respectively. We refer to (3) and (28) for more information about these numbers.

Kolmogorov and entropy numbers are tightly related by the following result in (2).

Proposition 4.1. *Let $(b_n)_{n\geq 1}$ be an increasing sequence tending to infinity and satisfying*

$$\sup_{n\geq 1} \frac{b_{2n}}{b_n} := \kappa < \infty\,.$$

Then there is a constant $c = c(\kappa) > 0$ such that for all compact operators u, we have

$$\sup_{n\geq 1} b_n\, e_n(u) \leq c \cdot \sup_{n\geq 1} b_n\, d_n(u)\,.$$

Let (T, ρ) be a compact metric space. Let $C(T)$ denote as usual the Banach space of continuous functions on T endowed with the norm

$$\|f\|_\infty := \sup_{t\in T} |f(t)|\,, \quad f \in C(T)\,.$$

If u is an operator from a Banach space E into $C(T)$, it is said to be H–Hölder for some $0 < H \leq 1$ provided there is a finite constant $c > 0$ such that

$$|(ux)(t_1) - (ux)(t_2)| \leq c \cdot \rho(t_1, t_2)^H \cdot \|x\|_E \tag{4.1}$$

for all $t_1, t_2 \in T$ and $x \in E$. The smallest possible constant c appearing in (4.1) is denoted by $|u|_{\rho,H}$ and we write $|u|_H$ whenever the metric ρ is clearly understood. Basic properties of H–Hölder operators may be found in (3).

Before stating the basic result about Kolmogorov numbers of Hölder operators we need some quantity to measure the size of the compact metric space (T, ρ). Given $n \in \mathbb{N}$, the n–th entropy number of T (with respect to the metric ρ) is defined by

$$\varepsilon_n(T) := \inf\{\varepsilon > 0 : \exists\, n\ \rho\text{–balls of radius } \varepsilon \text{ covering } T\}\,.$$

Now we may formulate Theorem 5.10.1 in (3) which will be crucial later on. We state it in the form as we shall use it.

Theorem 4.2. *Let $\mathcal{H}$ be a Hilbert space and let (T,ρ) be a compact metric space such that*

$$\varepsilon_n(T) \le \kappa \cdot n^{-\nu}\,, \quad n \in \mathbb{N}\,, \tag{4.2}$$

for a certain $\kappa > 0$ and $\nu > 0$. Then, if $u : \mathcal{H} \to C(T)$ is H–Hölder for some $H \in (0,1]$, then

$$d_n(u) \le c \cdot \max\left\{\|u\|\,, |u|_H\right\} \cdot n^{-1/2-H\,\nu}\,, \quad n \in \mathbb{N}\,, \tag{4.3}$$

where $c > 0$ depends on H, ν and κ. Here, $\|u\|$ denotes the usual operator norm of u.

For our purposes it is important to know how the constant c in (4.3) depends on the number κ appearing in (4.2).

Corollary 4.3. *Under the assumptions of Theorem 4.2, it follows that*

$$d_n(u) \le c \cdot \max\left\{\|u\|\,, \kappa^H\ |u|_H\right\} \cdot n^{-1/2-H\,\nu}\,, \quad n \in \mathbb{N}\,,$$

with $c > 0$ independent of κ.

Proof: We set

$$\widetilde{\rho}(t_1,t_2) := \kappa^{-1} \cdot \rho(t_1,t_2)\,, \quad t_1,t_2 \in T\,.$$

If $\widetilde{\varepsilon}_n(T)$ are the entropy numbers of T with respect to $\widetilde{\rho}$, then $\widetilde{\varepsilon}_n(T) = \kappa^{-1} \cdot \varepsilon_n(T)$, hence, by (4.2) we have $\widetilde{\varepsilon}_n(T) \le n^{-\nu}$, for $n \in \mathbb{N}$. Consequently, an application of Theorem 4.2 yields

$$d_n(u) \le c \cdot \max\left\{\|u\|\,, |u|_{\widetilde{\rho},H}\right\} \cdot n^{-1/2-H\,\nu}\,, \quad n \in \mathbb{N}\,, \tag{4.4}$$

where now $c > 0$ is independent of κ. Observe that a change of the metric does not change the operator norm of u. The proof of the corollary is completed by (4.4) and the observation $|u|_{\widetilde{\rho},H} = \kappa^H \cdot |u|_{\rho,H}$. □

4.2 Kolmogorov numbers of operators with values in $L_q(T,\mu)$

We now state and prove the main result of this section. Recall that $\sigma_\mu^{(H,q)}(n)$ was defined in (2.9).

Theorem 4.4. *Let μ be as before a Borel measure on $\mathbb{R}^N$ with compact support T and let u be an H–Hölder operator from a Hilbert space $\mathcal{H}$ into $C(T)$. Then for all $n,m \in \mathbb{N}$ and $q \in [1,\infty)$ we have*

$$d_{n+m}\big(u : \mathcal{H} \to L_q(T,\mu)\big) \le c \cdot |u|_H \cdot \sigma_\mu^{(H,q)}(m) \cdot n^{-H/N-1/2}\,. \tag{4.5}$$

Here $c > 0$ only depends on H, q and N. The Hölder norm of u is taken with respect to the Euclidean distance in $\mathbb{R}^N$.

Proof: Choose arbitrary compact sets $A_1,\ldots,A_m$ covering T, the support of μ. In each A_j we take a fixed element t_j, $1 \le j \le m$, and define operators $u_j : \mathcal{H} \to C(A_j)$ via

$$(u_j h)(t) := (uh)(t) - (uh)(t_j)\,, \quad t \in A_j,\ h \in \mathcal{H}\,.$$

Thus

$$\begin{aligned}\|u_j h\|_\infty &= \sup_{t\in A_j} |(u_j h)(t)| = \sup_{t\in A_j} |(uh)(t) - (uh)(t_j)| \\ &\le |u|_H \cdot \sup_{t\in A_j} |t-t_j|^H \cdot \|h\| \le \operatorname{diam}(A_j)^H \cdot |u|_H \cdot \|h\| \ ,\end{aligned}$$

i.e., the operator norm of u_j can be estimated by

$$\|u_j\| \le \operatorname{diam}(A_j)^H \cdot |u|_H \ . \tag{4.6}$$

Of course,

$$|u_j|_H \le |u|_H \ , \tag{4.7}$$

and, moreover, since $A_j \subseteq \mathbb{R}^N$,

$$\varepsilon_n(A_j) \le c \cdot \operatorname{diam}(A_j) \cdot n^{-1/N} \tag{4.8}$$

with some constant $c > 0$. We do not discuss here whether this constant depends on N or whether it can be chosen independent of the dimension because other parameters of our later estimates depend on N, anyway.

Let

$$w_j := \mu(A_j)^{1/q} \cdot u_j \ , \quad 1 \le j \le m \ . \tag{4.9}$$

An application of Corollary 4.3 with $\nu = 1/N$, together with (4.6), (4.7), (4.8) and (4.9), yields

$$d_n(w_j : \mathcal{H} \to C(A_j)) \le c \cdot |u|_H \cdot \operatorname{diam}(A_j)^H \cdot \mu(A_j)^{1/q} \cdot n^{-H/N-1/2} \ , \quad n \in \mathbb{N} \ . \tag{4.10}$$

Let E^q be the ℓ_q–sum of the Banach spaces $C(A_1), \ldots, C(A_m)$, i.e.,

$$E^q := \big\{(f_j)_{j=1}^m : f_j \in C(A_j)\big\}$$

and

$$\|(f_j)_{j=1}^m\|_{E^q} := \left(\sum_{j=1}^m \|f_j\|_\infty^q\right)^{1/q} \ .$$

Define $w^q : \mathcal{H} \to E^q$ by

$$w^q h := (w_1 h, \ldots, w_m h) \ , \quad h \in \mathcal{H} \ .$$

Proposition 4.2 in (17) applies, and (4.10) leads to

$$d_n(w^q) \le c \cdot |u|_H \cdot \left(\sum_{j=1}^m \operatorname{diam}(A_j)^{Hr} \cdot \mu(A_j)^{r/q}\right)^{1/r} \cdot n^{-H/N-1/2} \tag{4.11}$$

where

$$1/r = (H/N + 1/2) - 1/2 + 1/q = H/N + 1/q \ .$$

To complete the proof, set $B_1 = A_1$ and $B_j = A_j \setminus \bigcup_{i=1}^{j-1} A_i$, $2 \le j \le m$. If the operator Φ from E^q into $L_q(T,\mu)$ is defined by

$$\Phi\big((f_j)_{j=1}^m\big)(t) := \sum_{j=1}^m f_j(t) \cdot \frac{\mathbf{1}_{B_j}(t)}{\mu(A_j)^{1/q}} \ ,$$

then $\|\Phi\| \leq 1$ and

$$\Phi \circ w^q = u - u_0 \tag{4.12}$$

where

$$(u_0 h)(t) = \sum_{j=1}^{m} (uh)(t_j) \mathbf{1}_{B_j}(t)\,, \quad t \in T\,.$$

The operator u_0 from $\mathcal{H}$ into $L_q(T,\mu)$ has rank less or equal than m. Hence $d_{m+1}(u_0) = 0$ and therefore, from algebraic properties of the Kolmogorov numbers and (4.12) and (4.11), it follows that

$$\begin{aligned} d_{n+m}(u : \mathcal{H} \to L_q(T,\mu)) &\leq d_n(\Phi \circ w^q) + d_{m+1}(u_0) \leq d_n(w^q) \cdot \|\Phi\| \\ &\leq c \cdot |u|_H \cdot \left(\sum_{j=1}^{m} \operatorname{diam}(A_j)^{Hr} \cdot \mu(A_j)^{r/q} \right)^{1/r} \cdot n^{-H/N-1/2}\,. \end{aligned}$$

Taking the infimum over all coverings $A_1, \ldots, A_m$ of T yields (4.5). □

Corollary 4.5. *Let μ be a finite measure on $\mathbb{R}^N$ with compact support T and suppose that the operator u from the Hilbert space $\mathcal{H}$ into $C(T)$ is H–Hölder for some $H \in (0,1]$. If*

$$\sigma_\mu^{(H,q)}(n) \leq c \cdot n^{-\nu} \cdot (\log n)^\beta$$

for certain $c > 0$, $\nu \geq 0$ and $\beta \in \mathbb{R}$, then for $n \in \mathbb{N}$ we have

$$e_n(u : \mathcal{H} \to L_q(T,\mu)) \leq c' \cdot |u|_H \cdot n^{-\nu-H/N-1/2} \cdot (\log n)^\beta$$

with some constant $c' = c'(H,q,N,c,\nu,\beta) > 0$.

Proof: Apply Theorem 4.4 with $m = n$. The assertion follows from Proposition 4.1. □

4.3 Proof of Theorem 2.4

We start with some quite general remarks about Gaussian processes (cf. (21)). Let $X := (X(t),\, t \in T)$ be a centered Gaussian process and let us suppose that (T,ρ) is a compact metric space. Under quite mild conditions, e.g., if $\rho(t_n,t) \to 0$ in T implies $\mathbb{E}\,|X(t_n) - X(t)|^2 \to 0$, there are a (separable) Hilbert space $\mathcal{H}$ and an operator $u : \mathcal{H} \to C(T)$ such that

$$\mathbb{E} X(t)X(s) = \langle u^*\delta_t, u^*\delta_s \rangle_{\mathcal{H}} \tag{4.13}$$

where $u^* : C^*(T) \to \mathcal{H}$ denotes the dual operator of u and $\delta_t \in C^*(T)$ is the usual Dirac measure concentrated in $t \in T$. In particular, it follows that

$$\mathbb{E}\,|X(t) - X(s)|^2 = \|u^*\delta_t - u^*\delta_s\|_{\mathcal{H}}^2 = \sup_{\|h\| \leq 1} |(uh)(t) - (uh)(s)|^2\ , \quad t,s \in T\,.$$

Consequently, whenever u and X are related via (4.13), the operator u is H–Hölder if and only if

$$\left(\mathbb{E}\,|X(t) - X(s)|^2 \right)^{1/2} \leq c \cdot \rho(t,s)^H \tag{4.14}$$

for all $t, s \in T$. Moreover, $|u|_H$ coincides with the smallest $c > 0$ for which (4.14) holds.

Proof of Theorem 2.4: We start the proof by recalling a consequence of Theorem 5.1 in (11). Suppose that u and X are related via (4.13). Then for any finite Borel measure μ on T, any $q \in [1, \infty]$, $a > 0$ and $\beta \in \mathbb{R}$ the following are equivalent:

(i) There is a $c > 0$ such that for all $n \geq 1$

$$e_n(u : \mathcal{H} \to L_q(T, \mu)) \leq c \cdot n^{-1/a-1/2} (\log n)^{\beta} .$$

(ii) For some $c > 0$ it is true that

$$-\log \mathbb{P}\left(\|X\|_{L_q(T,\mu)} < \varepsilon\right) \leq c \cdot \varepsilon^{a} \cdot \log(1/\varepsilon)^{a\beta}$$

for all $\varepsilon > 0$.

Taking this into account, Theorem 2.4 is a direct consequence of Corollary 4.5 and the above stated equivalence of (4.14) with the H–Hölder continuity of the corresponding operator u. $\square$

5 Self–similar measures and sets

It is a challenging open problem to obtain suitable estimates for $\sigma_\mu^{(H,q)}$ and/or $\delta_\mu^{(H,q)}$ in the case of arbitrary compactly supported Borel measures μ on $\mathbb{R}^N$. As already mentioned, we even do not know how these quantities are related in the case $N > 1$. Yet if μ is self–similar, then suitable estimates for both of these quantities are available.

Let us briefly recall some basic facts about self–similar measures which may be found in (4) or (5). An affine mapping $S : \mathbb{R}^N \to \mathbb{R}^N$ is said to be a contractive similarity provided that

$$|S(t_1) - S(t_2)| = \lambda \cdot |t_1 - t_2| , \quad t_1, t_2 \in \mathbb{R}^N ,$$

with some $\lambda \in (0, 1)$. The number λ is called the contraction factor of S. Given (contractive) similarities $S_1, \ldots, S_m$ we denote by $\lambda_1, \ldots, \lambda_m$ their contraction factors. There exists a unique compact set $T \subseteq \mathbb{R}^N$ (the self–similar set generated by the S_j's) such that

$$T = \bigcup_{j=1}^{m} S_j(T) .$$

Let furthermore $\rho_1, \ldots, \rho_m > 0$ be weights, i.e., $\sum_{j=1}^m \rho_j = 1$. Then there is a unique Borel probability measure μ on $\mathbb{R}^N$ (μ is called the self–similar measure generated by the similarities S_j and the weights ρ_j) satisfying

$$\mu = \sum_{j=1}^{m} \rho_j \cdot (\mu \circ S_j^{-1}) .$$

Note that T and μ are related via $\operatorname{supp}(\mu) = T$.

We shall suppose that the similarities satisfy the strong open set condition, i.e., we assume that there exists an open bounded set $\Omega \subseteq \mathbb{R}^N$ with $T \cap \Omega \neq \emptyset$ such that

$$\bigcup_{j=1}^{m} S_j(\Omega) \subseteq \Omega \qquad \text{and} \qquad S_i(\Omega) \cap S_j(\Omega) = \emptyset,\ i \neq j\ . \tag{5.1}$$

It is known that then $T \subseteq \overline{\Omega}$; and since $T \cap \Omega \neq \emptyset$, we have $\mu(\Omega) > 0$, hence by the results in (7), we even have $\mu(\Omega) = 1$ and $\mu(\partial\Omega) = 0$. Let us note that under these assumptions, we have

$$\sum_{j=1}^{m} \lambda_j^N \leq 1\ . \tag{5.2}$$

Proposition 5.1. *Let μ be a self–similar measure generated by similarities S_j with contraction factors λ_j and weights ρ_j, $1 \leq j \leq m$. For $H \in (0,1]$ and $q \in [1,\infty)$, let $\gamma > 0$ be the unique solution of the equation*

$$\sum_{j=1}^{m} \lambda_j^{H\gamma} \rho_j^{\gamma/q} = 1\ . \tag{5.3}$$

Then, under the strong open set condition, we have

$$\sigma_\mu^{(H,q)}(n) \leq c \cdot \mathrm{diam}(\Omega)^H \cdot n^{-1/\gamma+1/r}$$

where as before $1/r = H/N + 1/q$.

Proof: By Hölder's inequality and (5.2), we necessarily have $\gamma \leq r$.

We say that α is a word of length p ($p \in \mathbb{N}$) over $\{1, \dots, m\}$, if $\alpha = (i_1, \dots, i_p)$ for certain $1 \leq i_j \leq m$. For each such word, we define (Ω being the set appearing in the open set condition)

$$\begin{aligned} S_\alpha &:= S_{i_1} \circ \cdots \circ S_{i_p}\ , \\ \Omega(\alpha) &:= S_\alpha(\Omega)\ , \\ \Lambda(\alpha) &:= (\lambda_{i_1} \cdots \lambda_{i_p})^H \cdot (\rho_{i_1} \cdots \rho_{i_p})^{1/q}\ . \end{aligned}$$

We need the following estimate.

Lemma 5.2. *For each real number $s > 0$, there exist $\ell = \ell(s)$ words $\alpha_1, \dots, \alpha_{\ell(s)}$ over $\{1, \dots, m\}$ (not necessarily of the same length) such that the following holds:*

$$T \subseteq \bigcup_{i=1}^{\ell(s)} \overline{\Omega(\alpha_i)}\ , \tag{5.4}$$

$$\max_{1 \leq i \leq \ell(s)} \Lambda(\alpha_i) \leq \mathrm{e}^{-s}\ , \tag{5.5}$$

$$\ell(s) \leq c_1 \cdot \mathrm{e}^{\gamma s}\ , \tag{5.6}$$

where γ was defined by (5.3).

We postpone the proof of Lemma 5.2 for a moment, and proceed in the proof of Proposition 5.1. Recall that the strong open set condition implies $\mu(\partial\Omega) = 0$; hence for any word α, we have $\mu(\overline{\Omega(\alpha)}) = \mu(\Omega(\alpha))$. Accordingly,

$$J_\mu^{(H,q)}(\overline{\Omega(\alpha)}) = \Lambda(\alpha) \cdot J_\mu^{(H,q)}(\Omega) = \Lambda(\alpha) \cdot \operatorname{diam}(\Omega)^H \ . \tag{5.7}$$

Let $s > 0$ be given, and let $\alpha_1, \dots, \alpha_{\ell(s)}$ be words over $\{1, \dots, m\}$ satisfying (5.4), (5.5) and (5.6). By (5.7),

$$\sigma_\mu^{(H,q)}(\ell(s)) \le \ell(s)^{1/r} \cdot \mathrm{e}^{-s} \cdot \operatorname{diam}(\Omega)^H \ .$$

Given $n \in \mathbb{N}$, we define $s > 0$ via the equation $c_1 \mathrm{e}^{\gamma s} = n$, where c_1 is the constant in (5.6). Then $\ell(s) \le n$. Note that $n \mapsto \sigma_\mu^{(H,q)}(n)$ is non–increasing (since in the definition of $\sigma_\mu^{(H,q)}(n)$, one or several of the A_j can be empty). Therefore,

$$\sigma_\mu^{(H,q)}(n) \le \sigma_\mu^{(H,q)}(\ell(s)) \le c \cdot n^{1/r} \cdot n^{-1/\gamma} \cdot \operatorname{diam}(\Omega)^H = c \cdot \operatorname{diam}(\Omega)^H \cdot n^{-1/\gamma + 1/r}$$

as asserted. □

Proof of Lemma 5.2: Let $\mathbb{Z}_+^m$ be the set of vectors $x = (x_1, \dots, x_m)$ with $x_j \in \mathbb{Z}$ and $x_j \ge 0$. A path in $\mathbb{Z}_+^m$ of length p is a sequence $[x^0, \dots, x^p]$ with $x^k \in \mathbb{Z}_+^m$. It is said to be admissible provided that $x^0 = 0$ and for every $k \le p$, there exists $j_k \le m$ such that $x_{j_k}^k = x_{j_k}^{k-1} + 1$ while $x_j^k = x_j^{k-1}$ for all $j \in \{1, \dots, m\} \backslash \{j_k\}$. Let $\mathcal{P}$ be the set of all admissible paths of any finite length in $\mathbb{Z}_+^m$.

We define a linear function $L : \mathbb{Z}_+^m \to \mathbb{R}$ by

$$L(x) := \sum_{j=1}^m d_j \, x_j \ , \quad x = (x_1, \dots, x_m) \in \mathbb{Z}_+^m \ ,$$

with $d_j := -\log(\lambda_j^H \cdot \rho_j^{1/q})$, $1 \le j \le m$. Note that L takes values in $[0, \infty)$, and increases along an admissible path. For $s > 0$, let $\mathcal{P}_s$ be the set of those paths $[x^0, \dots, x^p]$ in $\mathcal{P}$ for which $L(x^{p-1}) < s \le L(x^p)$. It was shown in (23) that

$$\#(\mathcal{P}_s) \le c_1 \cdot \mathrm{e}^{\gamma s}$$

for a certain constant $c_1 > 0$ and with γ satisfying

$$1 = \mathrm{e}^{-d_1 \gamma} + \dots + \mathrm{e}^{-d_m \gamma} \ . \tag{5.8}$$

In view of the definition of d_j, the number γ in (5.8) coincides with the one defined in (5.3). To each path $[x^0, \dots, x^p]$ in $\mathbb{Z}_+^m$, we assign a word $\alpha = (j_1, \dots, j_p)$ as follows: j_k is such that $x_{j_k}^k = x_{j_k}^{k-1} + 1$. In this way, we obtain a one–to–one correspondence between $\mathcal{P}$ and the set of finite words over $\{1, \dots, m\}$. Moreover, a path belongs to $\mathcal{P}_s$ if and only if for the corresponding word α we have $\Lambda(\alpha) \le \mathrm{e}^{-s} < \Lambda(\bar{\alpha})$ with $\bar{\alpha} = (j_1, \dots, j_{p-1})$. We hereby set $\Lambda(\bar{\alpha}) = 1$ provided $\bar{\alpha}$ is the empty word. If we enumerate all words α corresponding to paths in $\mathcal{P}_s$, we get $\alpha_1, \dots, \alpha_{\ell(s)}$ with $\ell(s) \le c_1 \cdot \mathrm{e}^{\gamma s}$ and $\Lambda(\alpha_i) \le \mathrm{e}^{-s}$, $1 \le i \le \ell(s)$. Moreover, since $T \subseteq \overline{\Omega}$, it follows that

$$T = \bigcup_{i=1}^{\ell(s)} S_{\alpha_i}(T) \subseteq \bigcup_{i=1}^{\ell(s)} S_{\alpha_i}(\overline{\Omega}) = \bigcup_{i=1}^{\ell(s)} \overline{\Omega(\alpha_i)}$$

which completes the proof. $\square$

As a consequence of Proposition 5.1 and Theorem 4.4, we get the following.

Theorem 5.3. *Let μ be as in Proposition 5.1 with support $T \subset \mathbb{R}^N$. If u is an H–Hölder operator from a Hilbert space $\mathcal{H}$ into $C(T)$, then*

$$d_n(u : \mathcal{H} \to L_q(T,\mu)) \leq c \cdot \mathrm{diam}(\Omega)^H \cdot n^{-1/\gamma+1/q-1/2} ,$$

where γ is defined by equation (5.3). The entropy numbers of u can be estimated in the same way.

Corollary 5.4. *Let μ be a self–similar measure with support $T \subseteq \mathbb{R}^N$ and let $X := (X(t),\, t \in T)$ be a centered Gaussian process satisfying*

$$\mathbb{E}\,|X(t) - X(s)|^2 \leq c \cdot |t-s|^{2H} , \quad t, s \in T ,$$

for some $H \in (0,1]$. Then

$$-\log \mathbb{P}\left(\|X\|_{L_q(T,\mu)} < \varepsilon \right) \preceq \varepsilon^{-\gamma q/(q-\gamma)}$$

with γ defined by (5.3).

Our next aim is to find suitable lower estimates for $\delta_\mu^{(H,q)}(n)$ in the case of self–similar measures μ on $\mathbb{R}^N$. To this end, let Ω be the open bounded set satisfying (5.1), and consider

$$\delta_\mu^{(H,q)}(n) := \sup\left\{ \delta > 0 : \exists\, Q_1, \ldots, Q_n \subset \Omega,\; J_\mu^{(H,q)}(Q_i) \geq \delta \right\}$$

where the cubes Q_i are supposed to possess disjoint interiors.

Proposition 5.5. *Let $\gamma > 0$ be as in (5.3). Then*

$$\delta_\mu^{(H,q)}(n) \succeq n^{-1/\gamma} .$$

Proof: For an open subset $G \subseteq \mathbb{R}^N$ and $\delta > 0$, let $M_\mu = M_\mu(\delta, G)$ be the maximal number of cubes $Q_1, \ldots, Q_{M_\mu}$ in G with disjoint interiors and with $J_\mu^{(H,q)}(Q_j) \geq \delta$, $1 \leq j \leq M_\mu$. For Ω as above, define open sets Ω_i and measures μ_i on Ω_i by

$$\Omega_i := S_i(\Omega) \quad \text{and} \quad \mu_i := \rho_i \cdot (\mu \circ S_i^{-1}) , \quad 1 \leq i \leq m .$$

From (5.1), we derive

$$M_\mu(\delta, \Omega) \geq \sum_{i=1}^m M_{\mu_i}(\delta, \Omega_i) . \tag{5.9}$$

If $Q \subseteq \Omega_i$ is a cube, then by self–similarity, we have $J_{\mu_i}^{(H,q)}(Q) = \lambda_i^H \rho_i^{1/q} J_\mu^{(H,q)}(Q)$, hence $M_{\mu_i}(\delta, \Omega_i) = M_\mu(\beta_i\, \delta, \Omega)$ with $\beta_i := \lambda_i^{-H} \rho_i^{-1/q}$. By (5.9), $M_\mu(\delta, \Omega) \geq \sum_{i=1}^m M_\mu(\beta_i\, \delta, \Omega)$ for all $\delta > 0$. Applying Lemma 5.1 (first part) in (17) with $F(x) = M_\mu(x^{-1/\gamma})$ yields, for any $\delta_0 > 0$,

$$\inf_{\delta \leq \delta_0} \delta^\gamma M_\mu(\delta, \Omega) \geq c\, \delta_0^\gamma\, M_\mu(\delta_0, \Omega) \tag{5.10}$$

where γ is defined as the unique solution of $\sum_{i=1}^m \beta_i^{-\gamma} = 1$ (thus as in (5.3)) and the constant $c > 0$ only depends on $\beta_1, \dots, \beta_m$.

Since Ω is open and $\mu(\Omega) = 1$, there exists a non–empty cube $Q_0 \subseteq \Omega$ such that $\delta_0 := J_\mu^{(H,q)}(Q_0) > 0$. Thus $M_\mu(\delta, \Omega) \geq 1$ for $\delta \leq \delta_0$. In view of (5.10), we have

$$M_\mu(\delta, \Omega) \geq c_0\, \delta^{-\gamma}$$

whenever $0 < \delta \leq \delta_0$. Take $\delta := c_0^{-1/\gamma} n^{-1/\gamma}$. We have proved that when n is sufficiently large, there exist n cubes $Q_1, \dots, Q_n$ in Ω possessing disjoint interiors such that $J_\mu^{(H,q)}(Q_i) \geq \delta$. This completes the proof. $\square$

Remark: Propositions 5.1 and 5.5 imply that

$$n^{1/r} \delta_\mu^{(H,q)}(n) \succeq \sigma_\mu^{(H,q)}(n)$$

for self–similar measures μ.

As a consequence of Theorem 2.3 and Proposition 5.5 we get the following.

Corollary 5.6. *If μ is self–similar as before (in particular, the strong open set condition is assumed), then*

$$-\log \mathbb{P}\left(\|W_H\|_{L_q(\mathbb{R}^N,\mu)} < \varepsilon\right) \succeq \varepsilon^{-\gamma q/(q-\gamma)}$$

with γ defined by (5.3).

Combining Corollaries 5.4 and 5.6 finally gives us the following.

Theorem 5.7. *If μ is a self–similar measure on $\mathbb{R}^N$ as before, then*

$$-\log \mathbb{P}\left(\|W_H\|_{L_q(\mathbb{R}^N,\mu)} < \varepsilon\right) \approx \varepsilon^{-\gamma q/(q-\gamma)} .$$

Example: Suppose that the weights and the contraction factors in the construction of μ are related via

$$\lambda_i = \rho_i^s , \quad 1 \leq i \leq m ,$$

for some $s > 0$. Then it follows that $\gamma = (sH + 1/q)^{-1}$, hence

$$-\log \mathbb{P}\left(\|W_H\|_{L_q(\mathbb{R}^N,\mu)} < \varepsilon\right) \approx \varepsilon^{-1/(sH)} \tag{5.11}$$

for these special weights.

Of special interest is the case $s = 1/D$ where D denotes the similarity dimension of the self–similar set T, i.e.,

$$\sum_{i=1}^m \lambda_i^D = 1 . \tag{5.12}$$

Then (cf. (5)) μ is the (normalized) Hausdorff measure on T and D its Hausdorff dimension. Thus (5.11) becomes

$$-\log \mathbb{P}\left(\|W_H\|_{L_q(\mathbb{R}^N,\mu)} < \varepsilon\right) \approx \varepsilon^{-D/H} \tag{5.13}$$

for Hausdorff measures μ, as claimed in Theorem 1.1.

Remark: For the Lebesgue measure on $[0,1]^N$ and $q = 2$, a more precise asymptotic for (5.13) was recently evaluated in (25) by means of Hilbert space methods.

6 L_∞-norm

In the case $q=\infty$, the natural setting of our problem is as follows: given a metric space (T,ρ) and a centered Gaussian process $X:=(X(t),\, t\in T)$, evaluate

$$\mathbb{P}\left(\sup_{t\in T}|X(t)|\le\varepsilon\right),\qquad \varepsilon\to 0. \tag{6.1}$$

There is no reasonable place for a measure μ in this problem.

The main tool for working with (6.1) is provided by packing and entropy cardinalities defined as follows:

$$M(\varepsilon,T):=\max\left\{n:\exists\, t_1,\dots,t_n \text{ in } T \text{ such that } \rho(t_i,t_j)>\varepsilon,\ i\neq j\right\},$$

$$N(\varepsilon,T):=\min\left\{n:\exists\, t_1,\dots,t_n \text{ in } T \text{ such that } T\subset\cup_{j=1}^n B(t_j,\varepsilon)\right\},$$

where $B(t,\varepsilon)$ denotes the ball with center t and radius ε. Recall that the asymptotic behavior of $M(\cdot,T)$ and $N(\cdot,T)$ at zero is essentially the same, since

$$N(\varepsilon,T)\le M(\varepsilon,T)\le N(\frac{\varepsilon}{2},T).$$

In order to establish an upper bound for the small deviation probability in (6.1), let us use $v(t,\tau)$ and $v(A,\tau)$ as defined in (2.1) and (2.2), respectively. Let $\psi_1(\tau)=v(T,\tau)$. Denote $\psi_1^{-1}(\cdot)$ the inverse function of ψ_1. Then a trivial argument based on the non-determinism property (see Proposition 3 in (13), pp. 20–21) yields

$$\mathbb{P}\left(\sup_{t\in T}|X(t)|\le\varepsilon\right)\le\exp\left(-c_1 M\left(\psi_1^{-1}(\varepsilon),T\right)\right),$$

with some numerical constant $c_1>0$.

In order to establish a lower bound for the small deviation probability, we assume that a Hölder–type condition holds:

$$\mathbb{E}\left(|X(t)-X(s)|^2\right)\le\psi_2(\rho(s,t))^2\ ,\quad t,s\in T.$$

Then, under minimal regularity assumptions on $\psi_2(\cdot)$ and $N(\cdot,T)$, such as

$$c_2\, N\left(\psi_2^{-1}(\varepsilon),T\right)\le N\left(\psi_2^{-1}(\varepsilon/2),T\right)\le c_3\ N\left(\psi_2^{-1}(\varepsilon),T\right), \tag{6.2}$$

with some c_2 and $c_3>1$, Talagrand's lower bound (see the original result in (31) and a better exposition in (8)) applies and we have

$$\mathbb{P}\left(\sup_{s,t\in T}|X(t)-X(s)|\le\varepsilon\right)\ge\exp\left(-c_4 N\left(\psi_2^{-1}(\varepsilon),T\right)\right), \tag{6.3}$$

with a numerical constant $c_4>0$. Notice that, if necessary, the function $N(\psi_2(\cdot),T)$ can be replaced by any majorant in the regularity condition.

If we assume that there exists a non–decreasing function ψ regularly varying at zero such that

$$\psi_1 \approx \psi_2 \approx \psi, \tag{6.4}$$

then, by combining the upper and lower estimates, we get

$$-\log \mathbb{P}\left(\sup_{t\in T} |X(t)| \le \varepsilon\right) \approx N\left(\psi^{-1}(\varepsilon), T\right). \tag{6.5}$$

For the one–parameter version of this result related to Riemann–Liouville processes and fractional Brownian motions, we refer to (20).

For self–similar sets $T \subseteq \mathbb{R}^N$, (6.5) bears a particularly simple form, stated as follows.

Proposition 6.1. *Let $T \subseteq \mathbb{R}^N$ be a compact self–similar set such that the open set condition holds. If a Gaussian process X on $\mathbb{R}^N$ satisfies (6.4), then*

$$-\log \mathbb{P}\left(\sup_{t\in T} |X(t)| \le \varepsilon\right) \approx (\psi^{-1}(\varepsilon))^{-D} \tag{6.6}$$

where the constant $D > 0$ is defined by equation (5.12) and coincides with the Hausdorff dimension of T.

Proof: By Theorem 1 in (6), we have

$$N(\varepsilon, T) \approx \varepsilon^{-D}\ ,$$

and since ψ is regularly varying at zero, so is ψ^{-1}, and (6.2) is satisfied. Thus (6.6) follows from (6.5). □

Arguing as in the proof of Theorem 2.4, by (2.7) the preceding proposition applies to W_H and $\psi(\tau) = \tau^H$. Consequently, Proposition 6.1 leads to the following.

Corollary 6.2. *Let $T \subset \mathbb{R}^N$ be self–similar such that the open set condition holds. Then it follows that*

$$-\log \mathbb{P}\left(\sup_{t\in T} |W_H(t)| \le \varepsilon\right) \approx \varepsilon^{-D/H}$$

where as before $D > 0$ denotes the Hausdorff dimension of T.

Remark: In the case $T = [0,1]^N$ Corollary 6.2 recovers Theorem 1.2 of Shao and Wang (30) asserting

$$\exp\left(-\frac{c_1}{\varepsilon^{N/H}}\right) \le \mathbb{P}\left(\sup_{t\in[0,1]^N} |W_H(t)| \le \varepsilon\right) \le \exp\left(-\frac{c_2}{\varepsilon^{N/H}}\right).$$

We also mention interesting small deviation bounds in the sup–norm for *stationary* random fields in (19) and (32).

One question still has to be answered when comparing Theorem 2.4 with (6.3), namely, whether or not (6.3) may be viewed (if $\psi_2(\lambda) = c\,\lambda^H$) as an extension of Theorem 2.4 to the limit case

$q = \infty$. The answer is affirmative. If $T \subseteq \mathbb{R}^N$, $H > 0$, then a natural generalization of (2.9) to $q = \infty$ is as follows:

$$\sigma(n) = \sigma^{(H,\infty)}(n,T) := \inf \left\{ \left(\sum_{j=1}^{n} \operatorname{diam}(A_j)^N \right)^{H/N} : T \subseteq \bigcup_{j=1}^{n} A_j \right\} .$$

At a first glance, it is not clear how this quantity is related to $N(\varepsilon, T)$ or $M(\varepsilon, T)$. We will find a connection in terms of the inverse of $M(\,\cdot\,, T)$, i.e., in terms of the inner entropy numbers δ_n of T defined by

$$\begin{aligned} \delta_n &= \delta_n(T) := \sup\{\delta > 0 : \exists\, t_1, \ldots, t_n \in T\,, |t_i - t_j| > \delta\,, 1 \le i < j \le n\} \\ &= \inf\{\delta > 0 : M(\delta, T) \le n\}\,. \end{aligned}$$

The following proposition relates the sequences $(\sigma(n))$ and (δ_n).

Proposition 6.3. *There are positive constants c_1, c_2 and an integer κ depending only on H and N such that for $T \subseteq \mathbb{R}^N$,*

$$c_1 \cdot n^{H/N} \cdot \delta_{\kappa n}^H \le \sigma(n) \le c_2 \cdot n^{H/N} \cdot \delta_n^H\,. \tag{6.7}$$

Proof: Let κ be any fixed integer with $\kappa > 2^N$ and choose $t_1, \ldots, t_{\kappa n} \in T$ such that $|t_i - t_j| \ge \delta_{\kappa n}$ for $i \ne j$. Then the open balls $B(t_i, \delta_{\kappa n}/2)$ are disjoint. If $A_1, \ldots, A_n$ is any covering of T by compact sets, then for each $i \le \kappa n$ there is a $j \le n$ such that

$$B(t_i, \delta_{\kappa n}/2) \subseteq A_j + B(0, \delta_{\kappa n}/2)\,.$$

Let V_N be the volume of the N–dimensional Euclidean unit ball. Then

$$\begin{aligned} \kappa n \cdot V_N \cdot (\delta_{\kappa n}/2)^N &= \sum_{i=1}^{\kappa n} \operatorname{vol}_N(B(t_i, \delta_{\kappa n}/2)) \\ &= \operatorname{vol}_N\left(\bigcup_{i=1}^{\kappa n} B(t_i, \delta_{\kappa n}/2) \right) \\ &\le \operatorname{vol}_N\left(\bigcup_{j=1}^{n} (A_j + B(0, \delta_{\kappa n}/2)) \right) \\ &\le V_N \cdot \sum_{j=1}^{n} (\operatorname{diam}(A_j) + \delta_{\kappa n}/2)^N\,. \end{aligned}$$

By means of the elementary inequality $(a+b)^N \le 2^N(a^N + b^N)$ (for $a \ge 0$ and $b \ge 0$), this yields

$$\begin{aligned} \kappa n \cdot V_N \cdot (\delta_{\kappa n}/2)^N &\le 2^N \cdot V_N \cdot \sum_{j=1}^{n} \left[\operatorname{diam}(A_j)^N + (\delta_{\kappa n}/2)^N \right] \\ &= 2^N \cdot V_N \cdot \sum_{j=1}^{n} \operatorname{diam}(A_j)^N + V_N \cdot n \cdot \delta_{\kappa n}^N\,. \end{aligned}$$

Consequently,

$$n \cdot \left(\frac{\kappa}{2^N} - 1\right) \cdot 2^{-N} \cdot \delta_{\kappa n}^N \leq \sum_{j=1}^{n} \operatorname{diam}(A_j)^N .$$

This being true for all compact coverings $A_1, \ldots, A_n$ of T, the first inequality in (6.7) follows.

To prove the second inequality in (6.7), we take any $\delta > \delta_n$ and a maximal number m such that there exist $t_1, \ldots, t_m \in T$ with $|t_i - t_j| > \delta$, $1 \leq i \neq j \leq m$. From $\delta > \delta_n$ necessarily follows $m \leq n$. Moreover, by the maximality of m we have $T \subseteq \bigcup_{j=1}^m \overline{B(\delta, t_j)}$. Note that σ is decreasing, thus this implies

$$\sigma(n) \leq \sigma(m) \leq m^{N/H} \cdot (2\,\delta)^H \leq n^{N/H} \cdot (2\,\delta)^H .$$

Since $\delta > \delta_n$ was chosen arbitrarily, this completes the proof of (6.7). □

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