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Random Walks on Groups and Monoids with a Markovian Harmonic Measure

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Abstract

We consider a transient nearest neighbor random walk on a group G with finite set of generators Σ. The pair (G, Σ) is assumed to admit a natural notion of normal form words where only the last letter is modified by multiplication by a generator. The basic examples are the free products of a finitely generated free group and a finite family of finite groups, with natural generators. We prove that the harmonic measure is Markovian of a particular type. The transition matrix is entirely determined by the initial distribution which is itself the unique solution of a finite set of polynomial equations of degree two. This enables to efficiently compute the drift, the entropy, the probability of ever hitting an element, and the minimal positive harmonic functions of the walk. The results extend to monoids.

Keywords: Finitely generated group or monoid; free product; random walk; harmonic measure.

AMS classification (2000): Primary 60J10, 60B15, 31C05; Secondary 60J22, 65C40, 20F65.

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1 Introduction

Consider a Markov chain whose state space is the Cayley graph of a group $(G, *)$ with a finite set of generators Σ. If the Markov chain moves from a node g to a node $g * h$ with a probability $\mu(h)$ depending only on h, we say that it is a *randow walk*. If the support of μ is included in Σ, we say that we have a *nearest neighbor* random walk (with respect to Σ).

A remarkable and well known result is that the harmonic measure of a transient nearest neighbor random walk on a finitely generated free group is Markovian [6, 19, 25]. This can be viewed as a consequence of the multiplicative structure of the Green kernel. If one tries to extend the setting by assuming that the probability μ defining the random walk is not of nearest neighbor type but only of finite support, then the result fails to be true, see [17]. In other words, having a Markovian harmonic measure is a property which depends not only on the group but also on the choosen set of generators.

Consider now a nearest neighbor random walk on a free product of finitely many finite groups. It is easily seen that the Green kernel still has a multiplicative structure. The Markovian nature of the harmonic measure is therefore natural. What we prove is that it is a Markovian measure with a very specific structure that we call *Markovian multiplicative*. In particular, the Markovian measure is entirely determined by its initial distribution r, where r is the unique solution of a finite set of polynomial equations that we call the *Traffic Equations*. Understanding this point enables us to define precisely the setting in which the result holds: nearest neighbor random walks on a class of pairs (G, Σ) that we coin as *0-automatic pairs*.

A pair (G, Σ) formed by a group (group law $*$, unit element 1_G) and a finite set of generators is *0-automatic* if the set of words $L(G, \Sigma) = \{u_1 \cdots u_k \mid \forall i, u_i * u_{i+1} \notin \Sigma \cup 1_G\}$ is a cross-section of G. Such pairs were first considered by Stallings [27] under another name. Consider a group $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ which is a free product of a finitely generated free group and a finite family of finite groups, also called *plain group*. Consider the *natural* (but not necessarily minimal) set of generators $\Sigma = S \sqcup S^{-1} \sqcup_i G_i \backslash \{1_{G_i}\}$. Then the pair (G, Σ) is 0-automatic. Now consider an arbitrary 0-automatic pair (G, Σ). Then G is isomorphic to a plain group. On the other hand, Σ may be strictly larger than a natural set of generators of the group seen as a free product. (See Example 3.5.) And what is relevant in our context is the pair group - generators rather that the group itself, since the generators form the support of the measure defining the nearest neighbor random walk. The larger is the set of generators, the more general is the model.

With the above definition, we can describe more precisely the main result. Let (G, Σ) be a 0-automatic pair and let $(X_n)_n$ be a realization of a transient nearest neighbor random walk on G. The random walk can be viewed as evolving on $L(G, \Sigma)$. Let L^∞ be the set of right-infinite words whose finite prefixes belong to $L(G, \Sigma)$. Then $X_\infty = \lim_n X_n$ is a well-defined r.v. living on L^∞. The law μ^∞ of X_∞ is called the *harmonic measure* of the random walk. Intuitively, μ^∞ gives the directions in which the random walk escapes to infinity. We prove that the harmonic measure is a Markovian measure. Furthermore, let r and P be respectively the initial distribution and the transition matrix of the Markov chain defining μ^∞. Then, P is entirely determined by r, and r is itself characterized as the unique solution to the finite set of Traffic Equations.

Mutatis mutandis, the result extends to random walks on 0-automatic pairs formed by a monoid and a finite set of generators.

Apart from the structural aspect, the above mentioned result also provides a simple and direct approach for computational purposes. The drift and the entropy of the random walk exist as a direct consequence of Kingman's Subadditive Ergodic Theorem. But subadditive limits are in general hard to compute explicitly, and drift and entropy are no exceptions to the rule. Here, using the shape of the harmonic measure, we can express the drift and the entropy in a 'semi-explicit' way, as a simple function of r. In comparison, the 'classical' approach of [25] consists in expressing the drift as a functional of the *first-passage generating series* of the random walk. This provides also a potential method for explicit computations, but a less direct one (and which does not work for the entropy). Using the Traffic Equations yields more chances to solve the model completely and to get closed form formulas. Examples of drift computations are given in §6. In the companion paper [22], we perform a more precise and systematic study for free products of cyclic groups.

Let us clarify a couple of points.

• Following Dynkin & Malyutov [6], random walks on plain groups have been extensively studied in the 1980ies. General formulas for the harmonic measure are available, see in particular [30, Section 6]. For a detailed account and bibliographic references, see the monograph of Woess [31], and in particular the Chapters II-9, III-17, III-19, and IV-26. There are two main differences between the previous results and the work presented here. First, the previous results are usually stated in a more general framework, and are therefore less simple and less precise. Second, they are centered around the probabilities of hitting elements instead of the quantity r. We come back to this point at the end of §4.3.

• All the random walks considered here belong to the general setting of random walks on regular languages studied in [18]. In [18], local limit theorems are proved. Also, our random walks can be viewed as random walks on a tree with finitely many cone types in the sense of [24], but they are specific since they arise from a group, and since they have one-step moves at distance 1 and 2. In [24], it is proved that the harmonic measure is Markovian for random walks on a tree with finitely many cone types, but for one-step moves at distance 1. (See also [28].) Our method of proof is different from the one in [24], and the nature of the result is different as well (Markovian multiplicative versus Markovian). We discuss this point in §5.3.

• Some important aspects of the theory of random walks on discrete infinite groups are barely touched upon here, for instance, boundary theory, or the asymptotics of $P\{X_n = 1_G\}$. The interested reader should consult [14, 29, 31] and the references there. Of a different nature, but somehow related, are the 'homesick random walks' on Cayley graphs of groups, see [20].

• There exists a completely orthogonal approach to the study of random walks on monoids. The random walk is viewed as evolving in the egg-box diagram of the monoid. In general, any box of the diagram may contain monoid elements of arbitrary lengths (with respect to a given finite set of generators). For instance if the monoid is a group, then the egg-box diagram is trivial. See [13] for details.

The paper is organized as follows. Zero-automatic pairs are defined in §3. The main results for groups are then proved in §4, and the extension to monoids is done in §5.

The results presented here, as well as the ones from the companion paper [22], were announced without proofs in the Proceedings of the International Colloquium of Mathematics and Computer Science held in Vienna in Septembre 2004 [23].

Notations. Let $\mathbb{N}$ be the set of non-negative integers and let $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We also set $\mathbb{R}_+^* = \mathbb{R}_+\backslash\{0\}$. We denote the support of a random variable by supp. If μ is a measure on a group $(G, *)$, then μ^{*n} is the n-fold convolution product of μ, that is the image of the product measure $\mu^{\otimes n}$ by the product map $G \times \cdots \times G \to G$, $(g_1, \ldots, g_n) \mapsto g_1 * g_2 * \cdots * g_n$. The symbol $\sqcup$ is used for the disjoint union of sets. Given a finite set Σ, a vector $x \in \mathbb{R}^\Sigma$, and $S \subset \Sigma$, set $x(S) = \sum_{u \in S} x(u)$.

2 Finitely Generated Groups and Random Walks

Given a set Σ, the free monoid it generates is denoted by Σ^*. As usual, Σ is called the *alphabet*, the elements of Σ and Σ^* are called respectively *letters* and *words*, and the subsets of Σ^* are called *languages*. The empty word is denoted by 1 and we set $\Sigma^+ = \Sigma^*\backslash\{1\}$. The concatenation product of two elements u and v of Σ^* is written uv. The *length* (number of letters) of a word u is denoted by $|u|_\Sigma$. The longest prefix of two words u and v is denoted by $u \wedge v$.

Consider a finitely generated group $(G, *)$ with unit element 1_G. The inverse of the element u is denoted u^{-1}. Let $\Sigma \subset G$ be a finite set of generators of G. Throughout the paper, we always assume that $1_G \notin \Sigma$ and that $u \in \Sigma \implies u^{-1} \in \Sigma$. Denote by $\pi : \Sigma^* \to G$ the monoid homomorphism which associates to a word $a_1 \cdots a_k$ the group element $a_1 * \cdots * a_k$. A word $u \in \pi^{-1}(g)$ is called a *representative* of g.

The *length with respect to* Σ of a group element g is:

$$|g|_\Sigma = \min\{k \mid \ g = s_1 * \cdots * s_k, s_i \in \Sigma\} \,. \tag{1}$$

A representative u of g is a *geodesic* if $|u|_\Sigma = |g|_\Sigma$.

The *Cayley graph* $\mathfrak{X}(G, \Sigma)$ of a group G with respect to a set of generators Σ is the directed graph with set of nodes G and with an arc from g to h if $g^{-1} * h \in \Sigma$. As usual, an arc from g to h is represented graphically by $g \longrightarrow h$. It is often convenient to view $\mathfrak{X}(G, \Sigma)$ as a labelled graph with set of labels Σ (with $u \xrightarrow{a} v$ if $u * a = v$). For examples of Cayley graphs, see Figures 1 or 2. For simplicity, we draw a single undirected edge g — $g * a$ instead of the two directed arcs $g \rightleftarrows g * a$.

Observe that $|g|_\Sigma$ is the geodesic distance (in number of arcs) from 1_G to g in the Cayley graph.

Consider a relation $R \subset \Sigma^* \times \Sigma^*$, and let $\sim_R$ be the least congruence on Σ^* such that $u \sim_R v$ if $(u, v) \in R$. Let M be isomorphic to the quotient monoid $(\Sigma^*/ \sim_R)$. We say that $\langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ is a *monoid presentation* of M.

Given a set S, denote by $\mathbb{F}(S)$ the free group generated (as a group) by S. A monoid presentation of $\mathbb{F}(S)$ is

$$\langle\, S \sqcup S^{-1} \mid aa^{-1} = 1, a^{-1}a = 1,\ \forall a \in S \,\rangle\,. \tag{2}$$

Consider a relation $R \subset (S \sqcup S^{-1})^* \times (S \sqcup S^{-1})^*$. Let G be isomorphic to the quotient group $(\mathbb{F}(S)/\sim_R)$. We say that $\langle\, S \mid u = v, (u, v) \in R \,\rangle$ is a *group presentation* of G. A monoid presentation of G is $\langle\, S \sqcup S^{-1} \mid u = v, (u, v) \in R,\ aa^{-1} = 1, a^{-1}a = 1,\ \forall a \in S \,\rangle$.

We write, respectively, $M = \langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ and $G = \langle\, S \mid u = v, (u, v) \in R \,\rangle$, precising when necessary if we are dealing with a monoid or a group presentation.

Given two groups G_1 and G_2, let $G_1 \star G_2$ be the free product of G_1 and G_2. Roughly, the elements of $G_1 \star G_2$ are the finite alternate sequences of elements of $\Sigma_1 = G_1 \backslash \{1_{G_1}\}$ and $\Sigma_2 = G_2 \backslash \{1_{G_2}\}$, and the group law is the concatenation with simplification. More rigorously, set $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and define $R \subset \Sigma^* \times \Sigma^*$ by

$$R = \{(ab, c) \mid a, b \in \Sigma_i, c \in \Sigma_i \cup 1,\ \pi_i(ab) = \pi_i(c),\ i \in \{1, 2\}\}\,,$$

where $\pi_i : \Sigma_i^* \to G_i$ is the canonical monoid homomorphism. Then the monoid presentation $\langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ defines a group called the *free product of* G_1 *and* G_2 and denoted by $G_1 \star G_2$.

Following [12], define a *plain group* as the free product of a finitely generated free group and a finite family of finite groups. Let $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ be a plain group. Then

$$\Sigma = S \sqcup S^{-1} \sqcup_i \Sigma_i, \qquad \Sigma_i = G_i \backslash \{1_{G_i}\}\,, \tag{3}$$

is a set of generators for G that we call *natural generators*. (Even for a fixed decomposition $(\mathbb{F}(S), G_1, \ldots, G_k)$ of G, the set of natural generators is not unique as soon as $\#S \geq 2$, because of the flexibility in choosing the free generators of the free group.)

Random walks. Let μ be a probability distribution over Σ. Consider the Markov chain on the state space G with one-step transition probabilities given by: $\forall g \in G, \forall a \in \Sigma,\ P_{g,g*a} = \mu(a)$. This Markov chain is called the *(right) random walk* (associated with) (G, μ). It is a *nearest neighbor* random walk: one-step moves occur between nearest neighbors in the Cayley graph $\mathcal{X}(G, \Sigma)$.

Let $(x_n)_n$ be a sequence of i.i.d. r.v.'s distributed according to μ. Set

$$X_0 = 1_G,\ X_{n+1} = X_n * x_n = x_0 * x_1 * \cdots * x_n\,. \tag{4}$$

The sequence $(X_n)_n$ is a *realization* of the random walk (G, μ). The law of X_n is μ^{*n}.

Drift, entropy. The first natural question concerning the random walk (G, μ) is whether it is recurrent or transient. Assuming transience, the next natural question is the speed at which the random walk escapes all the finite sets.

For any $u, v \in G$, we have $|u * v|_\Sigma \leq |u|_\Sigma + |v|_\Sigma$. It implies immediately that $U_{n,n+m} = |x_n * \cdots * x_{n+m-1}|_\Sigma$ forms a doubly-indexed subadditive sequence of r.v.'s. Applying Kingman's

Subadditive Ergodic Theorem [16] yields the following (first noticed by Guivarc'h [11]). There exists $\gamma \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \frac{|X_n|_\Sigma}{n} = \gamma \ \text{ a.s. and in } L^p \ , \tag{5}$$

for all $1 \leq p < \infty$. We call γ the *drift* of the random walk. Intuitively, γ is the rate of escape to infinity of the walk.

Another quantity of interest is the entropy. The *entropy* of a probability measure μ with finite support S is defined by $H(\mu) = -\sum_{x\in S} \mu(x)\log[\mu(x)]$. The *entropy of the random walk* (G,μ), introduced by Avez [2], is

$$h = \lim_n \frac{H(\mu^{*n})}{n} = \lim_n -\frac{1}{n}\log \mu^{*n}(X_n) \ , \tag{6}$$

a.s. and in L^p, for all $1 \leq p < \infty$. The existence of the limits as well as their equality follow again from Kingman's Subadditive Ergodic Theorem, see [2, 4].

3 Zero-Automatic Pairs

Let $(G,*)$ be a group with finite set of generators $\Sigma \subset G$. A language L of Σ^* is a *cross-section* of G (over the alphabet Σ) if the restriction of π to L defines a bijection, that is if every element of G has a unique representative in L. A word of L is then called a *normal form word*. The map $\phi : G \to L$ which associates to a group element its unique representative in L is called the *normal form map*.

Definition 3.1. *Let G be a group with finite set of generators Σ. Define the language $L(G,\Sigma) \subset \Sigma^*$ of* locally reduced words *by:*

$$L(G,\Sigma) = \{u_1 \cdots u_k \mid \forall i \in \{1,\dots,k-1\},\ u_i * u_{i+1} \notin \Sigma \cup 1_G\} \ . \tag{7}$$

We say that the pair (G,Σ) is 0-automatic *if $L(G,\Sigma)$ is a cross-section of G.*

Such pairs were introduced and studied by Stallings in [27] under the name *unique factorization groups*, a terminology that we did not adopt for several reasons. It is convenient to introduce the sets:

$$\forall a \in \Sigma, \quad \mathrm{Next}(a) = \big\{b \in \Sigma \mid a * b \notin \Sigma \cup \{1_G\}\big\}, \quad \mathrm{Prev}(a) = \big\{b \in \Sigma \mid b * a \notin \Sigma \cup \{1_G\}\big\} \ . \tag{8}$$

Then we can reformulate (7) as:

$$\begin{aligned} L(G,\Sigma) &= \{u_1 \cdots u_k \mid \forall i \in \{2,\dots,k\},\ u_i \in \mathrm{Next}(u_{i-1})\} \\ &= \{u_1 \cdots u_k \mid \forall i \in \{1,\dots,k-1\},\ u_i \in \mathrm{Prev}(u_{i+1})\} \ . \end{aligned}$$

Here are some consequences of Definition 3.1. First, let $\phi : G \to L(G,\Sigma)$ be the normal form map. Then: $\forall g \in G$ s.t. $\phi(g) = u_1 \cdots u_k,\ \forall a \in \Sigma$,

$$\phi(g * a) = \begin{cases} u_1 \cdots u_{k-1} & \text{if } a = u_k^{-1} \\ u_1 \cdots u_{k-1} v & \text{if } u_k * a = v \in \Sigma \\ u_1 \cdots u_{k-1} u_k a & \text{if } u_k * a \notin \Sigma \cup 1_G \end{cases} \ , \tag{9}$$

and the analog for $\phi(a * g)$ also holds.

Second, the language $L(G, \Sigma)$ is regular and recognized by the following automaton: Set of states: $\Sigma \cup 1_G$, initial state: 1_G, final states: $\Sigma \cup 1_G$; Transitions: $a \xrightarrow{b} b$ if $a * b \notin \Sigma \cup 1_G$.

Third, the Cayley graph $\mathcal{X}(G, \Sigma)$ has a tree-like structure. In particular, if G is infinite then $\mathcal{X}(G, \Sigma)$ has uniform node-connectivity 1, that is, the removal of any node disconnects the graph.

Fourth, all the group elements have a unique geodesic representative with respect to Σ, and the set of these geodesic representatives is precisely $L(G, \Sigma)$.

Fifth, the set of simple circuits going through the node 1_G in $\mathcal{X}(G, \Sigma)$ is finite. In fact, this last property is equivalent to the property that $L(G, \Sigma)$ be a cross-section, see [12].

Let us now describe the pairs which are 0-automatic. The following is easy to prove.

Proposition 3.2. *Consider a plain group G and a set of natural generators Σ (see §2). Then the pair (G, Σ) is 0-automatic.*

Let $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ and $\Sigma = S \sqcup S^{-1} \sqcup_i \Sigma_i,\ \Sigma_i = G_i \backslash \{1_{G_i}\}$. The sets $\mathrm{Next}(\cdot), \mathrm{Prev}(\cdot)$, defined in (8), can be explicited:

$$\forall a \in S \sqcup S^{-1},\ \mathrm{Next}(a) = \mathrm{Prev}(a) = \Sigma \backslash \{a^{-1}\}, \quad \forall a \in \Sigma_i,\ \mathrm{Next}(a) = \mathrm{Prev}(a) = \Sigma \backslash \Sigma_i\,.$$

Conversely, next Proposition can be proved using results from Stallings [27]:

Proposition 3.3. *Let (G, Σ) be a 0-automatic pair. Then G is isomorphic to a plain group.*

The proof of Stallings is constructive and provides more precise information. Let φ be any isomorphism from G to the plain group $\widetilde{G}$. Then there exists a set S of natural generators of $\widetilde{G}$ such that $S \subset \varphi(\Sigma)$. The inclusion may be strict, but for $u \in \varphi(\Sigma)$, we must have $|u|_S = 1, 2$, or 3. This is illustrated in Example 3.5.

Plain groups are hyperbolic in the sense of Gromov [10] and automatic in the sense of Epstein & al [7]. Besides, (G, Σ) is 0-automatic iff $(\Sigma, L(G, \Sigma))$ is an automatic pair (in the sense of [7]) satisfying the 0-fellow traveller property. This is our justification for the chosen terminology.

Example 3.4. Consider the group $G = \mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$. Let a, b, and c be the respective generators of $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/4\mathbb{Z}$. Set $S = \{a, a^{-1}, b, c, c^{-1}\}$. We have represented the Cayley graph of G with respect to S in Figure 1.

Observe that (G, S) is not 0-automatic. For instance, $c * c$ has two representatives in $L(G, S)$: cc and $c^{-1}c^{-1}$; hence $L(G, S)$ is not a cross-section. Now, let us switch from the minimal set of generators S to the natural (but non-minimal) set of generators $\Sigma = \{a, a^{-1}, b, c, c^2, c^3 = c^{-1}\}$. According to Proposition 3.2, (G, Σ) is 0-automatic. This shows that 0-automaticity is not intrinsic: it depends on the set of generators.

Example 3.5. Consider the group $G = \langle a, b \mid abab = 1 \rangle$. (This is the dihedral Artin group $\langle a, b \mid abab = baba \rangle$ quotiented by its center.) Set $\Sigma = \{a, b, ab = (ab)^{-1}, ba = (ba)^{-1}, aba = b^{-1}, bab = a^{-1}\}$. Then (G, Σ) is a 0-automatic pair. Here, $\mathrm{Next}(x) = \{a, ab, aba\}$ if $x \in \{a, ba, aba\}$ and $\mathrm{Next}(x) = \{b, ba, bab\}$ if $x \in \{b, ab, bab\}$.

Now, the group G is isomorphic to $\mathbb{F}(a) \star \langle ab \rangle \sim \mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, see Figure 2. Set $S = \{a, a^{-1} = bab, ab\}$ for the corresponding set of natural generators. We have for instance $|b|_S = 2$ and $|ba|_S = 3$. Concentrating on the right of Figure 2, it is not obvious that (G, Σ) is 0-automatic.

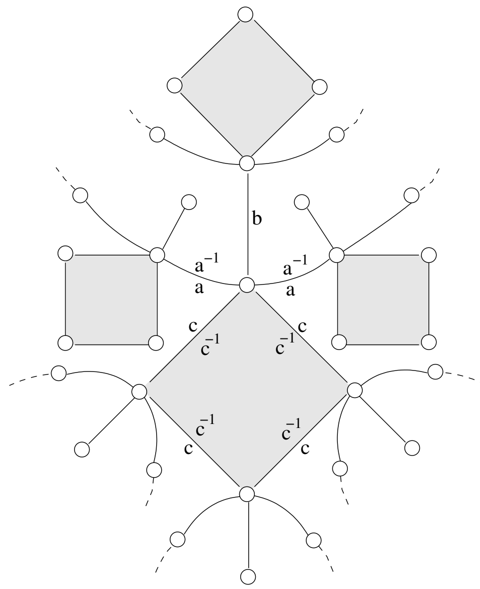

Figure 1: The Cayley graph of the group $\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$.

Zero-automatic pairs are studied in detail in [3, 21]. In [21], we compile several characterizations of 0-automatic pairs, some known and some new. The following natural question is also answered in [3, 21]: Given a plain group G and a finite set of generators Σ, is the pair (G, Σ) 0-automatic?

Another simple result proved in [21], and which will be used several times below, is the following. The *graph of successors* $\mathcal{N}\mathrm{ext}(G, \Sigma)$ of a 0-automatic pair (G, Σ) is the directed graph with:

$$\text{nodes: } \Sigma, \quad \text{arcs: } u \to v \text{ if } v \in \mathrm{Next}(u)\ . \tag{10}$$

Lemma 3.6. *Let (G, Σ) be a 0-automatic pair with G infinite and not isomorphic to $\mathbb{Z}$. The graph of successors $\mathcal{N}ext(G, \Sigma)$ is strongly connected.*

4 Random Walks on Zero-Automatic Pairs

Consider a 0-automatic pair (G, Σ). Assume that G is infinite and not isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$. Let μ be a probability measure on Σ which generates the whole group, that is, $\cup_n \mathrm{supp}(\mu^{*n}) = G$. Consider the random walk (G, μ).

This random walk is always transient. Indeed, in a non-amenable group, any random walk supported by the whole group is transient and has a strictly positive drift (see [11] and [31, Chapter 1.B] for details). And it is easily shown that the only plain groups which are amenable are $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$. (Observe that nearest-neighbor random walks on $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ are elementary.)

Let $L(G, \Sigma)$ and ϕ be defined as in §3. Define the set of *infinite normal form* words $L^\infty \subset \Sigma^{\mathbb{N}}$

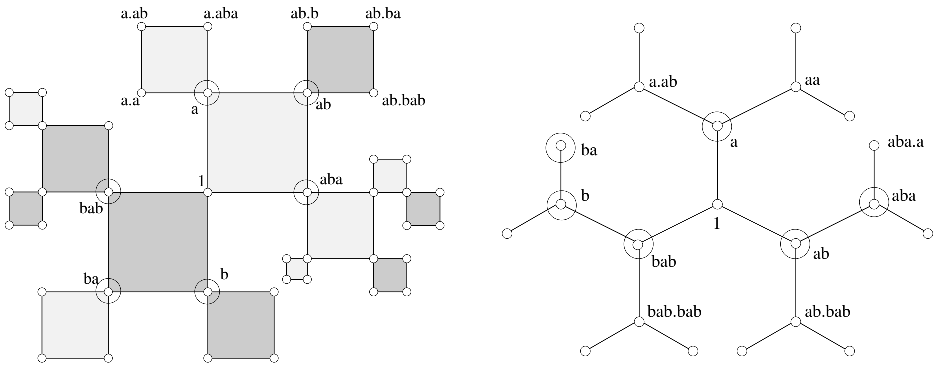

Figure 2: The Cayley graph of $\langle\, a, b \mid abab = 1 \,\rangle$ with respect to $\{a, a^{-1}, b, b^{-1}\}$ (left), and $\{a, a^{-1}, ab\}$ (right).

by

$$L^\infty = \{u_0 u_1 \cdots u_k \cdots \in \Sigma^{\mathbb{N}},\ \forall i \in \mathbb{N}, u_{i+1} \in \text{Next}(u_i)\}\,. \tag{11}$$

A word belongs to L^∞ iff all its finite prefixes belong to $L(G, \Sigma)$. The set L^∞ is the end boundary, as well as the hyperbolic boundary, of G. Using the symbolic dynamic terminology, it is a one-sided subshift of finite type.

Consider the map $\Sigma \times L^\infty \to L^\infty, (a, \xi) \mapsto a \cdot \xi$, with $a \cdot \xi = a\xi_0\xi_1 \cdots$ if $a \in \text{Prev}(\xi_0)$, $a \cdot \xi = (a * \xi_0)\xi_1 \cdots$ if $a * \xi_0 \in \Sigma$, and $a \cdot \xi = \xi_1\xi_2 \cdots$ if $a = \xi_0^{-1}$. Equip $\Sigma^{\mathbb{N}}$ with the Borel σ-algebra associated with the product topology. This induces a σ-algebra on L^∞. Given a measure ν^∞ on L^∞ and $a \in \Sigma$, define the measure $a\nu^\infty$ by: $\int f(\xi) d(a\nu^\infty)(\xi) = \int f(a \cdot \xi) d\nu^\infty(\xi)$. A probability measure ν^∞ on L^∞ is *μ-stationary* if

$$\nu^\infty(\cdot) = \sum_{a \in \Sigma} \mu(a)[a\nu^\infty](\cdot)\,. \tag{12}$$

Proposition 4.1. *Let $(X_n)_n$ be a realization of the random walk (G, μ) and set $Y_n = \phi(X_n)$. There exists a r.v. Y^∞ valued in L^∞ such that a.s.*

$$\lim_{n \to \infty} Y_n = Y^\infty\,,$$

meaning that the length of the common prefix between Y_n and Y^∞ goes to infinity a.s. Let μ^∞ be the distribution of Y^∞. The measure μ^∞ is μ-stationary and is the only μ-stationary probability on L^∞.

We call μ^∞ the *harmonic measure* of (G, μ). Intuitively, the harmonic measure gives the direction in which $(X_n)_n$ goes to infinity.

Proposition 4.2. *The drift and the entropy of the random walk are given by:*

$$\gamma \;=\; \sum_{a \in \Sigma} \mu(a) \Big[-\mu^\infty(a^{-1}\Sigma^{\mathbb{N}}) + \sum_{b \in Next(a)} \mu^\infty(b\Sigma^{\mathbb{N}}) \Big] \tag{13}$$

$$h \;=\; -\sum_{a \in \Sigma} \mu(a) \int \log\Big[\frac{da^{-1}\mu^\infty}{d\mu^\infty}(\xi)\Big] d\mu^\infty(\xi)\,, \tag{14}$$

where $dx^{-1}\mu^\infty/d\mu^\infty$ is the Radon-Nikodym derivative of $x^{-1}\mu^\infty$ with respect to μ^∞.

In words, γ is the expected change of length of an infinite normal form distributed according to μ^∞, when left-multiplied by an element distributed according to μ. The entropy h is the mutual information of Y_1 and Y^∞, or equivalently, the relative entropy of the law of (Y_1, Y^∞) with respect to the product law of Y_1 and Y^∞.

In the context of the free group, Propositions 4.1 and 4.2 are proved for instance in [19, Theorem 1.12, Theorem 4.10, Corollary 4.5]. Concerning the entropy, see also [5, 15]. The proofs adapt easily to the present setting. In adapting the proof of [19, Theorem 1.12], one has to argue that L^∞ is the unique closed G-invariant and non-empty subset included in L^∞. But this follows from Lemma 3.6.

Several of the key arguments for Propositions 4.1 and 4.2 go back to Furstenberg [8, 9], see [19] for precise references.

4.1 Markovian multiplicative measure

Define $\mathring{\mathcal{B}} = \{x \in \mathbb{R}^\Sigma \mid \forall u \in \Sigma, x(u) > 0,\ \sum_{u\in\Sigma} x(u) = 1\}$. Consider $r \in \mathring{\mathcal{B}}$. Define the matrix P of dimension $\Sigma \times \Sigma$ by

$$P_{u,v} = \begin{cases} r(v)/r(\mathrm{Next}(u)) & \text{if } v \in \mathrm{Next}(u) \\ 0 & \text{otherwise} \end{cases} . \tag{15}$$

It is the transition matrix of a Markov Chain on the state space Σ. According to Lemma 3.6, this Markov chain is irreducible.

Let $(U_n)_n$ be a realization of the Markov chain with transition matrix P and starting from U_1 such that $P\{U_1 = x\} = r(x)$. Set $U^\infty = \lim_n U_1 \cdots U_n$, and let ν^∞ be the distribution of U^∞. Clearly the support of ν^∞ is included in L^∞. For $u_1 \cdots u_k \in L(G, \Sigma)$, we have

$$\begin{aligned} \nu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) &= r(u_1) P_{u_1,u_2} \cdots P_{u_{k-1},u_k} \\ &= r(u_1) \frac{r(u_2)}{r(\mathrm{Next}(u_1))} \cdots \frac{r(u_k)}{r(\mathrm{Next}(u_{k-1}))} \\ &= \frac{r(u_1)}{r(\mathrm{Next}(u_1))} \cdots \frac{r(u_{k-1})}{r(\mathrm{Next}(u_{k-1}))} r(u_k) . \end{aligned} \tag{16}$$

We call ν^∞ the *Markovian multiplicative* probability measure associated with r.

The measure ν^∞ is in general non-stationary with respect to the translation shift $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}, (x_n)_n \mapsto (x_{n+1})_n$. Indeed, the distribution of the first marginal is r which is different in general from the stationary distribution of P.

4.2 Markovian harmonic measure

Definition 4.3 (Traffic Equations). *The* Traffic Equations *associated with (G, μ) are the equations of the variables $x(a), a \in \Sigma$, defined by: $\forall a \in \Sigma$,*

$$x(a) = \mu(a) \sum_{u \in Next(a)} x(u) + \sum_{u * v = a} \mu(u)x(v) + \sum_{u \in Prev(a)} \mu(u^{-1}) \frac{x(u)}{\sum_{v \in Next(u)} x(v)} x(a) . \tag{17}$$

The Traffic Equations are closely related to the harmonic measure of (G, μ).

Lemma 4.4. *If the harmonic measure μ^∞ is the Markovian multiplicative measure associated with $r \in \mathring{\mathcal{B}}$, then r is a solution to the Traffic Equations (17). Conversely, if the Traffic Equations admit a solution $r \in \mathring{\mathcal{B}}$, then the harmonic measure μ^∞ is the Markovian multiplicative measure associated with r.*

Proof. Assume that the harmonic measure μ^∞ is the Markovian multiplicative measure associated with some $r \in \mathring{\mathcal{B}}$. According to Proposition 4.1, μ^∞ is μ-stationary. So, applying (12), we get, for all $a \in \Sigma$,

$$\mu^\infty(a\Sigma^{\mathbb{N}}) = \mu(a) \sum_{x \in \mathrm{Next}(a)} \mu^\infty(x\Sigma^{\mathbb{N}}) + \sum_{u*v=a} \mu(u)\mu^\infty(v\Sigma^{\mathbb{N}}) + \sum_{x \in \mathrm{Prev}(a)} \mu(x^{-1})\mu^\infty(xa\Sigma^{\mathbb{N}}) \,.$$

Simplifying using (16), we obtain

$$r(a) = \mu(a) \sum_{x \in \mathrm{Next}(a)} r(x) + \sum_{u*v=a} \mu(u)r(v) + r(a) \sum_{x \in \mathrm{Prev}(a)} \mu(x^{-1}) \frac{r(x)}{\sum_{y \in \mathrm{Next}(x)} r(y)} \,. \tag{18}$$

Therefore, r is a solution to the Traffic Equations.

Conversely, let $r \in \mathring{\mathcal{B}}$ be a solution to the Traffic Equations. Let ν^∞ be the Markovian multiplicative measure associated with r. According to Proposition 4.1, the measure ν^∞ is the harmonic measure if and only if it is μ-stationary, i.e. iff it satisfies: for all $u = u_1 \cdots u_k \in L(G, \Sigma), k \geq 2$,

$$\nu^\infty(u\Sigma^{\mathbb{N}}) = \mu(u_1)\nu^\infty(u_2 \cdots u_k\Sigma^{\mathbb{N}}) + \sum_{x*y=u_1} \mu(x)\nu^\infty(yu_2 \cdots u_k\Sigma^{\mathbb{N}}) + \sum_{x \in \mathrm{Prev}(u_1)} \mu(x^{-1})\nu^\infty(xu\Sigma^{\mathbb{N}}) \,.$$

Set $q(a) = r(a)/r(\mathrm{Next}(a))$ for all a. The previous equation holds if and only if:

$$\begin{aligned} q(u_1) \quad &= \quad \mu(u_1) \frac{q(u_2) \cdots q(u_{k-1})r(u_k)}{q(u_2) \cdots q(u_{k-1})r(u_k)} + \sum_{x*y=u_1} \mu(x) \frac{q(y)q(u_2) \cdots q(u_{k-1})r(u_k)}{q(u_2) \cdots q(u_{k-1})r(u_k)} \\ &\quad + \sum_{x \in \mathrm{Prev}(u_1)} \mu(x^{-1}) \frac{q(x)q(u_1) \cdots q(u_{k-1})r(u_k)}{q(u_2) \cdots q(u_{k-1})r(u_k)} \,. \end{aligned}$$

This is equivalent to:

$$r(u_1) = \mu(u_1)r(\mathrm{Next}(u_1)) + \sum_{x*y=u_1} \mu(x)r(y) \frac{r(\mathrm{Next}(u_1))}{r(\mathrm{Next}(y))} + \sum_{x \in \mathrm{Prev}(u_1)} \mu(x^{-1}) \frac{r(x)}{r(\mathrm{Next}(x))} r(u_1). \tag{19}$$

Now let us prove that

$$\mathrm{Next}(x * y) = \mathrm{Next}(y) \,. \tag{20}$$

By definition, $a \in \mathrm{Next}(y)$ iff $\phi(y * a) = ya$. Using that $x * y \in \Sigma$ and applying (9), we deduce that $\phi(x * (y * a)) = (x * y)a$. In particular $a \in \mathrm{Next}(x * y)$. So we have $\mathrm{Next}(y) \subset \mathrm{Next}(x * y)$. Now using that $y = x^{-1} * (x * y)$, we conclude easily that (20) holds.

It follows that $r(\mathrm{Next}(u_1)) = r(\mathrm{Next}(y))$. So Equation (19) is equivalent to:

$$r(u_1) = \mu(u_1) r(\mathrm{Next}(u_1)) + \sum_{x*y=u_1} \mu(x) r(y) + \sum_{x \in \mathrm{Prev}(u_1)} \mu(x^{-1}) \frac{r(x)}{r(\mathrm{Next}(x))} r(u_1) \,.$$

This last equation holds since r is a solution to the Traffic Equations. We conclude that ν^∞ is indeed the harmonic measure. $\square$

Since the harmonic measure is uniquely defined, a corollary of the above is that the Traffic Equations (17) have at most one solution $x \in \mathring{\mathcal{B}}$. We are now ready to state the main result.

Theorem 4.5. *Let G be an infinite group with finite set of generators Σ, such that (G, Σ) is 0-automatic. Assume that G is not isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$. Let μ be a probability measure on Σ such that $\bigcup_{n \in \mathbb{N}^*} \mathrm{supp}\ \mu^{*n} = G$. Then the Traffic Equations (17) have a unique solution $r \in \mathring{\mathcal{B}}$. The harmonic measure of the random walk is the Markovian multiplicative measure associated with r.*

When specializing Theorem 4.5 to the case $G = \mathbb{F}(\Sigma)$, the free group generated by Σ, we recover the classical results of [6, 19, 25].

Corollary 4.6. *Under the assumptions of Theorem 4.5, the drift is given by*

$$\gamma = \sum_{a \in \Sigma} \mu(a) \big[-r(a^{-1}) + r(Next(a)) \big] \,, \tag{21}$$

and the entropy is given by

$$h = -\sum_{a \in \Sigma} \mu(a) \Big[\log\big[\frac{1}{q(a^{-1})}\big] r(a^{-1}) + \sum_{b | a*b \in \Sigma} \log\big[\frac{q(a*b)}{q(b)}\big] r(b) + \log[q(a)] r(Next(a)) \Big] \,, \tag{22}$$

where $\forall a \in \Sigma,\ q(a) = r(a)/r(Next(a))$.

In particular, if the probabilities $\mu(a), a \in \Sigma$, are algebraic numbers, then the drift and the entropy are also algebraic numbers.

We are going to prove Theorem 4.5 using a fixed point argument.

Proof of Theorem 4.5. Define the application $\Phi : (\mathbb{R}_+^*)^\Sigma \longrightarrow (\mathbb{R}_+^*)^\Sigma$ as follows. For $a \in \Sigma$ and for $X \in (\mathbb{R}_+^*)^\Sigma$,

$$\Phi(X)(a) = \underbrace{\mu(a) \Big(\sum_{u \in \mathrm{Next}(a)} X(u) \Big)}_{A} + \underbrace{\sum_{u*v=a} \mu(u) X(v)}_{B} + \underbrace{X(a) \sum_{u \in \mathrm{Prev}(a)} \mu(u^{-1}) \frac{X(u)}{\sum_{v \in \mathrm{Next}(u)} X(v)}}_{C} \,. \tag{23}$$

Hence, a solution to the Traffic Equations is a vector $r \in \mathring{\mathcal{B}}$ such that $r = \Phi(r)$.

Let us prove that $\Phi(\mathring{\mathcal{B}}) \subset \mathring{\mathcal{B}}$. Consider $X \in \mathring{\mathcal{B}}$. We consider separately the contributions from the terms A, B, and C in $\sum_{a\in\Sigma} \Phi(X)(a)$. We have

$$\sum_{a\in\Sigma} \Phi(X)(a) = \big[\sum_{a\in\Sigma} \Phi(X)(a)\big]_A + \big[\sum_{a\in\Sigma} \Phi(X)(a)\big]_B + \big[\sum_{a\in\Sigma} \Phi(X)(a)\big]_C ,$$

with

$$\begin{aligned}
\big[\sum_{a\in\Sigma} \Phi(X)(a)\big]_A &= \sum_{u\in\Sigma} \mu(u) \sum_{v \mid u*v \notin \Sigma\cup 1_G} X(v) \\
\big[\sum_{a\in\Sigma} \Phi(X)(a)\big]_B &= \sum_{a\in\Sigma} \sum_{u*v=a} \mu(u) X(v) = \sum_{u\in\Sigma} \mu(u) \sum_{v \mid u*v\in\Sigma} X(v) ,
\end{aligned}$$

and

$$\begin{aligned}
\big[\sum_{a\in\Sigma} \Phi(X)(a)\big]_C &= \sum_{a\in\Sigma} X(a) \sum_{u^{-1}\in \mathrm{Prev}(a)} \mu(u) \frac{X(u^{-1})}{\sum_{v\in \mathrm{Next}(u^{-1})} X(v)} \\
&= \sum_{u\in\Sigma} \mu(u) \frac{X(u^{-1})}{\sum_{v\in \mathrm{Next}(u^{-1})} X(v)} \sum_{a \mid u^{-1}\in \mathrm{Prev}(a)} X(a) \; = \; \sum_{u\in\Sigma} \mu(u) X(u^{-1})
\end{aligned}$$

Collecting the three contributions, we obtain $\sum_{a\in\Sigma} \Phi(X)(a) = \sum_{a\in\Sigma} \mu(a) = 1$. We have proved that $\Phi(\mathring{\mathcal{B}}) \subset \mathring{\mathcal{B}}$.

Define $\mathcal{B} = \{x \in \mathbb{R}^\Sigma \mid \forall i, x_i \geq 0, \ \sum_i x_i = 1\}$. The set $\mathcal{B}$ is a convex compact subset of $\mathbb{R}^\Sigma$, which suggests to look for a fixed point argument. However, Φ cannot in general be extended continuously on $\mathcal{B}$. Indeed, set $\Phi(X)_a = A(X) + B(X) + C(X)$ with the same decomposition as in (23). Clearly the maps $A(X)$ and $B(X)$ can be extended continuously on $\mathcal{B}$. On the other hand, $C(x)$ can be extended iff $\sum_{v\in \mathrm{Next}(u)} X(v) \neq 0$ for all $u \in \mathrm{Prev}(a)$.

For $x \in \mathcal{B}\backslash\mathring{\mathcal{B}}$, let $\Phi(x) \subset \mathcal{B}$ be the set of possible limits of $\Phi(x_n), x_n \in \mathring{\mathcal{B}}, x_n \to x$. We have extended Φ to a correspondence $\Phi : \mathcal{B} \twoheadrightarrow \mathcal{B}$. Clearly this correspondence has a closed graph and nonempty convex values. Therefore, we are in the domain of application of the Kakutani-Fan-Glicksberg Theorem, see [1, Chapter 16]. The correspondence has at least one fixed point: $\exists x \in \mathcal{B}$ such that $x \in \Phi(x)$. Now let us prove that a fixed point x satisfies $x \in \mathring{\mathcal{B}}$. Using the shape of the Traffic Equations, we get that:

$$x(a) > 0 \implies [\forall b \in \mathrm{Prev}(a), \ x(b) > 0] .$$

The graph $\mathcal{G}$ with nodes Σ and arcs $a \to b, b \in \mathrm{Prev}(a)$, is the graph obtained from the graph $\mathcal{N}\mathrm{ext}(G, \Sigma)$ by reverting the direction of the arcs. Hence $\mathcal{G}$ is strongly connected according to Lemma 3.6. We deduce that x must necessarily belong to $\mathring{\mathcal{B}}$.

Finally, according to Lemma 4.4, there is at most one fixed point in $\mathring{\mathcal{B}}$, by uniqueness of the harmonic measure. Therefore there is exactly one r in $\mathring{\mathcal{B}}$ such that $\Phi(r) = r$ and the Markovian multiplicative measure associated with r is the harmonic measure. $\square$

4.3 Probability of ever hitting a group element

Asymptotic properties of random walks on plain groups have been extensively studied via the Green kernel, see [31, Chapters II-9 and III-17]. The purpose of this section is to clarify the link between this classical approach and the results in §4.2.

For all $g \in G$, let $q(g)$ be the probability of ever hitting g:

$$q(g) = P\{\exists n \mid X_n = g\} \, . \tag{24}$$

Clearly, $0 < \mu^{\infty}(\phi(g)\Sigma^{\mathbb{N}}) < q(g) < 1$. If $\phi(g) = u_1 \cdots u_k \in L(G, \Sigma)$, by the strong Markov property, we have

$$q(g) = q(u_1)q(u_2) \cdots q(u_k) \, .$$

Therefore, all we need to compute are the quantities $q(a), a \in \Sigma$. Set $\Sigma_a = \{b \in \Sigma \mid \exists c \in \Sigma \cup 1_G, \ a * c = b\}$. Now, $(q(a))_{a\in\Sigma}$ has to be a solution of the set of equations: $\forall a \in \Sigma$,

$$x(a) \;\; = \;\; \mu(a) + \sum_{u*v=a} \mu(u)x(v) + x(a) \sum_{c \in \Sigma\setminus\Sigma_a} \mu(c)x(c^{-1}) \, . \tag{25}$$

Let us justify it. The first two terms on the right-hand side are more or less obvious. Now assume that the random walk starts with an initial step of type $c \in \Sigma\setminus\Sigma_a$. Given the tree-like structure of the Cayley graph, it has to go back to 1_G before possibly reaching a. Now, the probability of ever hitting 1_G starting from c is equal to the probability of ever hitting c^{-1} starting from 1_G. This accounts for the third right-hand term in (25).

The Equations (25) are close enough to the Traffic Equations (17). Let $(r(a))_{a\in\Sigma}$ be the unique solution in $\mathring{\mathcal{B}}$ of the Traffic Equations (Theorem 4.5). We have:

$$\begin{aligned} & r(a) = \mu(a)r(\mathrm{Next}(a)) + \sum_{u*v=a} \mu(u)r(v) + r(a) \sum_{u\in\Sigma\setminus\Sigma_a} \mu(u)\frac{r(u^{-1})}{r(\mathrm{Next}(u^{-1}))} \\ \Longrightarrow \qquad & \frac{r(a)}{r(\mathrm{Next}(a))} = \mu(a) \sum_{u*v=a} \mu(u)\frac{r(v)}{r(\mathrm{Next}(v))} + \\ & \qquad\qquad \frac{r(a)}{r(\mathrm{Next}(a))} \sum_{u\in\Sigma\setminus\Sigma_a} \mu(u)\frac{r(u^{-1})}{r(\mathrm{Next}(u^{-1}))} \, , \end{aligned}$$

where we have used that $[u * v = a] \implies [\mathrm{Next}(v) = \mathrm{Next}(a)]$. Hence $\tilde{q}(a) = r(a)/r(\mathrm{Next}(a)), a \in \Sigma$, is a solution to (25). To conclude that $q = \tilde{q}$, it is enough to show that the Equations (25) have a unique solution.

Lemma 4.7. *The Equations (25) have a unique solution in $(0,1)^{\Sigma}$, which is*

$$\forall a \in \Sigma, \qquad q(a) = P\{\exists n \mid X_n = a\} = \frac{r(a)}{\sum_{v\in Next(a)} r(v)} \, . \tag{26}$$

Proof. Define the map $\Phi : \mathbb{R}^\Sigma \to \mathbb{R}^\Sigma$ as follows: $\Phi(x)(a)$ is equal to the right-hand term in (25). Observe that Φ is a non-decreasing function (for the coordinate-wise ordering). Besides, we have:

$$\Phi(0,\ldots,0) = (\mu(a))_{a\in\Sigma}, \qquad \Phi(1,\ldots,1) = (1,\ldots,1)\,.$$

Consequently, $\Phi(\prod_{a\in\Sigma}[\mu(a),1]) \subset \prod_{a\in\Sigma}[\mu(a),1]$. According to the above, Φ also has a fixed point in $\prod_{a\in\Sigma}(\mu(a),1)$ which is $\tilde{q} = (\tilde{q}(a))_{a\in\Sigma}$. Now let us prove that $\tilde{q}$ and $(1,\ldots,1)$ are the only fixed points of Φ in $\prod_{a\in\Sigma}[\mu(a),1]$.

We have $\Phi(x) = \mu + Ax + Q(x)$, where A is a non-negative matrix and $Q(.)$ is a quadratic map with non-negative coefficients. Set $\mathbf{1} = (1,\ldots,1)$. Assume that there exist two fixed points u and v of Φ in $\prod_{a\in\Sigma}[\mu(a),1]$ such that $u \leq v \leq \mathbf{1}$ and $u, v, \mathbf{1}$ are all different. Then we have:

$$\Phi(v) - \Phi(u) = A(v-u) + Q(v) - Q(u) = v - u \implies Q(v) = Q(u) + (I-A)(v-u)\,,$$

and similarly $Q(\mathbf{1}) = Q(u) + (I-A)(\mathbf{1}-u)$. It implies that the quadratic map Q has to be a linear map, which is not the case.

Now assume that there exist two different fixed points of Φ in $\prod_{a\in\Sigma}(\mu(a),1)$, say u and v. According to the above, there exists i,j, such that $u_i < v_i,\ u_j > v_j$. Set $w = u \wedge v$, where the minimum is taken coordinate-wise. Since Φ is increasing, we have $\Phi(w) \leq \Phi(v) = v$ and $\Phi(w) \leq \Phi(u) = u$. So $\Phi(w) \leq w$. Therefore, we have $\Phi(\prod_{a\in\Sigma}[\mu(a),w(a)]) \subset \prod_{a\in\Sigma}[\mu(a),w(a)]$. By Brouwer Fixed Point Theorem, the continuous map Φ has a fixed point x in $\prod_{a\in\Sigma}[\mu(a),w(a)]$ and $x \leq u$, which contradicts the first part of the argument. This completes the proof. $\square$

A consequence of Lemma 4.7 is the following way of defining the harmonic measure μ^∞: $\forall u = u_1\cdots u_k \in L(G,\Sigma)$,

$$\mu^\infty(u\Sigma^{\mathbb{N}}) = q(u_1)\cdots q(u_{k-1})r(u_k) = P\{\exists n \mid \phi(X_n) = u_1\cdots u_{k-1}\}r(u_k)\,.$$

Inverting the formulas (26) to get $(r(a))_a$ as a function of $(q(a))_a$ does not provide easy to handle formulas in general. But it does for some examples. For the free group $\mathbb{F}(S)$ and the free product $\mathbb{Z}/2\mathbb{Z}\star\cdots\star\mathbb{Z}/2\mathbb{Z}$, we get respectively

$$r(a) = \frac{q(a)(1-q(a^{-1}))}{1-q(a)q(a^{-1})}, \quad r(a) = \frac{q(a)}{1+q(a)}\,. \tag{27}$$

Harmonic functions, Martin boundary, and Poisson boundary. For $\xi \in L^\infty$, define $K_\xi : G \to \mathbb{R}_+$ by $K_\xi(x) = \lim_n K_{\pi(\xi[n])}(x) = \lim_n q(x^{-1} * \pi(\xi[n]))/q(\pi(\xi[n])$, where $\xi[n]$ is the length n prefix of ξ. Set $\xi = \xi_0\xi_1\cdots$. For $x \in G$ with $\phi(x) = x_0\cdots x_{n-1} \in L(G,\Sigma)$, set $k = |\phi(x)\wedge\xi|_\Sigma$, the length of the longest joint prefix of $\phi(x)$ and ξ. We have

$$K_\xi(x) = \begin{cases} \big[q(x_k^{-1})\cdots q(x_{n-1}^{-1})\big]/\big[q(\xi_0)\cdots q(\xi_{k-1})\big] & \text{if } x_k^{-1} \in \mathrm{Prev}(\xi_k) \\ \big[q(x_k^{-1} * \xi_k)q(x_{k+1}^{-1})\cdots q(x_{n-1}^{-1})\big]/\big[q(\xi_0)\cdots q(\xi_k)\big] & \text{otherwise} \end{cases}\,. \tag{28}$$

The minimal positive harmonic functions are the functions $K_\xi,\ \xi \in L^\infty$. (This is easily proved, for instance by adapting the argument in [19, Theorem 2.10]. Here, the only small novelty is to have the precise expression (28) in the framework of 0-automatic pairs.)

In particular, we recover the classical fact that the Martin boundary and minimal Martin boundary of the random walk coincide with the end boundary L^∞. (This holds in much more general contexts, see [31, Chapter IV-26] and the references therein.) The pair (L^∞, μ^∞) is the Poisson boundary.

Comparison with the literature. The importance of the Equations (25) in q is well-known. In the seminal paper of Dynkin & Malyutov [6], these equations are explicitely solved in the free group case, and the harmonic functions are then derived as above. In [25], the authors prove that μ^∞ is Markovian for the free group as follows: they use the expression for q obtained in [6], they define r as in (27), and then they prove that the measure defined by $\nu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) = q(u_1) \cdots q(u_{k-1}) r(u_k)$ is the harmonic measure. The approach of [30] is similar. For trees with finitely many cone types, the proof in [24] that the harmonic measure is Markovian is also centered around the analog of the Equations (25). The series version of (25) is still the main ingredient in getting central or local limit theorems. See for instance [25, 18, 24] for nearest neighbor random walks, and [17] for *finite range* random walks on free groups.

Here the proof that μ^∞ is Markovian multiplicative is different and based on the Traffic Equations (17) instead of the Equations (25). This is a more direct path and the only way to proceed in the general case since we cannot retrieve a solution to the Traffic Equations from a solution to (25). This point becomes even clearer in the context of monoids, see §5.

5 From Groups to Monoids and Beyond

At the cost of some added technicalities in the definitions, Theorem 4.5 holds when replacing groups by monoids. The main result is Theorem 5.3. The notion of zero-automaticity for monoids is discussed in more details in [21].

5.1 Zero-automaticity for monoids

Let $(M, *)$ be a finitely generated monoid with unit element 1_M. Let $\Sigma \subset M$ be a finite set of generators. We always assume that $1_M \notin \Sigma$. Let $\pi : \Sigma^* \to M$ be the corresponding monoid homomorphism.

Mutatis mutandis, the following notions and notations are defined as in §2: a representative word $u \in \pi^{-1}(m)$, a cross-section of M, a normal form word, the normal form map, the length $|\cdot|_\Sigma$, a geodesic, the Cayley graph $\mathfrak{X}(M, \Sigma)$. Define the language $L(M, \Sigma) \subset \Sigma^*$ of *locally reduced words* by:

$$L(M, \Sigma) = \{u_1 \cdots u_k \mid \forall i \in \{1, \ldots, k-1\},\ u_i * u_{i+1} \notin \Sigma \cup 1_M\} . \tag{29}$$

Define the sets $\mathrm{Next}(.), \mathrm{Prev}(.)$ as in (8).

We now want a notion of 0-automaticity for monoids. The condition that $L(M, \Sigma)$ is a cross-section of M is not sufficient to get an analog of Theorem 4.5. We propose the following definition:

Definition 5.1. *Let $(M, *)$ be a monoid with finite set of generators Σ. Assume that the language of locally reduced words $L(M,\Sigma)$ is a cross-section of M. Let $\phi : M \to L(M,\Sigma)$ be the corresponding normal form map. Assume that: $\forall u \in M$ s.t. $\phi(u) = u_1 \cdots u_k$, $\forall a \in \Sigma$,*

$$\phi(u*a) = \begin{cases} u_1 \cdots u_{k-1} & \text{if } u_k * a = 1_M \\ u_1 \cdots u_{k-1} v & \text{if } u_k * a = v \in \Sigma \\ u_1 \cdots u_{k-1} u_k a & \text{if } u_k * a \notin \Sigma \cup 1_M \end{cases}, \quad \phi(a*u) = \begin{cases} u_2 \cdots u_k & \text{if } a * u_1 = 1_M \\ v u_2 \cdots u_k & \text{if } a * u_1 = v \in \Sigma \\ a u_1 \cdots u_k & \text{if } a * u_1 \notin \Sigma \cup 1_M \end{cases}. \tag{30}$$

Assume furthermore that: $\forall a, b \in \Sigma$ such that $a * b \in \Sigma$,

$$Prev(a) = Prev(a * b), \qquad Next(b) = Next(a * b) . \tag{31}$$

Then we say that the pair (M, Σ) is 0-automatic.

Definition 5.1 is more complex than its analog for groups, Definition 3.1. For groups, the assumption that $L(G,\Sigma)$ is a cross-section implies both (30) and (31) (see Equations (9) and (20)).

As for groups, the above definition of zero-automatic pairs has several consequences. First, the cross-section $L(M,\Sigma)$ is precisely the set of geodesics. Second, if M is infinite, then $\mathcal{X}(M,\Sigma)$ has uniform node-connectivity 1.

Define a *plain monoid* to be the free product of a finitely generated free group, a finitely generated free monoid, and a finite family of finite monoids. Given a plain monoid $M = \mathbb{F}(S) \star \Sigma^* \star M_1 \star \cdots \star M_k$, the set of generators $S \sqcup S^{-1} \sqcup \Sigma \sqcup M_1 \backslash 1_{M_1} \sqcup \cdots \sqcup M_k \backslash 1_{M_k}$ is said to be a set of *natural generators*. Next result is the counterpart of Prop. 3.2.

Proposition 5.2. *The pair formed by a plain monoid and natural generators is 0-automatic.*

There is no counterpart to Proposition 3.3. That is, the family of plain monoids does not exhaust the monoids appearing in 0-automatic pairs. Consider for instance the monoids given by the following monoid presentations:

$$M_1 = \langle\, a, b \mid ab = 1 \,\rangle, \quad M_2 = \langle\, a, b, c \mid ab = 1 \,\rangle . \tag{32}$$

The monoid M_1 is known as the *bicyclic monoid*. Set $\Sigma_1 = \{a, b\}$ and $\Sigma_2 = \{a, b, c\}$. One can show that (M_1, Σ_1) and (M_2, Σ_2) are 0-automatic. However, neither M_1 nor M_2 is isomorphic to a plain monoid. We have represented the Cayley graphs of the two monoids in Figure 3.

The graph of successors. Let (M,Σ) be 0-automatic and define the graph of successors $\mathcal{N}\text{ext}(M,\Sigma)$ as in (10). The graph $\mathcal{N}\text{ext}(M,\Sigma)$ is not always strongly connected, i.e. there is no counterpart to Lemma 3.6. Consider for instance the monoids in (32). The graph $\mathcal{N}\text{ext}(M_2,\Sigma_2)$ is strongly connected, but not the graph $\mathcal{N}\text{ext}(M_1,\Sigma_1)$.

On the other hand, the pair formed by a plain monoid and a set of natural generators has a strongly connected graph of successors. In [26], random walks on the monoids $M = \langle\, a_1, b_1, \ldots, a_k, b_k \mid a_1 b_1 = 1, \ldots, a_k b_k = 1 \,\rangle$ are considered. Except for the case $k = 1$, the pair (M, Σ), $\Sigma = \{a_1, b_1, \ldots, a_k, b_k\}$, is 0-automatic and $\mathcal{N}\text{ext}(M,\Sigma)$ is strongly connected.

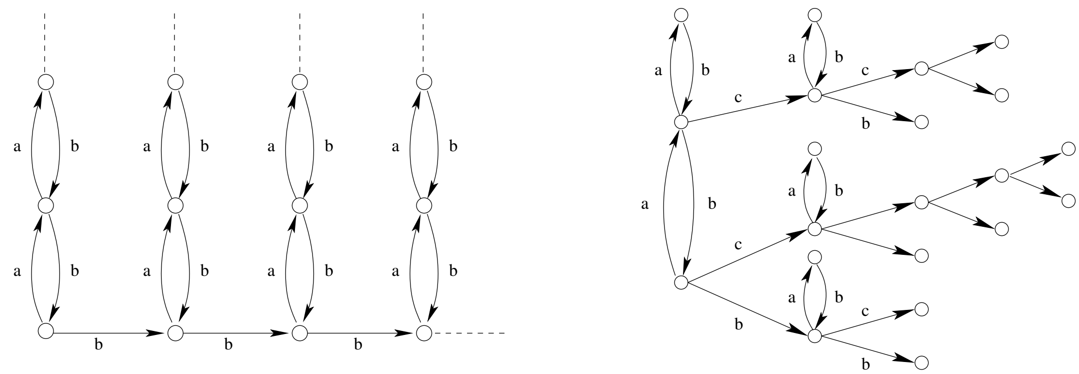

Figure 3: The Cayley graphs of $\langle\, a, b \mid ab = 1 \,\rangle$ (left) and $\langle\, a, b, c \mid ab = 1 \,\rangle$ (right).

5.2 Random walks

Let M be a monoid with finite set of generators Σ. Let μ be a probability measure on Σ such that $\bigcup_{n\in\mathbb{N}^*} \operatorname{supp} \mu^{*n} = M$. The *(right) random walk* (associated with) (M, μ) is defined as in the group case. The *drift* is defined as in (5) and the *entropy* as in (6).

¿From now on, assume that the pair (M, Σ) is 0-automatic. We make three additional assumptions. First, that the graph of successors $\mathcal{N}\mathrm{ext}(M, \Sigma)$ is strongly connected. Second, that M is infinite. Third, that the random walk (M, μ) is transient.

This last assumption is not restrictive. If M is in fact a group, see the discussion at the beginning of §4. If M is not a group, then (M, μ) is transient. Indeed, there must exist $b \in \Sigma$ with no inverse. Consider $c \in \Sigma$ such that $b * d = c$ for some $d \in \Sigma$. Clearly, c has no inverse either. Let $(Y_n)_n$ be a realization of the random walk on $L(M, \Sigma)$. In view of (30), if Y_n equals an element ending with the letter b, then we have $\forall m \geq n,\ |Y_m|_\Sigma \geq |Y_n|_\Sigma$. The transience follows easily.

Define L^∞ as in (11). Define a μ-stationary probability measure as in (12). The analog of Proposition 4.1 holds. Indeed, the proof of Proposition 4.1 given in [19, Theorem 1.12] translates to the new setting. In particular, to get the uniqueness of the μ-stationary measure, one uses the assumption that $\mathcal{N}\mathrm{ext}(M, \Sigma)$ is strongly connected.

So the harmonic measure μ^∞ of the random walk is well-defined. The formulas for the drift and the entropy are now:

$$\gamma = \sum_{a\in\Sigma} \mu(a) \sum_{b\in\Sigma} \big(|a * b|_\Sigma - |b|_\Sigma\big) \mu^\infty(b\Sigma^{\mathbb{N}}), \quad h = -\sum_{a\in\Sigma} \mu(a) \int \log\Big[\frac{d\mu^\infty(a * \cdot)}{d\mu^\infty}(\xi)\Big] d\mu^\infty(\xi)\,,$$

with obvious notations for the entropy. Once again, this is proved by slightly modifying the arguments in [19, Theorem 4.10, Corollary 4.5].

The *Traffic Equations* are defined by: $\forall a \in \Sigma$,

$$x(a) = \mu(a) \sum_{u\in \mathrm{Next}(a)} x(u) + \sum_{u*v=a} \mu(u)x(v) + \sum_{t*u=1_M, u\in \mathrm{Prev}(a)} \mu(t) \frac{x(u)}{\sum_{v\in \mathrm{Next}(u)} x(v)} x(a)\,. \quad (33)$$

The proof of Lemma 4.4 carries over. The only modification is that the equality [Next$(x * y) =$ Next(y)] which was proved in the group case (see (20)), is now true by definition (see (31)).

All the ingredients are in place to state the extension of Theorem 4.5 for monoids. The proof of Theorem 4.5 can be used, mutatis mutandis, to prove Theorem 5.3.

Theorem 5.3. *Let M be an infinite monoid with finite set of generators Σ, such that (M, Σ) is 0-automatic. Assume that the graph of successors $\mathcal{N}ext(M, \Sigma)$ is strongly connected. Let μ be a probability measure on Σ. Assume that $\bigcup_{n \in \mathbb{N}^*} supp\ \mu^{*n} = M$ and that the random walk (M, μ) is transient. Then the Traffic Equations have a unique solution $r \in \mathring{\mathcal{B}}$. The harmonic measure of the random walk is the Markovian multiplicative measure associated with r. The drift is given by:*

$$\gamma = \sum_{a \in \Sigma} \mu(a) \big[- \sum_{b | a * b = 1_M} r(b) + r(Next(a)) \big] \,,$$

and the entropy is given by:

$$h = - \sum_{a \in \Sigma} \mu(a) \Big[\sum_{b | a * b = 1_M} \log \big[\frac{1}{p(b)} \big] r(b) + \sum_{b | a * b \in \Sigma} \log \big[\frac{p(a * b)}{p(b)} \big] r(b) + \log[p(a)] r(Next(a)) \Big] \,,$$

where $\forall a \in \Sigma,\ p(a) = r(a)/r(Next(a))$.

Define $q(a)$, the probability of ever hitting a, as in (24). The results of §4.3 do *not* extend to the monoid case. Indeed there is no direct link anymore between the Traffic Equations and the equations defining q. This is best understood on an example.

Let $\mathbb{B} = \langle\ a \mid a^2 = a\ \rangle$ be the Boolean monoid, let G be a non-trivial finite group ($\Sigma_1 = G \backslash \{1_G\}$), and consider the free product $\mathbb{B} \star G$. Here are the respective equations satisfied by $r(a)/r(\text{Next}(a)) = r(a)/r(\Sigma_1)$ and by $q(a)$:

$$\begin{aligned} r(a)/r(\Sigma_1) &= \mu(a) + \mu(a) r(a)/r(\Sigma_1) + r(a)/r(\Sigma_1) \sum_{b \in \Sigma_1} \mu(b) r(b^{-1})/r(a) \\ q(a) &= \mu(a) + q(a) \sum_{b \in \Sigma_1} \mu(b) q(b^{-1}) \,. \end{aligned}$$

There is an additional term in the first equation with respect to the second.

5.3 Beyond monoids

To delimit the precise role of the group or monoid assumption in Theorems 4.5 and 5.3, we may consider the following model.

Let Σ be a finite set. Consider a map $f : \Sigma \times \Sigma \to 1 \cup \Sigma \cup \Sigma^2$ such that $f(a,b) \in \Sigma^2 \implies f(a,b) \in a\Sigma$. Define the map $\Psi : \Sigma^* \to \Sigma^*$ recursively as follows: (i) $\forall x \in 1 \cup \Sigma, \Psi(x) = x$; (ii) if $\Psi(u_1 \cdots u_n) = v_1 \cdots v_k$ then $\Psi(u_1 \cdots u_n u_{n+1}) = v_1 \cdots v_{k-1} f(v_k, u_{n+1})$.

Let μ be a probability measure on Σ, let $(x_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. r.v.'s with law μ. Define the Σ^*-valued Markov chain $(Y_n)_n$ by $Y_n = \Psi(x_0 \cdots x_{n-1})$. This is a generalization of the previous group and monoid random walks, with f replacing the group or monoid law.

Assume that f and μ are such that the Markov chain $(Y_n)_n$ is transient, i.e. $\lim_n |Y_n|_\Sigma = \infty$ a.s. Define the $\Sigma^{\mathbb{N}}$-valued r.v. $Y^\infty = \lim_n Y_n$ and let μ^∞ be its law. (The limit exists since the chain is transient and since the dynamic affects only the last letter of a word).

Proposition 5.4. *The measure μ^∞ is a Markovian measure. That is, there exist a vector r of dimension Σ and a stochastic matrix P of dimension $\Sigma \times \Sigma$ such that $\mu^\infty(u_1 \cdots u_n \Sigma^{\mathbb{N}}) = r(u_1) P_{u_1,u_2} \cdots P_{u_{n-1},u_n}$.*

Proof. Consider the realization $(Y_n)_n$ of the random walk defined on the canonical product space $\Sigma^{\mathbb{N}}$ equipped with the product measure $\mu^{\otimes \mathbb{N}}$. By definition, $Y_n(w) = \Psi(w[n])$, where $w[n]$ is the prefix of length n of $w \in \Sigma^{\mathbb{N}}$. Set $Y^\infty(w) = \lim_n Y_n(w)$.

The idea of the proof is simple and goes as follows. If $Y^\infty(w) = u_1 \cdots u_i \cdots$, then there is a *last* integer n_i such that $Y_{n_i}(w) = u_1 \cdots u_i$, and we can parse the trajectories according to the sequence $(n_i)_i$. We use in an essential way the fact that Y_n and Y_{n+1} differ only by the last letter.

Fix $u = u_1 \cdots u_k \in \Sigma^k, k \geq 2$, such that $\mu^\infty(u\Sigma^{\mathbb{N}}) > 0$. Define

$$S = \big\{ w \in \Sigma^{\mathbb{N}} \mid Y^\infty(w) \in u_1 \cdots u_k \Sigma^{\mathbb{N}} \big\}, \qquad S_x = \big\{ w \in \Sigma^{\mathbb{N}} \mid Y^\infty(w) \in u_1 \cdots u_k x \Sigma^{\mathbb{N}} \big\} .$$

Observe that $S = \bigsqcup_{x \in \Sigma} S_x$: the set S is the disjoint union of the sets S_x. Besides,

$$\mu^\infty(ux\Sigma^{\mathbb{N}}) / \mu^\infty(u\Sigma^{\mathbb{N}}) = \mu^{\otimes \mathbb{N}}(S_x) / \mu^{\otimes \mathbb{N}}(S) . \tag{34}$$

Define also

$$T = \big\{ w \in \Sigma^* \mid \Psi(w) = u_1 \cdots u_k \big\} .$$

Now, define for all $a \in \Sigma \cup 1$ and $b \in \Sigma$,

$$U_{a,b} = \big\{ w \in \Sigma^* \mid \Psi(w) = b, \ \forall 1 \leq n \leq |w|_\Sigma, \ a\Psi(w[n]) \in a\Sigma^+ \big\} .$$

At last, define for all $a \in \Sigma$,

$$V_a = \big\{ w \in \Sigma^{\mathbb{N}} \mid \forall n \geq 1, \ a\Psi(w[n]) \in a\Sigma^+ \big\} .$$

Observe that $V_a = \sqcup_b U_{a,b} V_b$. Consider $w \in S$ and set $\tau = \sup\{ n \mid \Psi(w[n]) = u_1 \cdots u_k \}$. We have $w \in (T \cap \Sigma^\tau) V_{u_k}$. Besides, the sets $[(T \cap \Sigma^n) V_{u_k}]_n$ are clearly disjoint. Consider now $w \in S_x$. Set $\tau' = \sup\{ n \mid \Psi(w[n]) = u_1 \cdots u_k x \}$. We have $w \in (T \cap \Sigma^\tau)(U_{u_k,x} \cap \Sigma^{\tau' - \tau}) V_x$. The sets $[(T \cap \Sigma^{n_1})(U_{u_k,x} \cap \Sigma^{n_2}) V_x]_{n_1,n_2}$ are disjoint. Hence we get

$$S = \bigsqcup_{n \in \mathbb{N}} (T \cap \Sigma^n) V_{u_k}, \qquad S_x = \bigsqcup_{n_1,n_2 \in \mathbb{N}} (T \cap \Sigma^{n_1})(U_{u_k,x} \cap \Sigma^{n_2}) V_x .$$

Therefore,

$$\begin{aligned}
\mu^{\otimes \mathbb{N}}(S_x) &= \sum_{n_1,n_2 \in \mathbb{N}} \mu^{\otimes \mathbb{N}}\big((T \cap \Sigma^{n_1})(U_{u_k,x} \cap \Sigma^{n_2}) V_x \big) \\
&= \sum_{n_1,n_2 \in \mathbb{N}} \mu^{\otimes \mathbb{N}}(T \cap \Sigma^{n_1}) \mu^{\otimes \mathbb{N}}(U_{u_k,x} \cap \Sigma^{n_2}) \mu^{\otimes \mathbb{N}}(V_x) \ = \ \mu^{\otimes \mathbb{N}}(T) \mu^{\otimes \mathbb{N}}(U_{u_k,x}) \mu^{\otimes \mathbb{N}}(V_x) \\
\mu^{\otimes \mathbb{N}}(S) &= \sum_{n \in \mathbb{N}} \mu^{\otimes \mathbb{N}}\big((T \cap \Sigma^n) V_{u_k} \big) = \sum_{n \in \mathbb{N}} \mu^{\otimes \mathbb{N}}(T \cap \Sigma^n) \mu^{\otimes \mathbb{N}}(V_{u_k}) = \mu^{\otimes \mathbb{N}}(T) \mu^{\otimes \mathbb{N}}(V_{u_k}) .
\end{aligned}$$

Using (34), we get

$$\frac{\mu^\infty(ux\Sigma^{\mathbb{N}})}{\mu^\infty(u\Sigma^{\mathbb{N}})} = \frac{\mu^{\otimes\mathbb{N}}(S_x)}{\mu^{\otimes\mathbb{N}}(S)} = \frac{\mu^{\otimes\mathbb{N}}(U_{u_k,x})\mu^{\otimes\mathbb{N}}(V_x)}{\mu^{\otimes\mathbb{N}}(V_{u_k})} = P_{u_k,x} \ ,$$

and, with a similar argument, for all $a \in \Sigma$,

$$\mu^\infty(a\Sigma^{\mathbb{N}}) = \frac{\mu^{\otimes\mathbb{N}}(U_{1,a})\mu^{\otimes\mathbb{N}}(V_a)}{\sum_{x\in\Sigma}\mu^{\otimes\mathbb{N}}(U_{1,x})\mu^{\otimes\mathbb{N}}(V_x)} = r(a) \ .$$

This completes the proof. □

As opposed to the group or monoid case, the measure μ^∞ is Markovian but not necessarily Markovian multiplicative. Hence the crucial link, via Lemma 4.4, with the Traffic Equations is cut. It is not possible anymore to characterize μ^∞ using the Traffic Equations. To illustrate this last point, consider the following example.

Example 5.5. Consider the monoid $M = \langle\, a,b,c \mid ac = a, cb = b, c^2 = c \,\rangle$, the generators $\Sigma = \{a,b,c\}$, and a transient random walk on (M,Σ). Here $L(M,\Sigma)$ is a cross-section and (30) holds but not (31): $\{b\} = \text{Prev}(c) \neq \text{Prev}(c*b) = \text{Prev}(b) = \{a,b\}$. Therefore, the pair (M,Σ) is not 0-automatic. However, we are in the domain of application of Proposition 5.4. Furthermore, $|Y_{n+1}|_\Sigma \geq |Y_n|_\Sigma$. Consequently μ^∞ can be explicitly determined using elementary arguments. In particular, we have

$$\mu^\infty(a\Sigma^{\mathbb{N}}) = \mu(a), \quad \mu^\infty(b\Sigma^{\mathbb{N}}) = \mu(b) + \mu(c)\frac{\mu(b)}{\mu(a)+\mu(b)}, \quad \mu^\infty(c\Sigma^{\mathbb{N}}) = \mu(c)\frac{\mu(a)}{\mu(a)+\mu(b)} \ ,$$

and for instance

$$\mu^\infty(aa\Sigma^{\mathbb{N}}) = \mu(a)\frac{\mu(a)}{\mu(a)+\mu(b)} \neq \mu^\infty(a\Sigma^{\mathbb{N}})\frac{\mu^\infty(a\Sigma^{\mathbb{N}})}{\mu^\infty(a\Sigma^{\mathbb{N}})+\mu^\infty(b\Sigma^{\mathbb{N}})} \ .$$

This last equation shows that μ^∞ is not Markovian multiplicative.

Trees with finitely many cone types. In [24], the authors prove that a transient Nearest Neighbor (NN) random walk on a Tree with Finitely Many Cone Types (TFMCT) has a Markovian 'harmonic measure'. Let us compare more closely the result from [24] and Theorems 4.5 and 5.3.

Let (M,Σ) be a 0-automatic pair, and let $\phi : M \to L(M,\Sigma)$ be the normal form map. Define the directed and labelled tree $T(M,\Sigma)$ with set of nodes M and arcs:

$$u \xrightarrow{a} v, a \in \Sigma, \ \text{ if } \ \phi(v) = \phi(u)a \ \text{ or } \ \phi(u) = \phi(v)b, \ b*a = 1_M \ .$$

Let $T_x(M,\Sigma), x \in M$, be the *cone* rooted at x, that is, the subtree of $T(M,\Sigma)$ with nodes $\{u \mid \phi(u) \in \phi(x)\Sigma^*\}$. Then $T_x(M,\Sigma)$ and $T_y(M,\Sigma)$ are isomorphic as soon as $\phi(x)$ and $\phi(y)$ end up with the same letter. In particular, $T(M,\Sigma)$ is a TFMCT.

Now consider the random walk (M,μ) where μ is a probability on Σ. There are three possible types of one-step moves, see (30): the lines 1 and 3 in (30) correspond to one-step moves between adjacent nodes in $T(M,\Sigma)$, the line 2 corresponds to a one-step move between nodes at distance 2 in $T(M,\Sigma)$.

To summarize, we have a non-NN random walk on a specific type of TFMCT. It is not straightforward to adapt the proof of Markovianity of the harmonic measure in [24] to such a random walk. There are two difficulties: (i) the steps at distance 2; (ii) the fact that the TFMCT is not *irreducible* in the sense of [24]. On the other hand, the approach via the Traffic Equations has no chance to apply to a general NN random walk on a TFMCT. Indeed, the Traffic Equations yield a Markovian multiplicative harmonic measure, whereas the harmonic measure is Markovian but not Markovian multiplicative in [24]. In the same spirit, see also Proposition 5.4.

6 Examples

In small examples, the Traffic Equations can be completely solved to provide closed form expressions for the harmonic measure, the drift, or the entropy. We illustrate this on a couple of examples. The computations have been carried out using Maple.

Denote the cyclic group of order 2 by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and the Boolean monoid by $\mathbb{B}$ (monoid presentation: $\mathbb{B} = \langle x \mid x^2 = x \rangle$). Consider the free product $\mathbb{Z}_2 \star \mathbb{B} \star \mathbb{B}$, and denote the respective generators by a, b, and c. Let μ be a non-degenerate probability measure on $\{a,b,c\}$. The Traffic Equations can easily be solved. The unique solution in $\mathring{\mathcal{B}}$, which defines the harmonic measure, is:

$$r(a) = \frac{\mu(a)}{1+\mu(a)}, \quad r(b) = \frac{\mu(b)}{1-\mu(a)^2}, \quad r(c) = \frac{\mu(c)}{1-\mu(a)^2} .$$

The drift is then:

$$\gamma = \frac{1 - 2\mu(a)^2(1-\mu(a)) - \mu(a)^2 - \mu(b)^2 - \mu(c)^2}{1-\mu(a)^2} .$$

Consider now a free product of the form $M = M_a \star M_b \star M_c \star M_d$, where M_i is equal either to $\mathbb{Z}_2$ or to $\mathbb{B}$. Let a_i be the generator of M_i. We consider the simple random walk on M, that is the random walk defined by $\mu : \forall i, \mu(a_i) = 1/4$. In Fig. 4, we have represented this random walk for $\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{B}$. (Recall that the undirected edges — correspond to two directed arcs $\rightleftarrows$.)

The values of the drift for all the possible cases are given in the Table below.

	$\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2$	$\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{B}$	$\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{B} \star \mathbb{B}$	$\mathbb{Z}_2 \star \mathbb{B} \star \mathbb{B} \star \mathbb{B}$	$\mathbb{B} \star \mathbb{B} \star \mathbb{B} \star \mathbb{B}$
γ	$1/2$	$(12+3\sqrt{2})/28$	$(6+\sqrt{3})/12$	$7/10$	$3/4$

The values $1/2, 7/10$, and $3/4$, can be obtained by elementary arguments without having to solve the Traffic Equations. For instance consider the case $M = \mathbb{Z}_2 \star \mathbb{B} \star \mathbb{B} \star \mathbb{B}$. Denote by z, resp. b, the set of elements of M whose normal form representatives end with the letter a_1 (the generator of $\mathbb{Z}_2$), resp. with one of the letters a_2, a_3, a_4 (the generators of the copies of

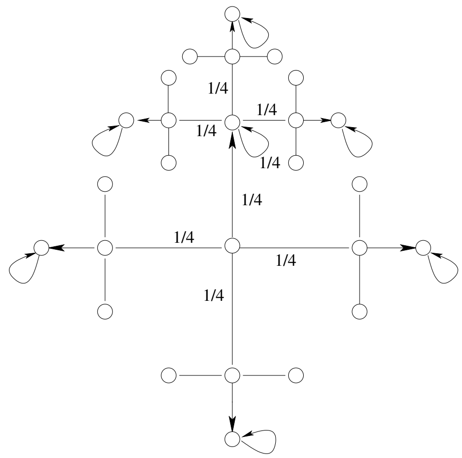

Figure 4: The simple random walk on $\mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{Z}_2 \star \mathbb{B}$.

$\mathbb{B}$). When we are far from 1_M, the random walk on M induces a Markov chain on $\{z, b\}$ with transition matrix P defined by: $P(b,b) = 3/4, P(b,z) = 1/4, P(z,b) = 1, P(z,z) = 0$. The corresponding stationary distribution π is given by: $\pi(b) = 4/5, \pi(z) = 1/5$. According to the Ergodic Theorem for Markov chains, we have: $\gamma = \pi(b) \times (3/4) + \pi(z) \times (1/2) = 7/10$.
On the other hand, the values $(12 + 3\sqrt{2})/28$ and $(6 + \sqrt{3})/12$ cannot be retrieved using such elementary arguments. Here it is necessary to solve the Traffic Equations.

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