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THE JOINT LAW OF AGES AND RESIDUAL LIFETIMES FOR TWO SCHEMES OF NESTED REGENERATIVE SETS

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Abstract We are interested in the component intervals of the complements of a monotone sequence $\mathcal{R}_n \subseteq \cdots \subseteq \mathcal{R}_1$ of regenerative sets, for two natural embeddings. One is based on Bochner's subordination, and one on the intersection of independent regenerative sets. For each scheme, we study the joint law of the so-called ages and residual lifetimes.

Keywords Multivariate renewal theory, regenerative set, subordinator, random covering intervals.

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1 Introduction

The classical renewal theory is concerned with the so-called age and residual lifetime

$$A(t) = \inf\{s > 0 : t - s \in \mathcal{R}\}, \qquad R(t) = \inf\{s > 0 : s + t \in \mathcal{R}\},$$

where $\mathcal{R}$ is a regenerative set, that is the closed range of some subordinator. In particular, these are two useful notions for investigating the partition induced by $\mathcal{R}$, that is the family of connected components of the complement of $\mathcal{R}$ (cf e.g. [18, 20]). Clearly, if $\mathcal{R}_1 \subseteq \mathcal{R}_2$, the partition induced by $\mathcal{R}_1$ is finer than that induced by $\mathcal{R}_2$, in the sense that it is obtained by breaking each interval of the partition induced by $\mathcal{R}_2$ into smaller intervals. Hence, a nested family of regenerative sets can be used to construct fragmentation or coalescent processes. This idea has been used in particular in [6], where a connection with the Bolthausen-Sznitman coalescent [8] is established (see also [4], [19] and the references therein for some related works).

The purpose of this paper is to determine the joint distribution of the sequence of ages and residual lifetimes associated with certain nested sequences of regenerative sets. Specifically, following [6], we shall consider two natural schemes to produce such families. One is based on Bochner's subordination, and the other on the intersection of independent regenerative sets. We shall focus on two natural sub-classes, stable and stationary regenerative sets, whose distributions are invariant under scaling and translation, respectively. The main reason for studying these special cases is that these are precisely the two types of distributions that arise in classical limit theorems for regenerative sets (typically the Dynkin-Lamperti theorem [7] and the renewal theorem [10]).

An interesting feature in our results is they show that in general, the joint law of the ages and residual lifetimes depends on the nesting scheme. This contrasts with [2], where it is established that the joint law of the ages alone only depends on the individual distributions of the regenerative sets.

The probability measure $\mathbb{P}^{\cap}$ will refer to the nesting scheme based on the intersection of independent regenerative sets. The probability measure $\mathbb{P}^{\star}$ will refer to the nesting scheme based on Bochner's subordination. In the stationary case, we show that the n-tuple $(A_1, R_1, \dots, A_n, R_n) \doteq (A_1(0), R_1(0), \dots, A_n(0), R_n(0))$ is an inhomogeneous Markov chain whose law is specified. In particular under $\mathbb{P}^{\star}$, this Markov chain has independent increments.

In the stable case, we give the semigroup of the bivariate Markov processes $(A_\alpha, R_\alpha; 0 < \alpha < 1) \doteq (A_\alpha(1), R_\alpha(1); 0 < \alpha < 1)$, both under $\mathbb{P}^{\cap}$ and $\mathbb{P}^{\star}$, and we specify its Lévy kernel under $\mathbb{P}^{\cap}$.

In the next section, we introduce some notation. The following two sections treat successively the stable case and the stationary case. The last section is devoted to some technical proofs.

2 Preliminaries

2.1 Basics about subordinators

Denote by $\vec{\sigma} = (\vec{\sigma}(t), t \geq 0)$ a subordinator started at 0, and by $\vec{\mathcal{R}}$ its closed range

$$\vec{\mathcal{R}} = \{x \geq 0 : \vec{\sigma}_t = x \text{ for some } t \geq 0\}^{\mathrm{cl}}.$$

To focus on the most interesting setting, we rule out the case of compound Poisson processes, so that in particular the regenerative set $\vec{\mathcal{R}}$ is non-lattice. The Laplace exponent ϕ of $\vec{\sigma}$ is characterized by

$$\mathbb{E}(\exp(-\lambda\vec{\sigma}_t)) = \exp(-t\phi(\lambda)), \qquad \lambda, t \geq 0.$$

It is specified by the Lévy-Khinchin formula

$$\phi(q) = \delta q + \int_0^\infty (1 - \mathrm{e}^{-qx})\Pi(dx) = q\left(\delta + \int_0^\infty \mathrm{e}^{-qx}\bar{\Pi}(x)dx\right), \qquad q \geq 0, \tag{1}$$

where $\bar{\Pi}(x) = \Pi(x, \infty)$, $x > 0$, is the tail of the Lévy measure.

Definition 2.1 *A subordinator is called stable when its Laplace exponent is a power function ($\phi(\lambda) = \lambda^\alpha$, for some $\alpha \in (0,1)$).*

The following equivalences are well-known

$$\int_0^\infty \bar{\Pi}(x)dx = \int_0^\infty x\Pi(dx) < \infty \Longleftrightarrow \lim_{q\to 0+} q^{-1}\phi(q) < \infty \Longleftrightarrow \mathbb{E}(\vec{\sigma}_1) < \infty.$$

We then say that $\vec{\mathcal{R}}$ is positive-recurrent and define

$$\mu \doteq \delta + \int_0^\infty x\Pi(dx) = \lim_{q\to 0+}\frac{\phi(q)}{q} = \mathbb{E}(\vec{\sigma}_1).$$

We denote by $M(\phi)$ the law of the pair $(UZ, (1-U)Z)$, where U is uniform on $(0,1)$ and Z is a nonnegative r.v. independent of U such that

1. $\mathbb{P}(Z = 0) = \delta/\mu$,
2. $\mathbb{P}(Z \in dz) = \mu^{-1}z\Pi(dz)$, $z > 0$.

The Laplace transform of the probability measure $M(\phi)$ is then

$$\int_0^\infty\int_0^\infty M(\phi)(dx\,dy)\exp(-\alpha x - \beta y) = \mu^{-1}\frac{\phi(\alpha) - \phi(\beta)}{\alpha - \beta}, \qquad \alpha, \beta > 0, \alpha \neq \beta. \tag{2}$$

In order to define stationary (i.e. translation invariant) regenerative sets, follow [14] and [21], and set

Definition 2.2 *The two-sided subordinator $\overleftrightarrow{\sigma} = (\overleftrightarrow{\sigma}(t), t \in \mathbb{R})$ is called stationary if*

- *The pair $(-\overleftrightarrow{\sigma}(0^-), \overleftrightarrow{\sigma}(0))$ follows $M(\phi)$,*
- *The processes $\sigma^+ = (\overleftrightarrow{\sigma}(t) - \overleftrightarrow{\sigma}(0), t \geq 0)$ and $\sigma^- = (-\overleftrightarrow{\sigma}(-t^-) + \overleftrightarrow{\sigma}(0^-), t \geq 0)$ are independent and both distributed as $(\vec{\sigma}(t), t \geq 0)$.*

We then denote by $\overleftrightarrow{\mathcal{R}}$ the closed range of $\overleftrightarrow{\sigma}$. We point out that stationary regenerative sets with σ-finite distributions can be defined when $\mu = \infty$ (null-recurrent case), as in [11].
For any closed subset $\mathcal{R}$ of $\mathbb{R}$, and any real number t, define the age $A(t)$ and the residual lifetime $R(t)$ by

$$A(t) = \inf\{s > 0 : t - s \in \mathcal{R}\}, \qquad R(t) = \inf\{s > 0 : s + t \in \mathcal{R}\}.$$

For convenience, we will sometimes use the following notation

$$G(t) = \sup\{s < t : s \in \mathcal{R}\} = t - A(t), \qquad D(t) = \inf\{s > 0 : s \in \mathcal{R}\} = t + R(t).$$

Let also s_t stand for the scaling operator, and θ_t for the shift operator

$$\mathrm{s}_t(\mathcal{R}) = \{s : ts \in \mathcal{R}\}, \qquad \theta_t(\mathcal{R}) = \{s : s + t \in \mathcal{R}\}.$$

The Dynkin-Lamperti theorem then states that for any regenerative set $\overrightarrow{\mathcal{R}}$ whose Laplace exponent is regularly varying at $0+$ with index α, $t^{-1}(A(t), R(t)) = (A(1), R(1)) \circ \mathrm{s}_t$ converges in distribution as $t \to \infty$ to the pair $(A(1), R(1))$ associated to a stable regenerative set of index α.
The renewal theorem states that for any positive-recurrent regenerative set $\overrightarrow{\mathcal{R}}$, $(A(t), R(t)) = (A(0), R(0)) \circ \theta_t$ converges in distribution as $t \to \infty$ to the pair $(A(0), R(0))$ of the associated stationary regenerative set, that is, the pair of variables with law $M(\phi)$.
We will now focus on these two possible limiting distributions, the scaling-invariant one, and the translation-invariant one.
(i) In the stable case, we will consider the age $A \doteq A(1)$ and the residual lifetime $R \doteq R(1)$ for stable regenerative sets, that we shall simply denote by $\mathcal{R}$ (instead of $\overrightarrow{\mathcal{R}}$).
(ii) In the positive-recurrent case, we will consider the age $A \doteq A(0)$ and the residual lifetime $R \doteq R(0)$ for stationary regenerative sets, that we shall simply denote by $\mathcal{R}$ (instead of $\overleftrightarrow{\mathcal{R}}$).

2.2 Two constructions of nested regenerative sets

Let $n \geq 1$. We here define the two schemes of n nested regenerative sets mentioned in the Introduction. We consider probability measures $\mathbb{P}^\star$ and $\mathbb{P}^\cap$ on the Cartesian product E^n, where E denotes the space of closed subsets of the line, endowed with Matheron's topology [17]. In the stable case, every set starts at zero (and is a subset of $\mathbb{R}_+$). In the stationary case, we let $\mathbb{P}_0$ stand for the regeneration law, ie the law of the shifted n-tuple of sets by any finite stopping time T w.r.t. $(\mathcal{F}_t)_{t\geq 0}$, where $T \in \mathcal{R}_k$ p.s., $k = 1, \ldots, n$, and $\mathcal{F}_t = \sigma((\mathcal{R}_1, \ldots, \mathcal{R}_n) \cap (-\infty, t]^n)$ (regeneration property w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$). We stress that for simplicity the integer n may sometimes equal 2 in the sequel without further change of notation.

In the intersection scheme, let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be independent regenerative sets, all stable subsets of $\mathbb{R}_+$ in the stable case, and stationary subsets of $\mathbb{R}$ in the stationary case. Define $\mathcal{R}_k = \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_k$, $k = 1, ..., n$. The probability measure $\mathbb{P}^\cap$ then denotes the law of $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$, and $\mathbb{P}_0^\cap$ its regeneration law.

We next define the measure $\mathbb{P}^\star$ related to Bochner's subordination. Consider n independent subordinators $\tau_1, \ldots, \tau_n$, all stable subordinators indexed by $\mathbb{R}_+$ in the stable case, and subordinators indexed by $\mathbb{R}$ in the stationary case. Define $\mathcal{R}_k$ as the closed range of $\tau_1 \circ \cdots \circ \tau_k$,

$k = 1, \dots, n$. The probability measure $\mathbb{P}^\star$ then denotes the law of $(\mathcal{R}_1, \dots, \mathcal{R}_n)$, and $\mathbb{P}_0^\star$ its regeneration law. For a deep study on this topic, see e.g. [9].

We have to assume that the coordinates $\mathcal{R}_k$, $k = 1, \dots, n$, are not trivial (that is, reduced to $\{0\}$ in the stable case, empty or discrete in the stationary case). Note that this always holds under $\mathbb{P}^\star$.
As for $\mathbb{P}^\cap$, the potential measures of the $\mathcal{S}_k$ have to satisfy in the positive-recurrent case some technical assumptions (see [3]). In the stable case, we recall that for independent stable regenerative sets $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ of indexes α and β respectively, $\mathcal{S}_\alpha \cap \mathcal{S}_\beta$ is a stable regenerative set of index $\alpha + \beta - 1$ if $\alpha + \beta > 1$, and is reduced to $\{0\}$ otherwise.

To make things clear, we state the following elementary

Lemma 2.1 *The following assertions hold under $\mathbb{P}^\cap$ and under $\mathbb{P}^\star$.*
In the stable (resp. stationary) case, each coordinate $\mathcal{R}_k$ is a stable (resp. stationary) regenerative set, $k = 1, \dots, n$.

This lemma is proved in the appendix, as well as both following statements.

Proposition 2.2 *Under $\mathbb{P}^\cap$ as under $\mathbb{P}^\star$, and for both the stable and stationary cases, $((A_k, R_k), 1 \le k \le n)$ is a (inhomogeneous) Markov chain with values in $[0, \infty)^2$.*

The following result is an extension of the renewal theorem. Let $\overrightarrow{\theta}_t$ be the positive shift operator, that is for any real set $\mathcal{R}$

$$\overrightarrow{\theta}_t(\mathcal{R}) = \{s \ge 0 : s + t \in \mathcal{R}\},$$

and let $\overleftarrow{\theta}_t$ be the negative shift operator

$$\overleftarrow{\theta}_t(\mathcal{R}) = \{s \ge 0 : -s + t \in \mathcal{R}\}.$$

We here denote by $X_n(t)$ the vector $(A_1(t), R_1(t), \dots, A_n(t), R_n(t))$.

Proposition 2.3 *Let $\mathbb{P}$ stand for the law of a regenerative n-tuple of non-discrete stationary embedded regenerative sets with regeneration law $\mathbb{P}_0$. Then for any bounded measurable h_d on E^n, h_g on E^n with compact support, f on $[0, \infty)^{2n}$,*

$$\lim_{t \to \infty} \mathbb{E}_0 (h_g \circ \overleftarrow{\theta}_{G_n(t)} f(X_n(t)) h_d \circ \overrightarrow{\theta}_{D_n(t)}) = \mathbb{E}(h_g \circ \overleftarrow{\theta}_{G_n(0)} f(X_n(0)) h_d \circ \overrightarrow{\theta}_{D_n(0)})$$

3 The stable case

In this section, we study stable regenerative sets $\mathcal{R}_\alpha$ of index $1 - \alpha$, $\alpha \in (0, 1)$. In this situation, it will be easier to consider the quantities $G_\alpha = 1 - A_\alpha$ and $D_\alpha = 1 + R_\alpha$, rather than A_α and R_α.
The distribution of (G_α, D_α) (resp. G_α, resp. D_α) has density r_α (resp. p_α, resp. q_α) where

(see [7])

$$
\begin{aligned}
r_\alpha(x,y) &= \frac{1-\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} x^{-\alpha}(y-x)^{\alpha-2}, \qquad 0<x<1<y,\\
p_\alpha(x) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} x^{-\alpha}(1-x)^{\alpha-1}, \qquad 0<x<1,\\
q_\alpha(y) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} y^{-1}(y-1)^{\alpha-1}, \qquad y>1.
\end{aligned}
$$

In the first subsection, we will consider $\alpha, \beta \in (0,1)$ such that $\gamma \doteq \alpha+\beta < 1$. It is known that if $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are two independent stable regenerative sets of indices $1-\alpha$ and $1-\beta$, respectively, then $\mathcal{R}_\gamma \doteq \mathcal{R}_\alpha \cap \mathcal{R}_\beta$ is a stable regenerative set of index $1-\gamma$ embedded in $\mathcal{R}_\alpha$.

In the second subsection, we will consider two independent stable subordinators τ_α and τ_β with indices $1-\alpha$ and $1-\beta$, respectively. Writing $1-\gamma \doteq (1-\alpha)(1-\beta)$, it is known that the closed range $\mathcal{R}_\gamma$ of $\tau_\alpha \circ \tau_\beta$ is a stable regenerative set of index $1-\gamma$ which is embedded in the closed range $\mathcal{R}_\alpha$ of τ_α.

As said in the Preliminaries, the law of $(\mathcal{R}_\alpha, \mathcal{R}_\gamma)$ will be denoted by $\mathbb{P}^\cap$ in the first construction, and by $\mathbb{P}^\star$ in the second one.

3.1 The intersection scheme

In the next theorem, we give the joint law of $(G_\alpha, D_\alpha, G_\gamma, D_\gamma)$ under $\mathbb{P}^\cap$. At the end of the present subsection, we will more generally consider the process $(G_\alpha, D_\alpha; 0<\alpha<1)$ under $\mathbb{P}^\cap$, and we will determine the Lévy kernel of the Markov process $(G_\alpha, D_\alpha; 0<\alpha<1)$ (its semigroup is given by the next theorem).

Theorem 3.1 *Conditional on $(G_\alpha = g,\ D_\alpha = d)$, $(1/G_\gamma, D_\gamma)$ is distributed under $\mathbb{P}^\star$ as*

$$
\left(\frac{1}{g} + \left(\frac{1}{m} - \frac{1}{g}\right)\frac{1}{\Gamma},\ d + (M-d)\Delta\right),
$$

where Γ and Δ are two independent r.v.'s, independent of (m, M), with densities p_γ and q_γ, respectively, and

$$
\mathbb{P}(m > u, M < v \mid G_\alpha = g, D_\alpha = d) = \left(\frac{(g-u)(v-d)}{g(v-u)}\right)^{\gamma-\alpha} \qquad u \in (0,g), v \in (d,\infty).
$$

In order to compute the conditional density of (G_γ, D_γ), we recall that for $0<\mu<\nu$, $\lambda \in \mathbb{R}$, $z \mapsto F(\lambda,\mu,\nu,z)$ is the hypergeometric function defined by

$$
F(\lambda,\mu,\nu,z) = \frac{1}{B(\mu,\nu-\mu)} \int_0^1 t^{\mu-1}(1-t)^{\nu-\mu-1}(1-tz)^{-\lambda}dt, \qquad z \in \mathbb{R}, \tag{3}
$$

where B stands for the beta function.

The proof of the following corollary is to be found in the last section.

Corollary 3.1.1 *For any* $0 < x < g < 1 < d < y$, $0 < \gamma < \alpha < 1$, *with* $\beta = \gamma - \alpha$,

$$\mathbb{P}^{\cap}(G_\gamma \in dx, D_\gamma \in dy \mid G_\alpha = g, D_\alpha = d)/dx\,dy$$

$$= B(\beta, 1-\gamma)^{-2}(d-g)^{1-\gamma} g^{\alpha} x^{-\gamma} \frac{((g-x)(y-d))^{\beta-1}}{(y-x)^{1-\alpha}} H\left(\frac{(g-x)(y-d)}{(d-g)(y-x)}\right),$$

where H *is the function defined by*

$$H(x) = 1 + \frac{(1-\alpha)(1-\gamma)}{\beta} \int_0^x dz\, F(2-\alpha, \gamma, 1+\beta, -z).$$

The proof of Theorem 3.1 uses a representation of stable regenerative sets by random intervals extracted from [12]. These intervals are generated by Poisson measures on $(0,\infty)^2$, which were also used in [6] to construct the intersection embedding.

Consider a Poisson point process $(t, l_t; t \geq 0)$ taking values in $(0,\infty)^2$ with intensity $dx\,\mu(dy)$. To each point (t, l_t) of $(0,\infty)^2$, we associate the open interval $I_t = (t, t + l_t)$. For any Borel set B in $(0,\infty)^2$, we define $\mathcal{C}(B)$ as the open subset covered by the random intervals associated to the points 'fallen' in B

$$\mathcal{C}(B) = \bigcup_{(t,l_t)\in B} I_t,$$

and consider the random set left uncovered by all these intervals

$$\mathcal{R} = [0,\infty) \setminus \bigcup_{t\geq 0} I_t = [0,\infty) \setminus \mathcal{C}((0,\infty)^2).$$

If $\bar{\mu}$ denotes the tail of μ (that is $\bar{\mu}(x) = \mu(x,\infty)$, $x > 0$), then a theorem extracted from [12] states that

$$\int_0^1 \exp\left(\int_t^1 \bar{\mu}(s)ds\right)dt < \infty$$

$$\Downarrow$$

$$\mathcal{R} \text{ is a perfect regenerative set.}$$

For instance, take $\bar{\mu}_\alpha(x) = \alpha x^{-1}$, for some $\alpha \in (0,1)$, and $\mathcal{R}$ is then a stable regenerative set with index $1-\alpha$. These tools are easily adapted to our situation, since for two independent Poisson measures with respective intensities $dx\,\mu_\alpha(dy)$ and $dx\,\mu_\beta(dy)$, $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are two independent regenerative sets such that

$$\mathcal{R}_\alpha \cap \mathcal{R}_\beta = [0,\infty) \setminus (\mathcal{C}_\alpha((0,\infty)^2) \cup \mathcal{C}_\beta((0,\infty)^2)).$$

In words, $\mathcal{R}_\alpha \cap \mathcal{R}_\beta$ is the set of points left uncovered by the intervals associated to the point process with characteristic tail $\bar{\mu}^{\cap}(x) = \bar{\mu}_\alpha(x) + \bar{\mu}_\beta(x) = (\alpha+\beta)x^{-1}$, $x > 0$, and one recovers the fact that $\mathcal{R}_\alpha \cap \mathcal{R}_\beta$ is stable with index $1-\gamma$, where $\gamma = \alpha + \beta$.

The following two lemmas will be useful for the proof of the theorem.

Lemma 3.2 *For $0 < a < b$, set*

$$V(a,b) = \{(x,y) \in (0,\infty)^2 : x+y > b, x > a\}.$$

Then for any $\gamma \in (0,1)$, the following equality in distribution holds

$$\inf([b,\infty) \setminus \mathcal{C}_\gamma(V(a,b))) \overset{(d)}{=} a + (b-a)\Delta,$$

where Δ has density q_γ.

Proof of Lemma 3.2. By translation invariance of Lebesgue measure, it suffices to show the result for $a = 0$. Since all the intervals associated with the complement of $V(0,b)$ are a.s. included in $(0,b)$, the first point left uncovered by $V(0,b)$ is the first point greater than b in the complement of $\mathcal{C}_\gamma((0,\infty)^2)$, and consequently it has the same law as $D_\gamma(b)$. Conclude by recalling that $D_\gamma(b) \overset{(d)}{=} b\,D_\gamma$, and that D_γ has density q_γ. □

Lemma 3.3 *For $0 < a < b$, set*

$$U(a,b) = \{(x,y) \in (0,\infty)^2 : x+y < b, x < a\}.$$

Then for any $\gamma \in (0,1)$, the following equality in distribution holds

$$\sup((0,a] \setminus \mathcal{C}_\gamma(U(a,b))) \overset{(d)}{=} \left(\frac{1}{b} + (\frac{1}{a} - \frac{1}{b})\frac{1}{\Gamma}\right)^{-1},$$

where Γ has density p_γ.

Proof of Lemma 3.3. The main step is proving that the image of $\mathcal{C}_\gamma(U(a,b))$ by the mapping $x \mapsto x^{-1}$ has the same law as $\mathcal{C}_\gamma(V(b^{-1},a^{-1}))$. We will then use the previous lemma to conclude. For convenience, we write

$$\tilde{G} = \sup((0,a] \setminus \mathcal{C}_\gamma(U(a,b))).$$

Define the mapping $\psi : (0,\infty)^2 \to (0,\infty)^2$ by

$$\psi(x,y) = (\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y}), \qquad x,y > 0,$$

and notice that $\psi(U(a,b)) = V(b^{-1},a^{-1})$. If $\varepsilon_{(x,y)}$ denotes the Dirac mass at (x,y), then the counting measure $\sum_{(t,l_t)\in U(a,b)} \varepsilon_{(t,l_t)}$ is a Poisson measure on $U(a,b)$ with intensity $\rho(dx\,dy) = \gamma y^{-2} dx\,dy$ $(x,y>0)$. As the image of $\mathcal{C}_\gamma(U(a,b))$ by the mapping $x \mapsto x^{-1}$ is

$$\bigcup_{(t,l_t)\in U(a,b)} (\frac{1}{t+l_t}, \frac{1}{t}),$$

it is thus distributed as

$$\bigcup_{(t,l'_t)\in V(b^{-1},a^{-1})} (t, t+l'_t),$$

where $\sum_{(t,l'_t)\in V(b^{-1},a^{-1})} \varepsilon_{(t,l'_t)}$ is a Poisson measure on $V(b^{-1},a^{-1})$ with intensity $\rho\circ\psi^{-1}$. A straightforward calculation now shows that $\rho\circ\psi^{-1}=\rho$, and thanks to Lemma 3.2,

$$\begin{aligned}\frac{1}{\tilde{G}} &\overset{(d)}{=} \inf([a^{-1},\infty)\setminus \mathcal{C}_\gamma(V(b^{-1},a^{-1})))\\ &\overset{(d)}{=} b^{-1}+(a^{-1}-b^{-1})\Delta,\end{aligned}$$

where Δ has density q_γ. We conclude with the observation that if Δ has density q_γ, then $\Gamma=\Delta^{-1}$ has density p_γ. □

We now provide the

Proof of Theorem 3.1. We work under the conditional probability given $(G_\alpha=g, D_\alpha=d)$. Let Z be the following subset of $(0,\infty)^2$

$$Z=\{(x,y): x\le d, x+y\ge g\}.$$

Then a.s. under the conditional probability, $\mathcal{C}_\alpha(Z)=(g,d)$. On the other hand, since the characteristic measure μ_β is integrable on every interval (ε,∞) $(\varepsilon>0)$, but not integrable at $0+$, there are r.v.'s $0<m<g$, and $d<M<\infty$, such that

$$\mathcal{C}_\beta(Z)\cap(0,g]=(m,g],$$

$$\mathcal{C}_\beta(Z)\cap[d,\infty)=[d,M),$$

with

$$\begin{aligned} m&=\min\{t:(t,l_t)\in Z\},\\ M&=\max\{t+l_t:(t,l_t)\in Z\},\end{aligned}$$

where (t,l_t) refer to the points of the Poisson process with characteristic measure μ_β. Following what is explained before the statement of the theorem, recall that for any Borel set B, $\mathcal{C}_\alpha(B)\cup\mathcal{C}_\beta(B)=\mathcal{C}_\gamma(B)$, and in particular $\mathcal{C}_\gamma(Z)=(m,M)$. Before computing the (conditional) law of (m,M), note then that conditional on $(G_\alpha,D_\alpha)=(g,d)$, and $(m,M)=(u,v)$,

$$\mathcal{C}_\gamma((U(u,g)\cup V(d,v))^c)=(u,v),$$

with the notation of the preceding lemmas, that is

$$U(u,g)=\{(x,y)\in(0,\infty)^2: x+y<g, x<u\},$$

$$V(d,v)=\{(x,y)\in(0,\infty)^2: x+y>v, x>d\}.$$

As a consequence, $\mathcal{C}_\gamma((0,\infty)^2)=\mathcal{C}_\gamma(U(u,g))\cup\mathcal{C}_\gamma(V(d,v))\cup(u,v)$, so that

$$G_\gamma=\sup((0,1]\setminus\mathcal{C}_\gamma((0,\infty)^2))=\sup((0,u]\setminus\mathcal{C}_\gamma((0,\infty)^2))=\sup((0,u]\setminus\mathcal{C}_\gamma(U(u,g))),$$

and similarly

$$D_\gamma=\inf([v,\infty)\setminus\mathcal{C}_\gamma(V(d,v))).$$

Recall that to disjoint Borel sets correspond independent counting measures. Since the event $\{(G_\alpha,D_\alpha)=(g,d),(m,M)=(u,v)\}$ is measurable w.r.t. the counting measure on Z, the independence property between the counting measures on $U(u,g)$ and $V(d,v)$ (which have empty intersection with Z and are disjoint) still holds under the conditional law. Hence conditional on $(G_\alpha, D_\alpha) = (g,d)$, and $(m,M) = (u,v)$, according to Lemma 3.3,

$$\frac{1}{G_\gamma} \stackrel{(d)}{=} \frac{1}{g} + (\frac{1}{u} - \frac{1}{g})\frac{1}{\Gamma},$$

where Γ has density p_γ. Similarly, the conditional law of D_γ according to Lemma 3.2 is

$$D_\gamma \stackrel{(d)}{=} d + (v-d)\Delta,$$

where Δ has density q_γ. But $U(u,g) \cap V(d,v) = \emptyset$, hence Γ and Δ are (conditionally) independent. Since none of the distributions of Γ and Δ involves u or v, an integration w.r.t. the law of (m,M) yields the independence between Γ and Δ, and between (Γ,Δ) and (m,M), and completes this part of the proof.

It thus remains to compute the (conditional) law of (m,M). It is clear that if

$$Z' = \{(x,y) \in Z : x \geq u, x+y \leq v\}, \qquad u < g, v > d,$$

then

$$\begin{aligned}
\mathbb{P}(m > u, M < v) &= \mathbb{P}(\text{ the Poisson measure associated to } \mu_\beta \text{ has no atom in } Z\backslash Z') \\
&= \exp\left\{-\beta \int_{Z\backslash Z'} dx\, dy\, y^{-2}\right\} \\
&= \exp\left\{-\beta \left[\int_0^u dx \int_{g-x}^\infty dy\, y^{-2} + \int_u^d dx \int_{v-x}^\infty dy\, y^{-2}\right]\right\} \\
&= \exp\left\{-\beta \left[\int_0^u \frac{dx}{g-x} + \int_u^d \frac{dx}{v-x}\right]\right\} \\
&= \exp\left\{-\beta[-\ln(g-u) + \ln(g) - \ln(v-d) + \ln(v-u)]\right\} \\
&= \left(\frac{(g-u)(v-d)}{g(v-u)}\right)^\beta,
\end{aligned}$$

which is the expected expression. □

As announced in the beginning of this subsection, we now deal with a monotone family $(\mathcal{R}_\alpha, 0 < \alpha < 1)$, where $\mathcal{R}_\alpha$ is a regenerative set of index $1-\alpha$ $(0 < \alpha < 1)$ such that for $\alpha < \gamma$

$$\mathcal{R}_\gamma \stackrel{(d)}{=} \mathcal{R}_\alpha \cap \bar{\mathcal{R}}, \tag{4}$$

where $\bar{\mathcal{R}}$ is some independent stable regenerative set of index $1-(\gamma-\alpha)$. We know that the process $(G_\alpha, D_\alpha; 0 < \alpha < 1)$ is Markovian, and its semigroup is given by Theorem 3.1. We first describe the construction [6] of the family $(\mathcal{R}_\alpha, 0 < \alpha < 1)$, and then state a theorem giving the Lévy kernel of $(G_\alpha, D_\alpha; 0 < \alpha < 1)$.

Recall that if (x_i, y_i), $i \in I$, are the atoms of a Poisson measure on $(0,\infty)^2$ with characteristic measure $\mu_\alpha(dx\,dy) = \alpha y^{-2} dx\,dy$, then the closed set $[0,\infty) \setminus \cup_{i\in I}(x_i, x_i + y_i)$ is a stable regenerative set of index $1-\alpha$. As in Construction 5 in [6], generate a Poisson measure on $(0,\infty)^2 \times (0,1)$ with intensity $y^{-2}dx\,dy\,da$, and define $\mathcal{R}_\alpha$ as the set left uncovered by intervals (x_i, x_i+y_i), $i \in I$ corresponding to the atoms (x_i, y_i, a_i) of the Poisson measure such that $a_i \leq \alpha$. In particular, the following hold. The set $\mathcal{R}_\alpha$ is a stable regenerative set of index $1-\alpha$, the $\mathcal{R}_\alpha$'s are nested, meaning that $\mathcal{R}_\gamma \subseteq \mathcal{R}_\alpha$ for $0<\alpha<\gamma<1$, and if $\bar{\mathcal{R}}$ is the set left uncovered by intervals (x_i, x_i+y_i), $i \in I$, corresponding to points such that $\alpha \leq a_i < \gamma$, then $\bar{\mathcal{R}}$ satisfies equation (4).

We now focus on the process $(G_\alpha, D_\alpha; 0<\alpha<1)$. Recall from [6, Proposition 6], that there is the equality of finite-dimensional distributions

$$(1-G_\alpha, 0<\alpha<1) \overset{(d)}{=} (\frac{\Gamma_\alpha}{\Gamma_1}, 0<\alpha<1), \tag{5}$$

where $(\Gamma_\alpha, 0<\alpha<1)$ stands for a gamma subordinator.
On the other hand, we know from [22] that for fixed $\alpha \in (0,1)$, the distribution of $\mathcal{R}_\alpha$ is invariant under the action of $x \mapsto x^{-1}$, and consequently

$$G_\alpha \overset{(d)}{=} \frac{1}{D_\alpha}.$$

A slight modification of the proof of Lemma 3.3 shows in fact that for our construction $(\mathcal{R}_\alpha, 0 < \alpha < 1)$,

$$(G_\alpha, 0<\alpha<1) \overset{(d)}{=} (\frac{1}{D_\alpha}, 0<\alpha<1).$$

For convenience, we thus study the process $((G_\alpha)^{-1}, D_\alpha; 0<\alpha<1)$. We recall that the Lévy kernel N describes the distribution of the jumps of this (inhomogeneous) Markov process. More precisely, it is the mapping that associates to $(\alpha, g^{-1}, d) \in (0,1)\times(1,\infty)^2$ a σ-finite measure $N(\alpha, g^{-1}, d; \cdot)$ on $(0,\infty)^2$, such that for any nonnegative Borel function h on $(1,\infty)^4$ and nonnegative $(H_\alpha, 0<\alpha<1)$ predictable w.r.t. the natural filtration generated by $(G_\alpha, D_\alpha; 0 < \alpha < 1)$,

$$\begin{aligned} &\mathbb{E}^\cap \sum_{0<\alpha<1} H_\alpha\, h(\frac{1}{G_{\alpha-}}, D_{\alpha-}, \frac{1}{G_\alpha}, D_\alpha) \\ &= \mathbb{E}^\cap \int_0^1 d\alpha\, H_\alpha \int_0^\infty \int_0^\infty N(\alpha, \frac{1}{G_\alpha}, D_\alpha; dx\,dy) h(\frac{1}{G_\alpha}, D_\alpha, \frac{1}{G_\alpha}+x, D_\alpha+y), \end{aligned} \tag{6}$$

where the sum in the l.h.s. is taken over the (countable) jumps of $(G_\alpha^{-1}, D_\alpha; 0<\alpha<1)$.
Recall that F is the hypergeometric function defined by (3), and for $\beta \in (0,1)$, let p_β and q_β denote by abuse of notation the respective laws of G_β and D_β.

Theorem 3.4 *The Lévy kernel of $((G_\alpha)^{-1}, D_\alpha; 0<\alpha<1)$ defined by (6) can be expressed for* *any* $(\alpha, g^{-1}, d) \in (0,1) \times (1,\infty)^2$ *and nonnegative measurable function* f *on* $(0,\infty)^2$ *by*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty N(\alpha, g^{-1}, d; dx\, dy) f(x,y) \\
&\quad = \int_0^1 p_\alpha(d\Gamma) \int_1^\infty q_\alpha(d\Delta) \quad \times \quad \left\{ \int_0^g dx \int_d^\infty dy (y-x)^{-2} f\left(\left(\frac{1}{x} - \frac{1}{g}\right)\frac{1}{\Gamma}, (y-d)\Delta\right) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_0^g dx\, \frac{d-g}{(d-x)(g-x)} f\left(\left(\frac{1}{x} - \frac{1}{g}\right)\frac{1}{\Gamma}, 0\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \int_d^\infty dy\, \frac{d-g}{(y-d)(y-g)} f(0, (y-d)\Delta) \right\}.
\end{aligned}$$

Dropping α, g, d *for clarity, this yields*

$$N(dx\, dy) = N_0(dx\, dy) + N_1(dx)\varepsilon_0(dy) + \varepsilon_0(dx) N_2(dy), \qquad x, y > 0,$$

where ε_0 *stands for the Dirac mass at 0 and*

$$N_0(dx\, dy) = \frac{\alpha^2}{((g^{-1}+x)(d+y)-1)^{\alpha+1}} F\left(\alpha+1, 1-\alpha, 2, \frac{-xy}{(dg^{-1}-1)((g^{-1}+x)(d+y)-1)}\right) dx\, dy,$$

$$\begin{aligned}
N_1(dx) &= x^{-1}\left(1 + \frac{x}{g^{-1}-d^{-1}}\right)^{-\alpha} dx, \\
N_2(dy) &= y^{-1}\left(1 + \frac{y}{d-g}\right)^{-\alpha} dy.
\end{aligned}$$

In words, N_0 describes the simultaneous jumps of G_α and D_α, N_1 those of G_α alone, and N_2 those of D_α alone. Roughly speaking, the bivariate process $(G_\alpha, D_\alpha; 0 < \alpha < 1)$ jumps at time α iff there is a point of the tridimensional point process in $T_\alpha^{(0)} \cup T_\alpha^{(1)} \cup T_\alpha^{(2)}$, where

$$T_\alpha^{(0)} = \{(x,y,a) : x < G_{\alpha-}, x+y > D_{\alpha-}, a = \alpha\},$$

$$\begin{aligned}
T_\alpha^{(1)} &= \{(x,y,a) : x < G_{\alpha-}, G_{\alpha-} < x+y < D_{\alpha-}, a = \alpha\}, \\
T_\alpha^{(2)} &= \{(x,y,a) : G_{\alpha-} < x < D_{\alpha-}, x+y > D_{\alpha-}, a = \alpha\}.
\end{aligned}$$

Then α is a jump time of both coordinates, of the first coordinate only, or of the second one only, whether this point 'fell' into $T_\alpha^{(0)}$, $T_\alpha^{(1)}$, or $T_\alpha^{(2)}$.

Proof. Consider for $\alpha \in (0,1)$ the random subsets Z_α and Z'_α of $[0,\infty)^2 \times \{\alpha\}$

$$\begin{aligned}
Z_\alpha &= \{(x,y,a) : x < D_{\alpha-}, x+y > G_{\alpha-}, a = \alpha\}, \\
Z'_\alpha &= \{(x,y,a) : x > G_{\alpha-}, x+y < D_{\alpha-}, a = \alpha\}.
\end{aligned}$$

As in the proof of Theorem 3.1, note that β is a jump time of $(G_\alpha, D_\alpha; 0 < \alpha < 1)$ iff there is a point of the Poisson process in $Z_\beta \backslash Z'_\beta$. For such a point $(x_\beta, y_\beta, \beta)$, let

$$m_\beta = \min\{x_\beta, G_{\beta-}\} \qquad M_\beta = \max\{x_\beta + y_\beta, D_{\beta-}\}.$$

Referring to the proof of Theorem 3.1, by optional projection on the natural filtration generated by $(m_\alpha, M_\alpha; 0<\alpha<1)$ applied to the l.h.s. of (6), we get

$$\mathbb{E}\sum_{0<\alpha<1} H_\alpha h(\frac{1}{G_{\alpha-}}, D_{\alpha-}, \frac{1}{G_\alpha}, D_\alpha) = \mathbb{E}\sum_{0<\alpha<1} H_\alpha \int_0^1 p_\alpha(d\Gamma)\int_1^\infty q_\alpha(d\Delta)$$

$$\times h(\frac{1}{G_{\alpha-}}, D_{\alpha-}, \frac{1}{G_{\alpha-}} + \frac{1}{\Gamma}(\frac{1}{m_\alpha} - \frac{1}{G_{\alpha-}}), D_{\alpha-} + \Delta(M_\alpha - D_{\alpha-})).$$

Now the point process $(x_i, x_i+y_i, a_i; i\in I)$ indexed by its third component is a Poisson point process with characteristic measure $(v-u)^{-2} du\, dv$. By predictable projection on the natural filtration generated by $(G_\alpha, D_\alpha; 0<\alpha<1)$, the last displayed quantity equals

$$\mathbb{E}\int_0^1 d\alpha\, H_\alpha \int_0^1 p_\alpha(d\Gamma)\int_1^\infty q_\alpha(d\Delta)$$

$$\begin{aligned}
\times\ & \Big\{\int_0^{G_{\alpha-}} du \int_{D_{\alpha-}}^\infty dv\,(v-u)^{-2} h(\frac{1}{G_{\alpha-}}, D_{\alpha-}, \frac{1}{G_{\alpha-}} + (\frac{1}{u} - \frac{1}{G_{\alpha-}})\frac{1}{\Gamma}, D_{\alpha-} + (v - D_{\alpha-})\Delta) \\
+\ & \int_0^{G_{\alpha-}} du \int_{G_{\alpha-}}^{D_{\alpha-}} dv\,(v-u)^{-2} h(\frac{1}{G_{\alpha-}}, D_{\alpha-}, \frac{1}{G_{\alpha-}} + (\frac{1}{u} - \frac{1}{G_{\alpha-}})\frac{1}{\Gamma}, D_{\alpha-}) \\
+\ & \int_{G_{\alpha-}}^{D_{\alpha-}} du \int_{D_{\alpha-}}^\infty dv\,(v-u)^{-2} h(\frac{1}{G_{\alpha-}}, D_{\alpha-}, \frac{1}{G_{\alpha-}}, D_{\alpha-} + (v - D_{\alpha-})\Delta)\Big\}
\end{aligned}$$

and some elementary integration yields the r.h.s. in (6).
We skip the exact computation of N_0, N_1, N_2, which relies on the same arguments as in the proof of Corollary 3.1.1. □

3.2 The subordination scheme

Denote by $p_t^{(\alpha)}$ the density of $\tau_\alpha(t)$. In particular, the scaling property entails that

$$p_t^{(\alpha)}(x) = t^{-1/\alpha} p_1^{(\alpha)}(t^{-1/\alpha} x), \qquad t, x > 0.$$

Theorem 3.5 *The distribution of the quadruple $(G_\alpha, D_\alpha, G_\gamma, D_\gamma)$ under $\mathbb{P}^\star$ can be described as follows. For any $0<g<1<d$, $0<x<g, y>0$,*

$$\mathbb{P}^\star(G_\alpha - G_\gamma \in dx,\ D_\gamma - D_\alpha \in dy \mid G_\alpha = g, D_\alpha = d)/dx\,dy$$

$$= \frac{(1-\beta)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\beta)} g^\alpha \int_0^1 ds \int_1^\infty dt\, s^{-\beta}(t-s)^{\beta-2}\int_0^\infty dl\, p_{ls}^{(\alpha)}(g-x) p_{l(1-s)}^{(\alpha)}(x) p_{l(t-1)}^{(\alpha)}(y).$$

In particular, the pair $(G_\alpha - G_\gamma, D_\gamma - D_\alpha)$ is independent of D_α.

Before proving the theorem, we will need the following lemma, where for any subordinator τ started at 0

$$L \doteq L(1) = \inf\{t \geq 0 : \tau(t) > 1\}$$

is the local time.

Lemma 3.6 *Conditional on $L = l, G = g, D = d$, the pre-l process $(\tau(t), t \le l)$ and the post-l process $(\tau(t+l) - d, t \ge 0)$ are independent. Moreover, the post-l process is distributed as τ, and with the usual convention $\tau(0-) = 0$, the process $(g - \tau((l-t)-), t \le l)$ has the law of the bridge of τ from 0 to g over $[0, l]$.*

Proof of Theorem 3.5. We first work conditionally on $L_\alpha = l, G_\alpha = g, D_\alpha = d$. In particular, $g = \tau_\alpha(l-)$, $d = \tau_\alpha(l)$. Next notice that

$$L_\gamma = \inf\{t \ge 0 : \tau_\alpha \circ \tau_\beta(t) > 1\} = \inf\{t \ge 0 : \tau_\beta(t) > l\} = L_\beta(l).$$

Therefore

$$D_\gamma = \tau_\alpha(\tau_\beta(L_\gamma)) = \tau_\alpha(D_\beta(l)) \qquad \text{a.s.}$$

Similarly we get

$$G_\gamma = \tau_\alpha(G_\beta(l)) \qquad \text{a.s.}$$

Now by the scaling property, $(G_\beta(l),\, D_\beta(l)) \stackrel{(d)}{=} l(G_\beta,\, D_\beta)$. Then for any nonnegative measurable f and h,

$$\begin{aligned} &\mathbb{E}^\star\left(f(G_\alpha - G_\gamma)\, h(D_\gamma - D_\alpha) \mid L_\alpha = l,\, G_\alpha = g,\, D_\alpha = d\right) \\ &= \int_0^1 \int_1^\infty ds\, dt\, r_\beta(s,t) \mathbb{E}(f(g - \tau_\alpha(ls)) h(\tau_\alpha(lt) - d) \mid L_\alpha = l,\, G_\alpha = g,\, D_\alpha = d), \end{aligned}$$

Thanks to the previous lemma and from classical results on one-dimensional densities of bridges,

$$\begin{aligned} &\mathbb{P}^\star(G_\alpha - G_\gamma \in dx, D_\gamma - D_\alpha \in dy \mid L_\alpha = l,\, G_\alpha = g,\, D_\alpha = d)/dx\, dy \\ &= \int_0^1 \int_1^\infty ds\, dt\, r_\beta(s,t) \frac{p_{ls}^{(\alpha)}(g-x) p_{l(1-s)}^{(\alpha)}(x)}{p_l^{(\alpha)}(g)} p_{l(t-1)}^{(\alpha)}(y), \qquad 0 < x < g, y > 0. \end{aligned}$$

It only remains to integrate w.r.t. the conditional law of L_α. Recall from Theorem 4.1 in [13] that

$$\mathbb{P}(L_\alpha \in dl,\, G_\alpha \in dg,\, D_\alpha - G_\alpha \in dr) = \Pi_\alpha(dr) p_l^{(\alpha)}(g) dl\, dg, \tag{7}$$

where Π_α stands for the Lévy measure of τ_α, so we conclude since

$$\mathbb{P}(G_\alpha \in dg,\, D_\alpha - G_\alpha \in dr) = \Pi_\alpha(dr) u_\alpha(g) dg,$$

where $u_\alpha(g) = g^{-\alpha}/\Gamma(1-\alpha)$ is the potential density of τ_α at level g. $\square$

Proof of Lemma 3.6. The first assertion stems from the strong Markov property applied at the stopping time L. We next use the compensation formula, writing $\Delta\tau_L = D - G$ for the jump of τ at time L. For any nonnegative measurable functional F and function h,

$$\begin{aligned} &\mathbb{E}(F(\tau(L-) - \tau((L-s)-), s \le L) h(\Delta\tau_L)) \\ &= \mathbb{E} \sum_{t : \Delta\tau_t > 0} F(\tau(t-) - \tau((t-s)-), s \le t) h(\Delta\tau(t)) \mathbf{1}_{\tau(t-) < 1} \mathbf{1}_{\tau(t) > 1} \\ &= \mathbb{E} \int_0^\infty dt\, F(\tau(t-) - \tau((t-s)-), s \le t) \mathbf{1}_{\tau(t-) < 1} \int_0^\infty \Pi(dr) h(r) \mathbf{1}_{r > 1 - \tau(t-)}. \end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}(F(\tau(L-) - \tau((L-s)-), s \leq L), L \in dl,\, G \in dg,\, D-G \in dr) \\
&= dl\,\Pi(dr)\,\mathbb{P}(\tau(l) \in dg)\mathbb{E}(F(g - \tau((l-s)-), s \leq l) \mid \tau(l) = g) \\
&= dl\,\Pi(dr)\,\mathbb{P}(\tau(l) \in dg)\mathbb{E}(F(\tau(s), s \leq l) \mid \tau(l) = g),
\end{aligned}$$

where the last equality stems from the duality lemma for Lévy processes, which ensures that the bridge's distribution is invariant under time-reversal. By (7), we get that

$$\mathbb{E}(F(\tau(L-) - \tau((L-s)-), s \leq L) \mid L = l, G = g, D - G = r) = \mathbb{E}(F(\tau(s), s \leq l) \mid \tau(l) = g),$$

which ends the proof. □

4 The stationary case

Recall that by Proposition 2.3, under the hypothesis of non-triviality, the distribution of the shifted n-tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_n) \circ \theta_t$ under $\mathbb{P}_0^{\cap}$ (resp. $\mathbb{P}_0^{\star}$) converges as $t \to \infty$, to that of $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ under $\mathbb{P}^{\cap}$ (resp. $\mathbb{P}^{\star}$). In the next subsections, we specify these limiting distributions for each scheme (intersection and subordination).

4.1 The intersection scheme

We treat here the case of intersections of n independent stationary regenerative sets. Since by Proposition 2.2, $(A_k, R_k; 1 \leq k \leq n)$ is Markovian, we focus on the case $n = 2$, and change general notation within this subsection only. Namely, let $\mathcal{R}_0$ and $\mathcal{R}_1$ be two independent stationary regenerative sets and set

$$\mathcal{R}_2 \doteq \mathcal{R}_1 \cap \mathcal{R}_0.$$

Recall from the Preliminaries that $\mathcal{R}_2$ is assumed to be nonempty. The probability measure $\mathbb{P}^{\cap}$ then refers to the pair $(\mathcal{R}_1, \mathcal{R}_2)$. We here can no longer use the random covering intervals' representation, since it does not encompass all cases of regenerative sets.

We need to introduce U the potential (or renewal) measure of the generic subordinator σ, defined by

$$U(dx) = \int_0^\infty \mathbb{P}(\vec{\sigma}_t \in dx)\, dt, \qquad x \geq 0.$$

We then have the identity

$$\int_0^\infty U(dx) \exp(-qx) = \frac{1}{\phi(q)}, \qquad q \geq 0. \tag{8}$$

We further assume that U_0 is absolutely continuous with density u_0 (this is in particular the case when its drift is $\delta_0 > 0$). Thanks to Lemma 2.1, $\mathcal{R}_2$ is a stationary regenerative set which is the closed range of some two-sided subordinator σ_2 with Lévy measure Π_2. By [3, Corollary 12], the potential measure of $\mathcal{R}_2$ can be taken equal to

$$U_2(dx) = u_0(x) U_1(dx), \qquad x \geq 0.$$

A regenerative set is said to be heavy when it has positive drift. We stress that

$$\mathcal{R}_2 \text{ is heavy} \iff \mathcal{R}_0 \text{ and } \mathcal{R}_1 \text{ are heavy}.$$

Indeed if $\mathcal{R}_2$ is heavy, then it has positive Lebesgue measure a.s. The same holds for $\mathcal{R}_1 \supseteq \mathcal{R}_2$ and $\mathcal{R}_0 \supseteq \mathcal{R}_2$, hence by Proposition 1.8 in [1], $\mathcal{R}_0$ and $\mathcal{R}_1$ are heavy. For the converse, just recall that a regenerative set is heavy iff it has a potential density with a right limit at 0, which is then equal to the inverse of its drift coefficient. As a consequence, as $\mathcal{R}_1 \cap \mathcal{R}_0$ has potential density $u_0 u_1$, it is heavy with drift coefficient $\delta_0\delta_1$. It follows in particular thanks to (1) that the measure Π_2 is characterized by

$$\int_0^\infty \mathrm{e}^{-qx}\bar{\Pi}_2(x)dx = \frac{\phi_2(q)}{q} - \delta_2 = \left[q\int_0^\infty U_1(dx)u_0(x)\mathrm{e}^{-qx}\right]^{-1} - \delta_0\delta_1.$$

We now describe the distribution of (A_1, R_1, A_2, R_2) under the stationary law $\mathbb{P}^\cap$. The tools used to achieve this task are mainly extracted from [3] (see also [16] for earlier results on intersections of regenerative sets). We already know that (A_1, R_1) follows $M(\phi_1)$ (recall (2)).

Theorem 4.1 *Conditional on $A_1 + R_1 = Z$,*

$$(A_2 - A_1, R_2 - R_1) \textit{ is independent of } (A_1, R_1) \textit{ and follows } \nu_Z,$$

where for any $\Delta \geq 0$, ν_Δ is the unique solution of the convolution equation $W * \nu_\Delta = V_\Delta$, with

$$V_\Delta(dx\,dy) = \mu_0^{-1}u_0(x+y+\Delta)U_1(dx)U_1(dy),$$

and

$$W(dx\,dy) = u_0(x)u_0(y)U_1(dx)U_1(dy) = U_2(dx)U_2(dy).$$

In particular, (A_1, A_2) and (R_1, R_2) are identically distributed and $A_2 - A_1$ is (not only conditionally) independent of (A_1, R_1).

We stress that explicit (but rather complicated) formulas for ν_Δ will be provided after the proof of the theorem.

Remark. A direct consequence is that $(A_k + R_k; 1 \leq k \leq n)$ is again Markovian.

Proof of Theorem 4.1. Denote by $m(x)$ the first point in the intersection of the shifted regenerative set

$$m(x) = \inf\{y \geq x^+ : y \in (x + \mathcal{R}_1) \cap \mathcal{R}_0\}, \qquad x \in \mathbb{R}.$$

Then, since $\mathcal{R}_2$ is a.s. nonempty and nondiscrete, Proposition 2, Theorem 7 and Corollary 13 in [3] entail that

$$\mathbb{E}(\exp(-qm(x))) = \kappa_q(x)/\kappa_q(0), \qquad q \geq 0, x \in \mathbb{R},$$

where

$$\kappa_q(x) = \int_0^\infty \mathrm{e}^{-q(x+y)}u_0(x+y)U_1(dy), \qquad x \in \mathbb{R}.$$

Conditional on $A_1 = a_1, R_1 = z_1 - a_1$, we know that $(R_1(t + z_1 - a_1), t \geq 0)$ and $(R_1(-a_1 - t), t \geq 0)$ are independent and distributed as $(R_1(t), t \geq 0)$ and $(A_1(t), t \geq 0)$ starting from 0, respectively. Then notice that the law of (A_0, R_0) is $M(\phi_0)$, and therefore

$$\mathbb{E}^{\cap}\left(\exp\left[-\lambda(A_2 - A_1) - \mu(R_2 - R_1)\right] \mid A_1 = a_1, R_1 = z_1 - a_1\right)$$

$$\begin{aligned}
&= \int_0^\infty \int_{a_0}^\infty \mathbb{P}(A_0 \in da_0, R_0 \in dz_0 - a_0)\mathbb{E}[\exp(-\lambda(m(a_1 - a_0) - (a_1 - a_0)))] \\
&\times \mathbb{E}[\exp(-\mu(m(z_1 - a_1 - z_0 + a_0) - (z_1 - a_1 - z_0 + a_0)))] \\
&= \delta_0 \mu_0^{-1} \frac{\kappa_\lambda(a_1)}{\kappa_\lambda(0)} \exp(\lambda a_1) \frac{\kappa_\mu(z_1 - a_1)}{\kappa_\mu(0)} \exp(\mu(z_1 - a_1)) + \mu_0^{-1} \int_0^\infty da_0 \frac{\kappa_\lambda(a_1 - a_0)}{\kappa_\lambda(0)} \\
&\times \exp -\lambda(a_0 - a_1) \int_{a_0}^\infty \Pi_0(dz_0) \frac{\kappa_\mu(z_1 - a_1 - z_0 + a_0)}{\kappa_\mu(0)} \exp -\mu(z_0 - a_0 - z_1 + a_1) \\
&= (\mu_0 \kappa_\lambda(0)\kappa_\mu(0))^{-1} \int_0^\infty U_1(dx) \mathrm{e}^{-\lambda x} \int_0^\infty U_1(dy) \mathrm{e}^{-\mu y} f(x + a_1, y + z_1 - a_1),
\end{aligned}$$

where μ_0 comes from the definition (2) of $M(\phi_0)$, and for any positive s, t,

$$f(s,t) = \delta_0 u_0(s) u_0(t) + \int_0^\infty da_0 \int_{a_0}^\infty \Pi_0(dz_0) u_0(s - a_0) u_0(t - z_0 + a_0).$$

We next compute f by Laplace inversion. For any positive $q_1 \neq q_2$,

$$\int_0^\infty ds\, \mathrm{e}^{-q_1 s} \int_0^\infty dt\, \mathrm{e}^{-q_2 t} f(s,t)$$

$$\begin{aligned}
&= (\phi_0(q_1)\phi_0(q_2))^{-1} \left(\delta_0 + \int_0^\infty \Pi_0(dz_0) \int_0^{z_0} da_0\, \mathrm{e}^{-q_1 a_0} \mathrm{e}^{-q_2(z_0 - a_0)}\right) \\
&= (\phi_0(q_1)\phi_0(q_2))^{-1} \left(\delta_0 + \int_0^\infty \Pi_0(dz_0) \frac{\mathrm{e}^{-q_2 z_0} - \mathrm{e}^{-q_1 z_0}}{q_1 - q_2}\right) \\
&= \frac{1}{q_1 - q_2} \left(\frac{1}{\phi_0(q_2)} - \frac{1}{\phi_0(q_1)}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^\infty ds\, \mathrm{e}^{-q_1 s} \int_0^\infty dt\, \mathrm{e}^{-q_2 t} u_0(s + t) &= \int_0^\infty dz\, u_0(z) \frac{\mathrm{e}^{-q_2 z} - \mathrm{e}^{-q_1 z}}{q_1 - q_2} \\
&= \frac{1}{q_1 - q_2} \left(\frac{1}{\phi_0(q_2)} - \frac{1}{\phi_0(q_1)}\right),
\end{aligned}$$

hence for any nonnegative s and t, $f(s,t) = u_0(s+t)$, and the preceding conditional expectation equals

$$\mu_0^{-1} \int_0^\infty U_1(dx) \mathrm{e}^{-\lambda x} \int_0^\infty U_1(dy) \mathrm{e}^{-\mu y} \frac{u_0(x + y + z_1)}{\kappa_\lambda(0)\kappa_\mu(0)}.$$

But for any positive q,

$$\kappa_q(0) = \int_0^\infty U_1(dy) u_0(y) \mathrm{e}^{-qy} = \int_0^\infty U_2(dy) \mathrm{e}^{-qy} = \frac{1}{\phi_2(q)},$$

hence

$$\mathbb{E}^{\cap}\left(\exp\left[-\lambda(A_2-A_1)-\mu(R_2-R_1)\right]\mid A_1=a_1, R_1=z_1-a_1\right)= \\ =\phi_2(\lambda)\phi_2(\mu)\int_0^\infty\int_0^\infty V_{z_1}(dxdy)\mathrm{e}^{-\lambda x}\,\mathrm{e}^{-\mu y}.$$

The result follows thanks to (8). It only remains to prove the independence between $A_2 - A_1$ and (A_1, R_1). The conditional expectation is equal to

$$\begin{aligned} &= (\mu_0\kappa_\lambda(0))^{-1}\int_0^\infty U_1(dx)\,\mathrm{e}^{-\lambda x}(\kappa_\mu(0))^{-1}\int_0^\infty U_1(dy)u_0(x+y+z_1)\mathrm{e}^{-\mu y} \\ &= \mu_0^{-1}\phi_2(\lambda)\int_0^\infty U_1(dx)\,\mathrm{e}^{-\lambda x}\mathbb{E}(\exp-\mu(m(x+z_1)-x-z_1)). \end{aligned}$$

By dominated convergence, letting $\mu\to 0+$ yields

$$\mathbb{E}^{\cap}\left(\exp\left[-\lambda(A_2-A_1)\right]\mid A_1=a_1, R_1=z_1-a_1\right)=\frac{\phi_2(\lambda)}{\mu_0\phi_1(\lambda)},$$

and the proof is complete (moreover we recover the result of Theorem 1 in [2] for the particular case of intersection of independent regenerative sets). □

Using (1), we get the following expressions by convolution products.
If $\delta_2=0$, then for $x,y\geq 0$,

$$\nu_\Delta([0,x]\times[0,y])=\mu_0^{-1}\int_{[0,x]}U_1(ds)\bar{\Pi}_2(x-s)\int_{[0,y]}U_1(dt)\bar{\Pi}_2(y-t)u_0(s+t+\Delta).$$

When $\delta_2>0$, then $\delta_1>0$ and U_1 is absolutely continuous, which provides

$$\begin{aligned} \nu_\Delta([0,x]\times[0,y]) &= \mu_0^{-1}\int_{[0,x]}ds\,u_1(s)\bar{\Pi}_2(x-s)\int_{[0,y]}dt\,u_1(t)\bar{\Pi}_2(y-t)u_0(s+t+\Delta) \\ &+ \mu_0^{-1}\delta_2 u_1(x)\int_{[0,y]}dt\,u_1(t)\bar{\Pi}_2(y-t)u_0(x+t+\Delta) \\ &+ \mu_0^{-1}\delta_2 u_1(y)\int_{[0,x]}ds\,u_1(s)\bar{\Pi}_2(x-s)u_0(s+y+\Delta) \\ &+ \mu_0^{-1}\delta_2^2 u_1(x)u_1(y)u_0(x+y+\Delta). \end{aligned}$$

4.2 The subordination scheme

We study the stationary law $\mathbb{P}^\star$ defined in the Preliminaries. The next result gives the transition kernels of the Markov chain $((A_k,R_k),1\leq k\leq n)$. We recall that for $n=2$, τ_1 and τ_2 are two independent two-sided subordinators and $\mathbb{P}^\star$ stands for the law of $(\mathcal{R}_1,\mathcal{R}_2)$, where $\mathcal{R}_1$ is the closed range of $\sigma_1=\tau_1$, and $\mathcal{R}_2$ the closed range of $\sigma_2=\tau_1\circ\tau_2$.

Theorem 4.2 *Under $\mathbb{P}^\star$, $(A_k,R_k;1\leq k\leq n)$ is an inhomogeneous Markov chain with independent increments. The distribution of the increment is given by*

$$(A_2-A_1,R_2-R_1)\stackrel{(d)}{=}(\sigma_1^-(\alpha),\sigma_1^+(\rho)),$$

where σ_1^- and σ_1^+ are two independent subordinators both distributed as $\vec{\sigma}_1$, and (α,ρ) is an independent pair of variables following $M(\tau_2)$ (recall (2)).

Proof of Theorem 4.2. We show the result by induction on the number n of subordinators. Recall the general notation given in the Preliminaries. Let $n \geq 2$. By definition of the probability measure $\mathbb{P}^{\star}$, if $\mathcal{S}_k$ stands for the closed range of $\tau_2 \circ \cdots \circ \tau_k$, $k = 2, \ldots, n$, then $\mathcal{R}_1, \ldots, \mathcal{R}_{n-1}, \mathcal{R}_n$, are the respective ranges of $\mathbb{R}, \mathcal{S}_1, \ldots, \mathcal{S}_n$, by $\tau_1 = \sigma_1$. Denote by (a_k, r_k) the age and residual lifetime at 0 associated with $\mathcal{S}_k$, $k = 2, \ldots, n$. In particular, $(A_1, R_1) = (\tau_1(0-), \tau_1(0))$, and for $k = 2, \ldots, n$, with the notation bound to the definition of two-sided subordinator,

$$\begin{aligned} (A_k, R_k) &= (-\tau_1(-a_k-), \tau_1(r_k)) \\ &= (\tau_1^-(a_k) + A_1, \tau_1^+(r_k) + R_1), \end{aligned}$$

and therefore

$$(A_{k+1} - A_k, R_{k+1} - R_k) = (\tau_1^-(a_{k+1}) - \tau_1^-(a_k), \tau_1^+(r_{k+1}) - \tau_1^+(r_k)).$$

As a consequence,

$$\begin{aligned} &(A_1, R_1; A_2 - A_1, R_2 - R_1; \ldots; A_n - A_{n-1}, R_n - R_{n-1}) \\ &= (\tau_1(0-), \tau_1(0); \tau_1^-(a_2), \tau_1^+(r_2); \ldots; \tau_1^-(a_n) - \tau_1^-(a_{n-1}), \tau_1^+(r_n) - \tau_1^+(r_{n-1})). \end{aligned}$$

Applying the induction hypothesis to $(a_2, r_2; \ldots; a_n, r_n)$, we get the desired independence. Moreover, focusing on the pair $(A_2 - A_1, R_2 - R_1)$, we have

$$(A_2 - A_1, R_2 - R_1) \stackrel{(d)}{=} (\sigma_1^-(a_2), \sigma_1^+(r_2)),$$

where (a_2, r_2) follows $M(\tau_2)$. □

5 Appendix

5.1 Proofs of the three preliminary statements

Proof of Lemma 2.1. It is known that the scaling invariance and the regenerative property are preserved by intersection and Bochner's subordination. Hence, let us focus on the stationary case. For the intersection scheme, it is clear that the (nonempty by assumption) intersections of stationary sets are still stationary. For the subordination scheme, consider τ_1 and τ_2 two independent two-sided subordinators. The independence between $(\tau_1 \circ \tau_2(-t-) - \tau_1 \circ \tau_2(0-), t \geq 0)$ and $(\tau_1 \circ \tau_2(t) - \tau_1 \circ \tau_2(0), t \geq 0)$ follows from the independence of increments of τ_1 as well as the independence between τ_1^- and τ_1^+. It is easy to see that they are both distributed as $\vec{\tau}_1 \circ \vec{\tau}_2$. A straightforward calculation next shows that the law of $(-\tau_1 \circ \tau_2(0-), \tau_1 \circ \tau_2(0))$ is $M(\phi_2 \circ \phi_1)$.□

Proof of Proposition 2.2. For the sake of conciseness, we focus on the intersection scheme. We stress that the proof for the subordination scheme relies on similar arguments. Let $2 \leq k \leq n-1$.

We first deal with the stationary case. Denote by $\mathcal{F}_t^{(k)}$ the σ-algebra generated by $((-\infty, t] \cap \mathcal{S}_i, 1 \leq i \leq k)$. It is clear that the k-tuple $(\mathcal{S}_1, \ldots, \mathcal{S}_k)$ is regenerative w.r.t. the filtration

$\mathcal{F}^{(k)}$. Recall that D_k is the first passage time at $(0,\ldots,0)$. Since D_k is a $\mathcal{F}^{(k)}$-stopping time, the shifted k-tuple $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overrightarrow{\theta}_{D_k}$ is independent of $\mathcal{F}^{(k)}_{D_k}$ and has the same law as $(\overrightarrow{\mathcal{S}}_1,\ldots,\overrightarrow{\mathcal{S}}_k)$. In particular, $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap(-\infty,G_k]^k$ and $(G_1,D_1,\ldots,G_k,D_k)$ are jointly independent of $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overrightarrow{\theta}_{D_k}$. The construction of the stationary regenerative sets allows us to make the same reasoning in the negative direction. As a consequence, $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overleftarrow{\theta}_{G_k}$ and $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overrightarrow{\theta}_{D_k}$ are jointly independent of $(G_1,D_1,\ldots,G_k,D_k)$. Conclude by noting that the pair (G_{k+1},D_{k+1}) is a functional of (G_k,D_k), $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overleftarrow{\theta}_{G_k}$, $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overrightarrow{\theta}_{D_k}$, and of an independent regenerative set $\mathcal{S}_{k+1}$.

We next turn to the stable case. The regenerative property still applies at D_k, and we get that $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[0,G_k]^k$ and $(G_1,D_1,\ldots,G_k,D_k)$ are jointly independent of $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\circ\overrightarrow{\theta}_{D_k}$. From [22], we know that the distribution of every $\mathcal{S}_i$, $i=1,\ldots,k$ is invariant under the action of $\varphi: x\mapsto x^{-1}$. As they are all independent, this is again the case of the k-tuple $(\mathcal{S}_1,\ldots,\mathcal{S}_k)$. Next notice that $D_k\circ\varphi=1/G_k$. Applying the regenerative property to $(\varphi(\mathcal{S}_1),\ldots,\varphi(\mathcal{S}_k))$, we get that conditional on G_k, $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[0,G_k]^k$ is independent of $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[G_k,+\infty)^k$. As a consequence, conditional on (G_k,D_k), $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[0,G_k]^k$ and $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[D_k,+\infty)^k$ are jointly independent of $(G_1,D_1,\ldots,G_k,D_k)$. Conclude as previously by noting that (G_{k+1},D_{k+1}) is a functional of $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[0,G_k]^k$, $(\mathcal{S}_1,\ldots,\mathcal{S}_k)\cap[D_k,+\infty)^k$, and of an independent regenerative set $\mathcal{S}_{k+1}$. □

Proof of Proposition 2.3. Applying the regeneration property at $D_n(t)$ to the expression in the l.h.s., the latter is equal to

$$\mathbb{E}_0(h_g\circ\overleftarrow{\theta}_{G_n(t)}f(X_n(t)))\mathbb{E}_0(h_d).$$

The same property applied to $D_n(0)$ for the other side yields

$$\mathbb{E}(h_g\circ\overleftarrow{\theta}_{G_n(0)}f(X_n(0)))\mathbb{E}_0(h_d).$$

Therefore we need only show the result for $h_d\equiv 1$.

Another elementary remark allows us to assume that the drift coefficient of $\mathcal{R}_n$ is zero, that is $\mathbb{P}_0(X_n(t)=0)=0$ for any fixed t. We can thus apply the result of [5] to the strong Markov process $(X_n(t),t\geq 0)$ under $\mathbb{P}_0$. Let L_t stand for the local time at level 0 of X_n at time t, so that L^{-1} is a multiple of the subordinator τ_n whose closed range is $\mathcal{R}_n$. Next introduce T the a.s. positive and finite stopping time

$$T=\inf\{t>0: t>L_t\},$$

where L_t is normalized so as to have $\mathbb{E}_0(L_1^{-1})=1$. Then $\mathbb{P}_0(X_n(T)\in\cdot)$ is the invariant probability measure for the Markov process X_n.

For simplicity, let $\mathcal{R}$ stand for the generic n-tuple of regenerative sets. The idea is to apply an elementary coupling method by building up a new n-tuple of regenerative sets $\tilde{\mathcal{R}}$ defined conditional on T. The post-T part of $\tilde{\mathcal{R}}$ coincides with the post-T part of $\mathcal{R}$, that is $\tilde{X}_n(t)=X_n(t)$ for all $t\geq T$. The pre-T part of $\tilde{\mathcal{R}}$ is independent of $\mathcal{R}$, and such that $(\tilde{X}_n(T-t^-),t\geq 0)$ is equally distributed as $(X_n(t),t\geq 0)$ under $\mathbb{P}_0$. Hence

$$\mathbb{E}_0(h_g(\overleftarrow{\theta}_{G_n(t)}(\tilde{\mathcal{R}}))f(\tilde{X}_n(t)))=\mathbb{E}(h_g\circ\overleftarrow{\theta}_{G_n(0)}f(X_n(0))).$$

But it is known (see [17]) that there is some compact set K of $[0,\infty)^{2n}$ such that the support of h_g is included in $\{F \in E^n : F \subseteq K\}$. As a consequence, $\mathbb{P}_0$-a.s. for any sufficiently large t,

$$h_g(\overleftarrow{\theta}_{G_n(t)}(\tilde{\mathcal{R}}))f(\tilde{X}_n(t)) = h_g(\overleftarrow{\theta}_{G_n(t)}(\mathcal{R}))f(X_n(t)),$$

and the result follows conditional on T. Since the limiting distribution does not involve T, we conclude by dominated convergence. $\square$

5.2 Proof of Corollary 3.1.1

Recall the pair (m, M) in Theorem 3.1 and denote by $\zeta(g,d;\cdot)$ its conditional law on $(G_\alpha, D_\alpha) = (g,d)$. According to this theorem, for any Borel function f on $(0,1)\times(1,\infty)$,

$$\mathbb{E}(f(G_\gamma, D_\gamma) \mid G_\alpha = g, D_\alpha = d) = B(\gamma, 1-\gamma)^{-2}\int_0^g\int_d^\infty \zeta(g,d;du\,dv)$$

$$\times \int_0^1 ds\, s^{-\gamma}(1-s)^{\gamma-1}\int_1^\infty dt\, t^{-1}(t-1)^{\gamma-1}f((\frac{1}{g}+(\frac{1}{u}-\frac{1}{g})\frac{1}{s})^{-1},\ d+(v-d)t).$$

Changing variables by $(x,y) = ((\frac{1}{g}+(\frac{1}{u}-\frac{1}{g})\frac{1}{s})^{-1}, d+(v-d)t)$, the last quantity remains equal to

$$B(\gamma,1-\gamma)^{-2}\int_0^g\int_d^\infty \zeta(g,d;du\,dv)\int_0^u \frac{dx}{g-x}g^\gamma\left(\frac{g-u}{u-x}\right)^{1-\gamma}x^{-\gamma}\int_v^\infty \frac{dy}{y-d}\left(\frac{v-d}{y-v}\right)^{1-\gamma}f(x,y),$$

and referring to Theorem 3.1 for the expression of ζ, it is again equal to (we wrote $(s,t) = (g-u, v-d)$)

$$B(\gamma,1-\gamma)^{-2}\int_0^g \frac{dx}{g-x}g^\alpha x^{-\gamma}\int_d^\infty \frac{dy}{y-d}f(x,y)$$

$$\times\int_0^{g-x}ds\int_0^{y-d}dt\left[-\frac{\partial^2}{\partial s\partial t}\left\{\left(\frac{st}{s+t+d-g}\right)^{\gamma-\alpha}\right\}\right](st)^{1-\gamma}(g-x-s)^{\gamma-1}(y-d-t)^{\gamma-1}$$

$$= B(\gamma,1-\gamma)^{-2}\int_0^g \frac{dx}{g-x}g^\alpha x^{-\gamma}\int_d^\infty \frac{dy}{y-d}f(x,y)[(\gamma-\alpha)(\gamma-\alpha+1)(I)+(\gamma-\alpha)^2(d-g)(II)],$$

and the differentiation furnishes the following two expressions, where we set $X = g-x$ and $Y = y-d$,

$$\begin{aligned}(I) &= \int_0^X ds\int_0^Y dt\,\frac{(st)^{1-\alpha}}{(s+t+d-g)^{\gamma-\alpha+2}}(X-s)^{\gamma-1}(Y-t)^{\gamma-1}\\ (II) &= \int_0^X ds\int_0^Y dt\,\frac{(st)^{-\alpha}}{(s+t+d-g)^{\gamma-\alpha+1}}(X-s)^{\gamma-1}(Y-t)^{\gamma-1}.\end{aligned}$$

Changing variables by $(u,v) = (X^{-1}s, Y^{-1}t)$, we obtain

$$(I) = \frac{X^{\gamma-\alpha+1}}{Y}\int_0^1 du\, u^{1-\alpha}(1-u)^{\gamma-1}\int_0^1 dv\,\frac{v^{1-\alpha}(1-v)^{\gamma-1}}{(v+Y^{-1}(d-g+uX))^{\gamma-\alpha+2}}.$$

We apply formula 3.197(4) from [15], and then formulas 3.211 and 9.182(1), which entail

$$
\begin{aligned}
(I) &= \frac{X^{\gamma-\alpha+1}}{Y} B(2-\alpha,\gamma) \int_0^1 du\, u^{1-\alpha}(1-u)^{\gamma-1} \left(\frac{d-g+uX}{Y}\right)^{-\gamma} \left(1+\frac{d-g+uX}{Y}\right)^{\alpha-2} \\
&= B(2-\alpha,\gamma)^2 (d-g)^{-\gamma}(X+Y+d-g)^{\alpha-2}(XY)^{\gamma-\alpha+1} \\
&\times\ F(2-\alpha,\gamma,\gamma-\alpha+2,\frac{-XY}{(d-g)(X+Y+d-g)}).
\end{aligned}
$$

By similar calculations,

$$
(II) = B(1-\alpha,\gamma)^2(d-g)^{-\gamma}(X+Y+d-g)^{\alpha-1}(XY)^{\gamma-\alpha} F(1-\alpha,\gamma,\gamma-\alpha+1,\frac{-XY}{(d-g)(X+Y+d-g)}).
$$

It is then easy to get

$$
\begin{aligned}
&\mathbb{E}(f(G_1^{(\gamma)},D_1^{(\gamma)}) \mid G_1^{(\alpha)}=g, D_1^{(\alpha)}=d) = B(1-\gamma,\gamma-\alpha)^{-2}(d-g)^{1-\gamma}g^{\alpha} \\
&\times \int_0^g dx\, x^{-\gamma} \int_d^{\infty} dy\, f(x,y) \frac{((g-x)(y-d))^{\gamma-\alpha-1}}{(y-x)^{1-\alpha}} H\left(\frac{(g-x)(y-d)}{(d-g)(y-x)}\right),
\end{aligned}
$$

where for any real number z,

$$
\begin{aligned}
H(z) &= \frac{\Gamma(\gamma-\alpha)}{\Gamma(1-\alpha)\Gamma(\gamma)} \left[z(1-\alpha)\int_0^1 dt\, t^{\gamma-1}(1-t)^{1-\alpha}(1+tz)^{\alpha-2} \right. \\
&+ \left. (\gamma-\alpha)\int_0^1 dt\, t^{\gamma-1}(1-t)^{-\alpha}(1+tz)^{\alpha-1} \right] \\
&= \frac{\Gamma(\gamma-\alpha)}{\Gamma(1-\alpha)\Gamma(\gamma)} \left[-\int_0^1 dt\, t^{\gamma-1}(1-t)^{1-\gamma} \frac{\partial}{\partial t}\{(1-t)^{\gamma-\alpha}(1+tz)^{\alpha-1}\} \right],
\end{aligned}
$$

hence $H(0)=1$, and

$$
H'(z) = \frac{\Gamma(\gamma-\alpha)}{\Gamma(1-\alpha)\Gamma(\gamma)}(1-\alpha)\int_0^1 dt\, t^{\gamma-1}(1-t)^{1-\gamma}\frac{\partial}{\partial t}\{t(1-t)^{\gamma-\alpha}(1+tz)^{\alpha-2}\}.
$$

An integration by parts now yields

$$
\begin{aligned}
H'(z) &= \frac{\Gamma(\gamma-\alpha)}{\Gamma(1-\alpha)\Gamma(\gamma)}(1-\alpha)(1-\gamma)\int_0^1 dt\, t^{\gamma-1}(1-t)^{-\alpha}(1+tz)^{\alpha-2} \\
&= \frac{(1-\alpha)(1-\gamma)}{\gamma-\alpha} F(2-\alpha,\gamma,\gamma-\alpha+1,-z),
\end{aligned}
$$

which is the expected expression. □

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