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ASYMPTOTICS OF CERTAIN COAGULATION-FRAGMENTATION PROCESSES AND INVARIANT POISSON-DIRICHLET MEASURES

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Abstract We consider Markov chains on the space of (countable) partitions of the interval $[0,1]$, obtained first by size biased sampling twice (allowing repetitions) and then merging the parts with probability β_m (if the sampled parts are distinct) or splitting the part with probability β_s according to a law σ (if the same part was sampled twice). We characterize invariant probability measures for such chains. In particular, if σ is the uniform measure then the Poisson-Dirichlet law is an invariant probability measure, and it is unique within a suitably defined class of "analytic" invariant measures. We also derive transience and recurrence criteria for these chains.

Key words and phrases Partitions, coagulation, fragmentation, invariant measures, Poisson-Dirichlet

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1 Introduction and statement of results

Let Ω_1 denote the space of (ordered) partitions of 1, that is

$$\Omega_1 := \{p = (p_i)_{i\geq 1} \ : \ p_1 \geq p_2 \geq ... \geq 0, \ p_1 + p_2 + \ldots = 1\}.$$

By *size-biased* sampling according to a point $p \in \Omega_1$ we mean picking the j-th part p_j with probability p_j. The starting point for our study is the following Markov chain on Ω_1, which we call a *coagulation-fragmentation* process: size-bias sample (with replacement) two parts from p. If the same part was picked twice, split it (uniformly), and reorder the partition. If different parts were picked, merge them, and reorder the partition.

We call this Markov chain the *basic chain*. We first bumped into it in the context of triangulation of random Riemann surfaces [5]. It turns out that it was already considered in [15], in connection with "virtual permutations" and the Poisson-Dirichlet process. Recall that the Poisson-Dirichlet measure (with parameter 1) can be described as the probability distribution of $(Y_n)_{n\geq 1}$ on Ω_1 obtained by setting $Y_1 = U_1$, $Y_{n+1} = U_{n+1}(1 - \sum_{j=1}^{n} Y_j)$, and reordering the sequence $(Y_n)_n$, where $(U_n)_n$ is a sequence of i.i.d. Uniform[0,1] random variables. Tsilevich showed in [15] that the Poisson-Dirichlet distribution is an invariant probability measure for the Markov chain described above, and raised the question whether such an invariant probability measure is unique. While we do not completely resolve this question, a corollary of our results (c.f. Theorem 3) is that the Poisson-Dirichlet law is the unique invariant measure for the basic chain which satisfies certain regularity conditions.

Of course, the question of invariant probability measure is only one among many concerning the large time behavior of the basic chain. Also, it turns out that one may extend the definition of the basic chain to obtain a Poisson-Dirichlet measure with any parameter as an invariant probability measure, generalizing the result of [15]. We thus consider a slightly more general model, as follows.

For any nonnegative sequence $x = (x_i)_i$, let $|x| = \sum_i x_i$, the ℓ_1 norm of x, and $|x|_2^2 = \sum_i x_i^2$. Set

$$\Omega \ = \{p = (p_i)_{i\geq 1} \ : \ p_1 \geq p_2 \geq ... \geq 0, \qquad 0 < |p| < \infty\}$$

and $\Omega_{\leq} = \{p \in \Omega : |p| \leq 1\}$. Let $\mathbf{0} = (0, 0, \ldots)$ and define $\bar{\Omega} = \Omega \cup \{\mathbf{0}\}$ and $\bar{\Omega}_{\leq} = \Omega_{\leq} \cup \{\mathbf{0}\}$. Unless otherwise stated, we equip all these spaces with the topology induced from the product topology on $\mathbb{R}^{\mathbb{N}}$. In particular, $\bar{\Omega}_{\leq}$ is then a compact space.

For a topological space X with Borel σ-field $\mathcal{F}$ we denote by $\mathcal{M}_1(X)$ the set of all probability measures on $(X, \mathcal{F})$ and equip it with the topology of weak convergence. $\mathcal{M}_+(X)$ denotes the space of all (nonnegative) measures on $(X, \mathcal{F})$.

Define the following two operators, called the *merge* and *split* operators, on $\bar{\Omega}$, as follows:

$$\begin{aligned} M_{ij} \ &: \ \bar{\Omega} \to \bar{\Omega}, \quad M_{ij}p = \text{the nonincreasing sequence obtained by merging} \\ &\qquad\qquad\qquad p_i \text{ and } p_j \text{ into } p_i + p_j\,, i \neq j \\ S_i^u \ &: \ \bar{\Omega} \to \bar{\Omega}, \quad S_i^u p = \text{the nonincreasing sequence obtained by splitting } p_i \\ &\qquad\qquad\qquad \text{into } \ u\,p_i \ \text{ and } \ (1-u)p_i\,, 0 < u < 1 \end{aligned}$$

Note that the operators M_{ij} and S_i^u preserve the ℓ_1 norm. Let $\sigma \in \mathcal{M}_1((0, 1/2])$ be a probability measure on $(0, 1/2]$ (the *splitting measure*). For $p \in \bar{\Omega}_{\leq}$ and $\beta_m, \beta_s \in (0, 1]$, we then consider

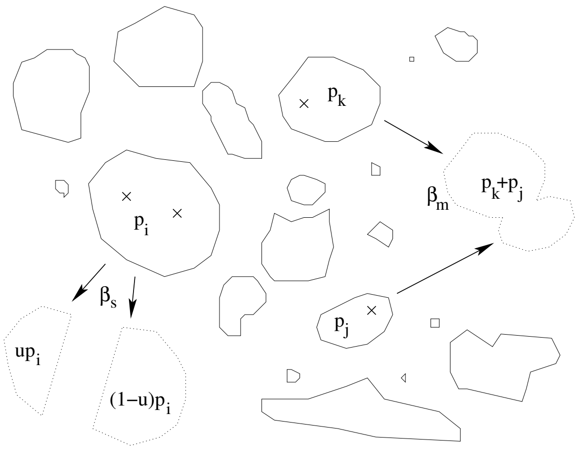

Figure 1: On the left side a part of size p_i has been chosen twice and is split with probability β_s. On the right side two different parts of sizes p_i and p_j have been chosen and are merged with probability β_m.

the Markov process generated in $\bar{\Omega}_{\leq}$ by the kernel

$$\begin{aligned} K_{\sigma,\beta_m,\beta_s}(p,\cdot) \;&:=\; 2\beta_m \sum_{i<j} p_i p_j \delta_{M_{ij}p}(\cdot) + \beta_s \sum_i p_i^2 \int \delta_{S_i^u p}(\cdot)\, d\sigma(u) \\ &\quad + \left(1 - \beta_m |p|^2 + (\beta_m - \beta_s)|p|_2^2\right) \delta_p(\cdot). \end{aligned}$$

It is straightforward to check (see Lemma 4 below) that $K_{\sigma,\beta_m,\beta_s}$ is Feller continuous. The basic chain corresponds to $\sigma = U(0,1/2]$, with $\beta_s = \beta_m = 1$.

It is also not hard to check (see Theorem 6 below) that there always exists a $K_{\sigma,\beta_m,\beta_s}$-invariant probability measure $\mu \in \mathcal{M}_1(\Omega_1)$. Basic properties of any such invariant probability measure are collected in Lemma 5 and Proposition 7. Our first result is the following characterization of those kernels that yield invariant probability measures which are supported on finite (respectively infinite) partitions. To this end, let $S := \{p \in \Omega_1 \mid \exists i \geq 2 : p_i = 0\}$ be the set of finite partitions.

Theorem 1 (Support properties) *For any $K_{\sigma,\beta_m,\beta_s}$-invariant $\mu \in \mathcal{M}_1(\Omega_1)$,*

$$\begin{aligned} \mu[S] = 1 \quad &\textit{if} \quad \int \frac{1}{x}\, d\sigma(x) \;<\; \infty \qquad \textit{and} \\ \mu[S] = 0 \quad &\textit{if} \quad \int \frac{1}{x}\, d\sigma(x) \;=\; \infty. \end{aligned}$$

Transience and recurrence criteria (which, unfortunately, do not settle the case $\sigma = U(0,1/2]$!) are provided in the:

Theorem 2 (Recurrence and transience) *The state $\bar{p} = (1, 0, 0, \ldots)$ is positive recurrent for $K_{\sigma,\beta_m,\beta_s}$ if and only if $\int 1/x \, d\sigma(x) < \infty$. If however*

$$\int_0^{1/2} \frac{1}{\sigma[(0,x]]} \, dx < \infty \tag{1}$$

then $\bar{p}$ is a transient state for $K_{\sigma,\beta_m,\beta_s}$.

We now turn to the case $\sigma = U(0, 1/2]$. In order to define invariant probability measures in this case, set $\pi : \Omega \to \Omega_1, \widehat{p} := \pi(p) = (p_i/|p|)_{i\geq 1}$. For each $\theta > 0$ consider the Poisson process on $\mathbb{R}_+$ with intensity measure $\nu_\theta(dx) = \theta x^{-1} e^{-x} \, dx$ which can be seen either as a Poisson random measure $N(A;\omega)$ on the positive real line or as a random variable $X = (X_i)_{i=1}^\infty$ taking values in Ω whose distribution shall be denoted by μ_θ, with expectation operator E_θ. (Indeed, $E_\theta|X| = E_\theta \int_0^\infty x \, N(dx) = \int_0^\infty x \, \nu_\theta(dx) < \infty$ while $P_\theta(|X| = 0) = \exp(-\nu_\theta[(0,\infty)]) = 0$, and thus $X \in \Omega$ a.s.). A useful feature of such a Poisson process is that for any Borel subset A of $\mathbb{R}_+$ with $0 < \nu_\theta(A) < \infty$, and conditioned on $\{N(A) = n\}$, the n points in A are distributed as n independent variables chosen each according to the law $\nu_\theta(\,\cdot\,|A)$. The Poisson-Dirichlet measure $\widehat{\mu}_\theta$ on Ω_1 is defined to be the distribution of $(\widehat{X}_i)_{i\geq 1}$. In other words, $\widehat{\mu}_\theta = \mu_\theta \circ \pi^{-1}$. In the case $\theta = 1$ it coincides with the previously described Poisson-Dirichlet measure. See [9], [10] and [3] for more details and additional properties of Poisson-Dirichlet processes.

We show in Theorem 3 below that, when $\sigma = U(0, 1/2]$, for each choice of β_m, β_s there is a Poisson–Dirichlet measure which is invariant for $K_{\sigma,\beta_m,\beta_s}$. We also show that it is, in this case, the unique invariant probability measure in a class $\mathcal{A}$, which we proceed to define. Set

$$\bar{\Omega}_<^k := \{(x_i)_{1\leq i\leq k} \; : \; x_i \geq 0, x_1 + x_2 + \ldots + x_k < 1\}$$

and denote by A_k the set of real valued functions on $\bar{\Omega}_<^k$ that coincide (leb^k-a.e.) with a function which has a real analytic extension to some open neighborhood of $\bar{\Omega}_<^k$. (Here and throughout, leb^k denotes the k-dimensional Lebesgue measure; all we shall use is that real analytic functions in a connected domain can be recovered from their derivatives at an internal point.) For any $\mu \in \mathcal{M}_1(\Omega_1)$ and each integer k, define the measure $\mu_k \in \mathcal{M}_+\left(\bar{\Omega}_<^k\right)$ by

$$\mu_k(B) = E_\mu \left[\sum_{\mathbf{j}\in\mathbb{N}_{\neq}^k} \left(\prod_{i=1}^k p_{j_i} \right) \mathbf{1}_B\left(p_{j_1}, \ldots, p_{j_k}\right) \right], \qquad B \in \mathcal{B}_{\bar{\Omega}_<^k}$$

(here $\mathbb{N}_{\neq}^k = \left\{\mathbf{j} \in \mathbb{N}^k \mid j_i \neq j_{i'} \text{ if } i \neq i' \right\}$). An alternative description of μ_k is the following one: pick a random partition p according to μ and then sample size-biased independently (with replacement) k parts $p_{i_1}, \ldots, p_{i_k}$ from p. Then,

$$\mu_k(B) = P(\text{the } i_j\text{-s are pairwise distinct, and } (p_{i_1}, \ldots, p_{i_k}) \in B) \,.$$

Part of the proof of part (b) of Theorem 3 below will consist in verifying that these measures $(\mu_k)_{k\geq 1}$ characterize μ (see [12, Th. 4] for a similar argument in a closely related context).

Set for $k \in \mathbb{N}$,

$$\mathcal{A}_k = \left\{ \mu \in \mathcal{M}_1(\Omega_1) \,\middle|\, \mu_k \ll \mathrm{leb}^k, m_k := \frac{d\mu_k}{d\,\mathrm{leb}^k} \in A_k \right\}.$$

Our main result is part (b) of the following:

Theorem 3 (Poisson-Dirichlet law) *Assume $\sigma = U(0,1/2]$ and fix $\theta = \beta_s/\beta_m$.*
(a) The Poisson-Dirichlet law of parameter θ belongs to $\mathcal{A} := \bigcap_{k=1}^{\infty} \mathcal{A}_k$, and is invariant (in fact: reversing) for the kernel $K_{\sigma,\beta_m,\beta_s}$.
(b) Assume a probability measure $\mu \in \mathcal{A}$ is $K_{\sigma,\beta_m,\beta_s}$-invariant. Then μ is the Poisson-Dirichlet law of parameter θ.

The structure of the paper is as follows: In Section 2, we prove the Feller property of $K_{\sigma,\beta_m,\beta_s}$, the existence of invariant probability measures for it, and some of their basic properties. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2 respectively, Section 5 studies the Poisson-Dirichlet measures and provides the proof of Theorem 3. We conclude in Section 6 with a list of comments and open problems.

2 Preliminaries

For fixed $\sigma \in \mathcal{M}_1((0,1/2])$, $\beta_m, \beta_s \in (0,1]$ and $p \in \bar{\Omega}_{\leq}$ we denote by $P_p \in \mathcal{M}_1(\bar{\Omega}_{\leq}^{\mathbb{N}\cup\{0\}})$ the law of the Markov process on $\bar{\Omega}_{\leq}$ with kernel $K_{\sigma,\beta_m,\beta_s}$ and starting point p, i.e. $P_p[p(0) = p] = 1$. Whenever $\mu \in \mathcal{M}_1(\bar{\Omega}_{\leq})$, the law of the corresponding Markov process with initial distribution μ is denoted by P_μ. In both cases, we use $(p(n))_{n\geq 0}$ to denote the resulting process.

Lemma 4 *The kernel $K_{\sigma,\beta_m,\beta_s}$ is Feller, i.e. for any continuous function $f : \bar{\Omega}_{\leq} \to \mathbb{R}$, the map $\bar{\Omega}_{\leq} \to \mathbb{R}$, $p \mapsto \int f \, dK_{\sigma,\beta_m,\beta_s}(p,\cdot)$ is continuous.*

Proof We have

$$\begin{aligned}\int f dK_{\sigma,\beta_m,\beta_s}(p,\cdot) &= 2\beta_m \sum_{i=1}^{\infty} p_i \sum_{j=i+1}^{\infty} p_j \left(f(M_{ij}p) - f(p)\right) \\ &\qquad + \beta_s \sum_{i=1}^{\infty} p_i^2 \int \left(f(S_i^u p) - f(p)\right) d\sigma(u) \quad + \; f(p) \\ &=: 2\beta_m \sum_{i=1}^{\infty} p_i g_i(p) + \beta_s \sum_{i=1}^{\infty} p_i^2 h_i(p) \; + f(p). \end{aligned} \tag{2}$$

One may assume that $f(p)$ is of the form $F(p_1, \ldots, p_k)$ with $k \in \mathbb{N}$ and $F \in C(\bar{\Omega}_{\leq}^k)$, since any $f \in C(\bar{\Omega}_{\leq})$ can be uniformly approximated by such functions, and denote accordingly $\|p\|_k$ the $\mathbb{R}^k$ norm of p's first k components. We shall prove the lemma in this case by showing that both sums in (2) contain finitely many nonzero terms, this number being uniformly bounded on some open neighborhood of a given q, and that g_i and h_i are continuous for every i.

For the second sum these two facts are trivial: $S_i^u p$ and p coincide in their first k components $\forall u \in (0,1/2]$, $\forall i > k$, since splitting a component doesn't affect the ordering of the larger ones, and thus $h_i \equiv 0$ for $i > k$. Moreover, h_i's continuity follows from equicontinuity of $(S_i^u)_{u\in(0,1)}$.

As for the first sum, given $q \in \bar{\Omega}_{\leq}$ with positive components (the necessary modification when q has zero components is straightforward), let $n = n(q) > k$ be such that $q_n < \frac{1}{4}\, q_k$ and consider q's open neighborhood $U = U(q) = \left\{p \in \bar{\Omega}_{\leq} \; : \; p_k > \frac{2}{3} q_k, \; p_n < \frac{4}{3} q_n\right\}$. In particular, for all $p \in U$,

$p_n < \frac{1}{2}p_k$ and thus, when $j > i > n$, $p_i + p_j \leq 2p_n < p_k$, which means that $M_{ij}p$ and p coincide in their first k components, or that $g_i(p) = 0$ for every $i > n(q)$ and $p \in U(q)$.

Finally, each g_i is continuous because the series defining it converges uniformly. Indeed, for $j > i$ and uniformly in p, $\|M_{ij}p - p\|_k \leq p_j \leq \frac{1}{j}$. For a given $\varepsilon > 0$, choose $j_0 \in \mathbb{N}$ such that $|F(y) - F(x)| < \varepsilon$ whenever $\|y - x\|_k < \frac{1}{j_0}$. Then

$$\left| \sum_{j=j_0}^{\infty} p_j \left(f(M_{ij}p) - f(p) \right) \right| < \varepsilon \sum_{j=j_0}^{\infty} p_j \leq \varepsilon$$

which proves the uniform convergence. $\square$

Lemma 5 *Let $\mu \in \mathcal{M}_1(\bar{\Omega}_{\leq})$ be $K_{\sigma,\beta_m,\beta_s}$-invariant. Then*

$$\int |p|_2^2 \; d\mu = \frac{\beta_m}{\beta_m + \beta_s} \int |p|^2 \; d\mu. \tag{3}$$

Furthermore, if we set for $n \geq 1$,

$$\nu_0 = \delta_{(1,0,0,\ldots)}, \quad \nu_n = \nu_{n-1} K_{\sigma,\beta_m,\beta_s} \quad \textit{and} \quad \bar{\nu}_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k \,, \tag{4}$$

then for all $n \geq 1$,

$$\int |p|_2^2 \; d\bar{\nu}_n \geq \frac{\beta_m}{\beta_m + \beta_s} \tag{5}$$

Proof Let $\varepsilon \in [0,1]$, and consider the random variable

$$X_\varepsilon := \sum_i 1_{\varepsilon < p_i}$$

on $\bar{\Omega}_{\leq}$ which counts the intervals longer than ε. We first prove (3). (The value $\varepsilon = 0$ is used in the subsequent proof of (5).) Assume that X_ε is finite which is always the case for $\varepsilon > 0$ since on $\bar{\Omega}_{\leq}$, $X_\varepsilon \leq 1/\varepsilon$ and is also true for $\varepsilon = 0$ if only finitely many p_i are non zero. Then the expected (conditioned on p) increment Δ_ε of X_ε after one step of the underlying Markov process is well-defined. It equals

$$\begin{aligned}
\Delta_\varepsilon &= \beta_m \sum_{i \neq j} p_i p_j (1_{p_i, p_j \leq \varepsilon < p_i + p_j} - 1_{\varepsilon < p_i, p_j}) \\
&\quad + \beta_s \sum_i p_i^2 1_{\varepsilon < p_i} \left(\int 1_{\varepsilon < x p_i} \; d\sigma(x) - \int 1_{\varepsilon \geq (1-x) p_i} \; d\sigma(x) \right) \\
&= \beta_m \sum_{i,j} p_i p_j (1_{p_i, p_j \leq \varepsilon < p_i + p_j} - 1_{\varepsilon < p_i, p_j}) \\
&\quad + \beta_s \sum_i p_i^2 1_{\varepsilon < p_i} \left(\sigma[(\varepsilon/p_i, 1/2]] - \sigma[[1 - \varepsilon/p_i, 1/2]] \right) \\
&\quad - \beta_m \sum_i p_i^2 \left(1_{p_i \leq \varepsilon < 2p_i} - 1_{\varepsilon < p_i} \right).
\end{aligned} \tag{6}$$

The right hand side of (6) converges as ε tends to 0 to

$$\lim_{\varepsilon\searrow 0}\Delta_\varepsilon = -\beta_m|p|^2+(\beta_m+\beta_s)|p|_2^2. \tag{7}$$

Since μ is $K_{\sigma,\beta_m,\beta_s}$-invariant we have $\int \Delta_\varepsilon \, d\mu = 0$ for all ε. Now (3) follows from (7) by dominated convergence since $|\Delta_\varepsilon| \leq 2$.

For the proof of (5) note that for all $n \geq 0$, ν_n has full measure on sequences $p \in \Omega_1$ for which the number X_0 of nonvanishing components is finite because we start with $X_0 = 1$ ν_0-a.s. and X_0 can increase at most by one in each step. Given such a $p \in \Omega_1$, the expected increment Δ_0 of X_0 equals (see (6), (7)) $\Delta_0 = -\beta_m + (\beta_m+\beta_s)|p|_2^2$. Therefore for $k \geq 0$,

$$\int X_0 \, d\nu_{k+1} - \int X_0 \, d\nu_k = -\beta_m + (\beta_m+\beta_s)\int |p|_2^2 \, d\nu_k.$$

Summing over $k = 0, \ldots, n-1$ yields

$$\int X_0 \, d\nu_n - \int X_0 \, d\nu_0 = -n\beta_m + (\beta_m+\beta_s)\sum_{k=0}^{n-1}\int |p|_2^2 \, d\nu_k. \tag{8}$$

The left hand side of (8) is nonnegative due to $\int X_0 \, d\nu_0 = 1$ and $\int X_0 \, d\nu_n \geq 1$. This proves (5). $\square$

Theorem 6 *There exists a $K_{\sigma,\beta_m,\beta_s}$-invariant probability measure $\mu \in \mathcal{M}_1(\Omega_1)$.*

Proof Define ν_n and $\bar\nu_n$ as in (4). Since $\bar\Omega_\leq$ is compact, $\mathcal{M}_1(\bar\Omega_\leq)$ is compact. Consequently, there are $\mu \in \mathcal{M}_1(\bar\Omega_\leq)$ and a strictly increasing sequence $(m_n)_n$ of positive integers such that $\bar\nu_{m_n}$ converges weakly towards μ as $n \to \infty$. This limiting measure μ is invariant under $K_{\sigma,\beta_m,\beta_s}$ by the following standard argument. For any continuous function $f : \bar\Omega_\leq \to \mathbb{R}$,

$$\begin{aligned}
\int f \, d(\mu K_{\sigma,\beta_m,\beta_s}) &= \int\int f \, dK_{\sigma,\beta_m,\beta_s}(p,\cdot)\, d\mu(p) \\
&= \lim_{n\to\infty}\int\int f \, dK_{\sigma,\beta_m,\beta_s}(p,\cdot)\, d\bar\nu_{m_n}(p) \qquad \text{[Lemma 4]} \\
&= \lim_{n\to\infty}\frac{1}{m_n}\sum_{k=0}^{m_n-1}\int\int f \, dK_{\sigma,\beta_m,\beta_s}(p,\cdot)\, d\nu_k(p) \\
&= \lim_{n\to\infty}\frac{1}{m_n}\sum_{k=0}^{m_n-1}\int f(p)\, d\nu_{k+1}(p) = \lim_{n\to\infty}\int f d\bar\nu_{m_n} = \int f d\mu\,.
\end{aligned}$$

Hence it remains to show that Ω_1 has full μ-measure, i.e. $\mu[|p|=1]=1$. To prove this observe that $|p|_2^2$ (unlike $|p|$) is a continuous function on $\bar\Omega_\leq$. Therefore by (3), weak convergence and (5),

$$1 \geq \int |p|^2 \, d\mu = \frac{\beta_m+\beta_s}{\beta_m}\int |p|_2^2 \, d\mu = \frac{\beta_m+\beta_s}{\beta_m}\lim_{n\to\infty}\int |p|_2^2 \, d\bar\nu_{m_n} \geq 1$$

by which the first inequality is an equality, and thus $|p| = 1$ $\mu - a.s.$ $\square$

Proposition 7 *If $\mu \in \mathcal{M}_1(\Omega_1)$ is $K_{\sigma_i,\beta_{m,i},\beta_{s,i}}$-invariant for $i = 1,2$, then $\sigma_1 = \sigma_2$ and $\theta_1 := \beta_{s,1}/\beta_{m,1} = \beta_{s,2}/\beta_{m,2} =: \theta_2$.*

Proof Let $k \geq 1$ be an integer and $\alpha \in \{1,2\}$. Given p, consider the expected increment $\Delta_{\alpha,k}$ of $\sum_i p_i^k$ after one step of the process driven by $K_{\sigma_\alpha,\beta_{m,\alpha}\beta_{s,\alpha}}$:

$$\begin{aligned}\Delta_{\alpha,k} &= \beta_{m,\alpha}\sum_{i\neq j} p_i p_j \left(-p_i^k - p_j^k + (p_i+p_j)^k\right) \\ &\quad + \beta_{s,\alpha}\sum_i p_i^2 \left(-p_i^k + \int (tp_i)^k + ((1-t)p_i)^k \; d\sigma_\alpha(t)\right).\end{aligned}$$

Note that $\int \sum_i p_i^k \; d\mu$ is finite because of $k \geq 1$. Therefore, by invariance, $\int \Delta_{\alpha,k} \; d\mu = 0$, which implies

$$\beta_{s,\alpha}\left[\int (t^k + (1-t)^k) \; d\sigma_\alpha(t) - 1\right] = \frac{\beta_{m,\alpha}\int \sum_{i\neq j} p_i p_j \left(p_i^k + p_j^k - (p_i+p_j)^k\right) \; d\mu}{\int \sum_i p_i^{2+k} \; d\mu}.$$

Hence, for any k,

$$\frac{\int (t^k + (1-t)^k) \; d\sigma_1(t) - 1}{\int (t^k + (1-t)^k) \; d\sigma_2(t) - 1} = \frac{\beta_{m,1}\beta_{s,2}}{\beta_{m,2}\beta_{s,1}} =: \gamma\,.$$

Taking $k \to \infty$ we conclude that $\gamma = 1$. This proves the second claim. In addition, we have

$$\int (t^k + (1-t)^k) \; d\sigma_1(t) = \int (t^k + (1-t)^k) \; d\sigma_2(t) \tag{9}$$

for all $k \geq 1$. Obviously, (9) also holds true for $k = 0$. Extend σ_α to probability measures on $[0,1]$ which are supported on $[0,1/2]$. It is enough for the proof of $\sigma_1 = \sigma_2$ to show that for all continuous real valued functions f on $[0,1]$ which vanish on $[1/2,1]$ the integrals $\int f(t) \; d\sigma_\alpha(t)$ coincide for $\alpha = 1,2$. Fix such an f and choose a sequence of polynomials

$$\pi_n(t) = \sum_{k=0}^{n} c_{k,n} t^k \qquad (c_{k,n} \in \mathbb{R})$$

which converges uniformly on $[0,1]$ to f as $n \to \infty$. Then $\pi_n(t) + \pi_n(1-t)$ converges uniformly on $[0,1]$ to $f(t) + f(1-t)$. Since $f(1-t)$ vanishes on the support of σ_1 and σ_2 we get for $\alpha = 1,2$,

$$\begin{aligned}\int f(t) \; d\sigma_\alpha(t) &= \int f(t) \; d\sigma_\alpha(t) + \int f(1-t) \; d\sigma_\alpha(t) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} c_{k,n} \int (t^k + (1-t)^k) \; d\sigma_\alpha(t)\end{aligned}$$

which is the same for $\alpha = 1$ and $\alpha = 2$ due to (9). $\square$

3 Support properties

Theorem 1 is a consequence of the following result.

Theorem 8 *Let $\mu \in \mathcal{M}_1(\Omega_1)$ be $K_{\sigma,\beta_m,\beta_s}$-invariant and denote $\bar{p} := (1,0,0,\ldots)$ and $(p(n))$'s stopping time $H := \min\{n \geq 1 \ : \ p(n) = p(0)\}$. Then*

$$\int \frac{1}{x}\ d\sigma(x) < \infty \Longleftrightarrow \mu[S] = 1 \Longleftrightarrow \mu[S] > 0 \Longleftrightarrow \mu[\{\bar{p}\}] > 0 \Longleftrightarrow E_{\bar{p}}[H] < \infty.$$

Proof We start by proving that $\int 1/x \ d\sigma(x) < \infty$ implies $\mu[S] = 1$. Fix an arbitrary $0 < \vartheta \leq 1/2$ and consider the random variables

$$W_n := \sum_{i \geq 1} p_i 1_{\vartheta^n < p_i} \quad (n \geq 1).$$

After one step of the process W_n may increase, decrease or stay unchanged. If we merge two intervals then W_n cannot decrease, but may increase by the mass of one or two intervals which are smaller than ϑ^n but become part of an interval which is bigger than ϑ^n. If we split an interval then W_n cannot increase, but it decreases if the original interval was larger than ϑ^n and at least one of its parts is smaller than ϑ^n. Thus given p, the expected increment Δ of W_n after one step of the process is

$$\begin{aligned}
\Delta &:= \Delta_+ - \Delta_-, \qquad \text{where} \\
\Delta_+ &:= \beta_m \sum_{i \neq j} p_i p_j \left(p_i 1_{p_i \leq \vartheta^n < p_j} + p_j 1_{p_j \leq \vartheta^n < p_i} + (p_i + p_j) 1_{p_i, p_j \leq \vartheta^n < p_i + p_j}\right) \qquad \text{and} \\
\Delta_- &:= \beta_s \sum_i p_i^2 \int \left(p_i 1_{(1-x)p_i \leq \vartheta^n < p_i} + x p_i 1_{x p_i \leq \vartheta^n < (1-x)p_i}\right)\ d\sigma(x).
\end{aligned}$$

We bound Δ_+ from below by

$$\begin{aligned}
\Delta_+ &\geq 2\beta_m \sum_{i,j} p_i^2 p_j 1_{p_i \leq \vartheta^n} \cdot 1_{\vartheta^n < p_j} \\
&\geq 2\beta_m \left(\sum_i p_i^2 1_{\vartheta^{n+1} < p_i \leq \vartheta^n}\right) \left(\sum_j p_j 1_{\vartheta^n < p_j}\right) \\
&\geq 2\beta_m \vartheta^{2n+2} W_n \sharp I_{n+1}
\end{aligned}$$

where

$$I_n := \left\{i \geq 1 \ : \ \vartheta^n < p_i \leq \vartheta^{n-1}\right\} \qquad (n \geq 1),$$

and Δ_- from above by

$$\begin{aligned}
\Delta_- &\leq \beta_s \sum_{i\geq 1} \int \left(p_i^3 1_{\vartheta^n < p_i \leq \vartheta^n/(1-x)} + p_i^3 x 1_{\vartheta^n < p_i} \cdot 1_{p_i \leq \vartheta^n/x}\right)\ d\sigma(x) \\
&\leq \beta_s \sum_{i\geq 1} p_i^3 1_{\vartheta^n < p_i \leq \vartheta^{n-1}} \qquad [\text{since } \vartheta \leq 1/2 \leq 1-x] \\
&\quad + \beta_s \sum_{i\geq 1} \int \sum_{j=0}^{n-1} p_i^3 x 1_{\vartheta^{n-j} < p_i \leq \vartheta^{n-j-1}} 1_{p_i \leq \vartheta^n/x}\ d\sigma(x) \\
&\leq \beta_s \sum_{i\geq 1} \vartheta^{3(n-1)} 1_{\vartheta^n < p_i \leq \vartheta^{n-1}} \\
&\quad + \beta_s \sum_{i\geq 1} \int \sum_{j=0}^{n-1} \vartheta^{3(n-j-1)} x 1_{\vartheta^{n-j} < p_i \leq \vartheta^{n-j-1}} 1_{x \leq \vartheta^j}\ d\sigma(x) \\
&\leq \beta_s \vartheta^{3(n-1)} \sharp I_n + \beta_s \sum_{j=0}^{n-1} \sum_{i\geq 1} \vartheta^{3(n-j-1)} \vartheta^j 1_{\vartheta^{n-j} < p_i \leq \vartheta^{n-j-1}} \sigma[(0,\vartheta^j]] \\
&\leq \beta_s \vartheta^{3(n-1)} \sharp I_n + \beta_s \vartheta^{3(n-1)} \sum_{j=0}^{n-1} \vartheta^{-2j} \sigma[(0,\vartheta^j]] \sharp I_{n-j} \\
&\leq 2\beta_s \vartheta^{3(n-1)} \sum_{j=0}^{n-1} \vartheta^{-2j} \sigma[(0,\vartheta^j]] \sharp I_{n-j}.
\end{aligned}$$

Since μ is invariant by assumption, $0 = \int \Delta\ d\mu = \int \Delta_+\ d\mu - \int \Delta_-\ d\mu$ and therefore

$$\begin{aligned}
2\beta_m \int W_n \sharp I_{n+1}\ d\mu &\leq 2\beta_s \vartheta^{3n-3-2n-2} \sum_{j=0}^{n-1} \vartheta^{-2j} \sigma[(0,\vartheta^j]] \int \sharp I_{n-j}\ d\mu \\
&= 2\beta_s \vartheta^{-5} \sum_{j=0}^{n-1} \vartheta^{-j} \sigma[(0,\vartheta^j]] \int \vartheta^{n-j} \sharp I_{n-j}\ d\mu.
\end{aligned}$$

Consequently ,

$$
\begin{aligned}
&\sum_{n\geq 1}\int W_n \sharp I_{n+1}\ d\mu\\
\leq\ & \frac{\vartheta^{-5}\beta_s}{\beta_m}\sum_{n\geq 1}\sum_{j=0}^{n-1}\vartheta^{-j}\sigma[(0,\vartheta^j]]\int \vartheta^{n-j}\sharp I_{n-j}\ d\mu\\
=\ & \frac{\vartheta^{-5}\beta_s}{\beta_m}\left(\sum_{j=0}^{\infty}\vartheta^{-j}\sigma[(0,\vartheta^j]]\right)\sum_{n\geq 1}\int \vartheta^{n}\sharp I_{n}\ d\mu\\
\leq\ & \frac{\vartheta^{-5}\beta_s}{(1-\vartheta)\beta_m}\left(\sum_{j=0}^{\infty}(\vartheta^{-j}-\vartheta^{-j+1})\sigma[(0,\vartheta^j]]\right)\sum_{n\geq 1}\int\sum_{i\in I_n} p_i\ d\mu\\
=\ & \frac{\vartheta^{-5}\beta_s}{(1-\vartheta)\beta_m}\left(\int\sum_{j=0}^{\infty}1_{\vartheta^{-j}\leq 1/x}(\vartheta^{-j}-\vartheta^{-j+1})\ d\sigma(x)\right)\int |p|\ d\mu\\
\leq\ & \frac{\vartheta^{-5}\beta_s}{(1-\vartheta)\beta_m}\int\frac{1}{x}\ d\sigma(x)
\end{aligned}
$$

which is finite by assumption. Therefore, $W_n \sharp I_{n+1}$ is summable and hence tends μ-a.s. to 0. However, W_n converges μ-a.s. to 1 as n tends to ∞. Thus even $\sharp I_{n+1}$ tends μ-a.s. to zero, which means that I_{n+1} is μ-a.s. eventually empty, that is $\mu[S]=1$.

Now we assume $\mu[S]>0$ in which case there exist some $i\geq 1$ and $\varepsilon>0$ such that $\delta:=\mu[p_i>\varepsilon, p_{i+1}=0]>0$. By i successive merges of the positive parts and μ's invariance we obtain

$$\mu[\{\bar{p}\}]=\mu[p_1=1]\geq(2\beta_m\varepsilon^2)^{i-1}\delta>0. \tag{10}$$

Next, we assume $\mu[\{\bar{p}\}]>0$ and note that $K_{\sigma,\beta_m,\beta_s}1_S=1_S$ and thus, defining $\bar{\mu}:=\mu/\mu[S]$, one obtains an invariant measure supported on S. The chain determined by $K_{\sigma,\beta_m,\beta_s}$ on S is $\delta_{\bar{p}}$-irreducible, and has $\bar{\mu}$ as invariant measure, with $\bar{\mu}[\{\bar{p}\}]>0$. Therefore, Kac's recurrence theorem [11, Theorem 10.2.2] yields $E_{\bar{p}}[H]<\infty$.

Finally, we assume $E_{\bar{p}}[H]<\infty$ and show $\int 1/x\ d\sigma(x)<\infty$. If $A:=\{\bar{p}=p(0)\neq p(1)\}$, then $P_{\bar{p}}[A]=\beta_s>0$, and when $p\in A$ we write $p(1)=p^{\xi}:=(1-\xi,\xi,0,\ldots)$, where ξ has distribution σ. Furthermore, restricted to A and conditioned on ξ, $H\geq\tau$ P_{p^ξ}–a.s., where in terms of the chain's sampling and merge/split interpretation, τ is the first time a marked part of size ξ is sampled, i.e. a geometric random variable with parameter $1-(1-\xi)^2\leq 2\xi$. Thus

$$\infty>E_{\overline{p}}[H]\geq P_{\overline{p}}[A]E_{\overline{p}}[H|A]\geq\beta_s\left(1+\int E_{p^\xi}[\tau]d\sigma(\xi)\right)\geq\beta_s\left(1+\int\frac{1}{2\xi}d\sigma(\xi)\right).$$

□

Corollary 9 *If $\int 1/x\ d\sigma(x)<\infty$ then there exists a unique $K_{\sigma,\beta_m,\beta_s}$-invariant probability measure $\mu\in\mathcal{M}_1(\Omega_1)$.*

Proof In view of Theorem 1, for the study of invariant measures it is enough to restrict attention to the state space S, where the Markov chain $(p(n))_n$ is $\delta_{\bar{p}}$-irreducible, implying, see [11, Chapter 10], the uniqueness of the invariant measure. □

4 Transience and recurrence

Proof of Theorem 2 The statement about positive recurrence is included in Theorem 8.

The idea for the proof of the transience statement is to show that under (1) the event that the size of the smallest positive part of the partition never increases has positive probability. By

$$n_0 := 0 \quad \text{and} \quad n_{j+1} := \inf\{n > n_j : \ p(n) \neq p(n-1)\} \quad (j \geq 0)$$

we enumerate the times n_j at which the value of the Markov chain changes. Denote by s_n the (random) number of instants among the first n steps of the Markov chain in which some interval is split. Since $j - s_{n_j}$ is the number of steps among the first n_j steps in which two parts are merged and since this number can never exceed s_{n_j} if $p(0) = \bar{p}$, we have that $P_{\bar{p}}$-a.s.,

$$s_{n_j} \geq \left\lceil \frac{j}{2} \right\rceil \quad \text{for all} \quad j \geq 0. \tag{11}$$

Let $(\tau_l)_{l\geq 1}$ denote the times at which some part is split. This part is split into two parts of sizes $\ell(l)$ and $L(l)$ with $0 < \ell(l) \leq L(l)$. According to the model the random variables $\xi_l := \ell(l)/(\ell(l) + L(l))$, $l \geq 1$, are independent with common distribution σ. Further, for any deterministic sequence $\xi = (\xi_n)_n$, let $P_{\xi,\bar{p}}[\ \cdot\ ]$ denote the law of the process which evolves using the kernel $K_{\sigma,\beta_m,\beta_s}$ except that at the times τ_l it uses the values ξ_l as the splitting variables. Note that

$$P_{\bar{p}}[\ \cdot\ ] = \int P_{\xi,\bar{p}}[\ \cdot\ ]\ d\sigma^{\mathbb{N}}(\xi)\,.$$

Now denote by $q(n) := \min\{p_i(n) : \ i \geq 1,\ p_i(n) > 0\}$ $(n \geq 0)$ the size of the smallest positive part at time n. We prove that for $N \geq 0$,

$$q(0) \geq \ldots \geq q(N) \quad \text{implies} \quad q(N) \leq \xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_{s_N}. \tag{12}$$

(Here and in the sequel, we take $\xi_1 \wedge \ldots \wedge \xi_{s_N} = \infty$ if $s_N = 0$). Indeed, we need only consider the case $s_N > 0$, in which case there exists a $1 \leq t \leq s_N$ such that $\xi_t = \xi_1 \wedge \ldots \wedge \xi_{s_N}$, and $\tau_t \leq N$. But clearly $q(\tau_t) \leq \xi_t$, and then the condition $q(1) \geq \cdots \geq q(N)$ and the fact that $\tau_t \leq N$ imply $q(N) \leq q(\tau_t) \leq \xi_t = \xi_1 \wedge \ldots \wedge \xi_{s_N}$, as claimed.

Next, fix some $\varepsilon \in (0, \beta_0/2]$ where $\beta_0 := \min\{\beta_m, \beta_s\}/2$. We will prove by induction over $j \geq 1$ that

$$P_{\xi,\bar{p}}[\varepsilon > q(1), q(0) \geq \ldots \geq q(n_j)] \geq \beta_s 1_{\xi_1 < \varepsilon} \prod_{k=1}^{j-1} \left(1 - \frac{\xi_1 \wedge \ldots \wedge \xi_{\lceil k/2 \rceil}}{\beta_0}\right). \tag{13}$$

For $j = 1$ the left hand side of (13) equals the probability that the unit interval is split in the first step with the smaller part being smaller than ε which equals $\beta_s 1_{\xi_1 < \varepsilon}$. Assume that (13) has been proved up to j. Then, with $\mathcal{F}_{n_j} = \sigma(p(n), n \le n_j)$,

$$\begin{aligned} & P_{\xi,\bar{p}}[\varepsilon > q(1),\ q(0) \ge \ldots \ge q(n_{j+1})] \\ = \ & E_{\xi,\bar{p}}\left[P_{\xi,\bar{p}}[q(n_j) \ge q(n_{j+1}) \mid \mathcal{F}_{n_j}], \varepsilon > q(1),\ q(0) \ge \ldots \ge q(n_j)\right]. \end{aligned} \tag{14}$$

Now choose k minimal such that $p_k(n_j) = q(n_j)$. One possibility to achieve $q(n_j) \ge q(n_{j+1})$ is not to merge the part $p_k(n_j)$ in the next step in which the Markov chain moves. The probability to do this is

$$\begin{aligned} 1 - \frac{2\beta_m \sum_{a:a\neq k} p_a(n_j)p_k(n_j)}{\beta_m \sum_{a\neq b} p_a(n_j)p_b(n_j) + \beta_s \sum_a p_a^2(n_j)} & \ge 1 - \frac{\beta_m q(n_j) \sum_a p_a(n_j)}{\beta_0 \sum_{a,b} p_a(n_j)p_b(n_j)} \\ & \ge 1 - \frac{q(n_j)}{\beta_0}. \end{aligned}$$

Therefore (14) is greater than or equal to

$$E_{\xi,\bar{p}}\left[(1 - q(n_j)/\beta_0), \varepsilon > q(1),\ q(0) \ge \ldots \ge q(n_j)\right].$$

By (12) this can be estimated from below by

$$E_{\xi,\bar{p}}\left[(1 - (\xi_1 \wedge \ldots \wedge \xi_{s_{n_j}})/\beta_0),\ \varepsilon > q(1),\ q(0) \ge \ldots \ge q(n_j)\right].$$

This is due to (11) greater than or equal to

$$(1 - (\xi_1 \wedge \ldots \wedge \xi_{\lceil j/2 \rceil})/\beta_0)\ P_{\xi,\bar{p}}\left[\varepsilon > q(1),\ q(0) \ge \ldots \ge q(n_j)\right].$$

Along with the induction hypothesis this implies (13) for $j+1$.

Taking expectations with respect to ξ in (13) yields

$$P_{\bar{p}}[q(n) \le \varepsilon \quad \text{for all } n \ge 1] \ge E_{\bar{p}}\left[\beta_s 1_{\xi_1 < \varepsilon} \prod_{k \ge 1}\left(1 - \frac{\varepsilon \wedge \xi_2 \wedge \ldots \wedge \xi_{\lceil k/2 \rceil}}{\beta_0}\right)\right]. \tag{15}$$

By independence of ξ_1 from $\xi_i,\ i \ge 2$, the right hand side of (15) equals

$$\beta_s \left(1 - \frac{\varepsilon}{\beta_0}\right)^2 P[\xi_1 < \varepsilon] E_{\bar{p}}\left[\prod_{k\ge 2}\left(1 - \frac{\varepsilon \wedge \xi_2 \wedge \ldots \wedge \xi_k}{\beta_0}\right)^2\right]. \tag{16}$$

Observe that (1) implies $P[\xi_1 < \varepsilon] = \sigma[(0,\varepsilon)] > 0$. By Jensen's inequality and monotone convergence, (16) can be estimated from below by

$$c_1 \exp\left(\sum_{k\ge 2} 2E_{\bar{p}}\left[\ln\left(1 - \frac{\varepsilon \wedge \xi_2 \wedge \ldots \wedge \xi_k}{\beta_0}\right)\right]\right)$$

with some positive constant $c_1 = c_1(\varepsilon)$. Since $\ln(1-x) \ge -2x$ for $x \in [0, 1/2]$ this is greater than

$$c_1 \exp\left(-\frac{4}{\beta_0}\sum_{k\ge 2} E_{\bar{p}}[\xi_2 \wedge \ldots \wedge \xi_k]\right) = c_1 \exp\left(-\frac{4}{\beta_0}\int_0^{1/2} \frac{P_{\bar{p}}[\xi_1 > t]}{P_{\bar{p}}[\xi_1 \le t]}\, dt\right) \tag{17}$$

where we used that due to independence

$$E_{\bar{p}}[\xi_2 \wedge \ldots \wedge \xi_k] = \int_0^{1/2} P_{\bar{p}}[\xi_1 > t]^{k-1}\, dt.$$

Due to assumption (1), (17) and therefore also the left hand side of (15) are positive. This implies transience of $\bar{p}$. $\square$

Remark: It follows from Theorem 2 that $c := \int x^{-1}\, d\sigma(x) < \infty$ implies $\int \sigma((0,x])^{-1}\, dx = \infty$. This can be seen also directly from $c \geq \int_0^x t^{-1}\, d\sigma(t) \geq \int_0^x x^{-1}\, d\sigma(t) = \sigma((0,x])/x$ for all $0 < x \leq 1/2$, which shows $\int \sigma((0,x])^{-1}\, dx \geq \int (cx)^{-1}\, dx$.

5 Poisson-Dirichlet invariant probability measures

Throughout this section, the splitting measure is the uniform measure on $(0,1/2]$. To emphasize this, we use K_{β_m,β_s} instead of $K_{\sigma,\beta_m,\beta_s}$ throughout. Recall that $\theta = \beta_s/\beta_m$.
It will be convenient to equip Ω (but not $\bar{\Omega}_{\leq}$) with the ℓ_1 topology (noting that the Borel σ-algebra is not affected by this change of topology), and to replace the kernel K_{β_m,β_s} by

$$\begin{aligned} K^H_{\beta_m,\beta_s}(p,\cdot) = \beta_m \sum_{i\neq j} \widehat{p}_i \widehat{p}_j \delta_{M_{ij}p}(\cdot) \quad &+ \quad \beta_s \sum_i \widehat{p}_i^2 \int_0^1 \delta_{S_i^u p}(\cdot)\, du \\ &+ \quad \left(1 - \beta_m + (\beta_m - \beta_s)\, |\widehat{p}|_2^2\right) \delta_p(\cdot). \end{aligned} \tag{18}$$

Both kernels coincide on Ω_1 (not on $\Omega_{\leq}$). However, $K^H_{\beta_m,\beta_s}$ has the advantage that it is well defined on all of Ω and is homogeneous (hence the superscript H) in the sense of the first of the following two lemmas, whose proof is straightforward and in which by a slight abuse of notation $K^H_{\beta_m,\beta_s}$ will denote both the kernel in Ω_1 and in Ω and also the operators induced by these kernels, the distinction being clear from the context.

Lemma 10 *For all $p \in \Omega$, $K^H_{\beta_m,\beta_s}(\pi p, \cdot) = K^H_{\beta_m,\beta_s}(p,\cdot) \circ \pi^{-1}$.*
More generally, denoting $(\Pi f)(p) = f(\pi(p))$, we have $K^H_{\beta_m,\beta_s}\Pi = \Pi K^H_{\beta_m,\beta_s}$.

In particular, if $\mu \in \mathcal{M}_1(\Omega)$ is invariant (resp. reversing) for $K^H_{\beta_m,\beta_s}$ then $\mu \circ \pi^{-1} \in \mathcal{M}_1(\Omega_1)$ is invariant (resp. reversing) for K_{β_m,β_s}.

Lemma 11 *The kernel $K^H_{\beta_m,\beta_s}$ maps continuous bounded functions to continuous bounded functions.*

Proof [Proof of Lemma 11] Note that we work with the ℓ_1 topology, and hence have to modify the proof in Lemma 4. The ℓ_1 topology makes the mapping $p \mapsto \widehat{p}$ continuous (when Ω_1 is equipped with the induced ℓ_1 topology). Fix $F \in C_b(\Omega)$. By (18) we have

$$\begin{aligned} K^H_{\beta_m,\beta_s} F(p) &= \beta_m \sum_{i\neq j} \widehat{p}_i \widehat{p}_j F\left(M_{ij}p\right) + \beta_s \sum_i \left(\widehat{p}_i\right)^2 \int_0^1 F\left(S_i^u p\right)\, du \\ &\qquad + \left(1 - \beta_m + (\beta_m - \beta_s)|\widehat{p}|_2^2\right) F(p) \\ &= \beta_m K_1(p) + \beta_s K_2(p) + K_3(p). \end{aligned} \tag{19}$$

Note that for $l = 1,2$, $K_l(p)$ is of the form $\langle T_l(p)\widehat{p}, \widehat{p}\rangle$, with $T_l(\cdot) \in C(\Omega; L(\ell_1, \ell_\infty))$, and $\langle \cdot, \cdot \rangle$ denoting the standard duality pairing. In stating this we have used the facts that F is continuous and bounded, and that all the mappings M_{ij} and S_i^u are contractive.

The continuity of K_l, $l = 1,2$, then follows from

$$\langle T_l(q)\widehat{q}, \widehat{q}\rangle - \langle T_l(p)\widehat{p}, \widehat{p}\rangle = \langle T_l(q)\widehat{q}, \widehat{q} - \widehat{p}\rangle + \langle (T_l(q) - T_l(p))\,\widehat{q}, \widehat{p}\rangle + \langle T_l(p)(\widehat{q} - \widehat{p}), \widehat{p}\rangle$$

after observing that $|\widehat{q}|$ and $\|T_l(q)\|$ remain bounded in any ℓ_1 neighborhoods of p.

The continuity of K_3 is obvious being the product of two continuous functions of p. It has thus been shown that $K^H_{\beta_m,\beta_s} F \in C(\Omega)$. □

Theorem 12 *The Poisson-Dirichlet measure $\widehat{\mu}_\theta \in \mathcal{M}_1(\Omega_1)$ is reversing for $K_{\sigma,\beta_m,\beta_s}$ with $\sigma = U(0,1/2]$.*

Proof By Lemma 10 it suffices to verify that $\mu_\theta \in \mathcal{M}_1(\Omega)$ is reversing for the kernel $K^H_{\beta_m,\beta_s}$, which for simplicity will be denoted by K for the rest of this proof.

We thus need to show that

$$E_\theta(G\,KF) = E_\theta(F\,KG) \qquad \text{for all } F, G \in B(\Omega). \tag{20}$$

Because $\mu_\theta \circ M_{ij}^{-1}$ and $\mu_\theta \circ (S_i^u)^{-1}$ are absolutely continuous with respect to μ_θ, it follows from (19) that if $F, \{F_n\}_n$ are uniformly bounded functions such that $\int |F_n - F|\mu_\theta(dp) \to_{n\to\infty} 0$, then $\int |KF_n - KF|\mu_\theta(dp) \to_{n\to\infty} 0$. Thus, by standard density arguments we may and shall assume F and G to be continuous.

Define for each $\varepsilon > 0$ the truncated intensity measure $\nu_\theta^\varepsilon \equiv \mathbf{1}_{(\varepsilon,\infty)}\nu_\theta$, and the corresponding Poisson measure μ_θ^ε, with expectation operator E_θ^ε. Alternatively, if X is distributed in Ω according the μ_θ, then μ_θ^ε is the distribution of $T^\varepsilon X := (X_i 1_{X_i > \varepsilon})_i$, that is, $\mu_\theta^\varepsilon = \mu_\theta \circ (T^\varepsilon)^{-1}$. Observe that $\forall \delta > 0$,

$$\mu_\theta(|T^\varepsilon X - X| > \delta) \le \delta^{-1} E_\theta |T^\varepsilon X - X| = \delta^{-1} E_\theta \sum_{p_i < \varepsilon} p_i = \delta^{-1} \int_0^\varepsilon x\,\nu_\theta(dx) \stackrel{\varepsilon\to 0}{\longrightarrow} 0 \ ,$$

implying that the measures μ_θ^ε converge weakly to μ_θ as $\varepsilon \to 0$.

To prove (20) we first write

$$\begin{aligned} |E_\theta(G\,KF) - E_\theta(F\,KG)| \ \le\ & |E_\theta^\varepsilon(G\,KF) - E_\theta(G\,KF)| + |E_\theta^\varepsilon(G\,KF) - E_\theta^\varepsilon(F\,KG)| \\ & + |E_\theta^\varepsilon(F\,KG) - E_\theta(F\,KG)| \end{aligned} \tag{21}$$

and conclude that the first and third terms in (21) converge to 0 as $\varepsilon \to 0$ by virtue of the weak convergence of μ_θ^ε to μ_θ and K's Feller property, established in Lemma 11. It thus remains to be shown that, for all $F, G \in B(\Omega)$ and $\varepsilon > 0$,

$$\lim_{\varepsilon\to 0} |E_\theta^\varepsilon(G\,KF) - E_\theta^\varepsilon(F\,KG)| = 0. \tag{22}$$

The truncated intensity ν_θ^ε has finite mass $V_\theta^\varepsilon = \theta\int_\varepsilon^\infty x^{-1}e^{-x}\,dx$, and thus $N(\mathbb{R}_+) < \infty$, μ_θ^ε–a.s. In particular each $F \in B(\Omega)$ can be naturally represented as a sequence $(F_n)_{n=0}^\infty$ of symmetric F_n's $\in B\left(\mathbf{R}_+^n\right)$, with $\|F_n\|_\infty \le \|F\|_\infty$ for each n. As a result, and in terms of the expectation operators $E^\varepsilon_{\theta,n}$ of μ^ε_θ conditioned on $\{N(\mathbb{R}_+)=n\}$, we may write

$$E^\varepsilon_\theta(G\,KF) - E^\varepsilon_\theta(F\,KG) = e^{-V^\varepsilon_\theta}\sum_{n=1}^\infty \frac{(V^\varepsilon_\theta)^n}{n!}\left[E^\varepsilon_{\theta,n}(G\,KF)\ - E^\varepsilon_{\theta,n}(F\,KG)\right], \tag{23}$$

while by the definition (18) of $K^H_{\beta_m,\beta_s}$ and the properties stated above of the Poisson random measure conditioned on $\{N(\mathbb{R}_+)=n\}$,

$$\begin{aligned}
&\frac{(V^\varepsilon_\theta)^n}{n!}\,E^\varepsilon_{\theta,n}(G\,KF) = \\
&\quad \frac{\beta_m\theta^n}{n!}\sum_{\substack{i,j=1\\ i\ne j}}^n \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \widehat{x}_i\,\widehat{x}_j F_{n-1}(M_{ij}\mathbf{x})G_n(\mathbf{x})e^{-|\mathbf{x}|}\frac{dx_1}{x_1}\ldots\frac{dx_n}{x_n} \\
&\quad + \frac{\beta_s\theta^n}{n!}\sum_{i=1}^n \int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \widehat{x}_i^2\left(\int_0^1 F_{n+1}(S^u_i\mathbf{x})G_n(\mathbf{x})\,du\right)e^{-|\mathbf{x}|}\frac{dx_1}{x_1}\ldots\frac{dx_n}{x_n} \\
&\quad + \frac{\theta^n}{n!}\int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \left(1-\beta_m+(\beta_m-\beta_s)\sum_{i=1}^n \widehat{x}_i^{\,2}\right)F_n(\mathbf{x})G_n(\mathbf{x})\,e^{-|\mathbf{x}|}\frac{dx_1}{x_1}\ldots\frac{dx_n}{x_n} \\
&\qquad =: I^{(1)}_n(F,G)+I^{(2)}_n(F,G)+I^{(3)}_n(F,G)\quad ,
\end{aligned}\tag{24}$$

where $\mathbf{x}=(x_1,\ldots,x_n)$. Our goal is to prove that this expression, after summing in n, is roughly symmetric in F and G (as stated precisely in (22)). Obviously $I^{(3)}_n(F,G)=I^{(3)}_n(G,F)$, and in addition we aim at showing that $I^{(2)}_{n-1}(G,F)\approx I^{(1)}_n(F,G)$ (with an error appropriately small as $\varepsilon\to 0$). This will be achieved by a simple change of variables, including the splitting coordinate u in $I^{(2)}$.

In the integral of the i–th term in $I^{(2)}_{n-1}(G,F)$ perform the change of variables $(u,x_1,\ldots,x_{n-1})\to(y_1,\ldots,y_n)$ given by $\mathbf{y}=S^u_i\mathbf{x}$ (or $(u,\mathbf{x})=(\frac{y_i}{y_0+y_i},M_{i\,n}\mathbf{y})$). More precisely, $\begin{cases} y_i = ux_i \\ y_j = x_j,\quad j\ne i \\ y_n=(1-u)x_i\end{cases}$
for which $|\mathbf{y}|=|\mathbf{x}|$ and $dy_1\ldots dy_n = x_i\,du\,dx_1\ldots dx_{n-1}$, so that

$$\begin{aligned}
&I^{(2)}_{n-1}(G,F) = \\
&\quad = \frac{\beta_s\theta^{n-1}}{(n-1)!}\sum_{i=1}^{n-1}\int_\varepsilon^\infty\cdots\int_\varepsilon^\infty G_n(\mathbf{y})F_{n-1}(M_{i\,n}\mathbf{y})\frac{e^{-|\mathbf{y}|}dy_1\ldots dy_n}{|\mathbf{y}|^2\;y_1\ldots\breve{y}_i\ldots y_{n-1}}+C^\varepsilon_n
\end{aligned}$$

(C^ε_n is as the term preceding it but with the dy_i and dy_n integrals taken in $[0,\varepsilon]$, and the notation $\breve{y}_i$ means that the variable y_i has been eliminated from the denominator)

$$= \frac{\beta_s\theta^{n-1}}{(n-2)!}\int_\varepsilon^\infty\cdots\int_\varepsilon^\infty G_n(\mathbf{y})F_{n-1}(M_{1\,n}\mathbf{y})\frac{e^{-|\mathbf{y}|}dy_1\ldots dy_n}{|\mathbf{y}|^2\;y_2\ldots y_{n-1}}+C^\varepsilon_n \tag{25}$$

(by F_{n-1}'s symmetry, the sum's $(n-1)$ terms are equal, hence the last equality).

On the other hand, and for the same reason of symmetry, the $n(n-1)$ terms in $I_n^{(1)}(F,G)$ are all equal so that

$$I_n^{(1)}(F,G) = \frac{\beta_m \theta^n}{(n-2)!} \int_\varepsilon^\infty \dots \int_\varepsilon^\infty F_{n-1}(M_{1\,2}\mathbf{x}) G_n(\mathbf{x}) \frac{e^{-|\mathbf{x}|}\, dx_1 \dots dx_n}{|\mathbf{x}|^2\, x_3 \dots x_n} \,. \tag{26}$$

Comparing (25) with (26), and observing that by definition $\beta_m \theta = \beta_s$, we conclude that there exists a $C > 0$ such that, for $n \geq 2$,

$$\begin{aligned} |C_n^\varepsilon| &:= \left| I_{n-1}^{(2)}(G,F) - I_n^{(1)}(F,G) \right| \\ &\leq \frac{\|F\|_\infty \|G\|_\infty \beta_s \theta}{(n-2)!} \int_0^\varepsilon \int_0^\varepsilon dy_1\, dy_n \, \frac{1}{((n-2)\varepsilon)^2} \left(\theta \int_\varepsilon^\infty \frac{e^{-y}}{y}\, dy \right)^{n-2} \\ &\leq C\, \frac{(V_\theta^\varepsilon)^{n-2}}{(n-1)!} \end{aligned} \tag{27}$$

Applying (27) via (24) in (23) twice, once as written and once reversing the roles of F and G, and noting that $I_1^{(1)}(F,G) = I_1^{(1)}(G,F) = 0$, we have

$$\begin{aligned} &|E_\theta^\varepsilon(G\, KF) - E_\theta^\varepsilon(F\, KG)| \\ &\leq e^{-V_\theta^\varepsilon} \left(\sum_{n=2}^\infty \left| I_{n-1}^{(2)}(G,F) - I_n^{(1)}(F,G) \right| + \sum_{n=2}^\infty \left| I_{n-1}^{(2)}(F,G) - I_n^{(1)}(G,F) \right| \right) \\ &\leq 2Ce^{-V_\theta^\varepsilon} \sum_{n=2}^\infty \frac{(V_\theta^\varepsilon)^{n-2}}{(n-1)!} \leq \frac{2C}{V_\theta^\varepsilon} \end{aligned}$$

from which (22) follows immediately since $V_\theta^\varepsilon \to_{\varepsilon \to 0} \infty$. $\square$

Proof of Theorem 3 (a) The Poisson-Dirichlet law $\mu = \widehat{\mu}_\theta$ is reversing by Theorem 12, and hence invariant. We now show that it belongs to $\mathcal{A}$. Note first that μ_k is absolutely continuous with respect to leb^k: for any $D \subset \Omega_<^k$ with leb$^k(D) = 0$, it holds that

$$\mu_k(D) \leq \int_{\mathbb{R}_+} \nu_\theta \left[\exists\, \mathbf{j} \in \mathbb{N}_{\neq}^k : (X_{j_1}, \dots, X_{j_k}) \in xD \right] d\gamma_\theta(x) = 0 \,,$$

where we used the fact that under μ_θ, $\pi(X) = X/|X|$ and $|X|$ are independent, with $|X|$ being distributed according to the Gamma law $\gamma_\theta(dx)$ of density $1_{x \geq 0} x^{\theta-1} e^{-x}/\Gamma(\theta)$ (see [9]). It thus suffices to compute the limit

$$p_k(x_1, \dots, x_k) := \lim_{\delta \to 0} \frac{E_{\widehat{\mu}_\theta} \left[\# \left\{ \mathbf{j} \in \mathbb{N}_{\neq}^k : p_{j_i} \in (x_i, x_i + \delta)\ , i = 1, \dots, k \right\} \right]}{\delta^k} \,,$$

where all x_i are pairwise distinct and nonzero, to have

$$m_k(x_1,\ldots,x_k) = p_k(x_1,\ldots,x_k)\prod_{i=1}^k x_i\,.$$

For such $x_1,\ldots,x_k$, set $I_i^\delta = (x_i, x_i+\delta)$ and $I^\delta = \cup_{i=1}^k I_i^\delta$. Define

$$L_X^\delta := \sum_i X_i 1_{\{X_i\notin I^\delta\}}\,,\quad N_{x_i}^\delta = \#\{j : X_j \in I_i^\delta\}\,.$$

By the memoryless property of the Poisson process, for any Borel subset $A\subset\mathbb{R}$,

$$\lim_{\delta\to 0} P(L_X^\delta \in A \,|\, N_{x_i}^\delta, i=1,\ldots,k) = P(|X|\in A) = \gamma_\theta(A)\,, \tag{28}$$

where (28), as above, is due to [9]. Further, recall that N and $(\widehat{X}_i)_i$ are independent. Recall that the density of the Poisson process at $(y_1,\ldots,y_k)$ is $\theta^k e^{-|y|}/\prod_{i=1}^k y_i$, where $|y| = y_1+\ldots+y_k$. Performing the change of variables $y_i/(z+|y|) = x_i$, one finds that the Jacobian of this change of coordinate is $(z+|y|)^k/(1-|x|)$ (in computing this Jacobian, it is useful to first make the change of coordinates $(y_1,\ldots,y_{k-1},|y|)\mapsto(\bar{x}_1,\ldots,\bar{x}_{k-1},|\bar{x}|)$ where $|y|,|\bar{x}|$ are considered as independent coordinates, and note the block-diagonal structure of the Jacobian). It follows that

$$m_k(x_1,\ldots,x_k) = \frac{\theta^k}{(1-|x|)}\int_0^\infty \exp\left(-z|x|/(1-|x|)\right)\gamma_\theta(dz) = \theta^k(1-|x|)^{\theta-1},$$

which is real analytic on $\{x\in\mathbb{R}^k : |x|<1\}$. Thus, $\widehat{\mu}_\theta\in\mathcal{A}$. In passing, we note that $m_k(\cdot)=1$ on $\bar{\Omega}_<^k$ when $\theta=1$.

(b) 1) First we show that the family of functions $(m_k)_{k\geq 1}$ associated with μ, determines μ. To this end, define for $\mathbf{j}\in\mathbb{N}^k$ $(k\in\mathbb{N})$ functions $g_{\mathbf{j}},\hat{g}_{\mathbf{j}}:\Omega_1\to[0,1]$ by

$$g_{\mathbf{j}}(p) := \sum_{\mathbf{i}\in\mathbb{N}_{\neq}^k}\prod_{\ell=1}^k p_{i_\ell}^{j_\ell}\quad\text{and}\quad \hat{g}_{\mathbf{j}}(p) := \prod_{\ell=1}^k Z_{j_\ell}(p)\quad\text{where}\quad Z_j(p) := \sum_i p_i^j.$$

Note that any function $\hat{g}_{\mathbf{j}}$ with $\mathbf{j}\in\mathbb{N}^k$ can be written after expansion of the product as a (finite) linear combination of functions $g_{\mathbf{h}}$ with $\mathbf{h}\in\mathbb{N}^n, n\geq 1$. Since we have by the definition of μ_k that

$$\int g_{\mathbf{j}}\,d\mu = \int_{\bar{\Omega}_<^k}\prod_{\ell=1}^k x_\ell^{j_\ell-1}\,d\mu_k(x) = \int_{\bar{\Omega}_<^k} m_k(x)\prod_{\ell=1}^k x_\ell^{j_\ell-1}\,dx\,, \tag{29}$$

the family $(m_k)_{k\geq 1}$ therefore determines the expectations $\int \hat{g}_{\mathbf{j}}\,d\mu$ $(\mathbf{j}\in\mathbb{N}^k, k\geq 1)$. Consequently, $(m_k)_{k\geq 1}$ determines also the joint laws of the random variables $(Z_1,\ldots,Z_k)$, $k\geq 1$, under μ. We claim that these laws characterize μ. Indeed, let $\bar{\mu}$ be the distribution of the random variable $\pi := (Z_n)_{n\geq 0}:\Omega_1\to[0,1]^{\mathbb{N}}$ under μ. Since π is injective it suffices to show that the distributions of $(Z_1,\ldots,Z_k)$, $k\geq 1$, under μ determine $\bar{\mu}$. But, since any continuous test function F on the compact space $[0,1]^{\mathbb{N}}$ can be uniformly approximated by the local function $F_k((x_n)_{n\geq 1}) := F(x_1,\ldots,x_k,0,\ldots)$, this is true due to

$$\int F\,d\bar{\mu} = \lim_{k\to\infty}\int F_k\,d\bar{\mu} = \lim_{k\to\infty}\int F_k(Z_1,\ldots,Z_k,0,\ldots)\,d\mu.$$

2) For $\mu \in \mathcal{A}$, the set of numbers

$$m_k^{(\mathbf{n})} := m_k^{(\mathbf{n})}(x_1, \ldots, x_k)\Big|_{0,0,\ldots,0} := \frac{\partial^{\mathbf{n}} m_k}{\partial x_1^{n_1} \cdots \partial x_k^{n_k}}\Big|_{0,0,\ldots,0} \tag{30}$$

with $k \geq 1$ and $n_1 \geq n_2 \geq \ldots \geq n_k \geq 0$ are enough to characterize $(m_k)_k$, and hence by the first part of the proof of b), to characterize μ. It is thus enough to prove that K_{β_m,β_s} uniquely determines these numbers. Toward this end, first note that

$$\int_0^1 m_1(x)\ dx = \mu_1[[0,1)] = 1. \tag{31}$$

To simplify notations, we define $m_0 \equiv 1$ and extend m_k to a function on $[0,1]^k$ by setting it 0 on the complement of $\bar{\Omega}^k_<$. For $k \geq 1$ we have

$$\int_0^1 m_k(x_1, \ldots, x_k) dx_1 = \left(1 - \sum_{i=2}^k x_i\right) m_{k-1}(x_2, \ldots, x_k). \tag{32}$$

Indeed, for $k = 1$ this is (31) while for $k \geq 2$, and arbitrary $B \in \mathcal{B}_{\bar{\Omega}^{k-1}_<}$,

$$\begin{aligned}
&\int_B \int_0^1 m_k(x_1, x_2, \ldots, x_k)\ dx_1\ dx_2 \ldots dx_k = \mu_k[[0,1] \times B] \\
&= E_\mu\left[\sum_{(j_2,\ldots,j_k)\in\mathbb{N}^{k-1}_{\neq}} \left(\prod_{i=2}^k p_{j_i}\right) 1_B(p_{j_2}, \ldots, p_{j_k}) \sum_{j_1 \notin \{j_2,\ldots,j_k\}} p_{j_1} 1_{[0,1]}(p_{j_1})\right] \\
&= \int 1_B(p_2, \ldots, p_k) \left(1 - \sum_{i=2}^k p_i\right) d\mu_{k-1} \\
&= \int_B \left(1 - \sum_{i=2}^k x_i\right) m_{k-1}(x_2, \ldots, x_k)\ dx_2 \ldots dx_k,
\end{aligned}$$

which implies (32). Now we fix $k \geq 1$, apply K_{β_m,β_s} to the test function $\#\{\mathbf{j} \in \mathbb{N}^k_{\neq} : p_{j_i} \in (x_i, x_i + \delta), i = 1, \ldots, k\}\delta^{-k}$, with $(x_1, \ldots, x_k) \in \Omega^k_<$ having pairwise distinct coordinates, and take $\delta \searrow 0$, which yields the basic relation

$$\begin{aligned}
&\beta_m \sum_{i=1}^k \int_0^{x_i} z(x_i - z) p_{k+1}(x_1, \ldots, x_{i-1}, z, x_i - z, x_{i+1}, \ldots, x_k) dz \\
&+2\beta_s \sum_{i=1}^k \int_{x_i}^1 z p_k(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_k) dz \\
= \ & 2\beta_m \left(\sum_{i=1}^k x_i\right) p_k(x_1, \ldots, x_k) + (\beta_s - \beta_m) \left(\sum_{i=1}^k x_i^2\right) p_k(x_1, \ldots, x_k) \\
&-\beta_m \left(\sum_{i=1}^k x_i\right)^2 p_k(x_1, \ldots, x_k).
\end{aligned}$$

Here the left hand side represents mergings and splittings that produce a new part roughly at one of the x_i-s; the right hand side represents parts near one of the x_i-s that merge or split. After multiplying by $x_1 \cdots x_k$, rearranging and using (32) to get rid of the integral with upper limit 1, we obtain the equality

$$\beta_m \sum_{i=1}^{k} x_i \int_0^{x_i} m_{k+1}(x_1, \ldots, x_{i-1}, z, x_i - z, x_{i+1}, \ldots, x_k) dz \tag{33}$$

$$-2\beta_s \sum_{i=1}^{k} x_i \int_0^{x_i} m_k(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_k) dz \tag{34}$$

$$+2\beta_s \sum_{i=1}^{k} x_i m_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \tag{35}$$

$$-2\beta_s \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} x_i x_j m_{k-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \tag{36}$$

$$= 2\beta_m \left(\sum_{i=1}^{k} x_i \right) m_k(x_1, \ldots, x_k) + (\beta_s - 2\beta_m) \left(\sum_{i=1}^{k} x_i^2 \right) m_k(x_1, \ldots, x_k) \tag{37}$$

$$-\beta_m \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} x_i x_j m_k(x_1, \ldots, x_k)\,. \tag{38}$$

We now proceed to show how (33) – (38) yield all the required information. As starting point for a recursion, we show how to compute $m_k(0, \ldots, 0)$ for all $k \geq 1$. Taking in (33) – (38) all $x_i \to 0$ except for x_1 and using the continuity of the functions m_k yields

$$\begin{aligned} & \beta_m \int_0^{x_1} m_{k+1}(z, x_1 - z, 0, \ldots, 0) dz - 2\beta_s \int_0^{x_1} m_k(z, 0, \ldots, 0) dz \\ & +2\beta_s m_{k-1}(0, \ldots, 0) \\ = \; & 2\beta_m m_k(x_1, 0, \ldots, 0) + (\beta_s - 2\beta_m) x_1 m_k(x_1, 0, \ldots, 0)\,. \end{aligned}$$

Letting $x_1 \to 0$ we get $\beta_m m_k(0, \ldots, 0) = \beta_s m_{k-1}(0, \ldots, 0)$. With $m_0 = 1$ as start of the recursion this implies

$$m_k(0, \ldots, 0) = \theta^k \quad (k \geq 0). \tag{39}$$

For the evaluation of the derivatives of m_k we proceed inductively. Recall the functions $m_k^{(\mathbf{n})}(x_1, \ldots, x_k)$ defined in (30), and write $m_k^{(n_1, n_2, \ldots, n_j)}$, $j < k$, for $m_k^{(n_1, n_2, \ldots, n_j, 0 \ldots, 0)}$. Fix $\mathbf{n}$ such that $n_1 \geq n_2 \geq \ldots \geq n_k$, with $n_1 \geq 2$. Our analysis rests upon differentiating (33) – (38) n_1 times with respect to x_1; to make this differentiation easy, call a term a *G term of degree* ℓ if it is a linear combination of terms of the form

$$x_1 \int_0^{x_1} m_{k+1}^{(\ell+1)}(z, x_1 - z, x_2, \ldots, x_k) dz$$

and

$$\int_0^{x_1} m_{k+1}^{(\ell)}(z, x_1 - z, x_2, \ldots, x_k) dz$$

and

$$m_{k+1}^{(\ell-1)}(x_1, 0, x_2, \ldots, x_k)$$

and

$$x_1 m_{k+1}^{(\ell)}(x_1, 0, x_2, \ldots, x_k)\,.$$

Note that (33) – (38) contains one G term of degree -1 in (33) and that differentiating a G term of degree ℓ once yields a G term of degree $\ell + 1$. Thus, differentiating the G term in (33) $n_1 \geq 2$ times and substituting $x_1 = 0$, we recover a constant multiple of $m_{k+1}^{(n_1-2)}(0, x_2, \ldots, x_k, 0)$. Similarly, call a term an *H term of degree* ℓ if it is a linear combination of terms of the form

$$m_k^{(\ell)}(x_1, \ldots, x_k) \quad \text{and} \quad x_1 m_k^{(\ell+1)}(x_1, \ldots, x_k) \quad \text{and} \quad x_1^2 m_k^{(\ell+2)}(x_1, \ldots, x_k).$$

Observe, that differentiating an H term of degree ℓ produces an H term of degree $\ell + 1$. If we differentiate twice the term $x_1 \int_0^{x_1} m_k(z, x_2, \ldots, x_k)dz$ in (34) we get an H term of degree 0. Therefore differentiating this term $n_1 \geq 2$ times results in an H term of degree $n_1 - 2$. Since the term $x_1^2 m_k(x_1, \ldots, x_k)$ in (37) is an H term of degree -2, differentiating this term n_1 times produces also an H term of degree $n_1 - 2$. Thus both terms produce after n_1-fold differentiation and evaluation at $x_1 = 0$ a constant multiple of $m_k^{(n_1-2)}(0, x_2, \ldots, x_k)$. The H term $x_1 m_k(x_1, \ldots, x_k)$ in (37) is treated more carefully. It is easy to see by induction that its n_1-th derivative equals $n_1 m_k^{(n_1-1)}(x_1, \ldots, x_k) + x_1 m_k^{(n_1)}(x_1, \ldots, x_k)$. Evaluating it at $x_1 = 0$ gives $n_1 m_k^{(n_1-1)}(0, x_2, \ldots, x_k)$.

Moreover, the terms in (35) and (36) for $i = 1$ vanish when differentiated twice with respect to x_1, while the term in (38), when differentiated with respect to x_1 $n_1 \geq 2$ times, and substituting $x_1 = 0$, produces terms of the form $(\sum_{j=2}^k x_j) m_k^{(n_1-1)}(0, x_2, \ldots, x_k)$.

Summarizing the above, we conclude by differentiating (33) – (38) $n_1 \geq 2$ times with respect to x_1 and subsequent evaluation at $x_1 = 0$ that there are some constants $C_i(n_1)$, such that

$$
\begin{aligned}
&2\beta_m n_1 m_k^{(n_1-1)}(0,x_2,\ldots,x_k) \qquad [(37a), i=1]\\
&= \quad C_1 m_{k+1}^{(n_1-2)}(0,x_2,\ldots,x_k,0) \qquad [(33), i=1]\\
&\quad +C_2 m_k^{(n_1-2)}(0,x_2,\ldots,x_k) \qquad [(37b), i=1+(34), i=1]\\
&\quad +C_3\left(\sum_{i=2}^k x_i\right) m_k^{(n_1-1)}(0,x_2,\ldots,x_k) \qquad [(38)]\\
&\quad -\left[2\beta_m\left(\sum_{i=2}^k x_i\right)+(2\beta_m-\beta_s)\left(\sum_{i=2}^k x_i^2\right)\right] m_k^{(n_1)}(0,x_2,\ldots,x_k) \qquad [(37)]\\
&\quad +\beta_m\sum_{i=2}^k x_i\int_0^{x_i} m_{k+1}^{(n_1)}(0,x_2,\ldots,x_{i-1},z,x_i-z,x_{i+1},\ldots,x_k)dz \qquad [(33)]\\
&\quad -2\beta_s\sum_{i=2}^k x_i\int_0^{x_i} m_k^{(n_1)}(0,x_2,\ldots,x_{i-1},z,x_{i+1},\ldots,x_k)dz \qquad [(34)]\\
&\quad +2\beta_s\sum_{i=2}^k x_i m_{k-1}^{(n_1)}(0,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_k) \qquad [(35)]\\
&\quad -2\beta_s n_1\sum_{i=2}^k x_i m_{k-1}^{(n_1-1)}(0,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_k) \qquad [(36), j=1]\\
&\quad -2\beta_s\sum_{i=2}^k\sum_{j=2,j\neq i}^k x_i x_j m_{k-1}^{(n_1)}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k) \qquad [(36)]\\
&\quad +\beta_m\sum_{i=2}^k\sum_{j=2,j\neq i}^k x_i x_j m_k^{(n_1)}(0,x_2,\ldots,x_k) \qquad [(38)]
\end{aligned}
$$

For $x_2=\ldots=x_k=0$ only the first three lines do not vanish and give a recursion which allows us to compute starting at (39) all derivatives $m_k^{(n)}(0,\ldots,0)$ $(n\geq 0)$.
Further differentiating with respect to $x_2,\ldots,x_k$, one concludes that there exist constants $D^i_{\mathbf{n},\mathbf{n}'}$ such that

$$
\begin{aligned}
2\beta_m n_1 m_k^{(n_1-1,n_2,\ldots,n_k)} &= \sum_{\mathbf{n}':|\mathbf{n}'|\leq|\mathbf{n}|-2,n'_i\leq n_i,n'_1<n_1}[D^1_{\mathbf{n},\mathbf{n}'}m_k^{(\mathbf{n}')}+D^2_{\mathbf{n},\mathbf{n}'}m_{k+1}^{(\mathbf{n}',0)}+D^3_{\mathbf{n},\mathbf{n}'}m_{k-1}^{(\mathbf{n}')}]\\
&+\sum_{\mathbf{n}':|\mathbf{n}'|\leq|\mathbf{n}|-1,n'_i\leq n_i,n_1=n'_1}[D^4_{\mathbf{n},\mathbf{n}'}m_k^{(\mathbf{n}')}+D^5_{\mathbf{n},\mathbf{n}'}m_{k+1}^{(\mathbf{n}',0)}+D^6_{\mathbf{n},\mathbf{n}'}m_{k-1}^{(\mathbf{n}')}]. \qquad (40)
\end{aligned}
$$

We now compute iteratively any of the $m_k^{(\mathbf{n})}$, with $n_1\geq n_2\geq\ldots\geq n_k$: first, substitute in (40) $n_1=n+1, n_2=1$ to compute $m_k^{(n,1)}$, for all n,k. Then, substitute $n_1=n+1, n_2=j$ $(j\leq n)$ to compute iteratively $m_k^{(n,j)}$ from the knowledge of the family $(m_k^{(\ell,j')})_{k,\ell,j'<j}$, etc. More generally, having computed the terms $(m_k^{(n_1,n_2,\ldots,n_j)})_{j\leq j_0<k}$, we compute first $m_k^{(n_1,\ldots,n_{j_0},1)}$ by substituting in (40) $\mathbf{n}=(n_1+1,n_2,\ldots,n_{j_0},1)$, and then proceed inductively as above. □

6 Concluding remarks

1) We of course conjecture (as did A. Vershik) the

Conjecture 13 *Part b) of Theorem 3 continues to hold true without the assumption* $\mu \in \mathcal{A}$.

We note that recently, [16] provided a further indication that Conjecture 13 may be valid, by proving that when $\theta = 1$, and one initiates the basic chain with the state $(1, 0, 0, \ldots)$, then the state of the chain sampled at a random, independent Binomial$(n, 1/2)$ time, converges in law to the Poisson-Dirichlet law of parameter 1.

It is tempting to use the technique leading to (3) in order to prove the conjecture by characterizing the expectations with respect to μ of suitable test functions. One possible way to do that is to consider a family of polynomials defined as follows. Let $\mathbf{n} = (n_2, n_3, \ldots, n_d)$ be a finite sequence of nonnegative integers, with $n_d \geq 1$. We set $|\mathbf{n}| = \sum_{j=2}^{d} j n_j$, i.e. we consider $\mathbf{n}$ as representing a partition of $|\mathbf{n}|$ having n_j parts of size j, and no parts of size 1. Recall next $Z_j = Z_j(p) = \sum_i p_i^j$ and the $\mathbf{n}$-*polynomial*

$$P_{\mathbf{n}}(p) = \prod_{j=2}^{d} Z_j^{n_j} : \Omega_1 \to \mathbb{R}.$$

$|\mathbf{n}|$ is the *degree* of $P_{\mathbf{n}}$, and, with $\mathbf{n}$ and d as above, d is the *maximal monomial degree* of $P_{\mathbf{n}}$. Because we do not allow partitions with parts of size 1, it holds that $P_{\mathbf{n}} \neq P_{\mathbf{n}'}$ if $\mathbf{n} \neq \mathbf{n}'$ (i.e, there exists a point $p \in \Omega_1$ such that $P_{\mathbf{n}}(p) \neq P_{\mathbf{n}'}(p)$). It is easy to check that the family of polynomials $\{P_{\mathbf{n}}\}$ is separating for $\mathcal{M}_1(\Omega)$. Letting $\Delta_{\mathbf{n}}$ denote the expected increment (conditioned on p) of $P_{\mathbf{n}}$ after one step of the process, we have that $\Delta_{\mathbf{n}}$ is uniformly bounded. Hence, by invariance of μ, $\int \Delta_{\mathbf{n}} d\mu = 0$. Expanding this equality, we get that

$$\begin{aligned}
&\frac{\beta_m}{\beta_s} E_\mu \left[\sum_{\alpha,\beta} p_\alpha p_\beta \sum_{k=2}^{d} \left(\prod_{j=2}^{k-1} (Z^q_{j,\alpha,\beta})^{n_j} \right) \left(\sum_{\ell=0}^{n_k-1} (Z_k)^\ell \binom{n_k}{\ell} q_{\alpha,\beta,k}^{n_k-\ell} \right) \left(\prod_{j=k+1}^{d} Z_j \right) \right] = \\
&-E_\mu \left[\sum_{\alpha} p_\alpha^2 \sum_{k=2}^{d} \int \left(\left(\prod_{j=2}^{k-1} \left(Z^f_{j,\alpha,x} \right)^{n_j} \right) \left(\sum_{\ell=0}^{n_k-1} Z_k^\ell \binom{n_k}{\ell} f_{\alpha,k,x}^{n_k-\ell} \right) \left(\prod_{j=k+1}^{d} Z_j^{n_j} \right) \right) d\sigma(x) \right] \\
&+\frac{\beta_m}{\beta_s} E_\mu \left[\sum_{\alpha} p_\alpha^2 \sum_{k=2}^{d} \left(\prod_{j=2}^{k-1} (Z_j + (2^j - 2) p_\alpha^j)^{n_j} \right) \right. \\
&\qquad \left. \left(\sum_{\ell=0}^{n_k-1} Z_k^\ell \binom{n_k}{\ell} \left((2^k - 2) p_\alpha^k \right)^{n_k-\ell} \right) \left(\prod_{j=k+1}^{d} Z_j^{n_j} \right) \right]
\end{aligned} \tag{41}$$

where

$$q_{\alpha,\beta,j} = (p_\alpha + p_\beta)^j - p_\alpha^j - p_\beta^j \geq 0, \; f_{\alpha,j,x} = [x^j + (1-x)^j - 1] p_\alpha^j \leq 0,$$
$$Z^q_{j,\alpha,\beta} = Z_j + q_{\alpha,\beta,j}, \; Z^q_{j,\alpha,x} = Z_j + f_{\alpha,j,x}.$$

Note that all terms in (41) are positive. Note also that the right hand side of (41) is a polynomial of degree $|\mathbf{n}|+2$, with maximal monomial degree $d+2$, whereas the left hand side is a polynomial of degree at most $|\mathbf{n}|+2$ and maximal monomial degree at most d. Let $\pi(k)$ denote the number of integer partitions of k which do not have parts of size 1. Then, there are $\pi(k)$ distinct polynomials of degree k, whereas (41) provides at most $\pi(k-2)$ relations between their expected values (involving possibly the expected value of lower order polynomials). Since always $\pi(k) > \pi(k-2)$, it does not seem possible to characterize an invariant probability measure $\mu \in \mathcal{M}_1(\Omega_1)$ using only these algebraic relations.

2) With a lesser degree of confidence we conjecture

Conjecture 14 *For any $\sigma \in \mathcal{M}_1((0,1/2])$ and any $\beta_m, \beta_s \in (0,1]$ there exists exactly one $K_{\sigma,\beta_m,\beta_s}$-invariant probability measure $\mu \in \mathcal{M}_1(\Omega_1)$.*

3) We have not been able to resolve whether the state $\bar{p} = (1,0,0,\ldots)$ is transient or null-recurrent for $K_{\sigma,1,1}$ with $\sigma = U(0,1/2]$.

4) There is much literature concerning coagulation-fragmentation processes. Most of the recent probabilistic literature deals with processes which exhibit either pure fragmentation or pure coagulation. For an extensive review, see [1], and a sample of more recent references is [2], [4] and [6]. Some recent results on coagulation-fragmentation processes are contained in [8]. However, the starting point for this and previous studies are the coagulation-fragmentation equations, and it is not clear how to relate those to our model. The functions m_k introduced in the context of Theorem 3 are related to these equations.

5) A characterization of the Poisson-Dirichlet process as the unique measure coming from an i.i.d. residual allocation model which is invariant under a *split and merge* transformation is given in [7]. J. Pitman has pointed out to us that a slight modification of this transformation, preceded by a size biased permutation and followed by ranking, is equivalent to our Markov transition $K_{\sigma,\beta_m,\beta_s}$. Pitman [13] then used this observation to give an alternative proof of part (a) of Theorem 3.

6) Yet another proof of part a) of Theorem 3 which avoids the Poisson representation and Theorem 12 can be obtained by computing the expectation of the polynomials $P_{\mathbf{n}}(p)$, defined in remark 1) above, under the Poisson-Dirichlet law. We prefer the current proof as it yields more information and is more transparent.

7) A natural extension of Poisson-Dirichlet measures are the *two parameter* Poisson-Dirichlet measures, see e.g. [14]. Pitman raised the question, which we have not addressed, of whether there are splitting measures σ which would lead to invariant measures from this family.

8) While according to Theorem 3 there is a reversing probability measure for $\sigma = U(0,1/2]$ this does not hold for general $\sigma \in \mathcal{M}_1((0,1/2])$. For instance, let us assume that the support of σ is finite. Then there exist $0 < a < b \le 1/2$ such that $\sigma[(a,b)] = 0$. To show that any invariant measure μ is not reversing it suffices to find $s,t \in \Omega_1$ such that the detailed balance equation

$$\mu[\{s\}]K_{\sigma,\beta_m,\beta_s}(s,\{t\}) = \mu[\{t\}]K_{\sigma,\beta_m,\beta_s}(t,\{s\}) \tag{42}$$

fails. Due to Theorem 8, $\mu[\{\bar{p}\}] > 0$. Now we first refine the partition $\bar{p}$ by successive splits until we reach a state $p \in \Omega_1$ with $p_1 < \varepsilon$, where $\varepsilon > 0$ is a small number. Since μ has finite support, $\mu[\{p\}] > 0$. Then we create from p by successive mergings some $s \in \Omega_1$ with $a < s_2/s_1 < b$, which is possible if ε was chosen small enough. Again, $\mu[\{s\}] > 0$. If we call now t the state which one gets from s by merging s_1 and s_2, then the left hand side of (42) is positive. On the other hand, the right hand side of (42) is zero because of $K(t,\{s\}) = 0$ due to the choice of a and b.

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References

[1] D. J. Aldous, Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists, *Bernoulli* **5** (1999), pp. 3–48.

[2] D.J. Aldous and J. Pitman, The standard additive coalescent, *Ann. Probab.*, **26** (1998), pp. 1703–1726.

[3] R. Arratia, A. D. Barbour and S. Tavaré, *Logarithmic Combinatorial Structures: a Probabilistic Approach*, book, preprint (2001).

[4] E. Bolthausen and A.-S. Sznitman, On Ruelle's probability cascades and an abstract cavity method, *Comm. Math. Phys.*, **197** (1998), pp. 247–276.

[5] R. Brooks, *private communication* (1999).

[6] S.N. Evans and J. Pitman, Construction of Markovian coalescents, *Ann. Inst. Henri Poincaré*, **34** (1998), pp. 339–383.

[7] A. Gnedin and S. Kerov, A characterization of GEM distributions ,*Combin. Probab. Comp.* **10**, no. 3, (2001), pp. 213–217.

[8] I. Jeon, Existence of gelling solutions for coagulation-fragmentation equations, *Comm. Math. Phys.* **194** (1998), pp. 541–567.

[9] J. F. C. Kingman, Random discrete distributions, *J. Roy. Statist. Soc. Ser. B* **37** (1975), pp. 1–22.

[10] J. F. C. Kingman, *Poisson Processes*, Oxford University Press, Oxford (1993).

[11] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, London (1993).

[12] J. Pitman, Random discrete distributions invariant under size–biased permutation, *Adv. Appl. Prob.* **28** (1996), pp. 525–539.

[13] J. Pitman, Poisson–Dirichlet and GEM invariant distributions for split-and-merge transformations of an interval partition, *Combin. Prob. Comp.*, to appear.

[14] J. Pitman and M. Yor, The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator, *Ann. Probab.*, **25** (1997), pp. 855–900.

[15] N. V. Tsilevich, Stationary random partitions of positive integers, *Theor. Probab. Appl.* **44** (2000), pp. 60–74.

[16] N. V. Tsilevich, On the simplest split-merge operator on the infinite-dimensional simplex, *PDMI PREPRINT 03/2001*, (2001). `ftp://ftp.pdmi.ras.ru/pub/publicat/preprint/2001/03-01.ps.gz`