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Strictly stable distributions on convex cones*

Youri Davydov
Université Lille 1
Laboratoire de Paule Painlevé
UFR de Mathématiques
Villeneuve d'Ascq Cedex
France
youri.davydov@univ-lille1.fr

Ilya Molchanov
University of Berne
Department of Mathematical Statistics
and Actuarial Science
Sidlerstrasse 5, CH-3012 Berne
Switzerland
ilya@stat.unibe.ch

Sergei Zuyev
Department of Statistics
and Modelling Science
University of Strathclyde
Glasgow G1 1XH
Scotland, U. K.
sergei@stams.strath.ac.uk

Abstract

Using the LePage representation, a symmetric α-stable random element in Banach space $\mathbb{B}$ with $\alpha \in (0,2)$ can be represented as a sum of points of a Poisson process in $\mathbb{B}$. This point process is union-stable, i. e. the union of its two independent copies coincides in distribution with the rescaled original point process. This shows that the classical definition of stable random elements is closely related to the union-stability property of point processes.

These concepts make sense in any convex cone, i. e. in a semigroup equipped with multiplication by numbers, and lead to a construction of stable laws in general cones by means of

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the LePage series. We prove that random samples (or binomial point processes) in rather general cones converge in distribution in the vague topology to the union-stable Poisson point process. This convergence holds also in a stronger topology, which implies that the sums of points converge in distribution to the sum of points of the union-stable point process. Since the latter corresponds to a stable law, this yields a limit theorem for normalised sums of random elements with strictly α-stable limit for $\alpha \in (0, 1)$.
By using the technique of harmonic analysis on semigroups we characterise distributions of strictly α-stable random elements and show how possible values of the characteristic exponent α relate to the properties of the semigroup and the corresponding scaling operation, in particular, their distributivity properties. It is shown that several conditions imply that a strictly stable random element admits the LePage representation. The approach developed in the paper not only makes it possible to handle stable distributions in rather general cones (like spaces of sets or measures), but also provides an alternative way to prove classical limit theorems and deduce the LePage representation for strictly stable random vectors in Banach spaces.

Key words: Character, convex cone, Laplace transform, LePage series, Lévy measure, point process, Poisson process, random measure, random set, semigroup, stable distribution, union-stability.

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1 Introduction

Stability of random elements is one of the basic concepts in probability theory. A random vector ξ with values in a Banach space $\mathbb{B}$ has a strictly stable distribution with characteristic exponent $\alpha \neq 0$ (notation StαS) if, for all $a, b > 0$,

$$a^{1/\alpha}\xi_1 + b^{1/\alpha}\xi_2 \overset{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi\,, \tag{1.1}$$

where ξ_1, ξ_2 are independent copies of ξ, and $\overset{\mathcal{D}}{=}$ denotes equality in distribution. Note that in [49] the notation SαS stands for *symmetric* α-stable, when ξ coincides in distribution with $-\xi$. The stability concept (in a more general form) was introduced by Paul Lévy and thereafter has been actively studied in relation to limit theorems for sums of random variables, see, e. g., [21; 49; 54] for the finite-dimensional case and [2; 49] for random elements in Banach spaces. The following basic results are available:

- a complete characterisation of StαS random elements in terms of characteristic functionals;
- a complete description in terms of LePage (or Khinchin–Lévy–LePage) expansions;
- a complete description of the domains of attraction using tail behaviour.

Recall that the LePage series representation of a SαS vector $\xi \in \mathbb{R}^d$ for $\alpha \in (0,2)$ (and for a StαS vector with $\alpha \in (0,1)$) is

$$\xi \overset{\mathcal{D}}{=} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}\varepsilon_k\,, \tag{1.2}$$

where $\Gamma_1, \Gamma_2, \dots$ are the successive times of jumps of a homogeneous Poisson process on the positive half-line, and $\varepsilon_1, \varepsilon_2, \dots$ are i. i. d. unit random vectors independent of the Γ's, see [33].

Definition (1.1) of stability makes sense in any space, where addition of elements and multiplication by positive scalars are defined, i. e. in any convex cone. So far this case has been thoroughly investigated for the cone of compact convex subsets of a separable Banach space with the topology generated by the Hausdorff metric and the main operation being the Minkowski (elementwise) addition, see, e. g., [12; 17; 18]. In this case the principal results (LePage representation, domains of attractions, etc.) are completely analogous to those well-known for general Banach spaces. The only exception is the lack of non-trivial strictly stable distributions for $\alpha > 1$, which is explained by impossibility to centre sums of sets and the fact that the exact dual operation to the Minkowski addition cannot be defined.

A wealth of information about stable probability measures on Euclidean spaces and locally compact groups can be found in [21], see also [20] and [23]. Strictly stable distributions are necessarily infinitely divisible. Infinite divisibility of random objects in positive convex cones was studied in [26]. Infinite divisible elements in semigroups were studied in [47; 48] and are comprehensively covered in the monograph [24]. As we show in this paper, the studies of stability of random elements in semigroups bring into the play further properties of semigroups, in particular, the relationships between the neutral element and the origin, distributivity laws and the metric structure on semigroups.

Max-stable random variables appear if the addition in (1.1) is replaced by the maximum operation and min-stable random variables in case of the minimum operation [44]. A max-stable

StαS laws exists for every positive α, while min-stable laws may have every $\alpha < 0$ as a possible value for the characteristic exponent. On the other hand, it is well known that possible values of α for a Banach space-valued random element fill $(0, 2]$. One of the aims of this paper is to identify basic algebraic and topological properties of the carrier space that control the range of the characteristic exponent α of strictly stable random elements. We also give answers to the questions that concern series representations and domains of attraction of stable laws. The key idea is the relationship between the stability concept formulated in (1.1) and the concept of stability of point processes (and random sets) with respect to the union [36; 39]. The obtained results cover not only the classical cases of linear spaces and extremes of random variables. They provide a unified framework for considering additive- and max-stable laws for random variables as special cases of semigroup-valued random elements. This framework, in particular, includes random closed sets stable with respect to Minkowski addition or union operations and random measures stable with respect to addition or convolution operations.

The content of this paper can be outlined as follows. Section 2 introduces the main algebraic concepts: the convex cone, the neutral element and the origin, and some metric properties that are important in the sequel.

Section 3 shows that the classical stability concept (1.1) is, in a sense, secondary to the union-stability. The LePage representation (1.2) of stable laws provides an expression of a StαS random vector ξ as the sum of points of a Poisson point process Π_α with support points

$$\operatorname{supp}\Pi_\alpha = \{\Gamma_k^{-1/\alpha}\varepsilon_k,\ k \geq 1\}\,.$$

The distribution of the random set $\kappa_\alpha = \operatorname{supp}\Pi_\alpha$ is union-stable, i. e.

$$a^{1/\alpha}\kappa'_\alpha \cup b^{1/\alpha}\kappa''_\alpha \overset{\mathcal{D}}{=} (a+b)^{1/\alpha}\kappa_\alpha\,, \tag{1.3}$$

where κ'_α and κ''_α are independent copies of κ_α. Distributions of union-stable random closed sets in $\mathbb{R}^d$ have been completely characterised in [39], see also [38, Ch. 4]. Because κ'_α and κ''_α possess common points with probability zero, (1.3) immediately translates into (1.1) by taking sums over the support points. It should be noted that (1.3) makes sense for all $\alpha \neq 0$ (see [39]), while (1.1) in a Banach space holds for $\alpha \in (0, 2]$ only. One of the reasons for this is that the series in (1.2) might diverge. In Section 3.3 it is shown that the LePage series absolutely converges for all $\alpha \in (0, 1)$ if the semigroup possesses a sub-invariant norm. This representation makes it possible to define Lévy processes with values in a cone, see Section 3.4.

It is well known that, under a regular variation type assumption, a normalised random sample (or binomial point process) converges in distribution to the Poisson point process Π_α, see, e. g., [44]. This result is generalised for point processes in Polish spaces. Furthermore, we show that in case $\alpha \in (0, 1)$ this convergence holds in a stronger topology that ensures the convergence of sums of points from point processes. This complements the result of [11], where this type of convergence was studied for point processes in $\mathbb{R}^d$. From this fact we derive that normalised sums of random elements converge in distribution to the LePage representation of the corresponding stable law, i. e. a limit theorem for normalised samples in cones. This also yields a new proof of the limit theorem for random elements in Banach spaces with StαS limits for $\alpha \in (0, 1)$. These results are described in Section 4.

Section 5 explores the distributions of stable random elements on semigroups, including the ranges of the stability parameter. We first define the Laplace transform of a random element as a functional that acts on the family of the characters, and confirm its uniqueness. The infinite divisibility property implies that the Laplace transform has an exponential form. The main result establishes the equivalence between the stability property and the homogeneity of the corresponding Laplace exponent. Further we describe several essential properties of cones and semigroups that have a particular bearing in view of the properties of stable distributions. Among these properties the most important are the distributivity properties of the multiplication by numbers and the relationship between the neutral element and the origin in a semigroup. The range of possible parameters for the stable law provides a new characteristic of a general cone. In particular, we describe the cases when the stability parameter α belongs to $(0, 2]$ as in the conventional case of linear spaces and when α is an arbitrary positive or arbitrary negative number.

Section 6 exploits techniques from harmonic analysis on semigroups, in particular, the representations of infinite divisible and negative definite functions in view of characterising the Laplace exponents of StαS random elements. In particular, we show that the corresponding Lévy measure is homogeneous, and characterise other ingredients of the integral representations: the linear functional and the quadratic form.

Finally, Section 7 aims to show that strictly stable random elements in a rather general cone admit the LePage representation. We address this question by comparing the integral representations of Laplace exponents with the formula for the probability generating functional of a stable Poisson process. It is shown that under rather weak conditions every StαS random element can be realised as the LePage series that corresponds to a Poisson process on the second dual semigroup. Its intensity measure is the Lévy measure of the corresponding StαS random element. The key issue here is to show that the Lévy measure is actually supported by the semigroup itself, which leads to the "conventional" LePage series similar to (1.2). Apart from the proof of the LePage representation in rather general semigroups, this also yields a new proof of the LePage representation for StαS random elements with $\alpha \in (0, 1)$ in reflexive Banach spaces.

To summarise, we first show that the sum of points of a union-stable Poisson point process follows StαS law, then demonstrate that convergence of point processes yields a limit theorem with StαS limiting distribution, and finally prove that in a rather general case any StαS random element with $\alpha \in (0, 1)$ can be represented as a sum of points of a Poisson process.

Section 8 describes a variety of examples drawing analogy or contrasting with classical Euclidean or Banach space valued StαS vectors. Clearly, if a cone is embeddable in a Banach space, then it is possible to use the results already available for stable distributions, see [49]. However even in this case we come up with new proofs that further our understanding of stable laws in linear spaces.

2 Convex cones

2.1 Basic definitions

Here we summarise several basic definitions related to convex cones and semigroups.

Definition 2.1. An abelian *topological semigroup* is a topological space $\mathbb{K}$ equipped with a commutative and associative continuous binary operation $+$. It is assumed that $\mathbb{K}$ possesses the *neutral element* $\mathbf{e}$ satisfying $x + \mathbf{e} = x$ for every $x \in \mathbb{K}$.

Consider a family of continuous automorphisms $D_a : \mathbb{K} \to \mathbb{K}$ indexed by positive real numbers $a > 0$. Assume that D_1 is the identical map and that $D_a D_b x = D_{ab} x$ for all $a, b > 0$ and $x \in \mathbb{K}$. In [43] such $\mathbb{K}$ is called an abelian semigroup over the operator domain $(0, \infty)$. The result of applying D_a to $x \in \mathbb{K}$ can be understood as the *multiplication* of x by a that yields the following equivalent reformulation of the properties of D.

Definition 2.2. A *convex cone* is an abelian topological semigroup $\mathbb{K}$ such that $\mathbb{K} \setminus \{\mathbf{e}\}$ is a Polish space, with a continuous operation $(x, a) \mapsto ax$ of multiplication by positive scalars a for $x \in \mathbb{K}$ so that the following conditions are satisfied:

$$a(x + y) = ax + ay\,, \quad a > 0,\ x, y \in \mathbb{K}, \tag{2.1}$$

$$a(bx) = (ab)x\,, \quad a, b > 0,\ x \in \mathbb{K}, \tag{2.2}$$

$$1x = x\,, \quad x \in \mathbb{K}\,, \tag{2.3}$$

$$a\mathbf{e} = \mathbf{e}\,, \quad a > 0\,. \tag{2.4}$$

$\mathbb{K}$ is called a *pointed cone* if there is a unique element $\mathbf{0}$ called the *origin* such that $ax \to \mathbf{0}$ as $a \downarrow 0$ for any $x \in \mathbb{K} \setminus \{\mathbf{e}\}$.

It should be emphasised that we do not always require the following distributivity condition

$$(a + b)x = ax + bx\,, \quad a, b > 0,\ x \in \mathbb{K}\,. \tag{2.5}$$

We often call (2.5) the *second distributivity law.* Although this condition is typically imposed in the literature on cones (see, e. g., [29]), this law essentially restricts the family of examples, e. g., it is not satisfied for the cone of compact (not necessarily convex) subsets of a Banach space (with Minkowski addition) or on $\mathbb{R}_+ = [0, \infty)$ with the maximum operation. Note that the condition (2.5) is not natural if the multiplication is generated by a family of automorphisms D_a, $a > 0$, as described above. In view of this, nx means $D_n x$ or x *multiplied* by n and *not* the sum of n identical summands being x.

If the addition operation on $\mathbb{K}$ is a group operation, then we simply say that $\mathbb{K}$ is a *group.* If also the second distributivity law holds, then the mapping $(a, x) \mapsto ax$ from $(0, \infty) \times \mathbb{K}$ can be extended to a mapping from $(-\infty, \infty) \times \mathbb{K}$ to $\mathbb{K}$ in such a way that $\mathbb{K}$ will satisfy the conventional axioms of a linear space. However, even if $\mathbb{K}$ is a group, the second distributivity law does not have to hold, e. g., if $\mathbb{K} = \mathbb{R}$ with the usual addition and the multiplication defined as $D_a x = a^\beta x$ with $\beta \neq 1$, see also Example 8.12.

2.2 Origin and neutral element

Unless stated otherwise we always assume that $\mathbb{K}$ is a pointed cone, i. e. it possesses the origin. The neutral element of $\mathbb{K}$ does not necessarily coincide with the origin. For instance, if $\mathbb{K}$ is the semigroup of compact sets in $\mathbb{R}^d$ with the union operation, the conventional multiplication by numbers and metrised by the Hausdorff metric, then $\mathbf{0} = \{0\}$, while the neutral element $\mathbf{e}$ is the empty set. In many other cases the neutral element does coincide with the origin, e. g., if $\mathbb{K}$ is a linear space. Note that unless in degenerated cases when either $\mathbb{K} = \{\mathbf{0}, \mathbf{e}\}$ or $x + y = \mathbf{e}$ for any $x, y \in \mathbb{K}$, the definition of the origin implies that $\mathbf{0} + \mathbf{0} = \mathbf{0}$ (by letting $a \downarrow 0$ in (2.1)), and $\mathbf{0}$ and $\mathbf{e}$ are the only elements of $\mathbb{K}$ satisfying $ax = x$ for all $a > 0$.

Lemma 2.3. *Let $\mathbb{K}$ be a pointed cone.*

(i) If the second distributivity law holds, then $\mathbf{e} = \mathbf{0}$.

(ii) If $\mathbb{K}$ is such that there are are two elements $x,\ y$ satisfying $x + y \neq \mathbf{e}$ and there exists $z \neq \mathbf{e}$ which possesses an inverse $(-z)$, i. e. $z + (-z) = \mathbf{e}$, then $\mathbf{e} = \mathbf{0}$.

Proof. *(i)* By (2.5), any $x \neq \mathbf{e}$ can be decomposed as

$$x = \frac{n-1}{n} x + \frac{1}{n} x\,, \quad n \geq 1\,. \tag{2.6}$$

By letting $n \to \infty$ and using the continuity of multiplication we arrive at $x = x + \mathbf{0}$ for any x. Thus, $\mathbf{0} = \mathbf{e}$ by the uniqueness of the neutral element.

(ii) By (2.1) and (2.4), $n^{-1}z + n^{-1}(-z) = \mathbf{e}$. The left-hand side converges as $n \to \infty$ to $\mathbf{0} + \mathbf{0}$. Letting $a \downarrow 0$ in (2.1) shows that this is $\mathbf{0}$. Hence $\mathbf{0} = \mathbf{e}$. $\square$

In particular, Lemma 2.3*(ii)* implies that $\mathbf{0} = \mathbf{e}$ if $\mathbb{K}$ is a non-degenerated group.

Definition 2.4. An element $z \in \mathbb{K}$ is called α-stable with $\alpha \neq 0$, if

$$a^{1/\alpha} z + b^{1/\alpha} z = (a+b)^{1/\alpha} z \tag{2.7}$$

for all $a, b > 0$ with convention $1/\infty = 0$ if $\alpha = \infty$.

Throughout this paper $\mathbb{K}(\alpha)$ denotes the set of α-stable elements of $\mathbb{K}$. Clearly, $\mathbf{e}, \mathbf{0} \in \mathbb{K}(\alpha)$ for any $\alpha \neq 0$. In particular, $\mathbb{K}(\infty)$ is the set of *idempotent* elements that satisfy $z + z = z$, and $\mathbb{K}(1)$ consists of all $z \in \mathbb{K}$ that satisfy (2.5).

Lemma 2.5. *If the second distributivity law (2.5) holds, then $\mathbb{K}(\alpha) = \{\mathbf{e}\}$ for any $\alpha \neq 1$.*

Proof. Putting $a = b$ in (2.7) and using (2.5) yields $2^{1-1/\alpha} z = z$ and $2^{1/\alpha - 1} z = z$. By iteration, we obtain that $\beta^n z = z$, where $\beta = 2^{1-1/\alpha}$ for $\alpha > 1$ and $\beta = 2^{1/\alpha - 1}$, for $\alpha < 1$. Passing to the limit, we have that $z = \mathbf{0}$ and thus $z = \mathbf{e}$ by Lemma 2.3*(i)*. $\square$

2.3 Norm and metric

Definition 2.6. A pointed cone $\mathbb{K}$ is said to be a *normed cone* if $\mathbb{K}$ (or $\mathbb{K} \setminus \{\mathbf{e}\}$ if $\mathbf{0} \neq \mathbf{e}$) is metrised by a metric d which is *homogeneous* at the origin, i. e. $d(ax, \mathbf{0}) = ad(x, \mathbf{0})$ for every $a > 0$ and $x \in \mathbb{K}$. The value $\|x\| = d(x, \mathbf{0})$ is called the *norm* of x.

In Sections 3 and 4 it is assumed that $\mathbb{K}$ is a normed cone. Note that the function $d(x, \mathbf{0})$ should be called a *gauge function* rather than a norm, since it is not assumed to be sub-linear, i. e. $d(x + y, \mathbf{0})$ is not necessarily smaller than $d(x, \mathbf{0}) + d(y, \mathbf{0})$. However, we decided to use the word *norm* in this context, because we employ the gauge function to define the balls and spheres in exactly the same way as the conventional norm is used. Most of further results can be reformulated for metrics that are homogeneous of a given order $r > 0$, i. e. $d(ax, \mathbf{0}) = a^r d(x, \mathbf{0})$.

It is obvious that $\|x\| = 0$ if and only if $x = \mathbf{0}$. Furthermore, $\|ax\| = a\|x\|$ for all $a > 0$ and $x \in \mathbb{K}$, so that if $x = ax$ for $x \neq \mathbf{0}, \mathbf{e}$ and $a \neq 1$, then $\mathbb{K}$ cannot be a normed cone. If $\mathbf{e} \neq \mathbf{0}$, then (2.4) implies that the *whole* $\mathbb{K}$ cannot be metrised with a homogeneous metric, since then $\|\mathbf{e}\|$ is necessarily infinite by (2.4). Thus $\mathbf{e}$ is the only element in a normed cone which may have an infinite norm and this is the case if and only if $\mathbf{e} \neq \mathbf{0}$. For instance, if $\mathbb{K}$ is the cone $[0, \infty]$ with the minimum operation, then the Euclidean distance from any $x \in \mathbb{R}_+$ to $\infty = \mathbf{e}$ is infinite.

If $\mathbb{K}$ is a linear space, the metric and the norm can be routinely constructed using a star-shaped neighbourhood of the origin. If this neighbourhood is convex, then $\mathbb{K}$ is a locally convex topological vector space and the corresponding norm is *sub-linear*, i. e.

$$\|x + y\| \leq \|x\| + \|y\| . \tag{2.8}$$

The open ball of radius r centred at $\mathbf{0}$ is denoted by

$$B_r = \{x \in \mathbb{K} : \ \|x\| < r\} .$$

The interior of its complement is given by

$$B^r = \{x : \ \|x\| > r\} .$$

If $\mathbf{e} \neq \mathbf{0}$, then $\mathbf{e} \in B^r$ for all $r > 0$. The set

$$\mathbb{S} = \{x : \ \|x\| = 1\}$$

is called the *unit sphere*. Note that $\mathbb{S}$ is complete with respect to the metric induced by the metric on $\mathbb{K}$. The existence of the origin implies that $\|x\| < \infty$ for all $x \in \mathbb{K} \setminus \{\mathbf{e}\}$, therefore $\mathbb{K}$ admits a *polar decomposition*. This decomposition is realised by the bijection $x \leftrightarrow (\|x\|, x/\|x\|)$ between

$$\mathbb{K}' = \mathbb{K} \setminus \{\mathbf{0}, \mathbf{e}\}$$

and $(0, \infty) \times \mathbb{S}$.

2.4 Sub-invariance

In addition to the homogeneity property of the metric d, we sometimes require that

$$d(x + h, x) \leq d(h, \mathbf{0}) = \|h\|, \quad x, h \in \mathbb{K} . \tag{2.9}$$

Then the metric (or the norm) in $\mathbb{K}$ is said to be *sub-invariant*. This technical condition makes it possible to control the continuity modulus of the function $h \mapsto \|x + h\|$ uniformly in x, i. e. closeness of $\|x\| = d(x, \mathbf{0})$ and $\|x + h\| = d(x + h, \mathbf{0})$ if $\|h\|$ is small. Indeed, by the triangular inequality, $d(x + h, \mathbf{0}) \leq d(x + h, x) + d(x, \mathbf{0})$, implying the sub-linearity of the norm (2.8) and also that $\|x + h\| - \|x\| \leq d(x + h, x)$. On the other hand, $d(x, \mathbf{0}) \leq d(x, x + h) + d(x + h, \mathbf{0})$, so that $-d(x, x + h) \leq \|x + h\| - \|x\|$. Therefore, in view of (2.9),

$$\big| \, \|x + h\| - \|x\| \, \big| \leq d(x + h, x) \leq \|h\| .$$

In particular,

$$\|x\| \leq \|x + h\| + \|h\|, \quad x, h \in \mathbb{K} . \tag{2.10}$$

The two '+' in the right-hand side of (2.10) should not be confused: the first one is the addition operation in $\mathbb{K}$ while the second one is the conventional sum of positive numbers. If $\mathbb{K}$ is a group, then an *invariant* (thus also sub-invariant) metric always exists, see [30, Ch. 6, p. 210], i. e. (2.9) holds with the equality sign. In general, (2.9) is too restrictive, e. g., for $\mathbb{R}_+$ with the maximum operation and Euclidean metric.

Lemma 2.7. *If $\mathbb{K}$ has a sub-invariant norm, then $\mathbf{0} = \mathbf{e}$ and, for any $\alpha \in (0,1)$, $\mathbf{e}$ is the only element that belongs to $\mathbb{K}(\alpha)$.*

Proof. Applying (2.9) with th instead of h and letting $t \downarrow 0$ implies that $x + \mathbf{0} = x$ for all x, whence $\mathbf{e} = \mathbf{0}$. The sub-invariant norm also satisfies (2.8), so that, for each $z \in \mathbb{K}(\alpha)$

$$(a+b)^{1/\alpha}\|z\| = \|(a+b)^{1/\alpha}z\| = \|a^{1/\alpha}z + b^{1/\alpha}z\| \le (a^{1/\alpha} + b^{1/\alpha})\|z\| .$$

Note that $\|z\| < \infty$ in our normed cone since $\mathbf{e} = \mathbf{0}$. If $z \ne \mathbf{0}$ then $(a+b)^{1/\alpha} \le a^{1/\alpha} + b^{1/\alpha}$ for all $a, b > 0$ is impossible for $\alpha \in (0,1)$. □

Typical examples of cones that fulfil our requirements are Banach spaces or convex cones in Banach spaces; the family of compact (or convex compact) subsets of a Banach space with Minkowski addition; the family of compact sets in $\mathbb{R}^d$ with the union operation; the family of all finite measures with the conventional addition operation and multiplication by numbers. Another typical example is the set $\mathbb{R}_+ = [0,\infty)$ with the maximum operation $x + y = x \vee y = \max(x,y)$. In order to distinguish this example from the conventional cone $(\mathbb{R}_+, +)$, we denote it by $(\mathbb{R}_+, \vee)$. These and other examples are discussed in Section 8.

We say that $\mathbb{K}$ can be isometrically embedded in a Banach space $\mathbb{B}$ if there exists an injection $I : \mathbb{K} \to \mathbb{B}$ such that $I(ax + by) = aI(x) + bI(y)$ for all $a, b > 0$ and $x, y \in \mathbb{K}$ and $d(x,y) = \|I(x) - I(y)\|$ for all $x, y \in \mathbb{K}$. However, this embedding is possible only under some conditions on the cone $\mathbb{K}$ and the corresponding metric.

Theorem 2.8. *A convex cone $\mathbb{K}$ with a metric d can be embedded into a Banach space if and only if the second distributivity law (2.5) holds and d is homogeneous and invariant, i. e. $d(ax, ay) = ad(x,y)$ and $d(x+z,\, y+z) = d(x,y)$ for all $a > 0$ and all $x, y, z \in \mathbb{K}$.*

Proof. Necessity is obvious. Sufficiency follows from Hörmander's theorem [25], see also [28]. □

3 LePage series on a cone

3.1 Point processes on a cone

Consider a normed cone $\mathbb{K}$ with its Borel σ-algebra $\mathcal{B}(\mathbb{K})$. Let $\mathcal{M}_{\mathbf{0}}$ (respectively $\mathcal{M}$) be the family of *counting measures* m on $\mathcal{B}(\mathbb{K})$ such that $m(B^r) < \infty$ (respectively $m(B_r) < \infty$) for every $r > 0$. Both families $\mathcal{M}_{\mathbf{0}}$ and $\mathcal{M}$ also contain the null-measure. Denote by δ_x the unit mass measure concentrated at $x \in \mathbb{K}$. Any counting measure can be represented as

$$m = \delta_{x_1} + \delta_{x_2} + \cdots = \sum_i \delta_{x_i} , \tag{3.1}$$

where $x_1, x_2, \dots$ is an at most countable collection of points such that only a finite number of x_i's lies in B^r if $m \in \mathcal{M}_{\mathbf{0}}$ or in B_r if $m \in \mathcal{M}$ for every $r > 0$. When considering $\mathcal{M}_{\mathbf{0}}$ we deviate from the typical setting in the theory of point processes (see, e. g., [10]), where the point sets are assumed to be locally finite. Our setting allows for a concentration point at the origin for measures from $\mathcal{M}_{\mathbf{0}}$. To cover these both cases with the same notation, let A_r denote B^r ($B_{r^{-1}}$, respectively) in case we consider measures from $\mathcal{M}_{\mathbf{0}}$ ($\mathcal{M}$, respectively). Then we always have $m(A_r) < \infty$ whenever $m \in \mathcal{M}$ or $m \in \mathcal{M}_{\mathbf{0}}$.

The counting measure m is said to be *simple* if the points $x_1, x_2, \dots$ from (3.1) are distinct. A simple counting measure is fully characterised by its support

$$\operatorname{supp} m = \{x \in \mathbb{K} : \ m(\{x\}) > 0\}\,.$$

A *point process* μ is a measurable map from some probability space into $\mathcal{M}_{\mathbf{0}}$ (or $\mathcal{M}$) with the σ-algebra generated by the sets of measures $m \in \mathcal{M}_{\mathbf{0}}$ (or $m \in \mathcal{M}$) such that $m(B) = n$ for Borel sets $B \subset \mathbb{K}$ and $n \geq 0$. The distribution of μ is denoted by $\mathbf{P}$. The process is *simple* if almost all its realisations are simple. The *probability generating functional* of μ is defined as

$$G_\mu(u) = \mathbf{E}\exp\Big\{\int_{\mathbb{K}} \log u(x)\,\mu(dx)\Big\} = \mathbf{E}\Big[\prod_{x_i \in \operatorname{supp}\mu} u(x_i)^{\mu(\{x_i\})}\Big]\,,$$

where $\mathbf{E}$ is the expectation with respect to $\mathbf{P}$ and $u : \mathbb{K} \mapsto (0,1]$ is a function that identically equals 1 on the complement of A_r for some $r > 0$, c. f. [10, Sec. 7.4].

If F is a Borel set, then $\int_F x\mu(dx)$ is the random element obtained as the sum of all the points from $F \cap \operatorname{supp}\mu$ taking into account possible multiplicities. If $\mu(F)$ is a. s. finite, this integral is a well-defined finite sum. Otherwise, e. g., if one considers

$$\int x\mu(dx) = \int_{\mathbb{K}} x\mu(dx)$$

with the integration over $F = \mathbb{K}$, the almost sure convergence of this integral is understood as the existence of the integral for almost all μ. The *absolute convergence* requires the existence of $\int \|x\|\mu(dx)$. A weaker condition is the convergence of the *principal value* which requires that $\int_{A_r} x\mu(dx)$ converges as $r \downarrow 0$. Similar definitions are applicable for the integral $\int g(x)\mu(dx)$, where $g : \mathbb{K} \mapsto \mathbb{K}$ is a measurable function.

3.2 Stable Poisson process

Let Λ be a measure on $\mathbb{K}$ which is finite on all A_r, $r > 0$. A point process Π is called a *Poisson process* with *intensity measure* Λ if, for any disjoint family of Borel sets $F_1, \dots, F_n$, the random variables $\Pi(F_1), \dots, \Pi(F_n)$ are jointly independent Poisson distributed with means $\Lambda(F_1), \dots, \Lambda(F_n)$, respectively. The Poisson process is simple if and only if its intensity measure is non-atomic.

Given the automorphisms $D_a : x \mapsto ax$ introduced in Section 2.1, $D_a m$ denotes the image of m, i. e. $(D_a m)(A) = m(D_a^{-1}A) = m(D_{a^{-1}}A)$ for every Borel A. If $m = \sum_i \delta_{x_i}$ is a counting measure, then $D_a m = \sum_i \delta_{ax_i}$, in particular, $\operatorname{supp}(D_a m) = a(\operatorname{supp} m)$ and

$$\int x D_a m(dx) = a \int x m(dx)\,. \tag{3.2}$$

An important property of a Poisson process Π is that $D_a\Pi$ is again a Poisson process driven by the intensity measure $D_a\Lambda$, if Λ is the intensity measure of Π.

Example 3.1 (Stable Poisson point process). Recall that $\mathbb{K}' = \mathbb{K} \setminus \{\mathbf{0}, \mathbf{e}\}$ can be identified with $(0, \infty) \times \mathbb{S}$ using the polar decomposition. Define a measure Λ (also denoted by $\Lambda_{\alpha,\sigma}$) on $\mathbb{K}'$ as the product of the measure θ_α on $(0, \infty)$ such that $\alpha \neq 0$ and

$$\begin{cases} \theta_\alpha((r,\infty)) = r^{-\alpha} & \text{if } \alpha > 0\,, \\ \theta_\alpha((0,r)) = r^{-\alpha} & \text{if } \alpha < 0\,, \end{cases} \qquad r > 0\,, \tag{3.3}$$

and a finite measure σ on the Borel σ-algebra $\mathcal{B}(\mathbb{S})$ induced on $\mathbb{S}$. The Poisson process on $\mathbb{K}'$ (and thus also on $\mathbb{K}$) with intensity measure $\Lambda_{\alpha,\sigma}$ is denoted by $\Pi_{\alpha,\sigma}$. We omit the subscript σ and write simply Π_α when no confusion occurs. The measure σ is called its *spectral measure.* If $\alpha > 0$, then $\Pi_\alpha \in \mathcal{M}_{\mathbf{0}}$, i. e. it has the concentration point at the origin. If $\alpha < 0$, then $\Pi_\alpha \in \mathcal{M}$, and if $\mathbf{e} \neq \mathbf{0}$, then the support points of Π_α have the concentration point at $\mathbf{e}$.

The importance of the process Π_α in our context stems from the fact that it is *stable* with respect to the addition operation applied to the corresponding counting measures.

Theorem 3.2. *Let Π'_α and Π''_α be two independent copies of Π_α. Then*

$$D_{a^{1/\alpha}}\Pi'_\alpha + D_{b^{1/\alpha}}\Pi''_\alpha \overset{\mathcal{D}}{=} D_{(a+b)^{1/\alpha}}\,\Pi_\alpha \tag{3.4}$$

for all $a, b > 0$.

Proof. By (3.3), the intensity measure $\Lambda = \Lambda_{\alpha,\sigma}$ of Π_α satisfies

$$(D_{a^{1/\alpha}}\Lambda)(A) = \Lambda(D_{a^{-1/\alpha}}A) = a\Lambda(A) \tag{3.5}$$

for any Borel A. Note that the left-hand side is the result of a contraction of the phase space, while the right-hand side is a measure obtained by multiplying the values of Λ by a number. Thus the processes on the both sides of (3.4) are Poisson with the same intensity measure $(a+b)\Lambda$. □

Property (3.4) can be reformulated as the stability property of Π_α with respect to the union operation applied to its support set. Let $\kappa_\alpha = \operatorname{supp}\Pi_\alpha$ and let κ'_α and κ''_α be two independent copies of κ_α. Since the intensity measure Λ of Π_α is non-atomic, $a^{1/\alpha}\kappa'_\alpha \cap b^{1/\alpha}\kappa''_\alpha = \emptyset$ with probability 1. Then (3.4) is equivalent to

$$a^{1/\alpha}\kappa'_\alpha \cup b^{1/\alpha}\kappa''_\alpha \overset{\mathcal{D}}{=} (a+b)^{1/\alpha}\kappa_\alpha \tag{3.6}$$

for all $a, b > 0$. This means that κ_α is a union-stable random closed set, i. e. a StαS random element in the cone of closed sets with the union operation, see [39].

The following theorem provides a useful representation of the process Π_α, especially for simulation purposes. Its proof relies on basic facts on transformations of a Poisson process.

Theorem 3.3. *Let $\{\zeta_k\}$ and $\{\varepsilon_k\}$ be two independent sequences of i. i. d. random variables, where ζ_k, $k \geq 1$, have exponential distribution with mean 1, and ε_k, $k \geq 1$ are distributed on*

$\mathbb{S}$ according to $\hat{\sigma}(\cdot) = \sigma(\cdot)/\sigma(\mathbb{S})$, where σ is a finite measure on $\mathbb{S}$. Define $c = \sigma(\mathbb{S})^{1/\alpha}$ and $\Gamma_k = \zeta_1 + \cdots + \zeta_k$, $k \geq 1$. Then, for any $\alpha \neq 0$,

$$\Pi_{\alpha,\sigma} \stackrel{\mathcal{D}}{=} \sum_{k=1}^{\infty} \delta_{\Gamma_k^{-1/\alpha} \varepsilon_k c} \,. \tag{3.7}$$

By conditioning with respect to the number of points of $\Pi_\alpha = \Pi_{\alpha,\sigma}$ in A_r, it is easy to calculate its probability generating functional

$$G_{\Pi_\alpha}(u) = \exp\left\{ - \int_{\mathbb{K}} (1 - u(x))\Lambda(dx) \right\}, \tag{3.8}$$

where $\Lambda = \Lambda_{\alpha,\sigma}$ and $u : \mathbb{K} \mapsto [0,1]$ is a function that is identically equal to 1 outside of A_r, see [10, Ex. 7.4(a)]. By passing to a limit, it is possible to show that (3.8) holds for any function u such that $1 - u$ is integrable with respect to Λ.

Lemma 3.4. *Let $\Pi_\alpha^{(\eta)}$ be the point process obtained by multiplying all points from $\Pi_{\alpha,\sigma}$ by i. i. d. realisations of a positive random variable η with $a = \mathbf{E}\,\eta^\alpha < \infty$. Then $\Pi_\alpha^{(\eta)}$ has spectral measure $a\sigma$ and coincides in distribution with the point process $D_{a^{1/\alpha}}\Pi_\alpha$.*

Proof. It suffices to show that the probability generating functionals of the both processes coincide. By conditioning with respect to the realisation of Π_α, one easily obtains that

$$G_{\Pi_\alpha^{(\eta)}}(u) = \exp\left\{ - \int_{\mathbb{K}} (1 - \mathbf{E}\, u(x\eta))\Lambda(dx) \right\}.$$

By changing variables $x\eta = y$ and using (3.5), it is easily seen that the exponent in the formula for the probability generating functional is

$$-\,\mathbf{E}\,\eta^\alpha \int_{\mathbb{K}} (1 - u(x))\Lambda(dx)$$

that corresponds to the probability generating functional of $D_{a^{1/\alpha}}\Pi_\alpha$. □

Remark 3.5. A stable Poisson process Π_α can be defined on any convex cone $\mathbb{K}$ without assuming that $\mathbb{K}$ possesses the origin or a norm. It suffices to consider a Poisson point process whose intensity measure Λ is homogeneous, i. e. satisfies (3.5).

3.3 LePage series

A $\mathbb{K}$-valued random element ξ is said to have StαS distribution if it satisfies (1.1) with the addition and multiplication operations defined on $\mathbb{K}$. Theorem 3.6 below provides a rich family of StαS distributions by their series decomposition.

Theorem 3.6. *Let $\{\zeta_k,\, k \geq 1\}$ be i. i. d. exponentially distributed random variables with mean 1 and let $\Gamma_k = \zeta_1 + \cdots + \zeta_k$, $k = 1, 2, \ldots$. Furthermore, let $\{\varepsilon_k\}$ be independent of $\{\zeta_k\}$ i. i. d. random elements on the unit sphere $\mathbb{S}$ in $\mathbb{K}$ with a common distribution $\hat{\sigma}$. If the principal value* *of the integral $\int x\,\Pi_{\alpha,\hat\sigma}(dx)$ is finite with probability 1, then for any $z \in \mathbb{K}(\alpha)$ and $c \geq 0$, the series*

$$\xi_\alpha = z + c\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}\varepsilon_k \tag{3.9}$$

converges almost surely and ξ_α has a StαS *distribution on $\mathbb{K}$.*

If the norm on $\mathbb{K}$ is sub-invariant, then the infinite sum in (3.9) converges absolutely a. s. for any $\alpha \in (0,1)$.

Proof. The case $c = 0$ is trivial. Without loss of generality assume that $z = \mathbf{e}$ in (3.9). Note that

$$c\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}\varepsilon_k = \int x\,\Pi_\alpha(dx)\,,$$

where $\Pi_\alpha = \Pi_{\alpha,\sigma}$ with $\sigma = c^\alpha\hat\sigma$. Recall that A_r denotes B^r if $\alpha > 0$ and $B_{r^{-1}}$ if $\alpha < 0$. By (3.4) and almost sure finiteness of $\Pi_\alpha(A_r)$ we have that

$$\int_{A_r} x\,D_{a^{1/\alpha}}\Pi'_\alpha(dx) + \int_{A_r} x\,D_{b^{1/\alpha}}\Pi''_\alpha(dx) \overset{\mathcal{D}}{=} \int_{A_r} x\,D_{(a+b)^{1/\alpha}}\Pi_\alpha(dx)\,.$$

Since the principal value of $\int x\Pi_{\alpha,\hat\sigma}(dx)$ is finite by the condition, (3.2) implies that the integrals above also converge. Then we can let $r \downarrow 0$ to obtain that

$$a^{1/\alpha}\xi'_\alpha + b^{1/\alpha}\xi''_\alpha \overset{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi_\alpha\,, \tag{3.10}$$

where ξ'_α and ξ''_α are two independent copies of ξ_α, i. e. ξ_α is StαS.

Now assume that the norm is sub-invariant, i. e. it satisfies (2.9). If $S_n = \sum_{k=1}^n \Gamma_k^{-1/\alpha}\varepsilon_k$, then

$$d(S_n, S_{n+m}) \leq \Big\|\sum_{k=n+1}^{n+m} \Gamma_k^{-1/\alpha}\varepsilon_k\Big\| \leq \sum_{k=n+1}^{n+m} \Gamma_k^{-1/\alpha}\,.$$

Note that the first sum corresponds to the semigroup addition in $\mathbb{K}$ while the second is the ordinary sum of positive numbers. The last expression vanishes almost surely as $n \to \infty$, since $\Gamma_k^{-1/\alpha} \sim k^{-1/\alpha}$ with probability 1 by the strong law of large numbers. Referring to the completeness of $\mathbb{K}$ confirms the a. s. convergence of the series. □

The distribution of ξ_α given by (3.9) is determined by its deterministic part z and the measure $\sigma = c^\alpha\hat\sigma$ on $\mathbb{S}$ called the *spectral measure*. The series (3.9), also written as

$$\xi_\alpha = z + \int x\Pi_\alpha(dx)\,, \tag{3.11}$$

is called the *LePage series* on $\mathbb{K}$. The convergence of the principal value of the integral in (3.11) is equivalent to the convergence of the series in (3.9). Theorem 3.6 implies that for $\alpha \in (0,1)$ under the sub-invariance condition, the integral in (3.11) converges absolutely a. s. The convergence property for other α depends on finer properties of the cone. For instance, in $(\mathbb{R}_+,+)$ the series (3.9) converges if and only if $\alpha \in (0,1)$. In a Banach space the properly centred (or symmetrised) series also converges for $\alpha \in [1,2)$, see [49, Sec. 1.5]. In some cones the series (3.9) converges for all positive or negative α as the following example shows.

Example 3.7. Consider the cone $(\mathbb{R}_+, \vee)$. If $\varepsilon_k = 1$ for all $k \geq 1$ and $\alpha > 0$, then $\xi_\alpha = c\Gamma_1^{-1/\alpha}$ has the Fréchet distribution which is max-stable. The Weibull distribution arises in $([0,\infty], \min)$ cone for $\alpha < 0$. Here any StαS-distribution for $\alpha > 0$ is trivial concentrated at 0.

Remark 3.8. The point z in (3.11) does not have to be α-stable itself in order to produce a StαS element ξ_α. For instance, ξ_α has the Fréchet distribution which is stable in the cone $(\mathbb{R}, \vee)$ for z being any negative number and $\varepsilon_k = 1$ for all $k \geq 1$, c. f. Corollary 5.15.

Remark 3.9. A convergent LePage series yields a (possibly degenerate) StαS random element in any convex cone $\mathbb{K}$. It is not essential that the topology on $\mathbb{K}$ is metrisable or $\mathbb{K}$ is equipped with a norm. The only assumption is that $\mathbb{K}$ possesses a polar decomposition.

A simple example of a StαS law in any Banach space is provided by the random element $\xi = \eta x_0$, where η is non-negative StαS random variable and x_0 is any non-zero vector. The same construction is applicable to any convex cone that satisfies the second distributivity law (2.5). However, if (2.5) does not hold, this construction no longer works. For instance, if $\mathbb{K} = (\mathbb{R}_+, \vee)$, then $\xi = \eta x_0$ with a non-negative StαS random variable η is not StαS, since for ξ_1 and ξ_2 being two independent copies of ξ, the random elements $\xi_1 + \xi_2 = (\eta_1 \vee \eta_2)x_0$ and $2^{1/\alpha}\xi$ have different distributions. In contrast, Theorem 3.6 yields that

$$Y_\alpha = \max_{k\geq 1} \Gamma_k^{-1/\alpha} x_0 = \Gamma_1^{-1/\alpha} x_0$$

is StαS and so provides an example of a StαS distribution in this cone.

3.4 Lévy process

In this section it is assumed that $\mathbb{K}$ has a sub-invariant norm. Let $\{\Gamma_k\}$ and $\{\varepsilon_k\}$ be as in Theorem 3.6. If $\alpha \in (0,1)$ and $\{\eta_k\}$ are i. i. d. copies of a non-negative random variable η with $c = \mathbf{E}\,\eta^\alpha < \infty$, then Theorem 3.6 and Lemma 3.4 imply that

$$Y_\alpha^{(\eta)} = \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \eta_k \varepsilon_k \tag{3.12}$$

absolutely converges almost surely and has StαS distribution on $\mathbb{K}$ with spectral measure $\sigma = c\hat{\sigma}$.

Representation (3.12) can be used to construct a $\mathbb{K}$-valued StαS Lévy process, i. e. $\mathbb{K}$-valued process with independent and stationary StαS increments. A $\mathbb{K}$-valued stochastic process X_t, $t \geq 0$, is said to have independent increments, if, for every $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, there exist jointly independent $\mathbb{K}$-valued random elements $\xi_{t_0t_1}, \ldots, \xi_{t_{n-1}t_n}$ (representing the increments) such that the joint distributions of $X_{t_0}, X_{t_1}, \ldots, X_{t_n}$ and $X_{t_0}, X_{t_0} + \xi_{t_0t_1}, \ldots, X_{t_0} + \xi_{t_0t_1} + \cdots + \xi_{t_{n-1}t_n}$ coincide.

Theorem 3.10. *Let $\{\varepsilon_k\}$ and $\{\Gamma_k\}$ be as in Theorem 3.6 and $\alpha \in (0,1)$. Let $\{\tau_k, k \geq 1\}$ be a sequence of i. i. d. random variables uniformly distributed on $[0,1]$ and independent of the sequences $\{\varepsilon_k\}$ and $\{\Gamma_k\}$. Then the process*

$$X_\alpha(t) = \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \mathbf{1}_{[0,t]}(\tau_k)\varepsilon_k\,, \quad t \in [0,1], \tag{3.13}$$

has independent increments given by

$$\xi_{ts} = \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \mathbf{1}_{(t,s]}(\tau_k)\varepsilon_k\,, \quad t < s, \tag{3.14}$$

which are StαS *distributed with the spectral measure* $(s-t)\hat{\sigma}$.

Proof. The proof follows immediately from the construction (3.12) applied to $\eta_k = \mathbf{1}_{(t,s]}(\tau_k)$, $k \geq 1$. □

Example 3.11. Consider $(\mathbb{R}_+, \vee)$. Since $\mathbb{S} = \{1\}$,

$$X_\alpha(t) = \Gamma_{k(t)}^{-1/\alpha}\,, \quad t \in [0,1]\,,$$

where $k(t) = \min\{k : \ \tau_k \leq t\}$. Then X_α is a Markov process with a. s. piecewise constant non-decreasing trajectories on $(\varepsilon, 1]$ for every $\varepsilon > 0$. This process has been studied in [45] under the name of a *super-extremal process* for $\mathbb{K}$ being the family of upper semicontinuous functions with pointwise maximum operation.

4 Convergence to stable laws

4.1 Weak convergence of point processes

A stable Poisson process Π_α has always infinite number of support points since its intensity measure is infinite. If $\alpha < 0$ this process has finite number of points in any bounded set (and so has realisations from $\mathcal{M}$), while for $\alpha > 0$ its concentration point is at the origin $\mathbf{0}$ and only a finite number of points lies outside any ball centred at $\mathbf{0}$. The first case corresponds to well-known σ-finite point processes. In order to study convergence of processes with possible concentration point at $\mathbf{0}$ (i. e. with realisations from $\mathcal{M}_\mathbf{0}$), we need to amend some conventional definitions.

Consider the family $\mathbb{C}$ of continuous bounded functions $f : \mathbb{K} \to \mathbb{R}$ such that $f(x) = 0$ for all $x \notin A_r$ with some $r > 0$. A sequence of counting measures $\{m_n, n \geq 1\}$ from $\mathcal{M}_\mathbf{0}$ (or from $\mathcal{M}$) is said to converge *vaguely* to m (notation $m_n \xrightarrow{\text{v}} m$) if

$$\int f(x)\, m_n(dx) \to \int f(x)\, m(dx) \quad \text{as } n \to \infty \tag{4.1}$$

for every $f \in \mathbb{C}$. For counting measures from $\mathcal{M}$ this definition turns into the conventional definition of vague convergence, see [37]. Using the inversion transformation $x \mapsto x/\|x\|^2$ it is easy to confirm that conventional properties of the vague convergence hold for point processes from $\mathcal{M}_\mathbf{0}$. In particular, the family $\mathcal{M}_\mathbf{0}$ with the vague topology is a Polish space, see [37, Prop. 1.15.5].

Note that the vague convergence is often formulated for counting measures in locally compact spaces, where the functions from $\mathbb{C}$ have *compact* supports, see, e. g., [44]. Counting measures and point processes in general Polish spaces have been systematically studied in [37] and [10].

The following result extends (4.1) for $\mathbb{K}$-valued functions.

Lemma 4.1. *If $m_n \xrightarrow{\mathrm{v}} m$ with a finite measure m, then*

$$\int g(x)\, m_n(dx) \to \int g(x)\, m(dx) \tag{4.2}$$

for any continuous function $g : \mathbb{K} \mapsto \mathbb{K}$ such that $g(x) = \mathbf{e}$ for all $x \in A_r$ for some $r > 0$.

Proof. Note that m is a finite measure on A_r. By [37, Sec. 1.15] it is possible to order the support points of m_n so that they converge to the corresponding support points of m. This implies (4.2) taking into account the continuity of g. □

Weak convergence $\mu_n \Rightarrow \mu$ of point processes is defined using the weak convergence of the corresponding probability distributions, i. e. $\mathbf{E}\, h(\mu_n) \to \mathbf{E}\, h(\mu)$ as $n \to \infty$ for any bounded and continuous in the vague topology function h that maps $\mathcal{M}$ or $\mathcal{M}_{\mathbf{0}}$ into $\mathbb{R}$.

Let $\mu|^r$ denote the restriction of μ onto A_r. It is shown in [37, Th. 3.1.13] that $\mu_n \Rightarrow \mu$ for a simple point process μ if and only if $\mu_n(B)$ weakly converges to $\mu(B)$ for all B from a certain subring of $\{B \in \mathcal{B}(\mathbb{K}) :\ \mu(\partial B) = 0 \text{ a.s.}\}$ such that for each closed F and each open neighbourhood $U \supset F$, one has $F \subset B \subset U$ for some B from this subring. Since any such set B may be chosen to be separated from the origin by a positive distance r (in case of $\mathcal{M}_{\mathbf{0}}$) or contained in a ball of radius r^{-1} (in case of $\mathcal{M}$), and $\mu(B) = \mu|^r(B)$, the following result holds.

Lemma 4.2. *A sequence $\{\mu_n, n \geq 1\}$ of point processes weakly converges to a simple point process μ if and only if $\mu_n|^r \Rightarrow \mu|^r$ as $n \to \infty$ for all $r > 0$ such that $\mu(\partial A_r) = 0$ almost surely.*

4.2 Convergence of binomial processes

Let $\{\xi_k,\ k \geq 1\}$ be a sequence of i. i. d. $\mathbb{K}$-valued random elements. For every $n \geq 1$, $\sum_{i=1}^n \delta_{\xi_i}$ is called the *binomial point process*. It is simple if and only if the distribution of ξ_k is non-atomic. The scaled versions of the binomial process are defined by

$$\beta_n = \sum_{k=1}^n \delta_{\xi_k / b_n}\,, \quad n \geq 1\,, \tag{4.3}$$

for a sequence $\{b_n,\ n \geq 1\}$ of normalising constants. We shall typically have

$$b_n = n^{1/\alpha} L(n)\,, \quad n \geq 1\,, \tag{4.4}$$

with $\alpha \neq 0$ and a slowly varying at infinity function L.

The following theorem shows that the Poisson point process Π_α with $\alpha > 0$ arises as a weak limit for binomial processes β_n defined by (4.3) if the ξ_k's have regularly varying tails.

Theorem 4.3. *Let $\xi, \xi_1, \xi_2, \ldots$ be i. i. d. $\mathbb{K}'$-valued random elements. Then $\beta_n \Rightarrow \Pi_\alpha$ as $n \to \infty$ for $\alpha > 0$ if and only if there exists a finite measure σ on $\mathcal{B}(\mathbb{S})$ such that*

$$\lim_{n\to\infty} n\mathbf{P}\Big\{\frac{\xi}{\|\xi\|} \in G,\ \|\xi\| > r b_n\Big\} = \sigma(G) r^{-\alpha} \tag{4.5}$$

for all $r > 0$ and $G \in \mathcal{B}(\mathbb{S})$ with $\sigma(\partial G) = 0$, where the b_n's are given by (4.4).

This result is similar to Proposition 3.21 in [44, p. 154] for $\mathbb{K}$ being a locally compact space. Note that the condition of type (4.5) is typical in limit theorems for sums of random elements, see [2, p. 167]. The condition of ξ being $\mathbb{K}'$-valued ensures that $0 < \|\xi\| < \infty$ a. s.

Proof of Theorem 4.3. Sufficiency. By Lemma 4.2, it suffices to show that $\beta_n|^r \Rightarrow \Pi_\alpha|^r$ for all $r > 0$. Since $\mathbb{S}$ is a Polish space, for every $\delta > 0$, there exists a compact set $\mathbb{S}_\delta \subset \mathbb{S}$ such that $\sigma(\mathbb{S} \setminus \mathbb{S}_\delta) < \delta$. Denote by $\beta_n|^{r,\delta}$ (respectively $\Pi_\alpha|^{r,\delta}$) the restriction of β_n (respectively Π_α) onto the set $[r, \infty) \times \mathbb{S}_\delta$.

Consider a continuous in the vague topology function $F : \mathcal{M}_{\mathbf{0}} \mapsto \mathbb{R}$ with the absolute value bounded by a. Since $[r, \infty) \times \mathbb{S}_\delta$ is a locally compact space, [44, Prop. 3.21] yields that $\mathbf{E}\, F(\beta_n|^{r,\delta}) \to \mathbf{E}\, F(\Pi_\alpha|^{r,\delta})$ as $n \to \infty$. Note that $r^{-\alpha}$ in (4.5) coincides with $\theta_\alpha((r, \infty))$.

Furthermore,

$$|\,\mathbf{E}\, F(\beta_n|^r) - \mathbf{E}\, F(\beta_n|^{r,\delta})| \le 2a\mathbf{P}\{\beta_n|^{r,\delta} \ne \beta_n|^r\}\,.$$

The latter probability is bounded by $n\mathbf{P}\{\|\xi\| \ge rb_n, \xi/\|\xi\| \notin \mathbb{S}_\delta\}$. By (4.5), its upper limit as $n \to \infty$ does not exceed $\delta r^{-\alpha}$, which can be made arbitrarily small for every fixed $r > 0$ by the choice of δ.

Similarly, the absolute difference between $\mathbf{E}\, F(\Pi_\alpha|^r)$ and $\mathbf{E}\, F(\Pi_\alpha|^{r,\delta})$ is bounded from above by

$$2a\mathbf{P}\{\Pi_\alpha|^{r,\delta} \ne \Pi_\alpha|^r\} \le 2a\Big(1 - \exp\{-\theta_\alpha(r,\infty)\,\sigma(\mathbb{S} \setminus \mathbb{S}_\delta)\}\Big) \le 2ar^{-\alpha}\delta\,.$$

Therefore $\mathbf{E}\, F(\beta_n|^r) \to \mathbf{E}\, F(\Pi_\alpha|^r)$ as $n \to \infty$.

Necessity. Let $B = (r, \infty) \times G$, where G is a Borel subset of $\mathbb{S}$ with $\sigma(\partial G) = 0$. Then $\mathbf{P}\{\Pi_\alpha(\partial B) > 0\} = 0$, whence by [37, Th. 3.1.13] $\beta_n(B)$ converges weakly to $\Pi_\alpha(B)$. The former has the binomial distribution with mean

$$\mathbf{E}\, \beta_n(B) = n\mathbf{P}\{\|\xi\| > rb_n,\ \xi/\|\xi\| \in G\}\,,$$

while the latter has the Poisson distribution with mean $r^{-\alpha}\sigma(G)$. Therefore,

$$n\mathbf{P}\{\|\xi\| > rb_n,\ \xi/\|\xi\| \in G\} \to r^{-\alpha}\sigma(G)\,.$$

$\square$

By applying the inversion transformation $x \mapsto x\|x\|^{-2}$ it is possible to convert all measures from $\mathcal{M}_{\mathbf{0}}$ to measures from $\mathcal{M}$. Therefore, an analogue of Theorem 4.3 holds for $\alpha < 0$.

Corollary 4.4. *Let $\xi, \xi_1, \xi_2, \dots$ be i. i. d. $\mathbb{K}'$-valued random elements. Then $\beta_n \Rightarrow \Pi_\alpha$ as $n \to \infty$ for $\alpha < 0$ if and only if there exists a finite measure σ on $\mathcal{B}(\mathbb{S})$ such that*

$$\lim_{n\to\infty} n\mathbf{P}\Big\{\frac{\xi}{\|\xi\|} \in G,\ \|\xi\| < rb_n\Big\} = \sigma(G)r^{-\alpha} \tag{4.6}$$

for all $r > 0$ and $G \in \mathcal{B}(\mathbb{S})$ with $\sigma(\partial G) = 0$, where the b_n's are given by (4.4).

4.3 Convergence of sums

Normalised sums of random elements in $\mathbb{K}$ can be represented as sums of points of the corresponding binomial processes β_n. In order to derive the limit theorem for the normalised sums from the convergence $\beta_n \Rightarrow \Pi_\alpha$ shown in Theorem 4.3 we need to prove that the convergence holds in a stronger topology than the vague topology described in Section 4.1. Indeed, the sum of points of a point process μ can be written as

$$\int g d\mu = \int g(x)\, \mu(dx) = \sum_{x \in \operatorname{supp} \mu} g(x)\,,$$

where $g(x) = x$ is a continuous function $\mathbb{K} \mapsto \mathbb{K}$ whose support is neither bounded nor separated from the origin. Therefore, the weak convergence $\mu_n \Rightarrow \mu$ does not imply the weak convergence of the integrals $\int g d\mu_n$ to $\int g d\mu$.

All point processes in this section are assumed to belong to $\mathcal{M}_\mathbf{0}$, i. e. they have almost surely at most a finite number of points outside B_r for every $r > 0$. Note that the sum and integrals of $\mathbb{K}$-valued functions are understood with respect to the addition operation in $\mathbb{K}$ and their absolute convergence with respect to the norm on $\mathbb{K}$.

Lemma 4.5. *Assume that $\mathbb{K}$ possesses a sub-invariant norm. Let $\mu_n \Rightarrow \mu$ for a point process μ such that $\mu(B^r) < \infty$ a. s. for all $r > 0$. Let $g : \mathbb{K} \mapsto \mathbb{K}$ be a continuous function such that $\int g d\mu$ converges absolutely and*

$$\limsup_n \mathbf{P}\left\{ \left\| \int_{B_r} g d\mu_n \right\| \geq \varepsilon \right\} \to 0 \quad \text{as } r \downarrow 0 \tag{4.7}$$

for each $\varepsilon > 0$. Then $\int g d\mu_n$ weakly converges to $\int g d\mu$.

Proof. Since the space $\mathcal{M}_\mathbf{0}$ is Polish, by the Skorohod theorem it is possible to define μ_n and μ on the same probability space so that $\mu_n \xrightarrow{\text{v}} \mu$ almost surely. In particular, $\mu_n|^r \xrightarrow{\text{v}} \mu|^r$ a. s. for any $r > 0$ such that $\mathbf{P}\{\mu(r\mathbb{S}) > 0\} = 0$. By the triangle inequality,

$$\begin{aligned} d\Big(\int g d\mu_n, \int g d\mu\Big) \\ \leq d\Big(\int g d\mu_n, \int g d\mu_n|^r\Big) + d\Big(\int g d\mu_n|^r, \int g d\mu|^r\Big) + d\Big(\int g d\mu|^r, \int g d\mu\Big)\,. \end{aligned}$$

By (2.9), the first summand is at most $\| \int_{B_r} g d\mu_n \|$. The second summand $\zeta_n(r)$ converges to zero a. s. as $n \to \infty$ for any $r > 0$ by Lemma 4.1 and continuity of g. The third summand $\gamma(r)$ converges to zero a. s. as $r \downarrow 0$, since $\int g d\mu$ exists. Thus,

$$\begin{aligned} \mathbf{P}\left\{ d\Big(\int g d\mu_n, \int g d\mu\Big) \geq \varepsilon \right\} &\leq \mathbf{P}\Big\{ \Big\| \int_{B_r} g d\mu_n \Big\| \geq \varepsilon - \zeta_n(r) - \gamma(r) \Big\} \\ &\leq \mathbf{P}\Big\{ \Big\| \int_{B_r} g d\mu_n \Big\| \geq \varepsilon/2 \Big\} + \mathbf{P}\{\zeta_n(r) + \gamma(r) \geq \varepsilon/2\}\,. \end{aligned}$$

By (4.7) the probability on the left can be made arbitrarily small by the choice of n and r. □

Now we prove that the result of Theorem 4.3 can be strengthened to show a stronger type of convergence if $\alpha \in (0,1)$. The corresponding topology in the special case of point processes in the Euclidean space was called the δ-topology in [11].

Theorem 4.6. *Assume that the norm on $\mathbb{K}$ is sub-invariant. If (4.5) holds with $\alpha \in (0,1)$, then $\int x\,\beta_n(dx)$ weakly converges to $\int x\,\Pi_\alpha(dx)$.*

Proof. Let $\rho = (\rho_1, \rho_2, \dots)$ be the decreasing sequence of the norms of the support points of Π_α and $\rho_n = (\rho_{1,n}, \rho_{2,n}, \dots)$ be the non-increasing infinite sequence of the norms of the support points of the process β_n filled with 0's starting from the index $n+1$.

Notice that (4.5) implies that

$$\lim_{n\to\infty} n\mathbf{P}\{\|\xi\| > rb_n\} = cr^{-\alpha}$$

with a constant c, which implies that $\|\xi\|$ belongs to the domain of attraction of α-stable one-sided law on $\mathbb{R}_+$. It is well known (see, e. g., [33, Lemma 1]) that in this case the finite-dimensional distributions of the sequences ρ_n converge to those of the sequence ρ. Using the Skorohod theorem, one can define ρ_n and ρ on the same probability space so that $\rho_{k,n} \to \rho_k$ for all $k = 1, 2, \dots$ almost surely.

Recall a well-known convergence criterion in the space $L^1(\mu)$ of positive functions integrable with respect to a σ-finite measure μ: if $f_n \geq 0$, $f_n \to f$ μ-almost everywhere and $\int f_n\,d\mu \to \int f\,d\mu$ then $f_n \to f$ in $L^1(\mu)$. Taking $\mu = \sum_k \delta_{\{k\}}$ to be a counting measure on $\{1, 2, \dots\}$, this translates into

$$\sum_{k=1}^{\infty} |\rho_{k,n} - \rho_k| \to 0 \quad \text{as } n \to \infty \text{ almost surely.} \tag{4.8}$$

Since the intensity measure of Π_α is non-atomic, with probability 1 none of the ρ_k's coincides with r. Conditionally on this event, in view of (4.8) we have that $\Pi_\alpha(B^r) = \beta_n(B^r) = k_0$ for all sufficiently large n, whence

$$J_n(r) = \left\|\int_{B_r} x\,\beta_n(dx)\right\| \leq \int_{B_r} \|x\|\,\beta_n(dx) = \sum_{k=k_0+1}^{\infty} \rho_{k,n} \to \sum_{k=k_0+1}^{\infty} \rho_k$$
$$= \int_{B_r} \|x\|\,\Pi_\alpha(dx) = I(r)$$

as $n \to \infty$ with probability 1. Hence, $\limsup_n J_n(r) \leq I(r)$ a. s. Given arbitrary $\delta > 0$, fix a small $r > 0$ so that

$$\mathbf{P}\{I(r) > \varepsilon\} < \delta\,.$$

This is possible since the integral $\int \|x\|\,d\Pi_\alpha$ with $\alpha \in (0,1)$ converges a. s. and thus in probability. Then by Fatou's lemma,

$$\limsup_n \mathbf{P}\{J_n(r) > \varepsilon\} \leq \mathbf{P}\{\limsup_n \mathbf{1}_{J_n(r)>\varepsilon}\}$$
$$= \mathbf{P}\{\limsup_n J_n(r) > \varepsilon\} \leq \mathbf{P}\{I(r) > \varepsilon\} < \delta\,.$$

Thus (4.7) holds and an application of Lemma 4.5 completes the proof. □

4.4 Domains of attraction for StαS laws

A random $\mathbb{K}$-valued element ζ is said to belong to the domain of attraction of a StαS random element ξ_α if, for a sequence $\{\zeta_n, n \geq 1\}$ of i. i. d. copies of ζ,

$$b_n^{-1}(\zeta_1 + \cdots + \zeta_n) \Rightarrow \xi_\alpha \,,$$

where $\{b_n, n \geq 1\}$ is a sequence of positive normalising constants and $\Rightarrow$ denotes the weak convergence of $\mathbb{K}$-valued random elements. The following result confirms that conventional sufficient conditions on the domain of attraction of StαS laws are also applicable for distributions on convex cones. It is proved by an immediate application of Theorem 4.6.

Theorem 4.7. *Assume that $\mathbb{K}$ has a sub-invariant norm. Let $\{\zeta_n\}$ be a sequence of independent copies of a random element $\zeta \in \mathbb{K}'$. If ζ satisfies (4.5) with $\alpha \in (0,1)$, then ζ belongs to the domain of attraction of* StαS *random element ξ_α with the spectral measure σ and representation (3.9) with $z = \mathbf{e}$.*

As a simple example, one sees that $\eta\varepsilon$ belongs to the domain of attraction of StαS law if η is a StαS positive random variable with $\alpha \in (0,1)$ and ε is any random element with values in $\mathbb{S}$ and independent of η.

The condition (4.5) for the domain of attraction appears also in [2, Th. 7.11] for $\mathbb{K}$ being a type p-Rademacher space. It also characterises the domains of attraction of multivariate max-stable distributions, see [32].

The following result shows that it is possible to deduce the LePage representation of a stable random element from the existence of the corresponding spectral measure.

Theorem 4.8. *Assume that $\mathbb{K}$ has a sub-invariant norm. Let ξ be a* StαS *random element in $\mathbb{K}'$, such that (4.5) holds with $\alpha \in (0,1)$, so that ξ possesses the spectral measure σ. Then ξ admits the LePage representation given by (3.9) with $z = \mathbf{e}$.*

Proof. By the stability property, ξ coincides in distribution with $n^{-1/\alpha}(\xi_1 + \cdots + \xi_n)$ for each $n \geq 1$. The latter is the sum of support points for the binomial process β_n (with all its points distributed as ξ), so by Theorem 4.6 it converges to the sum of points of Π_α being the LePage series (3.9). □

The existence of the spectral measure for StαS random elements in separable Banach spaces is a well-known fact, see [2, p. 152]. Together with Theorem 4.8, this provides an alternative way to deduce the LePage representation for StαS laws with $\alpha \in (0,1)$ in separable Banach spaces, c. f. [46]. In order to derive the existence of the LePage representation for more general semigroups, we use the technique of harmonic analysis on semigroups that is explained in the following sections.

5 Distributions of stable random elements

5.1 Characters on semigroups

Assume that the semigroup $\mathbb{K}$ is equipped with *involution*, i. e. a continuous map $\star : \mathbb{K} \mapsto \mathbb{K}$ satisfying $(x+y)^\star = y^\star + x^\star$ and $(x^\star)^\star = x$ for all $x, y \in \mathbb{K}$. Assume also that $(ax)^\star = ax^\star$ for every $x \in \mathbb{K}$ and $a > 0$. Note that the involution can be the identical map.

If $\mathbb{K}$ is a normed cone, we also assume that $\|x^\star\| = \|x\|$, i. e. the sphere $\mathbb{S}$ is invariant with respect to the involution operation. However, in Sections 5–7 we do not require that $\mathbb{K}$ is a normed cone unless stated otherwise. It is not even assumed that $\mathbb{K}$ is a pointed cone, i. e. $\mathbb{K}$ possesses the origin.

Definition 5.1. A function χ that maps $\mathbb{K}$ into the unit disk $\mathbb{D}$ on the complex plane is called a *bounded semicharacter* (shortly *character*) if $\chi(\mathbf{e}) = 1$, $\chi(x+y) = \chi(x)\chi(y)$ and $\chi(x^\star) = \overline{\chi(x)}$ (the complex conjugate of $\chi(x)$) for all $x, y \in \mathbb{K}$.

We often encounter the following three cases.

- If the involution is the identity, then the characters take real values from $[-1, 1]$.
- If $\mathbb{K}$ is an idempotent semigroup, i. e. $x + x = x$ for each $x \in \mathbb{K}$, then all characters take values 0 or 1.
- If the involution $x^\star$ is the inverse element to x, i. e. $x + x^\star = \mathbf{e}$ for all $x \in \mathbb{K}$, then the characters take values in the unit complex circle $\mathbb{T} = \{z : \ |z| = 1\}$.

It is also possible that the characters take values in the whole unit disk $\mathbb{D}$, for instance, if $\mathbb{K}$ is the semigroup of probability measures with the convolution operation, where the characters are given by characteristic functions, see Example 8.13. Note that one can also study not necessarily bounded homomorphisms between $\mathbb{K}$ and the whole complex plane, that are called semicharacters, see [5, Def. 4.2.1].

The set $\hat{\mathbb{K}}$ of all characters (with the pointwise multiplication operation) is called the *restricted dual semigroup* to $\mathbb{K}$. The character $\mathbf{1}$ (identically equal to one) is the neutral element in $\hat{\mathbb{K}}$. The involution on $\hat{\mathbb{K}}$ is the complex conjugate, i. e. $\chi^\star = \overline{\chi}$. The family of characters is endowed with the topology of pointwise convergence. Then the projection $\rho_x : \ \chi \mapsto \chi(x)$ becomes a continuous (and therefore measurable) function $\hat{\mathbb{K}} \mapsto \mathbb{D}$ with respect to the Borel σ-algebra on $\hat{\mathbb{K}}$ for each $x \in \mathbb{K}$. Note that Definition 5.1 imposes no continuity (nor even measurability) condition on the characters.

The multiplication by a in $\mathbb{K}$ induces the multiplication operation $\chi \mapsto a \circ \chi$ on $\hat{\mathbb{K}}$ given by $(a \circ \chi)(x) = \chi(ax)$ for all $x \in \mathbb{K}$. Note that $2 \circ \chi$ is not necessarily equal to χ^2, since $\chi(2x)$ is not necessarily equal to $\chi(x+x) = \chi(x)^2$ unless the second distributivity law (2.5) holds.

Definition 5.2. A family $\tilde{\mathbb{K}}$ of characters is said to be a *sub-semigroup* of $\hat{\mathbb{K}}$ if $\tilde{\mathbb{K}}$ contains the identity character $\mathbf{1}$ and is closed with respect to pointwise multiplication, i. e. $\chi_1\chi_2 \in \tilde{\mathbb{K}}$ for all $\chi_1, \chi_2 \in \tilde{\mathbb{K}}$. A sub-semigroup $\tilde{\mathbb{K}}$ is called a *cone* (of characters) if it is closed with respect to multiplication $\circ$, i. e. $a \circ \chi \in \tilde{\mathbb{K}}$ for all $\chi \in \tilde{\mathbb{K}}$ and $a > 0$.

Definition 5.3. A cone of characters $\tilde{\mathbb{K}} \subset \hat{\mathbb{K}}$ is called *separating*, if, for any two distinct elements $x, y \in \mathbb{K}$, there exists $\chi \in \tilde{\mathbb{K}}$ such that $\chi(x) \neq \chi(y)$; $\tilde{\mathbb{K}}$ is called *strictly separating* if, for any two distinct elements $x, y \in \mathbb{K}$, there exists $\chi \in \tilde{\mathbb{K}}$ such that $\chi(x') \neq \chi(y')$ for all x' and y' from some open neighbourhoods of x and y, respectively.

If the characters from $\tilde{\mathbb{K}}$ are continuous on $\mathbb{K}$, then the strict separation follows from the simple separation condition. It is known [22, Th. V.22.17] that every locally compact abelian group possesses a separating family of continuous characters. However, not all semigroups possess a separating family of characters. For instance, if $x+x = y+y$ and $x+x+x = y+y+y$ for some $x \neq y$, then x and y cannot be separated by any character, since every character χ necessarily satisfies $\chi(x)^2 = \chi(y)^2$ and $\chi(x)^3 = \chi(y)^3$, see Example 8.20.

5.2 Laplace transform

A sub-semigroup $\tilde{\mathbb{K}}$ of characters generates a $\tilde{\mathbb{K}}$-weak topology on $\mathbb{K}$ by declaring $x_n \xrightarrow{w} x$ if and only if $\chi(x_n) \to \chi(x)$ for all $\chi \in \tilde{\mathbb{K}}$. The $\tilde{\mathbb{K}}$-weak topology is the weakest topology that makes all characters from $\tilde{\mathbb{K}}$ continuous. Let $\mathcal{F}(\mathbb{K}; \tilde{\mathbb{K}})$ be the smallest σ-algebra on $\mathbb{K}$ that makes all $\chi \in \tilde{\mathbb{K}}$ measurable. This σ-algebra is generated by the cylindrical sets $\{x \in \mathbb{K} : \chi_i(x) \in F_i,\ i = 1, \dots, n\}$, $n \geq 1$, where $F_1, \dots, F_n$ are Borel subsets of $\mathbb{D}$ and $\chi_1, \dots, \chi_n \in \tilde{\mathbb{K}}$.

Denote by

$$\mathbb{K}^\sharp = \hat{\tilde{\mathbb{K}}}$$

the restricted dual semigroup to $\tilde{\mathbb{K}}$, i. e. the family of all characters on $\tilde{\mathbb{K}}$. Although $\mathbb{K}^\sharp$ depends on the choice of $\tilde{\mathbb{K}}$, we supress this dependence in our notation for brevity. We equip $\mathbb{K}^\sharp$ with the topology of pointwise convergence, which generates the corresponding Borel σ-algebra $\mathcal{B}(\mathbb{K}^\sharp)$. The *evaluation map* $\imath : \mathbb{K} \mapsto \mathbb{K}^\sharp$ is defined by associating every $x \in \mathbb{K}$ with $\rho = \rho_x \in \mathbb{K}^\sharp$ such that $\rho_x(\chi) = \chi(x)$ for all $\chi \in \tilde{\mathbb{K}}$, c. f. [31, Sec. 20]. The evaluation map $\imath$ is injective if and only if $\tilde{\mathbb{K}}$ is separating.

The *Laplace transform* of a $\mathbb{K}$-valued random element ξ is a complex-valued function $\chi \mapsto \mathbf{E}\,\chi(\xi)$, where χ is a Borel measurable character from $\hat{\mathbb{K}}$. The following result is well known for random elements in locally compact spaces with continuous characters, see [7, § IX.5.7]. However, it also holds in a more general framework.

Theorem 5.4. *If $\hat{\mathbb{K}}$ has a separating sub-semigroup $\tilde{\mathbb{K}}$ such that $\mathcal{F}(\mathbb{K}; \tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K})$, then every probability measure $\mathbf{P}$ on $\mathbb{K}$ (or the distribution of a random element ξ) is uniquely determined by its Laplace transform*

$$\mathbf{E}\,\chi(\xi) = \int_{\mathbb{K}} \chi(x)\mathbf{P}(dx)\,, \quad \chi \in \tilde{\mathbb{K}}\,. \tag{5.1}$$

Proof. The function $\mathbf{E}\,\chi(\xi)$ is a bounded positive definite function on $\tilde{\mathbb{K}}$. By [5, Th. 4.2.8], there exists one and only one Radon measure μ on the restricted dual semigroup to $\tilde{\mathbb{K}}$ (i. e. the second dual semigroup $\mathbb{K}^\sharp$), such that

$$\mathbf{E}\,\chi(\xi) = \int_{\mathbb{K}^\sharp} \rho(\chi)\mu(d\rho)\,.$$

The separation condition implies that the evaluation map $\imath$ is injective. By the condition, $\imath$ is $\mathcal{B}(\mathbb{K})$-measurable, since, for Borel sets $F_1, \dots, F_n \subset \mathbb{D}$,

$$\imath^{-1}(\{\rho \in \mathbb{K}^\sharp : \ \rho(\chi_i) \in F_i, \ i = 1, \dots, n\}) \\ = \{x \in \mathbb{K} : \ \chi_i(x) \in F_i, \ i = 1, \dots, n\} \in \mathcal{F}(\mathbb{K}; \tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K}).$$

This makes it possible to define the image measure μ' of $\mathbf{P}$ under the natural map $\imath$ by $\mu'(F) = \mathbf{P}(\imath^{-1}(F))$ for all Borel $F \subset \mathbb{K}^\sharp$. By definition, μ' is supported by $\imath(\mathbb{K})$. After substitution $\rho = \imath(x)$ (5.1) can be re-written as

$$\int_{\mathbb{K}^\sharp} \chi(\imath^{-1}(\rho))\, \mu'(d\rho) = \int_{\mathbb{K}^\sharp} \rho(\chi)\, \mu'(d\rho)$$

and the uniqueness property implies that $\mu = \mu'$. □

Remark 5.5. Note that $\mathcal{F}(\mathbb{K}; \tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K})$ if $\tilde{\mathbb{K}}$ is a separating *countable* family of $\mathcal{B}(\mathbb{K})$-measurable functions, see [53, Prop. I.1.4]. If $\tilde{\mathbb{K}}$ consists of *continuous* characters, then the separation condition already implies that $\mathcal{F}(\mathbb{K}; \tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K})$. For real-valued characters this fact is proved in [53, Th. I.1.2], while for complex-valued continuous characters it suffices to consider two separating families of real-valued functions generated by their modulus and argument. If $\mathbb{K}$ is a linear space, Theorem 5.4 turns into [53, Th. IV.2.2].

Remark 5.6. It should be noted that Theorem 5.4 is the pure uniqueness result. It does not assert that a positive definite function on $\tilde{\mathbb{K}}$ is the Laplace transform of a certain random element, i. e. an analogue of the Bochner theorem may not hold, see [6].

Example 5.7. For the cone $(\mathbb{R}_+, \vee)$, the collection of indicator functions $\mathbf{1}_{[0,a]}$ with $a \geq 0$ may be taken as a separating family $\tilde{\mathbb{K}}$. Then $\chi(x_n) \to \chi(x)$ if and only if $x_n \uparrow x$ as $n \to \infty$, i. e. the corresponding weakly-open sets in $\mathbb{R}_+$ are $(a, b]$ with $a < b$. They generate the same σ-algebra as the metric Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$, so that Theorem 5.4 applies. The Laplace transform of ξ is its cumulative distribution function $\mathbf{P}\{\xi \leq a\}$, $a \geq 0$.

A StαS random element ξ in $\mathbb{K}$ is necessarily *infinitely divisible*. The Laplace transform of ξ is a *positive definite* infinitely divisible function of χ, i. e. $(\mathbf{E}\,\chi(\xi))^{1/n}$ is a positive definite function of $\chi \in \tilde{\mathbb{K}}$ for every $n \geq 1$. The results on infinitely divisible functions in semigroups [5, Th. 3.2.2, Prop. 4.3.1] imply that

$$\mathbf{E}\,\chi(\xi) = \exp\{-\varphi(\chi)\}\,, \quad \chi \in \tilde{\mathbb{K}}\,, \tag{5.2}$$

where φ is a *negative definite* complex-valued function on $\mathbb{K}$ with $\operatorname{Re}\varphi \in [0, \infty]$ and $\varphi(\mathbf{1}) = 0$. We call φ the *Laplace exponent* of ξ. The dominated convergence theorem implies that $\varphi(\chi)$ is sequentially continuous with respect to pointwise convergence in $\tilde{\mathbb{K}}$.

Definition 5.8. A random element ξ is *idempotent* if ξ coincides in distribution with the sum of its two i. i. d. copies, i. e. $\xi \overset{\mathcal{D}}{=} \xi_1 + \xi_2$, see [19, p. 41].

It is easy to see from the definition that ξ is idempotent if and only if its Laplace transform assumes only values 0 and 1. The deterministic ξ being equal to an idempotent element of $\mathbb{K}$ clearly has an idempotent distribution. It is also the case if ξ is distributed according to a finite Haar measure on any subgroup of $\mathbb{K}$. In a sense, an idempotent random element is StαS with $\alpha = \infty$. The following two useful results show that in some cases non-trivial StαS random elements with a finite α cannot be idempotent.

Lemma 5.9. *Assume that $\mathbb{K}$ is a normed cone. If ξ is an idempotent* StαS *random element with a finite α, then $\xi = \mathbf{0}$ a. s. or $\xi = \mathbf{e}$ a. s.*

Proof. If ξ is idempotent and stable, then (1.1) implies that $\xi \stackrel{\mathcal{D}}{=} 2^{1/\alpha}\xi$, so that with probability 1 $\|\xi\|$ assumes only two values: 0 or ∞. If $\mathbf{e} = \mathbf{0}$ then $\xi = \mathbf{0}$ a.s. If $\mathbf{e} \neq \mathbf{0}$, then $\|\xi\| = \infty$ is equivalent to $\xi = \mathbf{e}$ and Definition 5.8 yields $\mathbf{P}\{\xi = \mathbf{e}\} = \mathbf{P}\{\xi_1 + \xi_2 = \mathbf{e}\} = \mathbf{P}\{\xi = \mathbf{e}\}^2$, so that $\mathbf{P}\{\xi = \mathbf{e}\}$ is either zero or one. □

Lemma 5.10. *Assume that $\mathbb{K}$ possesses a strictly separating family $\tilde{\mathbb{K}}$ of characters with values in $[0,1]$. Then each idempotent* StαS *random element ξ with $\alpha \neq \infty$ is necessarily a deterministic idempotent element in $\mathbb{K}$.*

Proof. Assume that the support of ξ contains at least two distinct points and consider a character χ that strictly separates their neighbourhoods. Then $\mathbf{E}\,\chi(\xi)$ is distinct from 0 and 1, which is impossible for an idempotent ξ. Hence ξ is deterministic. □

5.3 Characterisation of Laplace transforms for StαS elements

It should be noted that many semigroups do not possess any non-trivial continuous character. For example, the only continuous character in $(\mathbb{R}_+, \vee)$ is the one identically equal to 1, while non-trivial characters $\mathbf{1}_{[0,t)}$ and $\mathbf{1}_{[0,t]}$ are only semicontinuous. The following assumption imposes a weaker form of the continuity property on the characters.

(C) The cone of characters $\tilde{\mathbb{K}}$ possesses a strictly separating countable sub-family and there exist a cone $\tilde{\mathbb{K}}'$ whose elements are semicontinuous $[0,1]$-valued characters and a cone $\tilde{\mathbb{K}}''$ of continuous $\mathbb{D}$-valued characters such that every $\chi \in \tilde{\mathbb{K}}$ can be decomposed as the product

$$\chi = \chi'\chi'', \quad \text{where } \chi' \in \tilde{\mathbb{K}}',\ \chi'' \in \tilde{\mathbb{K}}''. \tag{5.3}$$

By Remark 5.5, if **(C)** is satisfied, then Theorem 5.4 holds, i. e. the Laplace transform on $\tilde{\mathbb{K}}$ uniquely determines the distribution of a $\mathbb{K}$-valued random element. Two particular important cases are

- $\tilde{\mathbb{K}}$ is a separating cone of continuous $\mathbb{D}$-valued characters. Then it is automatically strictly separating and also has a countable separating sub-family, see [53, Th. I.1.2].
- $\tilde{\mathbb{K}}$ is a cone of $[0,1]$-valued semicontinuous characters with a strictly separating countable subfamily.

Probability distributions in a topological linear space with a separating family of continuous characters have been studied in [9].

A non-degenerate random element ξ in a convex cone may have a *self-similar* distribution, i. e. $s\xi$ may coincide in distribution with ξ for some (or even all) $s > 0$. For instance, this is the case if $\mathbb{K}$ is the cone of closed sets in $\mathbb{R}_+$ and ξ is the set of zeros of the Wiener process. However, this is clearly impossible in a normed cone, since then the non-negative finite random variable $\|\xi\|$ would coincide in distribution with $s\|\xi\|$ for $s \neq 1$. The following result is applicable without assuming that $\mathbb{K}$ is a normed cone. Its proof follows the scheme of [38, Th. 4.1.12].

Theorem 5.11. *Assume that Condition **(C)** holds. If ξ is a non-idempotent* StαS *random element in a convex cone $\mathbb{K}$ such that for some $s > 0$, for all $\chi' \in \tilde{\mathbb{K}}'$ and all $\chi'' \in \tilde{\mathbb{K}}''$ from Condition **(C)** one has*

$$\mathbf{E}\,\chi'(s\xi) = \mathbf{E}\,\chi'(\xi)\,, \qquad \mathbf{E}\,\chi''(s\xi) = \mathbf{E}\,\chi''(\xi)\,, \tag{5.4}$$

then $s = 1$.

Proof. The non-idempotency condition implies that α is finite. Assume that (5.4) holds for $s > 1$. The case $s < 1$ is simply a reformulation of (5.4) for $s \circ \chi'$, $s \circ \chi''$ instead of χ', χ''. The StαS condition (1.1) immediately implies that $\xi_1 + \cdots + \xi_n \overset{\mathcal{D}}{=} a_n\xi$ for some $a_n > 0$ and all $n \geq 1$. Writing $a_n = \delta_n s^{k(n)}$ in powers of s with $\delta_n \in [1, s)$ we obtain that

$$\chi'(\xi_1)\cdots\chi'(\xi_n) \overset{\mathcal{D}}{=} \chi'(\delta_n s^{k_n}\xi)\,.$$

Now (5.4) applied to $\delta_n \circ \chi'$ leads to

$$(\mathbf{E}\,\chi'(\xi))^n = \mathbf{E}\,\chi'(\delta_n\xi)\,, \quad n \geq 1\,,$$

and a similar identity holds for χ''.

Without loss of generality assume that $\delta_n \to \delta$ as $n \to \infty$, otherwise, consider a convergent subsequence. Assume that χ' is lower semicontinuous. If $\mathbf{E}\,\chi'(\xi) < 1$, then

$$\mathbf{E}\,\chi'(\delta\xi) \leq \liminf_{n\to\infty} \mathbf{E}\,\chi'(\delta_n\xi) = \liminf_{n\to\infty} \big(\mathbf{E}\,\chi'(\xi)\big)^n = 0\,. \tag{5.5}$$

If $\mathbf{E}\,\chi'(\delta\xi) < 1$, then the strict separation condition implies that $\mathbf{E}\,\chi'(\delta_n\xi) < 1$ for sufficiently large n. Hence $\mathbf{E}\,\chi'(\xi) < 1$, and the above argument yields that $\mathbf{E}\,\chi'(\delta\xi) = 0$. Thus, $\mathbf{E}\,\chi'(\delta\xi)$ assumes only the values 0 or 1.

If χ' is upper semicontinuous, a similar argument applies to $\mathbf{E}\,\chi'(\delta_n^{-1}\xi) = (\mathbf{E}\,\chi'(\xi))^{1/n}$. Then

$$\mathbf{E}\,\chi'(\delta^{-1}\xi) \geq \limsup_{n\to\infty} \mathbf{E}\,\chi'(\delta_n^{-1}\xi) = \limsup_{n\to\infty} (\mathbf{E}\,\chi'(\xi))^{1/n}\,.$$

If $\mathbf{E}\,\chi'(\xi) > 0$, then $\mathbf{E}\,\chi'(\delta^{-1}\xi) = 1$. Furthermore, if $\mathbf{E}\,\chi'(\delta^{-1}\xi) > 0$, then $\mathbf{E}\,\chi'(\delta_n^{-1}\xi) > 0$ for sufficiently large n, whence $\mathbf{E}\,\chi'(\xi) > 0$ and consequently $\mathbf{E}\,\chi'(\delta^{-1}\xi) = 1$.

Now consider the character χ''. By continuity, $\mathbf{E}\,\chi''(\delta_n\xi) \to \mathbf{E}\,\chi''(\delta\xi)$. Thus taking limit in $(\mathbf{E}\,\chi''(\delta_n\xi))^2 = \mathbf{E}\,\chi''(\delta_{2n}\xi)$, we obtain that $\mathbf{E}\,\chi''(\delta\xi)$ is either 0 or 1.

If $\mathbf{E}\,\chi'(\xi) = 0$, then $\chi'(\xi) = 0$ a.s., so that by decomposition (5.3), $\chi(\xi) = 0$ a.s. and $\mathbf{E}\,\chi(\xi) = 0$. If $\mathbf{E}\,\chi'(\xi) = 1$, then $\chi'(\xi) = 1$ a.s., so that $\mathbf{E}\,\chi(\xi) = \mathbf{E}\,\chi''(\xi)$, which is either zero or one also in this case. Thus, $\mathbf{E}\,\chi(\xi)$ assumes only values 0 or 1 contradicting the non-idempotency assumption. ☐

Note that (5.4) is weaker than the equality $\mathbf{E}\,\chi(s\xi) = \mathbf{E}\,\chi(\xi)$ for all $\chi \in \tilde{\mathbb{K}}$.

Corollary 5.12. *Assume that **(C)** holds. If a non-idempotent random element ξ satisfies $s\xi \overset{\mathcal{D}}{=} \xi$ for some $s > 0$, then $s = 1$.*

Definition 5.13. A $\mathbb{K}$-valued random element is said to be *non-trivial* if $\mathbf{P}\{\xi = \mathbf{e}\} < 1$. A non-trivial $\mathbb{K}$-valued random element ξ is said to be *proper* if its Laplace transform does not vanish for any χ from the separating cone $\tilde{\mathbb{K}}$.

If the characters from $\tilde{\mathbb{K}}$ take values from $(0, 1]$, then all non-trivial random elements are proper. If ξ has an idempotent factor, i. e. ξ can be represented as a sum of an idempotent (not identically equal to $\mathbf{e}$) random element and another (possibly deterministic) random element, then ξ is not proper. The inverse implication holds for $\mathbb{K}$ being a locally compact group, where proper random elements can be characterised as those which do not possess idempotent factors, see [41, Th. IV.4.2]. In the studies of random sets [38, Sec. 4.1.2] a vanishing Laplace transform corresponds to the random set with fixed points, see Example 8.10. Below we summarise a few immediate properties of proper elements.

Lemma 5.14. *Assume that conditions of Theorem 5.4 are satisfied.*

(i) A proper random element ξ shares the same distribution with $\xi + c$ for a deterministic c if and only if $c = \mathbf{e}$.

(ii) A proper StαS *random element ξ satisfies $\xi \neq \mathbf{0}$ a. s. in case $\mathbf{e} \neq \mathbf{0}$.*

(iii) If ξ and η are two independent elements such that $\xi + \eta$ is proper, then both ξ and η are proper. If, additionally, ξ and $\xi + \eta$ are StαS*, then η is* StαS.

Proof. (i) It suffices to note that $\mathbf{E}\,\chi(\xi) = (\mathbf{E}\,\chi(\xi))\chi(c)$, so that $\chi(c) = 1$ for all χ.

(ii) By the separation property, $\mathbf{e}$ and $\mathbf{0}$ are separated by a certain character $\chi \in \tilde{\mathbb{K}}$. Since both $\mathbf{0}$ and $\mathbf{e}$ are idempotent and $\chi(\mathbf{e}) = 1$, we necessarily have $\chi(\mathbf{0}) = 0$. Then $\delta = \mathbf{P}\{\xi = \mathbf{0}\} \leq \mathbf{P}\{\chi(a\xi) = 0\}$ for all $a > 0$. The stability property implies that

$$\chi(a_n\xi_1)\cdots\chi(a_n\xi_n) \overset{\mathcal{D}}{=} \chi(\xi)$$

for certain $a_n > 0$ and each $n \geq 1$. Then $\chi(\xi) = 0$ when at least one of $\chi(a_n\xi_1), \dots, \chi(a_n\xi_n)$ vanishes. Since this happens with probability at least $\delta > 0$ for each of these independent factors and every $n \geq 1$, $\chi(\xi) = 0$ with probability 1, so that the Laplace transform of ξ vanishes, i. e. ξ is not proper.

(iii) Since $\mathbf{E}\,\chi(\xi + \eta) = \mathbf{E}\,\chi(\xi)\,\mathbf{E}\,\chi(\eta)$, both ξ and η are proper. Furthermore, $\mathbf{E}(c \circ \chi)(\xi) = \mathbf{E}\,\chi(c\xi)$ implies that any proper element multiplied by a constant is also proper. The definition of stability of $\xi + \eta$ implies that

$$\mathbf{E}\,\chi(a^{1/\alpha}\xi_1 + b^{1/\alpha}\xi_2)\ \mathbf{E}\,\chi(a^{1/\alpha}\eta_1 + b^{1/\alpha}\eta_2) = \mathbf{E}\,\chi((a+b)^{1/\alpha}\xi)\ \mathbf{E}\,\chi((a+b)^{1/\alpha}\eta)\,.$$

After cancelling the non-zero equal terms related to StαS element ξ, we get that

$$\mathbf{E}\,\chi(a^{1/\alpha}\eta_1 + b^{1/\alpha}\eta_2) = \mathbf{E}\,\chi((a+b)^{1/\alpha}\eta)$$

for all χ which, in view of Theorem 5.4, is equivalent to the fact that η is StαS. □

Corollary 5.15. *Let ξ be a proper* StαS *with representation (3.9) in a normed cone $\mathbb{K}$. Assume that the conditions of Theorem 5.4 are satisfied.*

(i) *Then z is not an idempotent element of $\mathbb{K}$ unless $z = \mathbf{e}$. In particular, $z \neq \mathbf{0}$ in case $\mathbf{0} \neq \mathbf{e}$.*

(ii) *If, in addition, the norm is sub-invariant and $0 < \alpha < 1$, then $z = \mathbf{e}$.*

Proof. *(i)*. Since $\chi(z) = 0$ or 1 for any idempotent element, then $z + \xi$ is proper if and only if $\chi(z) = 1$ for all χ. Then $z = \mathbf{e}$ by the separation condition.
(ii). By Lemma 5.14*(iii)*, $z \in \mathbb{K}(\alpha)$, but $\mathbb{K}(\alpha) = \{\mathbf{e}\}$ according to Lemma 2.7. □

The following result is the key characterisation theorem for Laplace transforms of proper StαS random elements. It also establishes the equivalence of the StαS property (1.1) and its "discrete" variant (5.6).

Theorem 5.16. *Let ξ be a proper random element in $\mathbb{K}$. Assume that **(C)** holds. Then the following conditions are equivalent.*

(i) *For every $n \geq 2$ there exists a positive constant $a_n \neq 1$ such that*

$$\xi_1 + \cdots + \xi_n \overset{\mathcal{D}}{=} a_n \xi \,, \tag{5.6}$$

where $\xi_1, \ldots, \xi_n$ are i. i. d. copies of ξ.

(ii) *ξ is* StαS *with finite α.*

(iii) *The Laplace transform of ξ is given by (5.2), where φ satisfies*

$$\varphi(s \circ \chi) = s^\alpha \varphi(\chi) \tag{5.7}$$

for all $s > 0$.

Proof. Note that *(ii)* immediately implies *(i)* and *(iii)* implies *(ii)* by Theorem 5.4 and (5.2).
It remains to prove that *(i)* implies *(iii)*. First, *(i)* implies that ξ is not idempotent. Furthermore, (5.6) yields that $n^{-1}\varphi(a_n \circ \chi) = \varphi(\chi)$. Define $a(s) = a_n/a_m$ for $s = n/m$. It is easy to see that $a(s)$ does not depend on the representation of its rational argument s,

$$s^{-1}\varphi(a(s) \circ \chi) = \varphi(\chi) \tag{5.8}$$

and $\varphi(a(s)a(s_1) \circ \chi) = \varphi(a(ss_1) \circ \chi)$. By Theorem 5.11, $a(ss_1) = a(s)a(s_1)$ for all rational $s, s_1 > 0$. To prove that $a(s) = s^{1/\alpha}$ for some $\alpha > 0$, it is now sufficient to show that $a(s)$ is continuous on the set of positive rational numbers.
Let $\chi = \chi'\chi''$ for $[0,1]$-valued character χ' and continuous $\mathbb{D}$-valued character χ'', see (5.3). Since χ' and χ'' are characters themselves, (5.8) holds for χ' and χ'' too. Thus, if s_n is a sequence of rational numbers that converges to 1, then $\mathbf{E}\,\chi'(a(s_n)\xi) \to \mathbf{E}\,\chi'(\xi)$ and $\mathbf{E}\,\chi''(a(s_n)\xi) \to \mathbf{E}\,\chi''(\xi)$.
Assume that $a(s_n)$ has a finite positive limit a. Since χ'' is continuous,

$$E\chi''(a\xi) = \lim \mathbf{E}\,\chi''(a(s_n)\xi) = \mathbf{E}\,\chi''(\xi)$$

by the dominated convergence. If χ' is lower semicontinuous, then by Fatou's lemma,

$$\mathbf{E}\,\chi'(a\xi) \leq \liminf \mathbf{E}\,\chi'(a(s_n)\xi) = \mathbf{E}\,\chi'(\xi)\,.$$

Furthermore, (5.8) written for $a(s_n)^{-1}$ yields that

$$\mathbf{E}\,\chi'(a^{-1}\xi) \le \liminf \mathbf{E}\,\chi'(a(s_n)^{-1}\xi) = \mathbf{E}\,\chi'(\xi)\,.$$

This also holds for $a \circ \chi'$, leading to $\mathbf{E}\,\chi'(\xi) \le \mathbf{E}\,\chi'(a\xi)$. Hence $\mathbf{E}\,\chi'(a\xi) = \mathbf{E}\,\chi'(\xi)$, i. e. (5.4) holds, so that $a = 1$ by Theorem 5.11. Similar arguments apply if χ' is an upper semicontinuous characters.

To finish the proof it suffices to consider either the case of $a(s_n) \to 0$ or $a(s_n) \to \infty$. Let $a_m > 1$ in (5.6) for some $m > 1$. Assume that $a(s_n) \to 0$ as $n \to \infty$. Then, for any $n \ge 1$, $a(s_n) = (a_m)^{-k(n)}\delta_n$, where $\delta_n \in [1, a_m)$ and $k(n) \to \infty$. Note that $(a_m)^{k(n)} = a_{m^{k(n)}}$. Hence

$$\delta_n\xi = (a_m)^{k(n)}a(s_n)\xi \overset{\mathcal{D}}{=} a(s_n)\xi_1 + \cdots + a(s_n)\xi_{m^{k(n)}}\,.$$

Therefore,

$$\mathbf{E}\,\chi(\delta_n\xi) = (\mathbf{E}\,\chi(a(s_n)\xi))^{m^{k(n)}} = (\mathbf{E}\,\chi(\xi))^{s_n m^{k(n)}}\,, \tag{5.9}$$

where χ stands for χ' or χ''. Taking if necessary a convergent subsequence, assume that $\delta_n \to \delta$ as $n \to \infty$. Assume that χ' is lower semicontinuous. If $\mathbf{E}\,\chi'(\delta\xi) < 1$, then the strict separation condition implies that $\mathbf{E}\,\chi'(\delta_n\xi) < 1$ for sufficiently large n. By (5.9), $\mathbf{E}\,\chi'(\xi) < 1$, which, in turn, implies that $\mathbf{E}\,\chi'(\delta_n\xi) \to 0$. By Fatou's lemma,

$$\mathbf{E}\,\chi'(\delta\xi) \le \liminf \mathbf{E}\,\chi'(\delta_n\xi) = 0\,.$$

Thus, $\mathbf{E}\,\chi'(\delta\xi) = 0$ is either zero or one. Applied to χ'', (5.9) yields that $\mathbf{E}\,\chi''(\delta\xi)$ is either zero or one. Similarly to the argument used in the proof of Theorem 5.11, we conclude that $\delta\xi$ is idempotent, whence ξ is idempotent too and so cannot be proper.

If $a_m < 1$ for all $m \ge 1$, then the assumption $a(s_n) \to \infty$ leads to a contradiction in the same manner. Therefore, $a(s_n) \to 1$ as $s_n \to 1$, whence $a(s)$ is continuous on the rational numbers. Thus $a(s) = s^{1/\alpha}$ for all $s > 0$ and (5.7) follows. □

Remark 5.17. In general, (5.6) either holds with $a_n = 1$ for all n (meaning that ξ is idempotent) or $a_n \ne 1$ for all $n \ge 2$. Indeed, if $a_n = 1$ for some $n \ge 2$, then $a_{2n-1} = 1$, and further $a_2 = 1$ implying that $a_n = 1$ for all n.

Theorem 5.16 can also be extended to any non-idempotent ξ, in which case the both sides of (5.7) are allowed to be infinite. From now on, we mainly consider proper random elements.

5.4 Possible values of α for StαS distributions

In the sequel we make use of the following condition:

(E) $\mathbb{K}$ is a pointed cone such that $\chi(sx) \to 1$ for all $\chi \in \tilde{\mathbb{K}}$ and $x \in \mathbb{K}'$, where $s \downarrow 0$ if $\mathbf{e} = \mathbf{0}$ or $s \to \infty$ if $\mathbf{e} \ne \mathbf{0}$.

Condition **(E)** means that sx $\tilde{\mathbb{K}}$-weakly converges to $\mathbf{e}$ for all $x \in \mathbb{K}'$. It clearly holds if $\mathbf{0} = \mathbf{e}$ and the characters are continuous. Conversely, if **(E)** holds and $\mathbb{K}$ is a group with the involution being the inversion and $\mathbf{e} = \mathbf{0}$, then $y_n \to y$ implies $y_n + y^\star \to \mathbf{e}$, so that $\chi(y_n)\chi(y^\star) = \chi(y_n)\chi^{-1}(y) \to 1$. Thus all characters are continuous, so that for a group $\mathbb{K}$, Condition **(C)** follows from **(E)** given that the characters have a countable strictly separating subfamily. Condition **(E)** has a further implication on the characterisation of proper random elements in $\mathbb{K}$.

Lemma 5.18. *If **(E)** holds, then every* StαS *random element in* $\mathbb{K}'$ *necessarily has a proper distribution.*

Proof. Depending on the sign of α and whether $\|\mathbf{e}\| = 0$ or $\|\mathbf{e}\| = \infty$, define $\eta_n = n^{1/\alpha}\xi$ or $\eta_n = n^{-1/\alpha}\xi$ so that $\|\eta_n\| \to \|\mathbf{e}\|$ a. s. Let $\mathbf{E}\,\chi(\xi) = 0$. The stability property implies that $\mathbf{E}\,\chi(\eta_n) = 0$. Now the dominated convergence theorem leads to a contradiction with the fact that $\mathbf{E}\,\chi(\eta_n) \to \mathbf{E}\,\chi(\mathbf{e}) = 1$. $\square$

The following important result shows that the relationship between the origin and the neutral element has a crucial influence on the range of possible values of the stability parameter α.

Theorem 5.19. *Assume that **(C)** and **(E)** hold. Then, for every proper* StαS *random element* ξ*, its characteristic exponent* α *is positive if and only if* $\mathbf{e} = \mathbf{0}$*.*

Proof. Assume that $\mathbf{e} = \mathbf{0}$. Then sx converges to $\mathbf{e}$ as $s \downarrow 0$. Since $(s \circ \chi)(x) = \chi(sx)$, **(E)** ensures that $s \circ \chi \to \mathbf{1}$ pointwise. The sequential continuity of the Laplace exponent φ implies that the left-hand side of (5.7) converges to $\varphi(\mathbf{1}) = 0$, so that $s^\alpha \to 0$ as $s \to 0$. Thus $\alpha > 0$.

Assume that $\mathbf{e} \neq \mathbf{0}$. By **(E)**, $\chi(sx) \to 1$ as $s \to \infty$ for all $x \neq \mathbf{0}$. Lemma 5.14*(ii)* implies that $\xi \neq \mathbf{0}$ a. s. Thus, $\mathbf{E}\,\chi(s\xi) \to 1$, whence the right-hand side of (5.7) converges to zero as $s \to \infty$, so that $\alpha < 0$. $\square$

Theorem 5.19 does not hold without assuming that ξ is proper, see Example 8.10. Neither it holds without assuming that $\mathbb{K}$ is a pointed cone, see Example 8.23. By Lemma 5.18, it is possible to omit the requirement that ξ is proper from Theorem 5.19 if ξ belongs to $\mathbb{K}'$ almost surely.

Since the function φ from (5.2) is negative definite, the properties of negative definite functions together with Theorem 5.16 yield the following result.

Theorem 5.20. *Assume that **(C)** and the second distributivity law (2.5) hold. Then*

- *(i) every proper* StαS *random element* ξ *has parameter* $\alpha \le 2$*;*
- *(ii) if the involution is identical, then* $\alpha \le 1$*.*

*If, in addition, **(E)** holds, then* $\alpha > 0$ *in (i) and (ii).*

Proof. *(i)* It follows from [5, Prop. 4.3.3] that

$$\sqrt{|\varphi(\chi_1\chi_2)|} \le \sqrt{|\varphi(\chi_1)|} + \sqrt{|\varphi(\chi_2)|}\,.$$

By letting $\chi_1 = \chi_2 = \chi$, we see that

$$|\varphi(\chi^2)| \le 4|\varphi(\chi)|\,.$$

The left-hand side is equal to $|\varphi(2 \circ \chi)|$ by (2.5), so that (5.7) implies that $2^\alpha \le 4$.

(ii) Since φ is negative definite, $\varphi(\chi_1) + \varphi(\chi_2) - \varphi(\chi_1\chi_2)$ is a positive definite kernel, see [5, Prop. 4.1.9]. In particular, its value is non-negative if $\chi_1 = \chi_2 = \chi$, whence $\varphi(2 \circ \chi) = \varphi(\chi^2) \le 2\varphi(\chi)$, i. e. $\alpha \le 1$.

Finally, note that Lemma 2.3 implies that $\mathbf{e} = \mathbf{0}$, thus $\alpha > 0$ by Theorem 5.19 given that **(E)** holds. □

If the second distributivity (2.5) does not hold, then α may have various ranges of possible values. For instance, in $(\mathbb{R}_+, \vee)$ any $\alpha > 0$ and for $([0,\infty], \min)$ any $\alpha < 0$ are possible, see Example 3.7. The following example shows that it is possible to define a cone where StαS laws exist with any α from a given interval $(0, \beta)$ or $(-\beta, 0)$.

Example 5.21. Consider the cone $\mathbb{R}_+$ with the addition operation given by $(x^\beta + y^\beta)^{1/\beta}$ with $\beta > 0$ and the conventional multiplication by positive numbers. Then $x + x = 2^{1/\beta}x$ for all $x \in \mathbb{K}$, so that a similar argument to Theorem 5.20 implies that the stability parameter is at most 2β. If ξ is StαS in $\mathbb{R}_+$ with the conventional addition and $\alpha \in (0,1)$, then $\eta = \xi^{1/\beta}$ is stable with parameter $\alpha\beta$ in the newly defined cone. The case $\beta = \infty$ corresponds to the maximum operation in $\mathbb{R}_+$, where the characteristic exponent α takes any value from $(0, \infty)$. The same construction with $\beta < 0$ gives a cone with negative range of α, c. f. Example 8.18.

In general, the range of possible parameters of stable laws may be used as a characteristic of a cone that, in a sense, assesses the extent by which the second distributivity law (2.5) is violated. The above construction shows that if α_0 is a possible value of the characteristic exponent, then any value in $(0, \alpha_0)$ in case $\alpha_0 > 0$ or in $(\alpha_0, 0)$ in case $\alpha_0 < 0$ is also possible.

The following results about possible values of α rely entirely on the properties of the norm if $\mathbb{K}$ is a normed cone. It does not refer to characters on $\mathbb{K}$.

Lemma 5.22. *Let $\mathbb{K}$ be a normed cone such that*

$$\|x + y\| \ge \|x\| \tag{5.10}$$

for all $x, y \in \mathbb{K}$. Then every StαS *law in $\mathbb{K}'$ has $\alpha > 0$.*

Proof. It follows from (1.1) and (5.10) that $2^{1/\alpha}\|\xi\|$ is stochastically greater than $\|\xi\|$, i. e. the cumulative distribution function of the first is not greater than of the second one. If $\alpha < 0$, this is only possible if $\|\xi\|$ vanishes or is infinite, while both these cases are excluded by requiring that ξ is $\mathbb{K}'$-valued for a normed cone $\mathbb{K}$. □

Note that (5.10) holds in such cones like the positive half-line with the sum or maximum operation, where the addition always increases the norm. It is interesting to note that a cone with $\mathbf{e} \ne \mathbf{0}$ that satisfies (5.10), **(C)** and **(E)** does not possess any non-trivial proper StαS random element. Indeed Theorem 5.19 implies that $\alpha < 0$. For instance, this is the case for the cone of compact sets with the union operation, see Example 8.10.

Lemma 5.23. *If $\mathbb{K}$ has a sub-invariant norm, then the range of possible values of the characteristic exponent α includes* $(0, 1)$.

Proof. It suffices to refer to Theorem 3.6 which provides an explicit construction of StαS random elements with $\alpha \in (0,1)$ by the LePage series. □

6 Integral representations of stable laws

6.1 Integral representations of negative definite functions

The theory of integral representations of negative definite functions [5, Ch. 4] makes it possible to gain further insight into the structure of the function φ from (5.2), i. e. the Laplace exponent of a StαS random element ξ. The random element is always assumed to be proper in this section, so that the corresponding Laplace exponent is finite. Let us first introduce several important ingredients of these integral representations specified for the cone $\tilde{\mathbb{K}}$ of characters with its dual $\mathbb{K}^\sharp$ introduced in Section 5.2.

A *Lévy measure* is a Radon measure on $\mathbb{K}^\sharp \setminus \{\mathbf{1}\}$ such that

$$\int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} (1 - \operatorname{Re} \rho(\chi))\, \lambda(d\rho) < \infty \tag{6.1}$$

for all $\chi \in \tilde{\mathbb{K}}$. Here $\mathbf{1}$ is the neutral element in $\mathbb{K}^\sharp$, i. e. the character identically equal to 1. A function $\ell : \tilde{\mathbb{K}} \mapsto \mathbb{R}$ is said to be $\star$-additive if

$$\ell(\chi_1\chi_2) = \ell(\chi_1) + \ell(\chi_2) \quad \text{and} \quad \ell(\overline{\chi}) = -\ell(\chi) \tag{6.2}$$

for all $\chi_1, \chi_2, \chi \in \tilde{\mathbb{K}}$ (recall that the involution of a character is its complex conjugate). The function $\chi \mapsto e^{i\ell(\chi)}$ is a character on $\tilde{\mathbb{K}}$, i. e. an element of $\mathbb{K}^\sharp$. A function $q : \tilde{\mathbb{K}} \mapsto \mathbb{R}$ is called a *quadratic form* if

$$2q(\chi_1) + 2q(\chi_2) = q(\chi_1\chi_2) + q(\chi_1\overline{\chi_2}) \tag{6.3}$$

for all $\chi_1, \chi_2 \in \tilde{\mathbb{K}}$. A real-valued function $L(\chi, \rho)$ defined on $\tilde{\mathbb{K}} \times \mathbb{K}^\sharp$ is called the *Lévy function* if L is $\star$-additive with respect to χ for each ρ, Borel measurable with respect to ρ for each χ, and

$$\int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} |1 - \rho(\chi) + iL(\chi, \rho)|\, \lambda(d\rho) < \infty \tag{6.4}$$

for each Lévy measure λ. It is shown in [8] that a Lévy function exists, can be chosen to be continuous with respect to its second argument and to satisfy $L(\chi, \overline{\rho}) = -L(\chi, \rho)$, see also [4, Th. 3.1]. We fix a certain Lévy function constructed according to [8] for the semigroup $\tilde{\mathbb{K}}$.

If the Laplace exponent φ from (5.2) is finite (i. e. ξ is proper), it can be represented as

$$\varphi(\chi) = i\ell(\chi) + q(\chi) + \int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} (1 - \rho(\chi) + iL(\chi, \rho))\, \lambda(d\rho)\,, \quad \chi \in \tilde{\mathbb{K}}\,, \tag{6.5}$$

for a unique triple (ℓ, q, λ) of $\star$-additive function, non-negative quadratic form and Lévy measure, see [5, Th. 4.3.19]. Note that $\varphi(\mathbf{1}) = 0$. Since the elements of the integral representation generally depend on the choice of $\tilde{\mathbb{K}}$, the family $\tilde{\mathbb{K}}$ is supposed to be fixed in the sequel. If $\mathbb{K}$ is a group, the quadratic form q corresponds to the Gaussian component of ξ, see [41, Sec. IV.6]. Following this terminology, we say that ξ does not have a *Gaussian component* if q in (6.5) vanishes. Since the addition operation is not invertible in general, the presence of a Gaussian component in ξ does not imply that ξ can be decomposed as the sum of independent Gaussian element and a certain remainder.

If the involution is identical, then ρ takes real values, the Lévy function and ℓ vanish, so that (6.5) turns into

$$\varphi(\chi) = q(\chi) + \int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} (1-\rho(\chi))\,\lambda(d\rho)\,, \tag{6.6}$$

where (6.3) for $q : \tilde{\mathbb{K}} \mapsto \mathbb{R}_+$ turns into

$$q(\chi_1\chi_2) = q(\chi_1) + q(\chi_2) \tag{6.7}$$

for all $\chi_1, \chi_2 \in \tilde{\mathbb{K}}$, see [5, Th. 4.3.20]. In this case q is also called a quadratic form. Furthermore, $e^{-q(\chi)}$ is a character from $\mathbb{K}^\sharp$. If it is possible to associate this character with a certain $z \in \mathbb{K}$ using the evaluation map, i. e. if $e^{-q(\chi)} = \chi(z)$ for all $\chi \in \tilde{\mathbb{K}}$, then the Gaussian component corresponds to the deterministic point z.

6.2 Lévy measures of StαS laws

Representations (6.5) and (6.6) hold for every infinitely divisible random element ξ with finite Laplace exponent. The StαS property of ξ can be used to characterise the elements of the triplet (ℓ, q, λ). For this, uplift the multiplication by numbers to $\mathbb{K}^\sharp$ from $\tilde{\mathbb{K}}$ by setting

$$(a \circ \rho)(\chi) = \rho(a \circ \chi)\,, \quad a > 0\,.$$

Theorem 6.1. *Assume that Condition **(C)** holds. If φ is the Laplace exponent of a proper* StαS *random element ξ, then the corresponding Lévy measure is homogeneous on $\mathbb{K}^\sharp$, i. e.*

$$\lambda(s \circ B) = s^{-\alpha}\lambda(B)\,, \quad s > 0\,, \tag{6.8}$$

for each Borel set B in $\mathbb{K}^\sharp$ and λ has infinite total mass.

Proof. Let μ_t be the distribution of $t^{1/\alpha}\xi$. It follows from Theorem 5.16*(iii)* that the Laplace transform of μ_t is given by $e^{-t\varphi(\chi)}$, so that $\{\mu_t,\ t > 0\}$ is the convolution semigroup associated with φ. By [5, Lemma 4.3.12], the Lévy measure λ is the vague limit as $t \downarrow 0$ of the images of $t^{-1}\mu_t$ under the evaluation map $\imath$. Since $t^{-1}\mu_t(sA) = s^{-\alpha}r^{-1}\mu_r(A)$ with $t = s^\alpha r$, the corresponding vague limit λ satisfies (6.8).

By taking B with $\lambda(B) > 0$ and letting $s \to \infty$ in (6.8) in case $\alpha < 0$ and $s \downarrow 0$ in case $\alpha > 0$, it follows that λ has infinite total mass. ☐

Since the left-hand side of (6.8) can be written as $(s^{-1}\circ\lambda)(B)$, we say that λ has the *homogeneity order* α. The following result provides an upper bound for the possible homogeneity order of the Lévy measure and thereupon can complement Theorem 5.20 even without using the second distributivity law. It relies instead on the local behaviour of characters near the neutral element.

Theorem 6.2. *Assume that Condition **(C)** holds. For some $\beta > 0$, all $\chi \in \tilde{\mathbb{K}}$ and all $\rho \in \mathbb{K}^\sharp\setminus\{\mathbf{1}\}$ define*

$$g(\chi, \rho) = \liminf_{t\downarrow 0} \frac{1 - \mathrm{Re}\,(t\circ\rho)(\chi)}{t^\beta}\,. \tag{6.9}$$

Assume that for all $\rho \neq \mathbf{1}$ there exists $\chi \in \tilde{\mathbb{K}}$ such that $g(\chi,\rho) > 0$. Then the Lévy measure λ of a proper StαS *random element ξ has the homogeneity order which is strictly less than β.*

If (6.9) holds for $t \to \infty$ and $\beta < 0$, then the same condition on g implies that the order of homogeneity is strictly greater than β.

Proof. If the homogeneity order is α, then for all sufficiently small $t > 0$,

$$\begin{aligned}\int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} (1-\operatorname{Re}\rho(\chi))\lambda(d\rho) &= \int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} (1-\operatorname{Re}(t\circ\rho)(\chi))\lambda(t\circ d\rho)\\ &= \int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} \frac{1-\operatorname{Re}(t\circ\rho)(\chi)}{t^\beta} t^{\beta-\alpha}\lambda(d\rho)\\ &\geq \int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} g(\chi,\rho)t^{\beta-\alpha}\lambda(d\rho)\,.\end{aligned}$$

Since, by Theorem 6.1, λ has infinite total mass, the obtained expression converges to infinity as $t \downarrow 0$ if $\alpha \geq \beta$ contrary to (6.1) unless λ is supported by $\{\rho :\ g(\chi,\rho) = 0\}$ for all χ. The latter is, however, impossible in view of the condition on g. The second statement is proved similarly. □

Define a measure Λ on $\mathcal{B}(\mathbb{K})$ as the inverse image under the natural map of λ restricted onto $\imath(\mathbb{K})$. The measure Λ is homogeneous on $\mathbb{K}$ with the same order as λ. Although it is generally impossible to examine (6.9) for *all* characters ρ from the second dual semigroup, it is possible to assess the homogeneity order of Λ by applying (6.9) to $\rho = \rho_x$ for $x \in \mathbb{K}$. If

$$g(\chi,x) = \liminf_{t\downarrow 0} \frac{1-\operatorname{Re}\chi(tx)}{t^\beta} > 0\,,$$

for all $x \neq \mathbf{e}$ and some $\chi \in \tilde{\mathbb{K}}$, then the homogeneity order of Λ is less than β, i.e. the maximal homogeneity order corresponds to the order of decrease of $1-\operatorname{Re}\chi(tx)$ as $t \downarrow 0$.

Corollary 6.3. *If **(C)** and the second distributivity law* (2.5) *hold, then the order of homogeneity of the Lévy measure of a proper* StαS *random element is strictly less than 2. If also the involution is identical, then the order of homogeneity is strictly less than 1.*

Proof. If the second distributivity law holds, then

$$(n^{-1}\circ\rho)(\chi) = \rho(n^{-1}\circ\chi) = \rho(\chi)^{1/n}\,.$$

The inequality

$$\frac{1}{n^2}(1-\operatorname{Re}(\rho(\chi)^n)) \leq \frac{\pi^2}{4}(1-\operatorname{Re}\rho(\chi)) \tag{6.10}$$

from [5, p. 109] implies that

$$n^2(1-\operatorname{Re}(n^{-1}\circ\rho)(\chi)) \geq \frac{4}{\pi^2}(1-\operatorname{Re}\rho(\chi))\,.$$

Then Theorem 6.2 is applicable with $\beta = 2$ and $g(\chi,\rho) \geq 1-\operatorname{Re}\rho(\chi)$.

Another inequality from [5, p. 109] reads that

$$\frac{1}{n}(1-|\rho(\chi)|^{2n}) \leq 1-\operatorname{Re}\rho(\chi\overline{\chi})\,.$$

If the involution is identical, then

$$n(1-(n^{-1}\circ\rho)(\chi)) \ge 1-|\rho(\chi)|\,,$$

whence the homogeneity order for the Lévy measure is strictly less than 1. $\square$

By combining Corollary 6.3 with Theorem 5.20*(i)* and the sequential continuity property of φ, we arrive at the following result.

Corollary 6.4. *If **(C)** and the second distributivity law* (2.5) *hold, then every proper* StαS *random element with $\alpha=2$ has the Laplace exponent given by*

$$\varphi(\chi)=i\ell(\chi)+q(\chi)\,,\quad \chi\in\tilde{\mathbb{K}}\,,$$

where ℓ is a $\star$-additive sequentially continuous function and q is a sequentially continuous non-negative quadratic form on $\tilde{\mathbb{K}}$.

If the involution is identical, then every proper StαS *random element with $\alpha=1$ has the Laplace exponent $\varphi(\chi)=q(\chi)$ being sequentially continuous non-negative linear functional of χ.*

Without the second distributivity law, the value $\alpha=2$ does not necessarily correspond to the Gaussian component, e. g., in $(\mathbb{R}_+,\vee)$.

6.3 Quadratic form and Lévy function

It remains to explore implications of the stability of ξ on the further ingredients of (6.5), namely the quadratic form q, the Lévy function and the function ℓ. The following result describes some cases when ξ does not possess a Gaussian component.

Theorem 6.5. *Assume that Condition **(C)** and the second distributivity law* (2.5) *hold. Let q be the quadratic form in the integral representation (6.5) of a proper* StαS *random element.*

(i) q vanishes unless $\alpha=1$ or $\alpha=2$. If $\mathbb{K}$ is a group, then q vanishes if $\alpha\neq 2$. If $\alpha=1$, then $q(\overline{\chi})=q(\chi)$ for all χ and q satisfies (6.7).

(ii) If the involution is identical, then q from representation (6.6) vanishes unless $\alpha=1$.

Proof. (i) The $\star$-additivity of $L(\cdot,\rho)$ implies that $L(\chi^2,\rho)=2L(\chi,\rho)$. By the second distributivity law, $2\circ\chi=\chi^2$, so that $\rho(\chi^2)=\rho^2(\chi)=(2\circ\rho)(\chi)$.

By Theorem 5.16, $\varphi(\chi^2)=\varphi(2\circ\chi)=2^\alpha\varphi(\chi)$, hence it is also finite. Furthermore,

$$\begin{aligned}\varphi(\chi^2)&=i\ell(\chi^2)+q(\chi^2)+\int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}}(1-\rho(\chi^2)+iL(\chi^2,\rho))\,\lambda(d\rho)\\ &=2i\ell(\chi)+q(\chi^2)+\int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}}(1-(2\circ\rho)(\chi)+2L(\chi,\rho))\lambda(d\rho)\,,\end{aligned}\tag{6.11}$$

while, by Theorem 6.1,

$$\begin{aligned}2^\alpha\varphi(\chi) &= 2^\alpha i\ell(\chi) + 2^\alpha q(\chi) + \int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} (1-\rho'(\chi) + iL(\chi,\rho'))\,\lambda(2^{-1}\circ d\rho') \\ &= 2^\alpha i\ell(\chi) + 2^\alpha q(\chi) + \int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} (1-(2\circ\rho)(\chi) + iL(\chi,2\circ\rho))\lambda(d\rho). \end{aligned} \tag{6.12}$$

Equating the right-hand sides of (6.11) and (6.12), we arrive at

$$2^\alpha i\ell(\chi) + 2^\alpha q(\chi) = 2i(\ell(\chi) - \tilde{\ell}(\chi)) + q(\chi^2)\,, \tag{6.13}$$

where the integral

$$\tilde{\ell}(\chi) = \frac{1}{2}\int_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}} (L(\chi,2\circ\rho) - 2L(\chi,\rho))\lambda(d\rho)$$

is finite and may be regarded as a real-valued $\star$-additive function on $\tilde{\mathbb{K}}$.

By comparing the coefficients of the real parts in (6.13), we see that

$$q(\chi^2) = 2^\alpha q(\chi)\,. \tag{6.14}$$

Next, (6.3) applied to a real-valued character χ, yields that $2q(\chi) = q(\chi^2)$. Since $\alpha \neq 1$, we have $q(\chi) = 0$ for every real-valued character χ. Since $\chi\overline{\chi}$ is a real-valued character, $q(\chi\overline{\chi}) = 0$. Now (6.3) applied to $\chi_1 = \chi_2 = \chi$ for a (not necessarily real-valued character) χ yields that $4q(\chi) = q(\chi^2) = 2^\alpha q(\chi)$. Since $\alpha \neq 2$, q vanishes identically.

If $\mathbb{K}$ is a group, then putting $\chi_1 = \chi_2 = \chi$ in (6.3) and noticing that $\overline{\chi} = \chi^{-1}$, we obtain that q vanishes unless $\alpha = 2$.

Assume that $\alpha = 1$. By applying (6.3) to $\chi_1 = \chi_2 = \chi$ and using (6.14) we obtain that $q(\chi\overline{\chi}) = 2q(\chi)$. Then

$$\begin{aligned} 2q(\chi_1) + 2q(\chi_2) &= \frac{1}{2}\big(2q(\chi_1\overline{\chi_1}) + 2q(\chi_2\overline{\chi_2})\big) \\ &= \frac{1}{2}\big(q(\chi_1\overline{\chi_1}\chi_2\overline{\chi_2}) + q(\chi_1\overline{\chi_1}\overline{\chi_2\overline{\chi_2}})\big) \\ &= q(\chi_1\overline{\chi_1}\chi_2\overline{\chi_2}) \\ &= 2q(\chi_1\chi_2)\,. \end{aligned}$$

Combining this with (6.3) yields that $q(\overline{\chi}) = q(\chi)$.

(ii) By Theorem 5.20*(ii)*, $\alpha \leq 1$. If $\alpha < 1$, then by [5, Th. 4.3.20],

$$q(\chi) = \lim_{n\to\infty}\frac{\varphi(\chi^n)}{n} = \lim_{n\to\infty}\frac{\varphi(n\circ\chi)}{n} = \lim_{n\to\infty}\frac{n^\alpha\varphi(\chi)}{n} = 0\,,$$

noticing that $\varphi(\chi)$ is finite. □

The following result concerns idempotent semigroups, where the second distributivity law never holds.

Lemma 6.6. *If $\mathbb{K}$ is an idempotent semigroup, then q in (6.6) vanishes identically.*

Proof. The idempotency of $\mathbb{K}$ implies that all characters take values 0 or 1. Thus, $\tilde{\mathbb{K}}$ is also an idempotent semigroup. By (6.7), $q(\chi) = q(\chi^2) = 2q(\chi)$, whence q vanishes. $\square$

In a number of cases the Lévy function vanishes or may be set to zero. Particular important instances of this are mentioned in the following theorem.

Theorem 6.7. *Assume that at least one of the following conditions holds.*

(i) The involution is identical.

(ii) For all $\chi \in \tilde{\mathbb{K}}$

$$\int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} (1 - \rho(\chi))\, \lambda(d\rho) < \infty\,. \tag{6.15}$$

Then the Laplace exponent of any proper StαS *random element ξ is given by*

$$\varphi(\chi) = i\ell(\chi) + q(\chi) + \int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} (1 - \rho(\chi))\, \lambda(d\rho)\,, \quad \chi \in \tilde{\mathbb{K}}\,, \tag{6.16}$$

*for a $\star$-additive function ℓ and a quadratic form q. If **(C)** and the second distributivity law* (2.5) *hold, then ℓ vanishes unless $\alpha = 1$.*

Proof. If the involution is identical, then the Lévy function vanishes. If (6.15) holds, then

$$\int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} L(\chi, \rho)\, \lambda(d\rho)$$

is a finite $\star$-additive functional of χ, so that it can be combined with $\ell(\chi)$ from (6.5).

By Theorem 6.1, the integral in (6.16) is homogeneous of order α. By the homogeneity and (2.5), $\varphi(\chi^2) = \varphi(2 \circ \chi) = 2^\alpha \varphi(\chi)$, so that $2\ell(\chi) = 2^\alpha \ell(\chi)$. Thus, ℓ vanishes unless $\alpha = 1$. $\square$

6.4 Symmetric random elements

Definition 6.8. A $\mathbb{K}$-valued random element is called *symmetric* if ξ coincides in distribution with its involution $\xi^\star$.

In case of the identical involution all random elements can be regarded as being symmetric. Recall that symmetric StαS random elements are traditionally called SαS. By generalising the *symmetrisation* procedure in Banach spaces, it is possible to obtain a symmetric element ξ as $\xi_1 + \xi_2^\star$ for i. i. d. random elements ξ_1 and ξ_2. By applying the involution to the both sides of (1.1) it is clear that if ξ is StαS then $\xi^\star$ also is.

The *principal value* of an integral over $\mathbb{K}^\sharp \setminus \{\mathbf{1}\}$ is defined as the limit of the integrals over a sequence $\{F_n\}$ of symmetric sets as $F_n \uparrow \mathbb{K}^\sharp \setminus \{\mathbf{1}\}$. The symmetry of $F \subset \mathbb{K}^\sharp$ is understood with respect to the involution $\rho^\star(\chi) = \rho(\overline{\chi}) = \overline{\rho}(\chi)$ on $\mathbb{K}^\sharp$.

Corollary 6.9. *If (C) holds, then the Laplace exponent of every proper* SαS *random element is given by*

$$\varphi(\chi) = q(\chi) + \int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} (1 - \rho(\chi))\, \lambda(d\rho)\,, \quad \chi \in \tilde{\mathbb{K}}\,, \tag{6.17}$$

where λ is a symmetric homogeneous measure on $\mathbb{K}^\sharp$, q is a non-negative quadratic form, and the principal value of the integral converges.

Proof. By repeating the argument from the proof of Theorem 6.1 it is easy to see that λ is symmetric with respect to the complex conjugate operation being the involution on $\mathbb{K}^\sharp$.

The integral in (6.5) converges and so does its principal value as the limit of the integrals over F_n as $F_n \uparrow \mathbb{K}^\sharp \setminus \{\mathbf{1}\}$. The symmetry property of F_n and λ implies that

$$\int_{F_n} L(\chi, \rho)\lambda(d\rho) = \int_{F_n} L(\chi, \overline{\rho})\lambda(d\overline{\rho}) = -\int_{F_n} L(\chi, \rho)\lambda(d\rho)\,,$$

whence

$$\int_{\mathbb{K}^\sharp \setminus \{\mathbf{1}\}} L(\chi, \rho)\lambda(d\rho) = 0$$

as the principal value. Note that $\mathbf{E}\,\chi(\xi^\star) = \mathbf{E}\,\overline{\chi}(\xi)$. Since the integral of the Lévy function vanishes and $\varphi(\chi) = \varphi(\overline{\chi})$, we obtain that $\ell(\chi) = 0$ in (6.16). $\square$

Consider now a Gaussian random element ξ in $\mathbb{K}$, i. e. assume that its Laplace transform is given by $\mathbf{E}\,\chi(\xi) = e^{-q(\chi)}$ for $\chi \in \tilde{\mathbb{K}}$.

Theorem 6.10. *Assume that $\tilde{\mathbb{K}}$ is a strictly separating cone of characters and $\mathcal{F}(\mathbb{K}; \tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K})$. If ξ is a Gaussian element, then ξ is symmetric, $\chi(\xi)$ is deterministic for every real-valued character χ and $\xi + \xi^\star$ is deterministic. In particular, ξ is deterministic if the involution is identical.*

Every character χ with values in the unit complex circle $\mathbb{T}$ is representable as $\chi(x) = e^{iu(x)}$ with an additive real-valued function u such that $u(\xi)$ has a normal distribution.

Proof. By applying (6.3) to $\chi_1 = \mathbf{1}$, we see that $q(\overline{\chi_2}) = q(\chi_2)$, i. e. q is involution symmetric. This immediately leads to the conclusion that $\mathbf{E}\,\chi(\xi) = \mathbf{E}\,\overline{\chi}(\xi) = \mathbf{E}\,\chi(\xi^\star)$ for all $\chi \in \tilde{\mathbb{K}}$, so that Theorem 5.4 yields that $\xi \overset{\mathcal{D}}{=} \xi^\star$.

It is easy to see that if at least one of χ_1, χ_2 is real-valued in (6.3), then $q(\chi_1\chi_2) = q(\chi_1) + q(\chi_2)$. Therefore, if χ is a real-valued character, then

$$\mathbf{E}\,\chi^2(\xi) = e^{-q(\chi^2)} = e^{-2q(\chi)} = (\mathbf{E}\,\chi(\xi))^2\,.$$

Hence $\chi(\xi)$ is deterministic for every real-valued χ. If the involution is identical, the strict separation condition implies that ξ itself is deterministic.

For a general $\chi \in \tilde{\mathbb{K}}$,

$$\mathbf{E}\,\chi(\xi + \xi^\star) = \mathbf{E}\,\chi(\xi)\overline{\chi}(\xi) = \mathbf{E}\,|\chi(\xi)|^2\,.$$

Since $|\chi|^2$ is a real-valued character, then, as it was just shown, $|\chi(\xi)|^2 = \chi(\xi + \xi^\star)$ is deterministic, so that the conclusion of the theorem follows from the strict separation condition.

Let χ be a $\mathbb{T}$-valued character. By iterating (6.3), it is seen that $q(\chi^n) = n^2 q(\chi)$ for each n. Interpreting this identity in terms of the Laplace transform of ξ yields that

$$\mathbf{E}\,\chi^s(\xi) = (\mathbf{E}\,\chi(\xi))^{s^2}$$

for every positive rational number s. Using the representation of χ, we have

$$\mathbf{E}\,e^{isu(\xi)} = (\mathbf{E}\,\chi(\xi))^{s^2}\,.$$

The symmetry property of ξ implies that $\mathbf{E}\,\chi(\xi)$ is a real number which does not exceed 1, so that $\mathbf{E}\,\chi(\xi) = e^{-a^2/2}$ for some a. Thus, $\mathbf{E}\,e^{isu(\xi)} = e^{-a^2 s^2/2}$ meaning that $u(\xi)$ is normally distributed with mean zero and variance a^2. $\square$

Note that in the absence of the second distributivity law, a Gaussian random element may be SαS with arbitrary α, neither it needs to be stable at all, see Example 8.22. If the Gaussian element ξ from Theorem 6.10 is SαS, then deterministic $\xi + \xi^\star$ is also SαS, hence an element of $\mathbb{K}(\alpha)$. Furthermore, $\xi \in \mathbb{K}(\alpha)$ if the involution is identical.

7 LePage series representation of StαS random elements

7.1 LePage series on the second dual semigroup

In Section 3.3 we have shown that the LePage series (3.9) (or the integral $\int x \Pi_\alpha(dx)$) defines a StαS random element. It is natural to ask if any StαS random element admits such a representation as it is the case for stable distributions in Banach spaces, see [46, Cor. 4.10]. We address this question by using the integral representations (6.5) and (6.6) for the Laplace exponent φ of a StαS random element. This idea is supported by the formula (3.8) for the probability generating functional of a Poisson process, which is quite similar to (6.6). The intensity measure of this Poisson process is the Lévy measure λ, so that this process lives on $\mathbb{K}^\sharp$ and therefore is denoted by $\Pi^\sharp$.

We first characterise the weak convergence of $\mathbb{K}^\sharp$-valued random elements.

Lemma 7.1. *A sequence $\xi_n^\sharp$ of random elements in $\mathbb{K}^\sharp$ with its Borel σ-algebra $\mathcal{B}(\mathbb{K}^\sharp)$ weakly converges to a random element $\xi^\sharp$ (notation $\xi_n^\sharp \Rightarrow \xi^\sharp$) if and only if $\mathbf{E}\,\xi_n^\sharp(\chi) \to \mathbf{E}\,\xi^\sharp(\chi)$ for all χ from a separating semigroup $\tilde{\mathbb{K}}$.*

Proof. Note that $\tilde{\mathbb{K}}$ is a separating family of characters on $\mathbb{K}^\sharp$ acting as $\chi(\rho) = \rho(\chi)$. Furthermore, $\mathcal{F}(\mathbb{K}^\sharp;\tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K}^\sharp)$, so that Theorem 5.4 is applicable. Each character $\chi : \mathbb{K}^\sharp \mapsto \mathbb{D}$ is continuous on $\mathbb{K}^\sharp$, since $\chi(\rho_n) = \rho_n(\chi) \to \rho(\chi) = \chi(\rho)$ if ρ_n converges to ρ pointwise.

The result now follows from the convergence of Laplace transforms $\mathbf{E}\,\xi_n^\sharp(\chi)$ together with the tightness condition that is clearly fulfilled because of the compactness of $\mathbb{K}^\sharp$, see [53, Th. IV.3.1]. $\square$

We also write $\xi_n^\sharp \Rightarrow \xi$ if $\xi_n^\sharp$ weakly converges to the $\mathbb{K}^\sharp$-valued random element $\imath(\xi)$ being the evaluation image of $\xi \in \mathbb{K}$. For this, the condition $\mathcal{F}(\mathbb{K};\tilde{\mathbb{K}}) = \mathcal{B}(\mathbb{K})$ of Theorem 5.4 should hold in order to be able to treat $\imath(\xi)$ as a $\mathbb{K}^\sharp$-valued random element.

Denote by $\mathbb{F}_\lambda(\mathbb{K}^\sharp)$ the family of Borel sets $F \subset \mathbb{K}^\sharp \setminus \{\mathbf{1}\}$ such that

$$\int_F (1-\rho(\chi))\lambda(d\rho) < \infty \tag{7.1}$$

for all $\chi \in \tilde{\mathbb{K}}$. With every F, the family $\mathbb{F}_\lambda(\mathbb{K}^\sharp)$ contains all its measurable subsets. If the involution is identical, then $\mathbb{K}^\sharp \setminus \{\mathbf{1}\} \in \mathbb{F}_\lambda(\mathbb{K}^\sharp)$.

If $F \in \mathbb{F}_\lambda(\mathbb{K}^\sharp)$, then the product

$$\xi_F^\sharp = \prod_{\rho \in F \cap \operatorname{supp} \Pi^\sharp} \rho$$

exists in $\mathbb{K}^\sharp$, i. e. it weakly converges in the topology of pointwise convergence on $\mathbb{K}^\sharp$ meaning that

$$\xi_{F\cap K_n}^\sharp(\chi) = \prod_{\rho \in F \cap K_n \cap \operatorname{supp} \Pi^\sharp} \rho(\chi)$$

weakly converges to $\xi_F^\sharp(\chi)$ for every $\chi \in \tilde{\mathbb{K}}$ as $K_n \uparrow \mathbb{K}^\sharp \setminus \{\mathbf{1}\}$ with $\lambda(K_n) < \infty$. This is easily seen by observing that the probability generating functional of $\Pi^\sharp$ restricted onto $F \cap K_n$ coincides with the expected value of $\xi_{F\cap K_n}^\sharp$ and using Lemma 7.1. Note that $\xi_{F\cap K_n}^\sharp$ is well defined because $\Pi^\sharp$ has only a finite number of points in K_n.

For any $F \in \mathbb{F}_\lambda(\mathbb{K}^\sharp)$ define

$$\gamma_F(\chi) = \exp\left\{-i\left(\ell(\chi) + \int_F L(\chi,\rho)\lambda(d\rho)\right)\right\}, \quad \chi \in \tilde{\mathbb{K}}. \tag{7.2}$$

Note that $\gamma_F \in \mathbb{K}^\sharp$.

Theorem 7.2. *Assume that Condition **(C)** holds. Let ξ be a proper $\mathbb{K}$-valued* StαS *random element without Gaussian component. Then there exists a unique measure λ on $\mathbb{K}^\sharp$ (which is then a homogeneous of order α Lévy measure) such that, for the Poisson process $\Pi^\sharp$ with intensity measure λ, one has*

$$\gamma_{F_n}\xi_{F_n}^\sharp \Rightarrow \xi$$

for any sequence $F_n \uparrow \mathbb{K}^\sharp \setminus \{\mathbf{1}\}$ such that $F_n \in \mathbb{F}_\lambda(\mathbb{K}^\sharp)$ for all n.

Proof. Let λ be the Lévy measure of ξ that stems from (6.5). Formula (3.8) for the probability generating functional applied to $\Pi^\sharp$ and the definition of γ_F imply that

$$\begin{aligned}\gamma_{F_n}(\chi)\,\mathbf{E}\,\chi(\xi_{F_n}^\sharp) &= \gamma_{F_n}(\chi)\,\mathbf{E}\left[\prod_{\rho\in F_n\cap\operatorname{supp}\Pi^\sharp}\rho(\chi)\right]\\ &= \exp\left\{-\left(i\ell(\chi) + \int_{F_n}(1-\rho(\chi)+iL(\chi,\rho))\,\lambda(d\rho)\right)\right\}.\end{aligned}$$

If $F_n \uparrow \mathbb{K}^\sharp \setminus \{\mathbf{1}\}$, the right-hand side converges to the Laplace transform of a stable random element ξ without Gaussian component, see (6.5). Finally, Lemma 7.1 shows that $\gamma_{F_n}\xi_{F_n}^\sharp$ weakly converges to $\xi^\sharp = \imath(\xi)$. ☐

In some cases (see, e. g., Theorem 6.7) the Lévy function vanishes or its contribution to the Laplace exponent may be subsumed in ℓ, so that $\gamma_{F_n}(\chi) = e^{-i\ell(\chi)}$. If ℓ vanishes, then the normalisation is not needed and

$$\xi^{\sharp}_{F_n} \Rightarrow \xi \,. \tag{7.3}$$

An important instance of this concerns symmetric random elements.

Corollary 7.3. *If **(C)** holds and ξ is a proper* SαS *random element without Gaussian component in a normed cone $\mathbb{K}$, then there exists a unique measure λ on $\mathbb{K}^{\sharp}$ (which is then necessarily symmetric and homogeneous of order α) such that*

$$\mathbf{E}\left(\prod_{\rho\in\operatorname{supp}\Pi^{\sharp}} \rho\right)(\chi) = \mathbf{E}\,\chi(\xi)\,, \quad \chi\in\tilde{\mathbb{K}}\,, \tag{7.4}$$

where $\Pi^{\sharp}$ is the Poisson process on $\mathbb{K}^{\sharp}$ with intensity measure λ. The left-hand side of (7.4) is defined as the principal value, i. e. as the limit of $\mathbf{E}\,\xi^{\sharp}_{F_n}(\chi)$ as $F_n\uparrow\mathbb{K}^{\sharp}\setminus\{\mathbf{1}\}$, where $\{F_n,\, n\geq 1\}$ are involution symmetric sets.

7.2 Support of the Lévy measure

Section 7.1 establishes that, under rather weak condition **(C)**, each proper StαS random element admits the LePage representation on the second dual semigroup $\mathbb{K}^{\sharp}$. The crucial further issue is to identify the elements of the second dual semigroup $\mathbb{K}^{\sharp}$ with elements of the original semigroup $\mathbb{K}$. However, this is not feasible in general, since (6.5) involves integration over *all* characters on $\tilde{\mathbb{K}}$, while it is seldom possible to describe all characters, even on the real line with the conventional addition.

If $\mathbb{K}$ is a locally compact group, then the integral in (6.5) can be taken over all continuous characters on $\mathbb{K}$, see [41]. The celebrated Pontryagin reflexivity property of locally compact groups, see [22, Th. V.24.8] establishes that if $\mathbb{K}^{\sharp}$ is the family of all continuous characters on the family of continuous characters $\tilde{\mathbb{K}}$ on a locally compact group $\mathbb{K}$, then $\mathbb{K}$ and $\mathbb{K}^{\sharp}$ are isomorphic. This means that the Lévy measures of random elements with values in locally compact groups maybe thought of as being supported by $\mathbb{K}$. However, the duality theory for semigroups is much poorer, and the results are available only in some special cases.

As noted in the proof of Theorem 6.1, the Lévy measure λ is the vague limit of the images of $\nu_t = t\mu_{t^{-1}}$ under the evaluation map $\imath : \mathbb{K}\mapsto\mathbb{K}^{\sharp}$ as $t\to\infty$. Although μ_t (the distribution of $t^{1/\alpha}\xi$) is supported by $\mathbb{K}$, the vague limit of the $\imath$-images of ν_t may be supported by the whole $\mathbb{K}^{\sharp}$. The following result establishes a condition under which the Lévy measure λ is supported by $\imath(\mathbb{K})$. In this case we say shortly that the Lévy measure is supported by $\mathbb{K}$ and write Λ for the measure on $\mathbb{K}$ being the inverse image of λ under the evaluation map.

We begin with a result on finiteness of the Lévy measure and related distributional properties of a StαS random elements.

Lemma 7.4. *Assume that Condition **(C)** is satisfied. Fix any set $F\subset\mathbb{K}$ such that*

$$\mathbf{1}\notin\operatorname{cl}(\imath(F))\,, \tag{7.5}$$

where the closure is taken in the topology of pointwise convergence in $\mathbb{K}^\sharp$. If ξ is a proper $\mathbb{K}'$-valued StαS *random element, then its Lévy measure is finite on* $\mathrm{cl}(\imath(F))$ *and there exists a constant $a > 0$ such that*

$$\limsup_{t\to\infty} t\mathbf{P}\{\xi \in rt^{1/\alpha}F\} = ar^{-\alpha}\,, \qquad r > 0\,. \tag{7.6}$$

Proof. The expression under the limit in (7.6) is $\nu_t(rF)$, where ν_t is defined above. By simple change of variable argument, the limit itself is a homogeneous in r function, say, $f(r)$, so that $f(r) = r^{-\alpha}f(1)$. The statement will be proved, if we show that $f(1) = a$ is finite.

The set $\mathrm{cl}(\imath(F))$ is closed and does not contain $\mathbf{1}$, so that it is compact in $\mathbb{K}^\sharp \setminus \{\mathbf{1}\}$. Since the Lévy measure λ is a Radon measure [5, Lemma 4.3.12], it is finite on compact sets, whence $\lambda(\mathrm{cl}(\imath(F))) < \infty$. According to the same reference, λ is the vague limit of the images of ν_t under the natural map $\imath$, thus

$$f(1) = \limsup_{t\to\infty} \nu_t(F) \le \lambda(\mathrm{cl}(\imath(F))) < \infty\,. \qquad \square$$

By Lemma 5.18, properness of ξ required in Lemma 7.4 can be guaranteed by imposing Condition **(E)**.

Condition (7.5) holds, in particular, if $\imath(F)$ is closed and $\mathbf{e} \notin F$. The following lemma describes an important case, when $\imath(F)$ is closed in $\mathbb{K}^\sharp$.

Lemma 7.5. *Condition (7.5) is equivalent to the fact that* $\mathbf{e}$ *does not belong to the $\tilde{\mathbb{K}}$-weak closure of F. In particular, if F is $\tilde{\mathbb{K}}$-weak (sequentially) compact, then $\imath(F)$ is closed in $\mathbb{K}^\sharp$ and (7.5) holds provided* $\mathbf{e} \notin F$.

Proof. The first statement is evident, since $\rho_{x_n}(\chi) \to 1$ for all $\chi \in \tilde{\mathbb{K}}$ is equivalent to $x_n \xrightarrow{w} \mathbf{e}$. Assume that $\rho_{x_n}(\chi) \to \rho(\chi)$ for all $\chi \in \tilde{\mathbb{K}}$, where $\{x_n\} \subset F$. By the compactness condition, $x_{n_k} \xrightarrow{w} x$ for a certain $x \in F$. Thus, $\chi(x_{n_k}) \to \chi(x)$, so that $\rho(\chi) = \chi(x)$ for all χ. $\square$

In the sequel we make use of the following condition:

(S) The neutral element $\mathbf{e}$ does not belong to the $\tilde{\mathbb{K}}$-weak closure of A_1.

In other words, Condition **(S)** means that for any sequence $\{x_n\}$ from A_1 there exists a character χ such that $\chi(x_n)$ does *not* converge to $1 = \chi(\mathbf{e})$. Recall that A_r denote B^r if $\alpha > 0$ and $B_{r^{-1}}$ otherwise. If $\mathbb{K}$ is a Banach space, then **(S)** becomes the Schur property [1, Sec. 16.6], which means that the collections of weakly compact and norm compact subsets coincide.

Theorem 7.6. *Assume that $\mathbb{K}$ is a normed cone with compact unit sphere $\mathbb{S}$ such that Conditions **(C)**, **(E)** and **(S)** hold. Then the Lévy measure of any proper* StαS *random element ξ is supported by $\mathbb{K}$.*

Proof. Fix $\varepsilon > 0$ and $r > 0$. Let $\nu_t|^r$ denote the restriction of ν_t onto A_r. Note that $t\mathbf{P}\{\xi \in rt^{1/\alpha}A_1\} = \nu_t(rA_1)$, where $rA_1 = A_r$ if $\alpha > 0$ and $rA_1 = A_{r^{-1}}$ if $\alpha < 0$. In view of Condition **(S)** and Lemma 7.5, Lemma 7.4 implies that the masses of all $\nu_t|^r$, $t \ge 1$, are totally bounded and there exists $R > 0$ such that $\nu_t(RA_1) \leq \varepsilon$ for all t large enough. Thus, $\{\nu_t|^r,\ t \geq 1\}$ is a tight family of measures. Using the diagonal procedure as in [34, Prop. 5.3.9] we come to the conclusion that there exists a measure Λ on $\mathbb{K}'$ such that $\nu_{t_n}|^r$ weakly converges to $\Lambda|^r$ as $t_n \to \infty$ for any $r > 0$. Thus, the Lévy measure is supported by $\mathbb{K}$, i. e. $\lambda = \Lambda \circ \imath^{-1}$. □

In some other cases where the unit sphere is not compact in $\mathbb{K}$, the following result is useful.

Theorem 7.7. *Assume that **(C)** holds and $\imath(\mathbb{K})$ is closed in the topology of pointwise convergence in $\mathbb{K}^\sharp$. Then the Lévy measure λ of a proper* StαS *random element in $\mathbb{K}$ is supported by $\imath(\mathbb{K})$ and is the $\imath$-image of a homogeneous Borel measure Λ on $\mathbb{K}$.*

Proof. By the Tikhonov theorem, $\mathbb{K}^\sharp$ equipped with the pointwise convergence is a compact space. By the condition, $\imath(\mathbb{K})$ is closed hence compact and the vague convergence definition immediately implies that the limiting measure λ in Theorem 6.1 is supported by $\imath(\mathbb{K})$. □

Corollary 7.8. *Assume that $\mathbb{K}$ is a normed cone such that A_1^c is $\tilde{\mathbb{K}}$-weak relatively compact and the following condition holds:*

(W) *If a sequence $\{x_n,\ n \geq 1\}$ in $\mathbb{K}$ is such that $\chi(x_n)$ converges for all $\chi \in \tilde{\mathbb{K}}$, then* $\sup \|x_n\| < \infty$ *in case $\mathbf{e} = \mathbf{0}$ or* $\liminf \|x_n\| > 0$ *in case $\mathbf{e} \neq \mathbf{0}$.*

*Then $\imath(\mathbb{K})$ is closed in $\mathbb{K}^\sharp$ and under Condition **(C)** the Lévy measure of any proper* StαS *random element is supported by $\mathbb{K}$.*

Proof. By rescaling, A_r^c is $\tilde{\mathbb{K}}$-weak relatively compact for all $r > 0$. If $\chi(x_n) \to \rho(\chi)$ for all $\chi \in \tilde{\mathbb{K}}$, Condition **(W)** yields that $x_n \in A_r^c$ for some $r > 0$ and all $n \geq 1$. The weak relative compactness condition implies that $\rho(\chi) = \chi(x)$ for some $x \in \mathbb{K}$ as in the proof of Lemma 7.5, so that $\imath(\mathbb{K})$ is closed. □

The above result is applicable if $\mathbb{K}$ is a reflexive Banach space, see Example 8.2. Although a general Banach space is also Pontryagin reflexive if the family of continuous functionals is equipped with the compact-open topology [52], we were not able to make use of this fact to show that the Lévy measure is always supported by $\mathbb{K}$.

Remark 7.9. One particular simple instance, when both Theorem 7.6 and Corollary 7.8 apply is when the $\tilde{\mathbb{K}}$-weak convergence is equivalent to the metric convergence in $\mathbb{K}$ and the unit sphere is compact. Compactness of the sphere is an important requirement here as Example 8.17 shows.

7.3 LePage series constructed from the Lévy measure

If the Lévy measure Λ of a proper StαS random element ξ is supported by $\mathbb{K}$, then the corresponding Laplace exponent (6.5) takes the form

$$\varphi(\chi) = i\ell(\chi) + q(\chi) + \int_{\mathbb{K}\setminus\{\mathbf{e}\}} (1 - \chi(x) + iL(\chi, x))\, \Lambda(dx)\,, \quad \chi \in \tilde{\mathbb{K}}\,. \tag{7.7}$$

We slightly abuse the notation here by writing $L(\chi, x)$ instead of $L(\chi, \rho_x)$ for $\rho_x = \imath(x)$. Note that the Lévy measure Λ satisfies

$$\int_{\mathbb{K}\setminus\{\mathbf{e}\}} (1 - \operatorname{Re}\chi(x))\,\Lambda(dx) < \infty \tag{7.8}$$

for all $\chi \in \tilde{\mathbb{K}}$.

In order to be able to consider the Poisson process with intensity measure Λ, we have to ensure that $\Lambda(A_r) < \infty$, i. e. Λ is finite on the sets in $\mathbb{K}'$ which are separated from $\mathbf{e}$ by a positive distance. Condition **(C)** on its own makes it possible to derive from (7.8) that Λ is locally finite on $\mathbb{K} \setminus \{\mathbf{e}\}$. Indeed, if $x \neq \mathbf{e}$, then the neighbourhoods of x and $\mathbf{e}$ can be strictly separated by some character χ. Therefore, $\operatorname{Re}\chi(y) \le 1 - \varepsilon$ for some $\varepsilon > 0$ and all y from a neighbourhood of x. It follows from (7.8) that Λ-measure of this neighbourhood is finite.

However, the local finiteness of Λ alone does not imply the finiteness of $\Lambda(A_r)$ unless $\mathbb{K}$ is locally compact. In general, the finiteness of $\Lambda(A_r)$ follows immediately from Lemma 7.4 if $\mathbf{e}$ does not belong to the $\tilde{\mathbb{K}}$-weak closure of A_r, since $\Lambda(A_r) \le \lambda(\operatorname{cl}(\imath(A_r)))$. In this case Λ is the product of the measure θ_α given by (3.3) with $\alpha \neq 0$ and a finite spectral measure σ on $\mathbb{S}$.

If $\Lambda(A_r)$ is finite, the Poisson process Π on $\mathbb{K}$ with intensity measure Λ a. s. has only a finite number of points in A_r for any $r > 0$. Define

$$\xi^{(r)} = \int_{A_r} x\,\Pi(dx)\,, \quad r > 0\,. \tag{7.9}$$

Similarly to Theorem 7.2, a suitable normalisation needed to ensure the convergence of $\xi^{(r)}$ as $r \downarrow 0$ is provided by integrals of the Lévy function.

Theorem 7.10. *Let ξ be a proper* StαS *random element in a normed cone $\mathbb{K}$ without Gaussian component such that its Lévy measure Λ is supported by $\mathbb{K}$. Assume that **(C)** holds and $\Lambda(A_r) < \infty$ for all $r > 0$.*

(i) If Π is a Poisson process on $\mathbb{K}$ with intensity measure Λ and

$$\gamma_r(\chi) = \exp\left\{-i\left(\ell(\chi) - \int_{A_r} L(\chi, x)\Lambda(dx)\right)\right\}\,, \quad \chi \in \tilde{\mathbb{K}}\,,\ r > 0\,, \tag{7.10}$$

then

$$\gamma_r(\chi)\,\mathbf{E}\,\chi(\xi^{(r)}) \to \mathbf{E}\,\chi(\xi) \quad \text{as} \quad r \downarrow 0\,. \tag{7.11}$$

(ii) If $\tilde{\mathbb{K}}$ consists of continuous characters and, for each $r > 0$, there exists $x_r \in \mathbb{K}$ such that $\gamma_r(\chi) = \chi(x_r)$ for all $\chi \in \tilde{\mathbb{K}}$, and if $x_r + \xi^{(r)}$ converges almost surely as $r \downarrow 0$, then the limit coincides in distribution with ξ.

Proof. The first statement is proved similarly to Theorem 7.2. By (7.11), $\mathbf{E}\,\chi(x_r+\xi^{(r)}) \to \mathbf{E}\,\chi(\xi)$. If $x_r + \xi^{(r)}$ a. s. converges to ζ, then its limit shares the Laplace transform with ξ. □

If $\gamma_r(\chi) = 1$ for all χ (e. g., if ξ is symmetric), then (7.11) implies that

$$\mathbf{E}\,\chi(\xi^{(r)}) \to \mathbf{E}\,\chi(\xi) \quad \text{as } r \downarrow 0\,.$$

If the characters from $\tilde{\mathbb{K}}$ are not necessarily continuous, the convergence of Laplace transforms implies that $\xi^{(r)}$ weakly converges to ξ with respect to $\tilde{\mathbb{K}}$-weak topology, see [53, Prop. IV.3.3]. However, in general this does not suffice to show that $\xi^{(r)}$ weakly converges to ξ.

7.4 LePage series representation on semigroups

The almost sure convergence of $\xi^{(r)}$ from (7.9) to a random element $\tilde{\xi} = \int x\Pi(dx)$ does not necessarily mean that $\tilde{\xi}$ and ξ, from which the $\xi^{(r)}$'s are derived, share the same distribution. Indeed, we only have that $\mathbf{E}\,\chi(\xi^{(r)}) \to \mathbf{E}\,\chi(\xi)$, while $\mathbf{E}\,\chi(\xi^{(r)})$ does not necessarily converge to $\mathbf{E}\,\chi(\tilde{\xi})$ if χ is not continuous. Indeed, discontinuous characters cannot be simply interchanged with infinite sums of elements from $\mathbb{K}$. The following definition singles out characters that are interchangeable with sums of series in $\mathbb{K}$.

Definition 7.11. A character χ is said to be *series continuous* if

$$\chi\Big(\sum_{k=1}^{\infty} x_k\Big) = \prod_{k=1}^{\infty} \chi(x_k) \tag{7.12}$$

for every convergent series $\sum x_k$ of elements from $\mathbb{K}$.

If the characters from $\tilde{\mathbb{K}}$ are series continuous, then Theorem 7.10*(ii)* holds without assuming the continuity of the characters. The following result shows the uniqueness of the ingredients of the LePage series.

Theorem 7.12. *If **(C)** holds with $\tilde{\mathbb{K}}$ that consists of series continuous characters, then any two a. s. convergent (as principal values) LePage series (3.9) with sums having proper distributions are identically distributed if and only if the corresponding parameters α and the spectral measures coincide.*

Proof. The LePage series (3.9) with a proper sum can be written as $\int x\Pi(dx)$ for the Poisson process Π on $\mathbb{K}$ with the intensity measure Λ. If ξ admits two different LePage representations, this means that the Laplace exponent of ξ has two representations with different Lévy measures. The uniqueness of (6.5) however implies that this is impossible. □

Consider relatively simple semigroups that are embeddable in a certain group $\mathbb{G}$ such that $\mathbb{G}$-valued proper StαS random elements admit the LePage series representation. For instance, this is the case for semigroups embeddable as cones in a Banach space $\mathbb{B}$, see Theorem 2.8. Then the following result holds if $\mathbb{B}$ has a separable dual space.

Theorem 7.13. *Let $\mathbb{G}$ be a group such that each proper* StαS *$\mathbb{G}$-valued random element with $\alpha \in (0,1)$ admits the LePage representation. Assume that there exists a countable family of continuous homomorphisms $f_n : \mathbb{G} \to \mathbb{R}$, $n \geq 1$, such that $\mathbb{K} = \{x \in \mathbb{G} : \ f_n(x) \geq 0, n \geq 1\}$. Then each proper* St$\alpha$S *random element in $\mathbb{K}$ with $\alpha \in (0,1)$ admits the LePage series representation (3.9).*

Proof. A StαS random element ξ in $\mathbb{K}$ is also StαS in $\mathbb{G}$. By the condition, ξ can be represented by (3.9) where ε_k are distributed on the unit sphere in $\mathbb{G}$. Now consider any homomorphism $f : \mathbb{G} \to \mathbb{R}$, i. e. a linear continuous functional on $\mathbb{G}$, such that $f(x) \geq 0$ for each $x \in \mathbb{K}$. By applying f to both sides of (3.9), using the fact that $f(\xi)$ is StαS random variable and the uniqueness of the LePage representation (see Theorem 7.12), we obtain that $f(\varepsilon_k) \geq 0$ a. s. Repeating this for a countable family of linear continuous functionals, yields that $\varepsilon_k \in \mathbb{K}$ a. s. □

In general, the main prerequisites for the existence of the LePage representation are the fact that Lévy measures are supported by $\mathbb{K}$ and finite on A_r, the continuity (or series continuity) of the characters and the convergence of the LePage series. The latter is particularly simple to derive if the norm is sub-invariant and $\alpha \in (0,1)$, see Theorem 3.6.

Theorem 7.14. *Assume that $\mathbb{K}$ has a sub-invariant norm and one of the following cases is applicable.*

*(i) Conditions **(C)**, **(E)** and **(S)** hold with $\tilde{\mathbb{K}}$ that consists of series continuous characters and the unit sphere $\mathbb{S}$ is compact in $\mathbb{K}$.*

*(ii) Conditions **(C)** and **(W)** hold with $\tilde{\mathbb{K}}$ that consists of continuous characters and the unit sphere $\mathbb{S}$ is $\tilde{\mathbb{K}}$-weak compact.*

Then every StαS random element in $\mathbb{K}'$ with $\alpha \in (0,1)$ and without Gaussian component can be represented as the LePage series (3.9) with $z = \mathbf{e}$.

Proof. First of all, note that $\mathbf{0} = \mathbf{e}$ by Lemma 2.7 and the weak compactness of the unit sphere implies that the closed unit ball (being A_1^c) is $\tilde{\mathbb{K}}$-weak compact.

Next, continuity of characters in *(ii)* implies that **(E)** holds in this case too. By Lemma 5.18, ξ has a proper distribution. By Theorem 7.6 or Corollary 7.8, the Lévy measure Λ is supported by $\mathbb{K}$. Let us show that also *(ii)* implies **(S)**. Since $\mathbf{0} = \mathbf{e}$, it suffices to consider a sequence $\{x_n\}$ with norm greater than 1. If this sequence $\tilde{\mathbb{K}}$-weakly converges, **(W)** implies that $\sup \|x_n\| = c$ is finite. Thus, the x_n's belong to the set $F = \{x \in \mathbb{K} : 1 \le \|x\| \le c\}$. The $\tilde{\mathbb{K}}$-weak compactness condition on $\mathbb{S}$ and the continuity of the characters implies that F is $\tilde{\mathbb{K}}$-weak compact and so $\tilde{\mathbb{K}}$-weak closed. The separation property implies the uniqueness of the weak limit, so that the weak limit of x_n also belongs to F, i. e. this weak limit is not $\mathbf{e}$. Thus **(S)** holds in the both cases *(i)* and *(ii)*, and Lemma 7.4 implies that $\Lambda(A_r) < \infty$ for all $r > 0$.

Let Π_α be the Poisson point processes with intensity measure Λ. By Theorem 3.6, the sum of its points converges absolutely, which implies that the principal value of the integral $G_{\Pi_\alpha}(\chi) = \int_{\mathbb{K}'}(1 - \chi(x))\Lambda(dx)$ converges. The last integral being (6.15) allows us to set the Lévy function to zero by Theorem 6.7.

Let ξ_α be the sum of the LePage series (3.9) with $z = \mathbf{e}$. By (3.8),

$$\mathbf{E}\,\chi(\xi_\alpha) = \exp\left\{-\int_{\mathbb{K}} (1 - \chi(x))\Lambda(dx)\right\}\,.$$

Writing a part of (6.5) as above, we arrive at

$$\mathbf{E}\,\chi(\xi) = e^{-(i\ell(\chi)+q(\chi))}\,\mathbf{E}\,\chi(\xi_\alpha)\,.$$

It suffices to show that ξ and ξ_α share the same Laplace transform. The quadratic form q vanishes, since ξ does not have a Gaussian component by the imposed condition. The function ℓ clearly vanishes in case of either the identical involution, or symmetric ξ or if the second distributivity law holds. Now we show that this also holds in general under the sub-invariance assumption for $\alpha \in (0,1)$.

Note that ℓ is the only imaginary part of the Laplace exponent of ξ. By applying χ to the both sides of (1.1) we obtain that

$$\mathbf{E}(a^{1/\alpha} \circ \chi)(\xi)\,\mathbf{E}(b^{1/\alpha} \circ \chi)(\xi) = \mathbf{E}((a+b)^{1/\alpha} \circ \chi)(\xi)\,,$$

whence, using the additivity property of ℓ we have for $a = b = 1/2$ that

$$\ell((2^{-1/\alpha} \circ \chi)(2^{-1/\alpha} \circ \chi) = \ell(\chi)\,.$$

This equality can be written shorter as $\ell(f(\chi)) = \ell(\chi)$, where $f(\chi)$ is a character that acts on $x \in \mathbb{K}$ as $(f(\chi))(x) = \chi(2^{-1/\alpha}(x+x))$. By iterating we obtain that

$$\ell(f^k(\chi)) = \ell(\chi)\,, \tag{7.13}$$

where $(f^k(\chi))(x) = 2^{-k/\alpha} S_{2^k})$ and S_n is the sum of n identical summands being x. By the sub-invariance property, $n^{-1}S_n$ has a norm bounded by $\|x\|$ which is finite for all x in a normed cone. Since $\alpha < 1$, we have $2^{-k/\alpha} S_{2^k} \to \mathbf{0} = \mathbf{e}$. Being the only imaginary part of the sequentially continuous Laplace exponent, the function ℓ is also sequentially continuous with respect to pointwise convergence of its argument. Since $(f^k(\chi))(x) \to \chi(\mathbf{e}) = 1$ for all x and **(E)** holds, $f^k(\chi)$ pointwisely converges to $\mathbf{1}$. By passing to the limit in (7.13) and noticing that $\ell(\mathbf{1}) = 0$, we obtain that $\ell(\chi) = 0$ for all $\chi \in \tilde{\mathbb{K}}$. ☐

If α does not belong to $(0,1)$ or if $\mathbb{K}$ does not have a sub-invariant norm, an analogue of Theorem 7.10 holds assuming that the LePage series converges almost surely. In a special case of semigroups with identical involution, we arrive at the following result.

Theorem 7.15. *Assume that $\mathbb{K}$ is a normed cone with identical involution such that conditions (i) or (ii) of Theorem 7.14 hold. Then every* StαS *random element in $\mathbb{K}'$ without Gaussian component admits the LePage representation* (3.11) *with $z = \mathbf{e}$, provided the principal value of the integral exists almost surely.*

If $\mathbb{K}$ is idempotent, then we do not have to require that ξ does not have a Gaussian component, c. f. Lemma 6.6.

8 Examples

8.1 Cones with the second distributivity law

These examples are the closest to "conventional" stable distributions on the line or in Banach spaces. The cone $\mathbb{K}$ is typically a group or is embeddable in a group. In particular, the second distributivity law implies that $\mathbf{0} = \mathbf{e}$ and $\mathbb{K}(\alpha) = \{\mathbf{e}\}$ for $\alpha \neq 1$. If **(C)** holds, then $\alpha \in (0,2]$ and the order of homogeneity of the Lévy measure is strictly less than 2 (respectively, 1 if the involution is identical), see Corollary 6.3. By Theorem 6.5, a proper StαS random element does not have a Gaussian component if $\alpha \neq 1, 2$ or $\alpha \neq 2$ if $\mathbb{K}$ is a group.

Example 8.1 (Positive half-line). Consider $\mathbb{R}_+$ with the arithmetic addition, Euclidean metric and identical involution. A separating family of continuous characters is given by $\chi_t(x) = e^{-tx}$

for $t \geq 0$. By Corollary 6.3, the Lévy measures are homogeneous with the order strictly less than 1. The only StαS laws with $\alpha = 1$ are deterministic distributions. Note that

$$\varphi(\chi) = -\log \mathbf{E}\, e^{-t\xi} = ct^{\alpha}$$

is a well-known representation of the Laplace transform of a strictly stable law on $\mathbb{R}_+$. The existence of the LePage representation follows from Theorem 7.14*(i)*.

Example 8.2 (Banach spaces). Let $\mathbb{K}$ be a Banach space. A separating semigroup $\tilde{\mathbb{K}}$ of continuous characters consists of $\chi_u(\cdot) = e^{iu(\cdot)}$ for all linear continuous functionals u. Since the norm is clearly invariant, the LePage series converges absolutely for $\alpha \in (0,1)$. Theorem 4.7 turns into [2, Th. 7.11].

The separation property holds by the Hahn-Banach theorem, so that **(C)** is satisfied. The $\tilde{\mathbb{K}}$-weak convergence is the conventional weak convergence in Banach spaces. The strong convergence implies the weak convergence, so that **(E)** holds, hence $\alpha \in (0,2]$. If the space is reflexive, then the unit sphere is weak sequentially compact, see, e. g., [14, Th. II.3.28]. Furthermore, the weak convergence implies the strong boundedness, so that **(W)** holds and by Corollary 7.8, the Lévy measure is supported by $\mathbb{K}$. Every StαS random element with $\alpha \in (0,1)$ admits the LePage representation by Theorem 7.14*(ii)*. For $1 \leq \alpha \leq 2$, one should use the methods that rely on symmetrisation arguments, see [34], e. g., to check the tightness of $x_r + \xi^{(r)}$ needed in Theorem 7.10*(ii)*.

Theorem 7.13 can be applied to any cone which is embeddable in a Banach space with separable dual, e. g., to the cone of non-negative continuous functions with addition operation.

Example 8.3 (Compact convex sets with Minkowski addition). Let $\mathbb{K}$ be the family $\operatorname{co}\mathcal{K}$ of nonempty compact convex sets K in $\mathbb{R}^d$ with the semigroup operation being the Minkowski (elementwise) addition. Note that $\mathbb{K}$ is not a group, since the Minkowski addition is not invertible. The involution corresponds to the symmetry with respect to the origin. The Hausdorff metric turns $\mathbb{K}$ into a normed cone with $\mathbf{e} = \mathbf{0} = \{0\}$ and the norm defined as $\|K\| = \sup\{\|x\| :\ x \in K\}$. Since the Hausdorff metric is invariant, i. e. the distance between $K_1 + L$ and $K_2 + L$ coincides with the distance between K_1 and K_2, the corresponding norm is clearly sub-invariant. The unit sphere in $\mathbb{K}$ is $\mathbb{S} = \{K \in \operatorname{co}\mathcal{K} :\ \|K\| = 1\}$; it is compact in the Hausdorff metric.

Before describing the characters on $\operatorname{co}\mathcal{K}$, consider the appropriate sub-family of "centred" sets. With every convex compact set K it is possible to associate its Steiner point

$$s(K) = \frac{1}{\kappa_d} \int_{S^{d-1}} h(K,u) u \, du \,,$$

where κ_d is the volume of the unit ball in $\mathbb{R}^d$, the integral is taken with respect to the $(d-1)$-dimensional Hausdorff measure on the unit sphere S^{d-1} in $\mathbb{R}^d$ and $h(K,u)$ is the support function of $K \in \operatorname{co}\mathcal{K}$, i. e. the supremum of the scalar product $\langle u, x\rangle$ over $x \in K$ for $u \in S^{d-1}$. It is known that the Steiner point is linear with respect to Minkowski addition and that $s(K) \in K$, see [50, p. 42]. Let $\operatorname{co}\mathcal{K}_0$ be the family of convex compact sets with their Steiner points located at the origin. This family can be considered as a convex cone with the identical involution and the Hausdorff metric. Condition **(C)** holds with a separating family of continuous characters given by

$$\chi_\nu(K) = \exp\Big\{-\int_{S^{d-1}} h(K,u)\, \nu(du)\Big\} \,, \tag{8.1}$$

where ν is a finite measure on S^{d-1}. In fact, it suffices to consider only counting measures ν. Note that it does not suffice to consider only the characters of the type $e^{-h(K,u)}$, since they do not form a closed family with respect to multiplication.

A random element X in $\mathbb{K} = \mathrm{co}\,\mathcal{K}$ is a *random compact set*, see, e. g., [38, Sec. 1.5]. All non-trivially equal to $\{0\}$ random compact sets are proper. By Theorem 5.20, a proper StαS random element in $\mathrm{co}\,\mathcal{K}_0$ necessarily has $\alpha \in (0,1]$. Otherwise X is not proper, i. e. $X = \{0\}$. If $\alpha = 1$, then by Corollary 6.4 the corresponding Laplace exponent is a linear continuous function of χ and the random set X is deterministic. A conventional argument for this relies on the fact that the support function in every direction is a non-negative StαS random variable, which is necessarily degenerated for $\alpha \geq 1$, see [17]. Similar arguments are applicable to the cone of convex compact sets that contain the origin (not necessarily having the Steiner point at the origin) and are equipped with the identical involution. Theorem 7.14*(i)* applies, implying the LePage representation of StαS random elements.

Now return to the cone of $\mathbb{K} = \mathrm{co}\,\mathcal{K}$ of all convex compact sets. A convex compact set K can be decomposed as $K = s(K) + K_0$. Therefore, the cone of compact convex sets can be decomposed into the sum of two cones: $\mathbb{R}^d$ with the conventional addition and the cone $\mathrm{co}\,\mathcal{K}_0$ of convex sets with the Steiner point at the origin. The first has the complex-valued characters, while the second has the identical involution and $[0,1]$-valued characters given by (8.1). By combining the two families of characters we obtain the separating family of continuous characters given by

$$\chi_{v,\nu}(K) = e^{i\langle v, s(K)\rangle} \exp\Big\{ - \int_{S^{d-1}} h(K - s(K), u)\, \nu(du)\Big\}\,, \tag{8.2}$$

where $v \in \mathbb{R}^d$. If X is a StαS random convex compact set, then $s(K)$ is a StαS random vector, so that $\alpha \in (0,2]$. Furthermore, $X - s(X)$ is StαS in $\mathrm{co}\,\mathcal{K}_0$, whence $\alpha \in (0,1]$. Therefore, a stable random set with $\alpha \in [1,2]$ is a sum of an StαS random vector ξ in $\mathbb{R}^d$ and a deterministic convex compact set, which is necessarily $\{0\}$ in case $\alpha > 1$, i. e. $X = \{\xi\}$. If the stability definition (1.1) is weakened by allowing an additive normalisation, then this deterministic part may be any convex compact set, see [17].

Theorem 5.16 implies that the Laplace exponent $\varphi(v,\nu) = \mathbf{E}\,\chi_{v,\nu}(X)$ of a StαS random set X satisfies

$$\varphi(tv, t\nu) = t^{\alpha}\varphi(v,\nu)$$

for all $t > 0$, v and ν. Since the unit sphere $\mathbb{S}$ is compact and **(S)** holds, every StαS random compact set with $\alpha \in (0,1)$ admits the absolutely convergent LePage series representation.

Not necessarily strictly stable random compact convex sets for any $\alpha \in (0,2]$ in a separable Banach space have been studied in [17]. It is shown in [17, Th. 1.14] that α-stable random compact convex sets with $\alpha \in (0,1)$ can be represented as a stochastic integral over $\mathbb{S}$ with respect to a certain independently scattered α-stable random measure. This representation coincides with the LePage representation in $\mathbb{K}$.

It should be noted that general semigroups representable as systems of compact convex sets have been characterised in [43].

Example 8.4 (Upper semicontinuous functions). Let $\mathbb{K}$ be the family of upper semicontinuous functions $u : \mathbb{R}^d \mapsto [0,1]$ such that $\cup_{t>0} F_t(u)$ is relatively compact where $F_t(u) = \{x :\ u(x) \geq t\}$. The metric between u and v is defined as the supremum of the Hausdorff distances between $F_t(u)$ and $F_t(v)$ over $t \in (0,1]$. The sum of u and v is defined to be the upper semicontinuous function

w such that $F_t(w)$ equals the Minkowski sum of $F_t(u)$ and $F_t(v)$ for all $t \in (0,1]$. This setting is similar to Example 8.3 reformulated for increasing set-valued function indexed by $[0,1]$, see also [40]. Accordingly, the conclusions of Example 8.3 can be transferred to this case practically without changes.

Example 8.5 (Finite random measures). Let $\mathbb{K}$ be the family of all finite measures m on a locally compact topological space E with a countable base. The operations are the conventional addition of measures and the multiplication of their values by numbers, while the involution is identical. The Prohorov metric (see, e. g., [42]) on $\mathbb{K}$ is sub-invariant and homogeneous at the origin. The corresponding norm $\|m\|$ is the total mass of m. The neutral element and the origin are both equal to the null-measure. A separating family of characters is given by

$$\chi_u(m) = \exp\Big\{-\int u\,dm\Big\} \tag{8.3}$$

for any continuous bounded function $u : E \mapsto \mathbb{R}_+$. Since the Prohorov metric metricises the weak convergence, these characters are continuous and the $\tilde{\mathbb{K}}$-weak convergence is equivalent to the metric convergence. The Laplace transform $\mathbf{E}\,\chi_u(\mu)$ is called the Laplace functional of the random measure μ, see, e. g., [10, (6.4.16)]. The corresponding Laplace exponent is given by

$$\varphi(u) = -\log\Big(\mathbf{E}\exp\Big\{-\int u\,d\mu\Big\}\Big)\,. \tag{8.4}$$

All non-trivial random measures have proper distributions. The random measure μ is StαS if

$$a^{1/\alpha}\mu_1(K) + b^{1/\alpha}\mu_2(K) \overset{\mathcal{D}}{=} (a+b)^{1/\alpha}\mu(K) \tag{8.5}$$

for all measurable K. By Theorem 5.20, $\alpha \in (0,1]$, which is also due to the fact that $\mu(K)$ is a non-negative StαS random variable, c. f. [49]. Note that $c \circ \chi_u$ corresponds to the character χ_{cu}. By Theorem 5.16, μ is α-stable with some $\alpha \in (0,1]$ if and only if its Laplace exponent $\varphi(u)$ satisfies $\varphi(su) = s^\alpha\varphi(u)$ for all u and $s > 0$. The integral representation (6.6) corresponds to the representation for the Laplace functional of infinitely divisible random measures, see [10, Prop. 9.2.VII]. The LePage series involves random measures distributed on the unit sphere $\mathbb{S}$ in $\mathbb{K}$, which is the family of all probability measures on E. Thus, a StαS measure is the weighted sum of i. i. d. random probability measures with weights $\Gamma_k^{-1/\alpha}$, $k \geq 1$. This representation is a particular case of [49, Th. 3.9.1] for strictly α-stable random measures.

If E is a compact space, then the unit sphere $\mathbb{S}$ is compact, so that each α-stable random measure with $\alpha \in (0,1)$ admits the LePage representation by Theorem 7.14*(i)*.

Example 8.6 (Locally finite random measures). Let $\mathbb{K}$ be the family of locally finite measures on $\mathbb{R}^d$ with the topology of vague convergence and the same operations as in Example 8.5. This space is important in the studies of point processes without accumulation points.

A separating family of continuous characters is given by (8.3), but with u being a measurable bounded function with bounded support. Conditions **(C)** and **(E)** hold, and all non-trivial random measures are proper. Any StαS random measure μ has the Laplace exponent that is homogeneous, i. e. $\varphi(su) = s^\alpha\varphi(u)$, where $\alpha \in (0,1]$ by Theorem 5.16.

The extension of the Prohorov metric typically used to metricise the vague convergence of locally finite measures (see, e. g., [10, Sec. A2.6] or [42, Sec. 10.2]) is not homogeneous, since it is constructed using the sums of the type $2^{-i}d_i/(1+d_i)$, where d_i is a certain distance between the restrictions of the measures onto the balls B_i, $i \geq 1$.

The unit sphere and the LePage series can be constructed for a sub-family of $\mathbb{K}$ that consists of locally finite measures m with a certain growth restriction at the infinity, e. g., those which satisfy

$$\int_0^\infty e^{-r} m(B_r) dr < \infty \,.$$

The value of this integral may serve as a norm of m with the corresponding sub-invariant metric given by the integral of the Prohorov distance between the measures restricted on B_r. In particular, scale the norm if necessary so that atomic measures δ_x belongs to the unit sphere $\mathbb{S}$ in $\mathbb{K}$ for all x from the unit sphere S^{d-1} in $\mathbb{R}^d$. Then the LePage series is $\mu_\alpha = \sum_k \Gamma_k^{-1/\alpha} \delta_{\varepsilon_k}$, where $\{\varepsilon_k,\, k \geq 1\}$ are i. i. d. random variables on S^{d-1}. Note that this representation does not coincide with the representation of a stable Poisson process Π_α from Theorem 3.3. The random measure μ_α is supported by a subset of the unit sphere S^{d-1}. It does not take integer values and has the total mass being the StαS random variable $\xi_\alpha = \sum_k \Gamma_k^{-1/\alpha}$ in $(\mathbb{R}_+, +)$.

8.2 Cones without the second distributivity law

These examples typically appear if $\mathbb{K}$ is not a group, but only a semigroup. In these cases one might have positive or negative stability parameter α unless an element of $\mathbb{K}$ possesses a non-trivial inverse, which implies, given **(C)** and **(E)**, that α is necessarily positive, see Theorem 5.19.

Example 8.7 (Positive cone in Euclidean space with coordinatewise maximum). For some $d \geq 1$, let $\mathbb{K}$ be $[0, \infty)^d$ with the coordinatewise maximum operation, i. e. $x \vee y = (x_1 \vee y_1, \dots, x_d \vee y_d)$, and the conventional multiplication by numbers. The Euclidean metric is sub-invariant. Hence in this case $\mathbf{0} = \mathbf{e} = 0$ and $\mathbb{K}(\alpha) = \{0\}$ for all $\alpha \neq 0$. The corresponding StαS laws are called *max-stable*, see [16; 32].

The semigroup $\mathbb{K}$ has the identical involution and moreover is idempotent, so that the characters take only values 0 or 1, and the Gaussian component is degenerated by Lemma 6.6. A separating family of series continuous (and also upper semicontinuous) characters is given by $\chi_z(x) = \mathbf{1}_{[0,z]}(x)$ for $z = (z_1, \dots, z_d) \in (0, \infty)^d$, where $[0, z] = [0, z_1] \times \cdots \times [0, z_d]$. If z has rational coordinates, these characters form a countable strictly separating family such that **(C)** and **(E)** hold. The multiplication operation acts on characters as $a \circ \mathbf{1}_{[0,z]} = \mathbf{1}_{[0,z/a]}$. The Laplace transform of ξ is then

$$\mathbf{E}\, \mathbf{1}_{[0,z]}(\xi) = \mathbf{P}\{\xi \leq z\} \,,$$

hence any non-trivial ξ with the support not separated from zero is proper. By Theorem 5.19, the corresponding parameter α is always positive.

The unit sphere is compact and **(S)** holds. The Lévy function and ℓ vanish, so that $\gamma_r = 1$, see (7.10). By Theorem 7.14*(i)*, each StαS random element with $\alpha \in (0, 1)$ admits the LePage representation. The LePage series absolutely converges a. s. for all $\alpha > 0$ and yields a max-stable random element. The corresponding representation of a max-stable law as the maximum of Poisson points is well known in the theory of multivariate extremes, see [44, Ch. 5].

Example 8.8 (Half-line with minimum operation). Consider the extended half-line $\mathbb{K} = [0, \infty]$ with the minimum operation and the conventional multiplication by numbers, so that $\mathbb{K} \setminus \{\mathbf{e}\} =$

$[0,\infty)$ with the Euclidean metric becomes a normed cone and $\mathbf{e}=\infty$. The origin $\mathbf{0}$ is the conventional zero 0. The upper semicontinuous characters are given by $\mathbf{1}_{[z,\infty]}(x)$ for $z\in[0,\infty]$. They are also series continuous and form a strictly separating family, so that Theorem 5.16 applies. The StαS random elements necessarily have $\alpha<0$ and the corresponding stable Poisson process Π_α has realisations from $\mathcal{M}$. The LePage series $\min_{k\geq 1}(\Gamma_k^{-1/\alpha}\varepsilon_k)$ with $\varepsilon_k=1$ evidently converges to $\Gamma_1^{-1/\alpha}$, so that each StαS random element has the LePage representation by Theorem 7.15.

In the same way it is possible to handle the coordinatewise minimum operation that leads to multivariate min-stable laws, c. f. Example 8.7.

Example 8.9 (Max-plus calculus). Let $\mathbb{K}$ be $[-\infty,\infty)$ with maximum as the semigroup operation and the multiplication by scalars $D_a x = x+\log a$. This object was studied under the name of a *max-plus* algebra in [3] with applications to modelling of queueing systems and as *idempotent calculus* in [35] and [13]. In both cases the conventional sum plays the role of the product in this algebra.

A homogeneous norm is defined by $\|x\|=e^x$, so that the unit sphere $\mathbb{S}$ is $\{0\}$. The corresponding StαS random variables satisfy $\max(\xi_1,\xi_2)\overset{\mathcal{D}}{=}\xi+b$ for some $b\in\mathbb{R}$, i.e. they are max-stable with respect to the shift. It is well known that such ξ has the Gumbel (or double exponential) distribution with the c.d.f. $\mathbf{P}\{\xi\leq x\}=\exp\{-\exp\{-(x-c)/\gamma\}\}$, see, e.g., [44]. The separating family of series continuous (and upper semicontinuous) characters consists of $\chi_z(x)=\mathbf{1}_{x\leq z}$ for $z\in[-\infty,\infty]$. In this case both the neutral element and the origin are $-\infty$, so that $\alpha>0$ for all StαS random variables. Since $\max(\xi_1,\xi_2)\overset{\mathcal{D}}{=}\xi+\frac{1}{\alpha}\log 2$, the value α^{-1} is the scaling parameter γ of the corresponding double exponential distribution.

The LePage series

$$\xi=\max_{k\geq 1}(-\frac{1}{\alpha}\log\Gamma_k)$$

converges for every $\alpha>0$ to the value $-\alpha^{-1}\log\Gamma_1$. The scheme of this example can be generalised to include random vectors with coordinatewise maximum or to random functions with pointwise maximum. In the latter case, the LePage series

$$\xi=\max_{k\geq 1}(\varepsilon_k-\frac{1}{\alpha}\log\Gamma_k)$$

is defined by i.i.d. $\{\varepsilon_k\}$ from the corresponding unit sphere, so that it coincides with the representation of max-stable processes by convolutions, see [15, p. 229].

Example 8.10 (Union-stable random compact sets). Let $\mathbb{K}$ be the family $\mathcal{K}$ of compact subsets $K\subset\mathbb{R}^d$ with the Hausdorff metric and the union operation that turns $\mathbb{K}$ into an idempotent semigroup. The multiplication by numbers is defined as the corresponding homothetical transformation of sets. The neutral element of $\mathbb{K}$ is the empty set, while $\mathbf{0}$ is the origin $\{0\}$. Note that the norm of the sum is always larger than the norm of each summand, so that every non-trivial StαS law should have $\alpha>0$ by Lemma 5.22.

A separating family $\tilde{\mathbb{K}}$ of series continuous characters consists of $\chi_G(K)=\mathbf{1}_{G\cap K=\emptyset}$ for all open sets G. It is possible to extract its countable separating sub-family by considering G that are unions of finite number of open balls with rational centres and radii, i. e. from the so-called separating class [38, Sec. 1.4]. The product of two characters χ_G and $\chi_{G'}$ is the character $\chi_{G\cup G'}$, so that these characters indeed build a semigroup $\tilde{\mathbb{K}}$.

The Laplace transform of a random compact set X is given by

$$\mathbf{E}\,\chi_G(X) = \mathbf{P}\{X \cap G = \emptyset\}\,,$$

which is usually called *the avoidance functional* of X and denoted by $Q_X(G)$. If X is not proper, then $Q_X(G) = 0$ for some G, i.e. X almost surely hits G. This implies that X has a fixed point, i.e. there exists $x \in \mathbb{R}^d$ such that $x \in X$ a.s., see [38, Lemma 4.1.8]. Thus proper random elements are exactly those that do not possess fixed points. Conditions **(C)** and **(E)** hold. By Theorem 5.16, the union-stable random compact sets can be characterised as those having homogeneous Laplace exponents $\varphi(G) = -\log Q_X(G)$, i.e. $\varphi(sG) = s^\alpha \varphi(G)$. This characterisation has been obtained in [38, Th 4.1.12] using direct proofs, but in a more general case for not necessarily compact random closed sets that possibly possess fixed points, i.e. with not necessarily proper distributions.

Since $\mathbf{e} \neq \mathbf{0}$, Theorem 5.19 implies that $\alpha < 0$, contrary to the above mentioned conclusion of Lemma 5.22. Thus, this cone does not possess any non-trivial *proper* StαS law. Indeed, the corresponding LePage series would involve the union of random compact sets scaled by arbitrarily large factors $\Gamma_k^{-1/\alpha}$, so that this union is no longer compact if $\alpha < 0$ and this union contains the origin (and thereupon is not proper) if $\alpha > 0$.

Example 8.11 (Union-stable random compact sets containing the origin). Let $\mathbb{K}$ be the cone of compact sets in $\mathbb{R}^d$ that contain the origin. We keep the same cone operations and the metric as in Example 8.10. However in this case both the neutral element and the origin are $\{0\}$, so that $\alpha > 0$ by Theorem 5.19 and also by Lemma 5.22. Since all elements of $\mathbb{K}$ contain the origin, the characters are given by $\mathbf{1}_{G\cap X=\emptyset}$ indexed by open sets G that do not contain the origin.

The Hausdorff metric on $\mathcal{K}$ is sub-invariant with the unit sphere $\mathbb{S} = \{K \in \mathbb{K} :\ \|K\| = 1\}$. The LePage series is

$$X = \bigcup_{k\geq 1} \Gamma_k^{-1/\alpha} X_k\,, \tag{8.6}$$

where $\{X_k,\, k \geq 1\}$ is any sequence of i.i.d. random compact sets from $\mathbb{S}$. This series converges for all $\alpha > 0$, although its convergence for $\alpha \geq 1$ does not follow from Theorem 3.6, but is easily seen by the direct proof. If $X_k = \{0, \varepsilon_k\}$ for i.i.d. ε_k distributed on the unit sphere in $\mathbb{R}^d$, then X is the support of the stable Poisson process Π_α with $\alpha > 0$.

Theorem 7.6 is applicable, so that the Lévy measures are supported by $\mathbb{K}$ and every proper StαS random element admits the LePage representation for each $\alpha > 0$, see Theorem 7.15. Note that the Gaussian component is always degenerated by Lemma 6.6.

Example 8.12 (Continuous functions with addition and argument rescaling). Let $\mathbb{K}$ be the family of continuous functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ such that $f(0) = 0$ and $\|x\|^{-1}|f(x)|$ is bounded in $x \in \mathbb{R}^d \setminus \{0\}$. The cone operations are the arithmetic addition and the rescaling of the argument, i.e. $(D_a f)(x) = f(ax)$ for all x. Define invariant and homogeneous metric by

$$d(f_1, f_2) = \sup_{x\in\mathbb{R}^d\setminus\{0\}} \|x\|^{-1}|f_1(x) - f_2(x)|\,. \tag{8.7}$$

Then $\|f\|$ is the supremum of $\|x\|^{-1}|f(x)|$, $x \neq 0$. The neutral element is the zero function, while the imposed condition $\|f\| < \infty$ implies that $\mathbf{0} = \mathbf{e}$. The involution is given by $f^\star = -f$. It is interesting to note that $\mathbb{K}$ is a group, where nonetheless the second distributivity law does not hold. Since the metric is invariant, the LePage series converges absolutely for $\alpha \in (0,1)$.

Define continuous characters on $\mathbb{K}$ as

$$\chi_\nu(f) = \exp\Big\{i \int f d\nu\Big\}\,, \tag{8.8}$$

where ν is a signed measure on $\mathbb{R}^d$ with bounded support and finite total variation. Their continuity follows from the fact that the convergence in norm implies the pointwise convergence, while the pointwise convergence is equivalent to the $\tilde{\mathbb{K}}$-weak convergence. Then $c \circ \chi_\nu = \chi_{c\circ\nu}$, where $(c \circ \nu)(A) = \nu(c^{-1}A)$ for measurable A. The Laplace exponent φ of a $\mathbb{K}$-valued random element ξ is given by $\mathbf{E}\,\chi_\nu(\xi) = e^{-\varphi(\nu)}$. Conditions **(C)** and **(E)** hold, so that Theorem 5.16 yields that StαS elements in $\mathbb{K}'$ are characterised by having homogeneous Laplace exponents, i. e. $\varphi(c \circ \nu) = c^\alpha \varphi(\nu)$, where $\alpha > 0$ by Theorem 5.19.

The unit ball in $\mathbb{K}$ is $\tilde{\mathbb{K}}$-weak compact, since $\|x\|^{-1} f_n(x) \le 1$, $n \ge 1$, implies that $\|x\|^{-1} f_n(x)$ has a pointwisely convergent subsequence, whence f_n has a pointwise convergent subsequence too. However Condition **(W)** does not hold. Neither is the unit sphere compact in the metric topology on $\mathbb{K}$. Thus the available results of Section 7.2 do not lead to the conclusion that the Lévy measure is supported by $\mathbb{K}$.

Example 8.13 (Integrable random probability measures with convolution operation). Let $\mathbb{K}$ be the family of probability measures m on $\mathbb{R}$ with finite first moment, the addition operation being the convolution of measures (denoted by $*$) and the multiplication by scalars given by $(D_a m)(K) = m(a^{-1}K)$. Assume that the corresponding random variables are defined on a non-atomic probability space. If ξ_1 and ξ_2 are independent random variables with distributions m_1 and m_2 respectively, then $(D_{c_1} m_1) * (D_{c_2} m_2)$ is the distribution of $c_1\xi_1 + c_2\xi_2$. The unit measure δ_0 concentrated at 0 is both the neutral element of $\mathbb{K}$ and the origin. The second distributivity law does not hold in this case, since the convolution of two measures (i. e. the distribution of the sum of two i. i. d. random variables) does not generally equal the rescaled measure (i. e. the distribution of the rescaled random variable). The involution corresponds to the central symmetry, i. e. $m^\star$ is the distribution of $-\xi$ if m is the distribution of ξ.

Note that $\mathbb{K}(\alpha)$ are all non-trivial for $1 < \alpha \le 2$ consisting of StαS probability distributions in $\mathbb{R}$. The integrability condition (that has not been used so far) implies the absence of non-trivial elements of $\mathbb{K}(\alpha)$ for $\alpha \in (0, 1]$. Indeed, the corresponding StαS laws are non-integrable, c. f. Example 8.24.

The problem of defining a sub-invariant norm on the family of probability measures can be easily solved using the tools from the theory of probability metrics, see [42; 55]. Recall that a probability metric d defines a distance between probability measures m_1 and m_2 or between the corresponding random variables ξ and η. The metric is called *simple*, if it does not take into account the joint distribution of these random variables. A metric d is called r-homogeneous if $d(c\xi, c\eta) = |c|^r d(\xi, \eta)$ for all $c \ne 0$; this metric is said to be regular if $d(\xi + \zeta, \eta + \zeta) \le d(\xi, \eta)$ for any random variable ζ that is independent of both ξ and η. The regularity property written using the addition in $\mathbb{K}$ (i. e. the convolution) turns into (2.9) meaning that the simple regular metric is sub-invariant. In the theory of probability metrics, an r-homogeneous and regular simple metric is called *ideal* of order r.

An ideal probability metric of order 1 between distributions of random variables ξ and η can be defined as

$$d(\xi, \eta) = \sup\{|\,\mathbf{E}(f(\xi) - f(\eta))| :\ f \in \mathrm{Lip}_1\}\,,$$

where Lip_1 is the family of functions with Lipschitz constant 1, i. e. $|f(x) - f(y)| \le |x - y|$ for all x and y. This is the so-called *Kantorovich metric* or *Wasserstein metric*, see, e. g., [55, Th. 1.3.1]. Note also that

$$d(\xi,\eta) = \int_{-\infty}^{\infty} |\mathbf{P}\{\xi \le x\} - \mathbf{P}\{\eta \le x\}| dx\,,$$

so that the corresponding norm of ξ (or of the measure m) is given by

$$\|\xi\| = \|m\| = \int_{-\infty}^{0} \mathbf{P}\{\xi \le x\} dx + \int_{0}^{\infty} \mathbf{P}\{\xi > x\} dx = \mathbf{E}\,|\xi|\,.$$

The norm of any integrable measure is clearly finite. The Kantorovich metric is complete, see [42, Th. 6.3.3]. Furthermore $d(\xi_k,\xi) \to 0$ if and only if $\xi_k \Rightarrow \xi$ weakly and $\mathbf{E}\,|\xi_k| \to \mathbf{E}\,|\xi|$, see [27, Th. 3.9.4].

Characters on $\mathbb{K}$ are the characteristic functions, i. e.

$$\chi_u(m) = \int e^{iux} m(dx)\,, \quad u \in \mathbb{R}\,.$$

They form a separating family of continuous characters with values in the unit complex disk $\mathbb{D}$, but they do not build a semigroup, since their products are not necessarily of the same type. However, if one defines

$$\chi_{u_1,\dots,u_k}(m) = \prod_{i=1}^{k} \chi_{u_i}(m) \tag{8.9}$$

for any finite set $\{u_1,\dots,u_k\} \subset \mathbb{R}$, $k \ge 1$, then this family does constitute a semigroup. Note that $a \circ \chi_{u_1,\dots,u_k} = \chi_{au_1,\dots,au_k}$. Conditions **(C)** and **(E)** hold, since the characters are continuous. The Laplace transform of a random measure μ from $\mathbb{K}$ is given by

$$\mathbf{E}\,\chi_{u_1,\dots,u_k}(\mu) = \int \cdots \int e^{i(u_1x_1+\cdots+u_kx_k)} \overline{\mu}_k(dx_1 \times \cdots \times dx_k)\,,$$

where $\overline{\mu}_k$ is the kth order moment measure of the random measure μ, see, e. g., [10; 37]. In other words, the Laplace exponent $\varphi(u_1,\dots,u_k) = -\log(\mathbf{E}\,\chi_{u_1,\dots,u_k}(\mu))$ is the log-characteristic function of the kth moment measure $\overline{\mu}_k$. A random probability measure μ is StαS if

$$\big((a^{1/\alpha}\mu_1) * (b^{1/\alpha}\mu_2)\big)(B) \overset{\mathcal{D}}{=} (a+b)^{1/\alpha}\mu(B) \tag{8.10}$$

for any Borel $B \in \mathcal{B}(\mathbb{R})$, where μ_1,μ_2 are two i. i. d. copies of μ. By Theorem 5.16, a non-trivial random measure is StαS if and only if $\varphi(au_1,\dots,au_k) = a^\alpha \varphi(u_1,\dots,u_k)$ for all $u_1,\dots,u_k \in \mathbb{R}$, $k \ge 1$, where $\alpha > 0$ by Theorem 5.19. Thus all moment measures have homogeneous log-characteristic functions. In particular, the first moment measure (also called the intensity measure) $\overline{\mu} = \overline{\mu}_1$ has the characteristic function of the form $e^{-c|u|^\alpha}$. This means that $\overline{\mu}$ is an StαS law in $\mathbb{R}$. Similarly, $\overline{\mu}_k$ is a stable law in $\mathbb{R}^k$. Thus, $0 < \alpha \le 2$, i. e. the stability properties of the cone of probability measures are similar to those of a linear space. Theorem 6.2 applied to $\rho = \rho_x$ yields that the order of homogeneity of the Lévy measure is necessarily smaller than 2.

However the introduced family of characters is still not rich enough, since m_n (or the corresponding random variables ξ_n) may $\tilde{\mathbb{K}}$-weakly converge to $\mathbf{e}$, whereas their norms are separated from zero. Thus, Condition **(S)** does not hold. Moreover a weakly convergent sequence may have unbounded norms, so that **(W)** neither holds. The unit sphere is not compact, since a weak convergent sequence of probability measures does not necessary converges in the Kantorovich metric. Consequently, the results of Section 7.2 does not lead to the conclusion that the Lévy measure is supported by $\mathbb{K}$. Thus, in general the LePage series representation of a StαS random probability measure might involve a point process on the second dual semigroup, see Section 7.1.

The sub-invariance property of the Kantorovich metric implies that the LePage series in $\mathbb{K}$ converges absolutely for $\alpha \in (0,1)$ and thereupon defines a StαS random probability measure. A simple example of a StαS random probability measure is obtained if we take ε_k in (3.9) to be a deterministic measure representing the probability distribution of a random variable ξ with $\mathbf{E}\,|\xi| = 1$. The corresponding LePage series defines a random probability measure μ that, for a given sequence Γ_k, is the distribution of $\sum \Gamma_k^{-1/\alpha}\xi_k$ with i. i. d. random variables $\xi_1, \xi_2, \dots$ distributed as ξ. If $\psi(u) = \log \mathbf{E}\, e^{iu\xi}$, then the characteristic function of every realisation of μ is given by

$$\chi_u(\mu(\omega)) = \exp\{\sum_{k=1}^{\infty} \psi(u\Gamma_k^{-1/\alpha}(\omega))\}\,.$$

In particular, if ξ is concentrated at a single point 1, i. e. if $\varepsilon_k = \delta_1$, then

$$\chi_u(\mu) = \exp\{iu\sum_{k=1}^{\infty}\Gamma_k^{-1/\alpha}\} = \exp\{iu\xi_\alpha\}\,,$$

meaning that μ is the unit mass measure concentrated at a realisation of StαS random variable ξ_α in $(\mathbb{R}_+, +)$. Similarly, if $\xi = \sqrt{\pi/2}\,\nu$, where ν is standard normal, then μ is the unit mass measure concentrated at a realisation of a normal random variable with mean 0 and variance $\xi_{\alpha/2}\pi/2$. Note that this random variable exists if $\alpha < 2$, so in this particular case the LePage series converges for $\alpha \in (0,2)$.

In contrast, a random probability measure $\mu = \delta_\nu$ concentrated at the standard normal distribution ν provides an example of a Gaussian element in $\mathbb{K}$. Indeed, its Laplace transform

$$\mathbf{E}\,\chi_{u_1,\dots,u_k}(\delta_\nu) = \mathbf{E}[e^{iu_1\nu}\cdots e^{iu_k\nu}] = \mathbf{E}\,e^{i(u_1+\cdots+u_k)\nu} = \exp\{-\frac{1}{2}(u_1+\cdots+u_k)^2\}$$

is the exponent of a quadratic form $q(\chi_{u_1,\dots,u_k})$.

In the same spirit, it is possible to consider the semigroup of probability measures where the addition operation corresponds to the maximum of independent random variables rather than the sum. As a distance, one can take any probability metric that is an ideal with respect to the maximum operation, see [55].

Example 8.14 (Uniformly integrable probability measures on $\mathbb{R}_+$). Consider the cone $\mathbb{K}$ which is a sub-family of integrable probability measures m on $\mathbb{R}_+$ such that $\mathbb{K}$ is closed with respect to the cone operations from Example 8.13, $\mathbb{K}$ is closed in the Kantorovich metric, and such that the unit sphere $\mathbb{S}$ constitutes a uniformly integrable family of probability measures with expectation (and Kantorovich norm) being 1. By the well-known sufficient condition [51, Lemma II.6.3], this is the case if

$$\sup_{m\in\mathbb{S}} \int G(x)\, m(dx) < \infty\,,$$

for some non-negative increasing function G such that $G(x)/x \to \infty$ as $x \to \infty$. For instance, it suffices to assume that for all random variables ξ corresponding to $m \in \mathbb{K}$ one has $\mathbf{E}\,\xi^{1+\varepsilon} \le C(\mathbf{E}\,\xi)^{1+\varepsilon}$ for some fixed constants $C, \varepsilon > 0$ and all ξ. The uniform integrability condition ensures that the unit sphere is compact.

Recall that $\mathbf{e} = \mathbf{0} = \delta_0$. Assume that the involution is identical. The cone $\tilde{\mathbb{K}}$ of continuous separating characters consists of the $[0,1]$-valued characters given by

$$\chi_{c,u_1,\dots,u_k}(m) = e^{-c\|m\|} \prod_{i=1}^{k} \int e^{-u_i x} m(dx)\,, \tag{8.11}$$

where $c, u_1, \dots, u_k \ge 0$ and $k \ge 1$. Then the $\tilde{\mathbb{K}}$-weak convergence is identical to the metric convergence. Furthermore, **(S)** holds, so that every StαS random probability measure from $\mathbb{K}$ admits the LePage representation by Theorem 7.14*(i)*.

8.3 Cones violating basic assumptions

The examples below clarify the influence of the crucial assumptions on $\mathbb{K}$ such as the existence of the origin, the norm, its sub-invariance property or the existence of a family of separating characters satisfying **(C)**. In some of these cases the LePage series converges, but its convergence usually has to be confirmed by means of methods specific to the particular situation.

Example 8.15 (Real line with maximum operation). Consider $\mathbb{K} = [-\infty, \infty)$ with the maximum operation, conventional multiplication by non-negative numbers and the Euclidean metric. The neutral element is $-\infty$ and the origin is 0. The corresponding norm is not sub-invariant, e. g., $d(\max\{-1,1\},-1) = 2$ is not smaller than $d(1,0) = 1$. However a direct argument shows that the LePage series with $\alpha < 0$ and $\varepsilon_k = -1$ defines a max-stable random element with the Weibull distribution. The same holds for $\alpha > 0$ and $\varepsilon_k = 1$. Semicontinuous characters are indicators $\mathbf{1}_{[-\infty,a)}$ and $\mathbf{1}_{[-\infty,a]}$. It should be noted that the Laplace transform of a non-negative random variable ξ vanishes for characters with $a < 0$, so that such ξ does not have a proper distribution.

Example 8.16 (Cylinder). Let $\mathbb{K} = \mathbb{R}_+ \times [0, 2\pi)$ with the Euclidean topology. The addition is defined coordinatewise with the second coordinates added modulo 2π. The multiplication defined as $D_a(x_1,x_2) = (ax_1,x_2)$ acts only on the first coordinate. Such a convex cone does not possess the origin, since $D_a(x_1,x_2) \to (0,x_2)$, i. e. the limit as $a \downarrow 0$ is not unique. Because of non-uniqueness of the origin, it is unclear how to define the unit sphere in this example. A natural replacement for the unit sphere is the set given by

$$\{(x_1,x_2):\ x_1 = 1\} = \{x:\ d(x, \lim_{c\downarrow 0} cx) = 1\}$$

for a (non sub-invariant) metric $d((x_1,x_2),(y_1,y_2)) = |x_1 - x_2| + |y_1 - y_2| \mod 2\pi$. However, the corresponding LePage series does not converge unless the spectral measure is concentrated on a singleton, since adding of summands of type $\Gamma_k^{-1/\alpha}\varepsilon_k$ changes the sum considerably even for a large k because the second coordinate is not scaled.

In contrast, if we look at the pair (x_1,x_2) as the polar coordinates of a point in $\mathbb{R}^2$ with the Euclidean distance metric, then all the points of the type $(0,x_2)$ are equivalent and this equivalence class can be used as the origin. All classical results are now valid for this cone showing the principal role uniqueness of the origin plays in our considerations.

Example 8.17 (Non-compact unit sphere). Take $\mathbb{K}$ to be the positive quadrant $\mathbb{R}^2_+$ with both axes $\{0\} \times (0, +\infty)$ and $(0, +\infty) \times \{0\}$ excluded. For some $\alpha \in (0, 1)$, consider two independent α-stable positive random variables ξ_1 and ξ_2. Then the vector (ξ_1, ξ_2) is StαS in $\mathbb{K}$, whereas its spectral measure is concentrated on two points $(1, 0)$ and $(0, 1)$ which are not part of $\mathbb{K}$. Consequently, no LePage representation is possible for (ξ_1, ξ_2). This example shows importance of compactness of the unit sphere even when all other sufficient conditions for existence of the LePage series are met, see, e. g., Theorem 7.14.

Example 8.18 (Half-line with harmonic mean as addition). Let $\mathbb{K}$ be the extended positive half-line $[0, \infty]$ with addition $x \oplus y = (x^{-1} + y^{-1})^{-1}$, usual multiplication and the Euclidean metric and norm on $[0, \infty)$. Then $\mathbf{0}$ is the conventional 0, while the neutral element is ∞. Although the norm is sub-linear, the corresponding metric d cannot be sub-invariant. Indeed, by the required continuity of operations, $x \oplus ch \to x \oplus 0$ as $c \downarrow 0$, so that the sub-linearity would imply $d(x \oplus ch, x) \leq c\|h\| \to 0$. However $d(x \oplus ch, x) \to d(x \oplus 0, x) = d(0, x) = \|x\|$, since $x \oplus 0 = 0$.

The LePage series is $\left(\sum_{k=1}^{\infty} \Gamma_k^{1/\alpha}\right)^{-1}$, thus for positive α it gives identical 0 while for $-1 < \alpha < 0$ it converges to $\xi_{-\alpha}^{-1}$, where $\xi_{-\alpha}$ is the $(-\alpha)$-stable random variable in $(\mathbb{R}_+, +)$. Continuous characters are given by $\chi_a(x) = e^{-a/x}$ for $a > 0$. The bijection $x \mapsto x^{-1}$ provides a homomorphism of semigroups $(\mathbb{R}_+, \oplus)$ and $(\mathbb{R}_+, +)$ that enables one to show directly that α takes values from $(-1, 0)$ and that any StαS random variable with $\alpha \in (-1, 0)$ admits a LePage representation.

Example 8.19 (Half-plane). Let $\mathbb{K} = \mathbb{R} \times \mathbb{R}_+$ be the upper half-plane. The addition is defined as the arithmetic addition of the first coordinates and as the maximum of the second coordinates of the points. Condition **(C)** holds, since the characters $\chi_{a,t}(x) = \mathbf{1}_{(-\infty,a]}(x_2)\, e^{itx_1}$, $x = (x_1, x_2) \in \mathbb{K}$, have the product form as in (5.3). The StαS laws in this case have the first coordinate which is stable in the conventional sense in $\mathbb{R}$, and the max-stable second coordinate. Thus, the parameter of such a stable law belongs to $(0, 2]$.

If the cone operation is altered, so that instead of the maximum we take the minimum of the second coordinates, then the second coordinate should have a negative parameter α unless the second coordinate is identical 0. The only non-trivial stable laws in this case are degenerate in the second coordinate and are stable with $\alpha \in (0, 2]$ in the first coordinate. The Euclidean metric is not sub-invariant for this addition operation. Indeed, adding a small $h = (h_1, h_2)$ to $x = (x_1, x_2)$ results in $(x_1 + h_1, \min(x_2, h_2))$, which may be quite far away from x if x_2 is large and h_2 is small.

Example 8.20 (Compact sets with Minkowski addition). Let us drop the convexity requirement in Example 8.3, so that $\mathbb{K}$ is the family $\mathcal{K}$ of nonempty compact sets K in $\mathbb{R}^d$ with the Hausdorff metric. Then (2.5) does not hold and $\mathbb{K}$ does not possess a family of separating characters. Indeed, $K + K = L + L$ and $K + K + K = L + L + L$ is possible, for instance if $K = [0, 1]$ and $L = [0, 0.4] \cup [0.6, 1]$ on the real line. As a result, this example cannot be investigated using the harmonic analysis tools. However the Hausdorff metric is sub-invariant and so the LePage series still defines a StαS random compact set in $\mathbb{R}^d$ with $\alpha \in (0, 1)$, thereby positively answering a question about a definition of a stochastic integral that defines non-convex stable random sets, see [17, p. 457]. Theorem 4.7 complements a similar result obtained in [17] for the convex case.

As an example, let $\varepsilon_k = \{0, 1\}$, $k \geq 1$, be deterministic. Then the LePage series (3.9) defines a StαS random compact set X in $[0, \infty)$ given by all sums of the type $\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} i_k$, where i_k is either 0 or 1, i.e. all sub-series sums of $\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}$.

Example 8.21 (Union-stable random closed sets). Let $\mathbb{K}$ be the family $\mathcal{F}$ of closed subsets $F \subset \mathbb{R}^d$ with the union operation as addition and homothety as multiplication by scalars. Equip $\mathcal{F}$ with the Fell topology, see [38, App. B]. Then $\mathbf{e} = \emptyset$, whereas no unique origin exists, since aF may have a limit as $a \downarrow 0$ that varies with the choice of F, e. g., it can be F itself if F is a cone in $\mathbb{R}^d$. Any cone in $\mathbb{R}^d$ is an element of $\mathbb{K}(\alpha)$ for each $\alpha \neq 0$, unlike the statement of Lemma 2.7.

An analogue of the LePage series in this case is constructed by means of a locally finite homogeneous measure Λ on $\mathcal{F}$ that defines a Poisson process on $\mathcal{F}$. The union of the sets from this Poisson process defines a StαS random element in $\mathbb{K}$, i. e. a union-stable random closed set. The union-stable random closed sets with possibly non-proper distributions have been extensively studied in [38, Ch. 4] and [39]. Note that $\mathbb{K}$ possesses the same family of separating characters as described in Example 8.10, so that the characterisation results from Section 5 (that do not require the existence of the norm) hold. Here StαS random elements (or union-stable random closed sets) exist for every $\alpha \neq 0$.

Example 8.22 (Gaussian distributions with unusual properties). Let $\mathbb{K}$ be the family of continuous functions on $\mathbb{R}$ with the pointwise addition and the multiplication by scalars corresponding to the rescaling of the argument as in Example 8.12. Consider the characters given by (8.8). If η is a standard Gaussian random variable, then the Laplace transform of the random function $\xi(x) = \eta f(x)$ is given by $\mathbf{E}\,\chi_\nu(\xi) = e^{-q(\nu)}$, where

$$q(\nu) = \frac{1}{2}\Big(\int f(x) d\nu\Big)^2$$

is a quadratic form, since $q(\nu_1+\nu_2)+q(\nu_1-\nu_2) = 2q(\nu_1)+2q(\nu_2)$. Thus ξ is a Gaussian element in $\mathbb{K}$.

If $f(x) = x^\beta$ for some $\beta > 0$, then ξ is StαS with $\alpha = 2\beta$, since $(D_{a^{1/\alpha}}\xi)(x) = \eta a^{1/2} x^\beta$ and (1.1) holds. By varying β it is possible to obtain StαS Gaussian elements in $\mathbb{K}$ with an arbitrary characteristic exponent α.

If $f(x)$ is non-homogeneous, e. g., $f(x) = \max\{0, 1-x^2\}$, then the corresponding quadratic form is not homogeneous either. Since the support of $D_{a^{1/\alpha}}\xi(x)$ is $[-a^{-1/\alpha}, a^{-1/\alpha}]$, such a random function ξ is not StαS as the supports in the right and left sides of (1.1) differ.

It should be noted however that the considered cone does not possess the origin and a norm.

Example 8.23 (Intrinsically stable random measures). Consider the family $\mathbb{K}$ of all locally finite measures on $\mathbb{R}^d \setminus \{0\}$ with the vague topology and the conventional addition operation as in Example 8.5, but with the multiplication defined as $(D_a m)(K) = m(a^{-1}K)$. The second distributivity law does not hold in this case, since $(m+m)$ is generally not $D_2 m$. The neutral element is zero measure, while a unique origin does not exist, since the rescaled measures $D_a m$ may have various limits as $a \to 0$. A random measure μ is called intrinsically stable if

$$\mu_1(a^{-1/\alpha}K) + \mu_2(b^{-1/\alpha}K) \overset{\mathcal{D}}{=} \mu((a+b)^{-1/\alpha}K) \tag{8.12}$$

for all measurable K. Note that this definition combines the addition operation as in Example 8.6 with the scaling used in Example 8.13.

The continuous $(0,1]$-valued characters χ_u are given by (8.3) for u being a continuous function with bounded support. The multiplication by scalars is uplifted to the characters as $(c \circ \chi_{u(\cdot)}) = \chi_{u(c^{-1}\cdot)}$. Since **(C)** holds, an intrinsically stable non-trivial random measure has a homogeneous

Laplace exponent $\varphi(u) = \varphi(\chi_u)$, i. e. $\varphi(u(s^{-1}\cdot)) = s^\alpha \varphi(u(\cdot))$ for all continuous u with bounded support and $s > 0$, c. f. Theorem 5.16. Theorem 5.19 is not applicable to show that α has a particular sign, since the origin is not defined.

In this case, any $\alpha \neq 0$ is a possible parameter for a StαS random measure. For instance, the counting measures Π_α for all $\alpha \neq 0$ are stable in this cone $\mathbb{K}$.

Example 8.24 (Random probability measures). Let $\mathbb{K}$ be the family of all probability measures on $\mathbb{R}$ with the same cone operations as in Example 8.13, convergence in distribution and the characters given by (8.9). Theorem 5.19 implies that $\alpha > 0$. In this case $\mathbb{K}(\alpha)$ are all non-trivial for $0 < \alpha \leq 2$ consisting of StαS probability distributions in $\mathbb{R}$. As a consequence of Lemma 2.7, it is not possible to define a sub-invariant metric in this cone unless assigning infinite norm for all elements of $\mathbb{K}(\alpha)$ with $\alpha < 1$. Recalling the discussion in Example 8.13, this shows that it is not possible to construct an ideal probability metric of order 1 that is finite on all random variables.

Convergence of the LePage series should be checked in each particular case. For this, it is simpler to work with random characteristic functions rather than with random probability measures, since the convolutions of measures become the product of characteristic functions. Let $\mathbb{K}$ be the family of characteristic functions $\theta(t)$, $t \in \mathbb{R}$, with addition being the product of functions and multiplication by numbers corresponding to the rescaling of the argument, i. e. $(D_a\theta)(t) = \theta(at)$. The LePage series yields the random characteristic function

$$\prod_{k=1}^{\infty} \theta_k(\Gamma_k^{-1/\alpha} t)\,, \tag{8.13}$$

where $\{\theta_k,\ k \geq 1\}$ are i. i. d. random characteristic functions from a unit sphere in $\mathbb{K}$. The unit sphere $\mathbb{S}$ in $\mathbb{K}$ can defined by using any polar decomposition of $\mathbb{K}$. For instance, a homogeneous (but not sub-linear) norm of a probability measure m with distribution function $F_m(x) = m\{(-\infty, x]\}$ and the corresponding percentiles $q_m(t) = \inf\{x : \ F_m(x) \geq t\}$ can be defined as

$$\|m\|^2 = q_m^2\big(F_m(0-)/2\big) + q_m^2\big((1 + F_m(0))/2\big)\,.$$

Notice that $\|m\| = 0$ implies $m = \delta_0$. In case of integrable centred probability measures, one can use the sub-invariant homogeneous (i. e. ideal) metric given by

$$d(\xi, \eta) = d(\theta_1, \theta_2) = \sup_{u \in \mathbb{R}} |u|^{-1} |\theta_1(u) - \theta_2(u)|\,,$$

where θ_1 and θ_2 are the characteristic functions of ξ and η respectively.

Assume that $\{\theta_k,\ k \geq 1\}$ are non-random characteristic functions of the strictly stable distribution with characteristic exponent $\beta \in (0, 1]$, i. e. $\theta_k(t) = e^{-|t|^\beta}$. In case $\beta = 1$ one obtains the symmetric Cauchy distribution. Convergence of the infinite product (8.13) is then equivalent to convergence of the series

$$\sum_{k=1}^{\infty} \Gamma_k^{-\beta/\alpha}\,.$$

This series converges a. s. if and only if $\beta > \alpha$, and its limit is a strictly stable random variable ζ with characteristic exponent α/β. Thus, the LePage series produces a random probability measure $D_\zeta m$ that corresponds to the strictly stable probability measure m with characteristic exponent $\beta \in (0,1)$ rescaled by a random α/β-stable coefficient ζ.

As another example, assume that $\{\theta_k,\ k \geq 1\}$ are characteristic functions of the Gamma-distributions with random shape parameter $p_k > 0$ and scale parameter 1. The corresponding measures are integrable, so that this case is covered by Example 8.13. Since the unit sphere consists of all probability measures on $\mathbb{R}_+$ with unit mean, the considered random measures (or random characteristic functions) need to be rescaled by p_k to lie on the unit sphere. Then (8.13) becomes

$$\prod_{k=1}^{\infty}(1 - i\Gamma_k^{-1/\alpha} t/p_k)^{-p_k} \,,$$

so that the LePage series converges for every $\alpha \in (0,1)$ and every distribution of the p_k's. This is also confirmed by the existence of a sub-invariant norm for integrable probability measures, see Example 8.13.

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