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Computation of Moments for the Length of the One Dimensional ISE Support

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abstract: We consider in this paper the support $[L', R']$ of the one dimensional Integrated Super Brownian Excursion. We determine the distribution of (R', L') through a modified Laplace transform. Then we give an explicit value for the first two moments of R' as well as the covariance of R' and L'.

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1. Introduction and results

The motivation of this work comes from the paper of Chassaing and Schaeffer [2]. They prove the rescaled radius of random quadrangulation converges (in law), as the number of faces goes to infinity, to the width $r' = R' - L'$ of the one dimensional Integrated Super Brownian Excursion (ISE) support $[L', R']$ (see [1] and the reference therein for the definition of the ISE). They also prove the convergence of moments. As pointed by Aldous in [1], little is known about the law of r'. It is of particular interest to compute therefore the law of $r' = R' - L'$ as well as its moments. We recall that $L' \leq 0 \leq R'$ a.s., and that by symmetry, R' and $-L'$ are equally distributed. More precisely we give a sort of Laplace transform of R' in the following proposition, which is proved in section 4.

Proposition 1. *For $\lambda > 0, b > 0$, we have*

$$\int_0^\infty \frac{dr}{r^{3/2}}\ \mathrm{e}^{-\lambda r}\, \mathbb{P}(R' > br^{-1/4}) = \frac{6\sqrt{\pi}\sqrt{\lambda}}{[\sinh((\lambda/2)^{1/4}b)]^2}. \tag{1}$$

For $b = 1$, from the uniqueness of the Laplace transform, we deduce the function $r \mapsto r^{-3/2}\mathbb{P}(R' > r^{-1/4})$, hence the law of R', is uniquely defined by the above equation. However, this does not allow us to give explicitly the law of R'. We derive in section 4, the first two moments of R'.

Corollary 2. *We have*

$$\mathbb{E}[R'] = 3\frac{2^{3/4}\,\Gamma(5/4)}{\sqrt{\pi}} \quad \textit{and} \quad \mathbb{E}[{R'}^2] = 3\sqrt{2\pi}.$$

Multiplying equation (1) by $b^{\alpha-1}$, with $\alpha > 2$, and integrating w.r.t. b on $(0, +\infty)$ gives the following corollary.

Corollary 3. *Let $\alpha > 2$. Then, we have*

$$\mathbb{E}[{R'}^\alpha] = 24\sqrt{\pi}\, \frac{\Gamma(\alpha+1)\ \zeta(\alpha-1)}{2^{3\alpha/4}\ \Gamma((\alpha-2)/4)}.$$

In section 5, we prove the following result.

Proposition 4. *For $\lambda > 0$, $b > 0$, $c > 0$, we have*

$$\int_0^\infty \frac{dr}{r^{3/2}}\ \mathrm{e}^{-\lambda r}[1 - \mathbb{P}(R' > br^{-1/4}, |L'| > cr^{-1/4})] = \sqrt{2\pi}\ [u_{\lambda,(c+b)/2}(\min(c,b)) - \sqrt{\lambda/2}],$$

where $u_{\lambda,r}$ is the unique non-negative solution of the non-linear differential equation

$$\begin{cases} u'' = 2u^2 - \lambda & \textit{in } (0,r), \\ u(0) = +\infty, \\ u'(r) = 0. \end{cases}$$

Then, one can derive the expectation of $R'\,|L'|$ (see section 5).

Corollary 5. *We have*

$$\mathbb{E}[R'\,|L'|] = -3\sqrt{2\pi} + 3\sqrt{\pi/2}\int_1^\infty \frac{dt}{\sqrt{t^3-1}} \int_1^\infty \frac{(u+1)\ du}{\sqrt{u^3-1}(u+\sqrt{u^2+u+1})},$$

and

$$\mathbb{E}[\min(R', |L'|)^2] = 6\sqrt{2\pi}[1 - \alpha_0^2/8],$$

where $\alpha_0 = \displaystyle\int_1^\infty \frac{du}{\sqrt{u^3-1}}$.

We give the following numerical approximation (up to 10^{-3}): $\mathbb{E}[R'] \simeq 2.580$, $\mathrm{Var}(R') \simeq 0.863$ and for $r' = R' - L'$, $\mathbb{E}[r'] \simeq 5.160$ and $\mathrm{Var}(r') \simeq 0.651$.

To prove those results, we use the fact that the ISE has the same distribution (up to a constant scaling) as the total mass of an excursion of the Brownian snake conditioned to have a duration σ of length 1. Then we compute under the excursion measure of the Brownian snake the joint law of σ, R' and L'. In particular we use the special Markov property of the Brownian snake and the connection between Brownian snake and PDE The next section is devoted to the presentation of the Brownian snake and its link with ISE.

2. Brownian snake and ISE

Let $\mathcal{W}$ be the set of stopped continuous paths (w, ζ_w) defined on $\mathbb{R}^+$ with values in $\mathbb{R}$. $\zeta_w \geq 0$ is called the lifetime of the path w, and w is a continuous path $w = (w(t), t \geq 0)$ defined on $\mathbb{R}^+$ with values in $\mathbb{R}$ and constant for $t \geq \zeta_w$. We sometimes write w for (w, ζ_w). We define

$$d(w, w') = |\zeta_w - \zeta_{w'}| + \sup_{t\geq 0} \left|w(t) - w'(t)\right|.$$

It is easy to check that d is a distance on $\mathcal{W}$, and that $(\mathcal{W}, d)$ is a Polish space.

We shall denote by $\mathbb{N}_x[dW]$ the excursion measure on $\mathcal{W}$ of the Brownian snake $W = (W_s, s \geq 0)$ started at $x \in \mathbb{R}$ with underlying process a linear Brownian motion. We refer to [5] for the definition and properties of the Brownian snake. We recall that under $\mathbb{N}_x$, the law of the lifetime process $\zeta = (\zeta_s, s \geq 0)$ is the Itô measure, n_+, on positive excursions of linear Brownian motion, where we take the normalization $\mathbb{N}_x[\sup_{s\geq 0} \zeta_s > \varepsilon] = \frac{1}{2\varepsilon}$. Under $\mathbb{N}_x$, conditionally on the lifetime process, W is a continuous $\mathcal{W}$-valued Markov process started at the constant path (with lifetime zero) equal to $x \in \mathbb{R}$. Conditionally, on the lifetime process and on $(W_u, u \in [0, s])$, the law of $W_{s'}$, with $s' \geq s$ is as follow: the two paths W_s and $W_{s'}$ coincide up to time $m = \inf_{u\in[s,s']} \zeta_u$, and $(W_{s'}(t+m), t \geq 0)$ is a linear Brownian motion, constant after $\zeta_{s'} - m$, which depends on $(W_u, u \in [0, s])$ only through its starting point $W_{s'}(m) = W_s(m)$.

We define $\sigma = \inf\{s > 0; \zeta_s = 0\}$ the duration of the excursion. From the normalization of $\mathbb{N}_x$, we deduce that σ is distributed according to

$$\mathbb{N}_x[\sigma \in [r, r + dr]] = n_+(\sigma \in [r, r + dr]) = \frac{dr}{2\sqrt{2\pi}\, r^{3/2}}, \quad \text{for } r > 0.$$

It is easy to check that for any $x \in \mathbb{R}$,

$$\mathbb{N}_x\left[1 - \mathrm{e}^{-\lambda\sigma}\right] = \sqrt{\lambda/2}. \tag{2}$$

The Brownian snake enjoys a scaling property: if $\lambda > 0$, the law of the process $W_s^{(\lambda)}(t) = \lambda^{-1} W_{\lambda^4 s}(\lambda^2 t)$ under $\mathbb{N}_x$ is $\lambda^{-2}\mathbb{N}_{\lambda^{-1}x}$.

We now recall the connection between ISE and Brownian snake. There exists a unique collection $\left(\mathbb{N}_0^{(r)}, r > 0\right)$ of probability measures on $C(\mathbb{R}^+, \mathcal{W})$ such that:

(1) For every $r > 0$, $\mathbb{N}_0^{(r)}[\sigma = r] = 1$.
(2) For every $\lambda > 0$, $r > 0$, F, nonnegative measurable functional on $C(\mathbb{R}^+, \mathcal{W})$,

$$\mathbb{N}_0^{(r)}\left[F(W^{(\lambda)})\right] = \mathbb{N}_0^{(\lambda^{-4}r)}\left[F(W)\right].$$

(3) For every nonnegative measurable functional F on $C(\mathbb{R}^+, \mathcal{W})$,

$$\mathbb{N}_0[F] = \int_0^\infty \frac{dr}{2\sqrt{2\pi} r^{3/2}} \mathbb{N}_0^{(r)}[F]. \tag{3}$$

We take the opportunity to stress a misprint in [3], where 1/2 is missing in the right member of formula (3). The measurability of the mapping $r \mapsto \mathbb{N}_0^{(r)}[F]$ follows from the scaling property 2. Under $\mathbb{N}_0^{(1)}$, the distribution of W is characterized as under $\mathbb{N}_0$, except that the lifetime process is distributed according to the normalized Itô measure of positive excursions.

Let $\mathcal{R} = \{W_s(t); 0 \leq t \leq \zeta_s, 0 \leq s \leq \sigma\}$ be the range of the Brownian snake. Since we are in dimension one, using the continuity of the paths, we get that $\mathbb{N}_x$-a.e., the range is an interval which we denote by $[L, R]$, with $L \leq x \leq R$. Notice we also have $\mathcal{R} = \{\hat{W}_s; 0 \leq s \leq \sigma\}$, where $\hat{W}_s = W_s(\zeta_s)$ is the end of the path W_s. We have henceforth

$$R = \sup_{0 \leq s \leq \sigma} \hat{W}_s, \quad \text{and} \quad L = \inf_{0 \leq s \leq \sigma} \hat{W}_s.$$

The law of the ISE is the law of the continuous tree associated to $\sqrt{2}W$, under $\mathbb{N}_0^{(1)}$ (see corollary 4 in [4] and [1], see also [7] section IV.6). In particular the law of the support of ISE is the law of $\sqrt{2}\mathcal{R}$ under $\mathbb{N}_0^{(1)}$, where we set $\lambda A = \{x; \lambda^{-1}x \in A\}$. Therefore, we deduce that in dimension one, the support of the ISE is an interval, say $[L', R']$. And we deduce that (L', R') is distributed as $(\sqrt{2}L, \sqrt{2}R)$ under $\mathbb{N}_0^{(1)}$.

From [5], it is well known that the function $\mathbb{N}_x[\mathcal{R} \not\subset [a, b]]$ is the maximal non-negative solution of $u'' = 4u^2$ in (a, b). But,as we said, we need the law of $\mathcal{R}$ under $\mathbb{N}_x^{(1)}$. Therefore, we need the joint law of $\mathcal{R}$ and σ. We shall compute $\mathbb{N}_x[1 - \mathbf{1}_{\mathcal{R} \subset [a,b]} \mathrm{e}^{-\lambda\sigma}]$ using Brownian snake techniques (see the key lemma 6 and its consequences lemmas 7 and 8). From this, thanks to scaling properties mentioned above, we will deduce a (kind of) Laplace transform for $\mathbb{N}_x^{(1)}[1 - \mathbf{1}_{\mathcal{R} \subset [a,b]}]$ (see lemma 8).

We prove in section 4 the following result: for $\lambda > 0$, $b > 0$,

$$\int_0^\infty \frac{dr}{r^{3/2}} \mathrm{e}^{-\lambda r} \mathbb{N}_0^{(1)}[R > br^{-1/4}] = \frac{6\sqrt{\pi}\sqrt{\lambda}}{[\sinh((2\lambda)^{1/4} b)]^2},$$

which determines the law of R. We also compute

$$\mathbb{N}_0^{(1)}[R] = 3 \, \frac{2^{1/4} \, \Gamma(5/4)}{\sqrt{\pi}} \quad \text{and} \quad \mathbb{N}_0^{(1)}[R^2] = 3\sqrt{\frac{\pi}{2}}.$$

In section 5, lemma 8, we compute for $\lambda > 0$, $b > 0$, $c > 0$,

$$\int_0^\infty \frac{dr}{r^{3/2}} \mathrm{e}^{-\lambda r} \mathbb{N}_0^{(1)} \left[1 - \mathbf{1}_{\{R \leq br^{-1/4}, \, |L| \leq cr^{-1/4}\}}\right],$$

and

$$\mathbb{N}_0^{(1)}[\min(R, |L|)^2] = 3\sqrt{2\pi}[1 - \alpha_0^2/8],$$
$$\mathbb{N}_0^{(1)}[R|L|] = -3\sqrt{\frac{\pi}{2}} + \frac{3\sqrt{2\pi}}{4} \int_1^\infty \frac{dt}{\sqrt{t^3 - 1}} \int_1^\infty \frac{(u+1) \, du}{\sqrt{u^3 - 1}(u + \sqrt{u^2 + u + 1})}.$$

3. Exit measure of the Brownian snake

We refer to [6] for general definition and properties of the exit measure of Brownian snake. Let $-\infty \leq a < x < b \leq +\infty$, and consider $X^{(a,b)}$ the exit measure of the Brownian snake of (a,b) under $\mathbb{N}_x$. We recall that $X^{(a,b)}$ is a random measure on $\{a,b\}$, defined by : for any nonnegative measurable function φ defined on $\mathbb{R}$,

$$\int \varphi(x)\ X^{(a,b)}(dx) = \int_0^\sigma \varphi(\hat{W}_s) dL_s^{(a,b)}.$$

The continuous additive functional of the Brownian snake $L_s^{(a,b)}$ is defined $\mathbb{N}_x$-a.e. for all $s \geq 0$ by

$$L_s^{(a,b)} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s ds'\ \mathbf{1}_{\{\tau(W_{s'}) < \zeta_{s'} < \tau(W_{s'}) + \varepsilon\}},$$

where $\tau(w) = \inf\{t \in [0, \zeta_w], w(t) \notin (a,b)\}$ is the first exit time of (a,b) for the path $w \in \mathcal{W}$. We use the convention $\inf \emptyset = +\infty$. Intuitively, $L_s^{(a,b)}$ increases when the path W_s dies as it reaches for the first time the boundary of (a,b). It is well known that in dimension 1, $\mathbb{N}_x$-a.e.,

$$\{\mathcal{R} \subset [a,b]\} = \{X^{(a,b)} = 0\}.$$

Recall also that the support of $X^{(a,b)}$ is a random subset of $\{a,b\} \cap \mathbb{R}$. We set $Y^{(a,b)} = \int X^{(a,b)}(dy)$ for the total mass of $X^{(a,b)}$.

We define the function defined for $\mu > 0, \lambda \geq 0, a < x < b$: by

$$v_{\mu,\lambda,a,b}(x) = \mathbb{N}_x \left[1 - \mathrm{e}^{-\mu Y^{(a,b)} - \lambda\sigma} \right].$$

The next lemma is proved in section 6 using the link between Brownian snake and non-linear PDE.

Lemma 6. *The function $v_{\mu,\lambda,a,b}$ solves*

$$\frac{1}{2} v'' = 2v^2 - \lambda \quad \text{in } (a,b). \tag{4}$$

Furthermore if $a > -\infty$ (resp. $b < \infty$), then we have the boundary condition $v(a) = \mu + \sqrt{\lambda/2}$ (resp. $v(b) = \mu + \sqrt{\lambda/2}$).

As an application of this lemma, we have the following result.

Lemma 7. *We define for $b > 0$ and $\lambda \geq 0$,*

$$w_\lambda(b) = \mathbb{N}_0 \left[1 - \mathbf{1}_{\{R \leq b\}}\, \mathrm{e}^{-\lambda\sigma} \right].$$

We have for $b > 0, \lambda > 0$,

$$w_\lambda(b) = \sqrt{\frac{\lambda}{2}} \left[3 \coth(2^{1/4} b \lambda^{1/4})^2 - 2 \right].$$

Proof. Since $\mathbb{N}_x$-a.e., for $x < b$,

$$\{R \leq b\} = \{\mathcal{R} \subset (-\infty, b]\} = \{X^{(-\infty,b)} = 0\} = \{Y^{(-\infty,b)} = 0\},$$

we have

$$w_\lambda(b) = \lim_{\mu \to \infty} \mathbb{N}_0 \left[1 - \mathrm{e}^{-\mu Y^{(-\infty,b)} - \lambda\sigma} \right] = \lim_{\mu \to \infty} v_{\mu,\lambda,-\infty,b}(0).$$

By translation and symmetry, it is clear that $v_{\mu,\lambda,-\infty,b}(0) = v_{\mu,\lambda,0,\infty}(b)$. As $w_\lambda(b)$ is the increasing limit of $v_{\mu,\lambda,0,\infty}(b)$ as $\mu \to \infty$, and since the set of nonnegative solutions of (4) is closed under pointwise convergence (see proposition 9 (iii) in section V.3 of [7], stated for $\lambda = 0$, which can be extended to the case $\lambda > 0$), we deduce that w_λ also solves (4) with $(a,b) = (0,+\infty)$. Notice that

$$w_\lambda(0) = \lim_{\mu\to\infty} v_{\mu,\lambda,0,\infty}(0) = \lim_{\mu\to\infty} \mu + \sqrt{\lambda/2} = +\infty$$

and that, since $\mathcal{R}$ is a compact set $\mathbb{N}_0$-a.e.,

$$w_\lambda(+\infty) = \mathbb{N}_0\left[1 - \mathrm{e}^{-\lambda\sigma}\right] = \sqrt{\lambda/2}.$$

Therefore w_λ is solution of

$$\begin{cases} \frac{1}{2}w'' = 2w^2 - \lambda \quad & \text{in } (0,+\infty), \\ w(0) = +\infty, \\ w(+\infty) = \sqrt{\lambda/2}. \end{cases}$$

This ordinary differential equation has a unique nonnegative solution, which is given by

$$w_\lambda(b) = \sqrt{\frac{\lambda}{2}}\left[3\coth(2^{1/4}b\lambda^{1/4})^2 - 2\right].$$

□

4. Proof of proposition 1 and corollary 2

Proof of proposition 1. By scaling, we get that the law of R under $\mathbb{N}_0^{(r)}$ is the law of $Rr^{1/4}$ under $\mathbb{N}_0^{(1)}$. We deduce from (3) and the above scaling property, that

$$\begin{aligned} w_\lambda(b) &= \mathbb{N}_0\left[1 - \mathbf{1}_{\{R\le b\}}\,\mathrm{e}^{-\lambda\sigma}\right] \\ &= \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,\mathbb{N}_0^{(r)}\left[1 - \mathbf{1}_{\{R\le b\}}\,\mathrm{e}^{-\lambda\sigma}\right] \\ &= \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,\mathbb{N}_0^{(1)}\left[1 - \mathbf{1}_{\{R\le br^{-1/4}\}}\,\mathrm{e}^{-\lambda r}\right] \\ &= \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,(1-\mathrm{e}^{-\lambda r}) + \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,\mathrm{e}^{-\lambda r}\left[1 - \mathbb{N}_0^{(1)}[R\le br^{-1/4}]\right] \\ &= \sqrt{\lambda/2} + \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,\mathrm{e}^{-\lambda r}\,\mathbb{N}_0^{(1)}[R > br^{-1/4}]. \end{aligned}$$

Eventually we define $H_\lambda(b)$ for $\lambda > 0, b > 0$ by

$$H_\lambda(b) = \int_0^\infty \frac{dr}{r^{3/2}}\,\mathrm{e}^{-\lambda r}\,\mathbb{N}_0^{(1)}[R > br^{-1/4}],$$

and we get

$$H_\lambda(b) = \frac{6\sqrt{\pi}\sqrt{\lambda}}{[\sinh((2\lambda)^{1/4}b)]^2}. \tag{5}$$

Since R' is distributed as $\sqrt{2}R$, this proves proposition 1. □

Proof of corollary 2. By monotone convergence, letting λ decrease to 0, we get :

$$H_0(b) = \int_0^\infty \frac{dr}{r^{3/2}}\,\mathbb{N}_0^{(1)}[R > br^{-1/4}] = \frac{3\sqrt{2\pi}}{b^2}. \tag{6}$$

Set $u = r^{-1/4}$ and $b = 1$, to get

$$4\int_0^\infty u\mathbb{N}_0^{(1)}[R > u]\, du = 3\sqrt{2\pi},$$

which implies

$$\mathbb{N}_0^{(1)}[R^2] = 3\sqrt{\frac{\pi}{2}}.$$

Since R' is distributed as $\sqrt{2}R$, we get the second equality of corollary 2.

We recall the following development near 0:

$$\frac{1}{\sinh(x)^2} = \frac{1}{x^2}(1 - \frac{x^2}{3} + O(x^4)).$$

Hence the function $\dfrac{1}{x^2} - \dfrac{1}{\sinh(x)^2}$ is integrable over $\mathbb{R}^+$. The function $H_0(b) - H_\lambda(b)$ is positive and integrable on $\mathbb{R}^+$. Setting $x = (2\lambda)^{1/4}b$, we get

$$\begin{aligned}
\int_0^\infty [H_0(b) - H_\lambda(b)]\, db &= 3\sqrt{2\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \left[\frac{1}{b^2} - \frac{\sqrt{2\lambda}}{\sinh((2\lambda)^{1/4}b)^2}\right] db \\
&= 3\sqrt{2\pi}(2\lambda)^{1/4}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \left[\frac{1}{x^2} - \frac{1}{\sinh(x)^2}\right] dx \\
&= 3\sqrt{2\pi}(2\lambda)^{1/4}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon} + 1 - \frac{\cosh(\varepsilon)}{\sinh(\varepsilon)} \\
&= 3\sqrt{2\pi}(2\lambda)^{1/4}.
\end{aligned}$$

On the other hand, by Fubini, we have

$$\begin{aligned}
\int_0^\infty [H_0(b) - H_\lambda(b)]\, db &= \int_0^\infty db\int_0^\infty \frac{dr}{r^{3/2}}(1 - \mathrm{e}^{-\lambda r})\mathbb{N}_0^{(1)}[R > br^{-1/4}] \\
&= \int_0^\infty \frac{dr}{r^{3/2}}(1 - \mathrm{e}^{-\lambda r})r^{1/4}\int_0^\infty \mathbb{N}_0^{(1)}[R > v]\, dv \\
&= \lambda^{1/4}\int_0^\infty \frac{dr}{r^{5/4}}(1 - \mathrm{e}^{-r})\mathbb{N}_0^{(1)}[R] \\
&= \lambda^{1/4}4\,\Gamma(3/4)\,\mathbb{N}_0^{(1)}[R].
\end{aligned}$$

We deduce that

$$\mathbb{N}_0^{(1)}[R] = \frac{3\sqrt{2\pi}\,2^{1/4}}{4\,\Gamma(3/4)}.$$

Since

$$\Gamma(1/4)\Gamma(3/4) = \int_0^\infty \frac{t^{-3/4}\,dt}{1+t} = 4\int_0^\infty \frac{dv}{1+v^4} = \pi\sqrt{2},$$

we have $\dfrac{1}{4\,\Gamma(3/4)} = \dfrac{\Gamma(1/4)}{4\pi\sqrt{2}} = \dfrac{\Gamma(5/4)}{\pi\sqrt{2}}$. And we get

$$\mathbb{N}_0^{(1)}[R] = 3\,\frac{2^{1/4}\,\Gamma(5/4)}{\sqrt{\pi}}.$$

Since R' is distributed as $\sqrt{2}R$, we get the first equality of corollary 2. $\square$

5. Proof of proposition 4 and corollary 5

We define for $-\infty < a < 0 < b < +\infty$ and $\lambda \geq 0$,

$$\tilde{w}_\lambda(a,b) = \mathbb{N}_0\left[1 - \mathbf{1}_{\{R\leq b, L\geq a\}}\,\mathrm{e}^{-\lambda\sigma}\right].$$

Since $\mathbb{N}_x$-a.e., for $a < x < b$, $\{R \leq b, L \geq a\} = \{X^{(a,b)} = 0\} = \{Y^{(a,b)} = 0\}$, we have with the notations of section 3,

$$\tilde{w}_\lambda(a,b) = \lim_{\mu\to\infty} \mathbb{N}_0\left[1 - \mathrm{e}^{-\mu Y^{(a,b)} - \lambda\sigma}\right] = \lim_{\mu\to\infty} v_{\mu,\lambda,a,b}(0).$$

Notice from lemma 6, that by symmetry, $v'_{\mu,\lambda,a,b}((a+b)/2) = 0$. By translation and symmetry, it is clear that $v_{\mu,\lambda,a,b}(0) = v_{\mu,\lambda,0,b+|a|}(\min(|a|,b))$. We set $w_{\lambda,r}(x)$ as the increasing limit of $v_{\mu,\lambda,0,2r}(x)$ for $x \in (0,2r)$ as $\mu \uparrow \infty$. Therefore, we have

$$\tilde{w}_\lambda(a,b) = w_{\lambda,r_0}(\min(|a|,b)), \quad \text{where we set} \quad r_0 = \frac{|a|+b}{2}.$$

As the set of nonnegative solutions of (4) is closed under pointwise convergence, we deduce from lemma 6 that $w_{\lambda,r}$ also solves (4) with $(a,b) = (0,2r)$. We have the boundary condition

$$w_{\lambda,r}(0) = \lim_{\mu\to\infty} v_{\mu,\lambda,0,2r}(0) = \lim_{\mu\to\infty} \mu + \sqrt{\lambda/2} = +\infty.$$

By symmetry, we deduce that $w'_{\lambda,r}(r) = 0$. The ordinary differential equation

$$\begin{cases} \frac{1}{2}w'' = 2w^2 - \lambda & \text{in } (0,r], \\ w(0) = +\infty, \\ w'(r) = 0. \end{cases} \tag{7}$$

has a unique nonnegative solution. However, we don't have an explicit formula for $w_{\lambda,r}$.

Arguing as in the proof of proposition 1, we get with $c = -a > 0$,

$$\begin{aligned}
\tilde{w}_\lambda(a,b) &= \mathbb{N}_0\left[1 - \mathbf{1}_{\{R\leq b, |L|\leq c\}}\,\mathrm{e}^{-\lambda\sigma}\right] \\
&= \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,\mathbb{N}_0^{(1)}\left[1 - \mathbf{1}_{\{R\leq br^{-1/4}, |L|\leq cr^{-1/4}\}}\,\mathrm{e}^{-\lambda r}\right] \\
&= \sqrt{\lambda/2} + \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^{3/2}}\,\mathrm{e}^{-\lambda r}\,\mathbb{N}_0^{(1)}\left[1 - \mathbf{1}_{\{R\leq br^{-1/4}, |L|\leq cr^{-1/4}\}}\right].
\end{aligned}$$

We set for $b > 0$, $c > 0$,

$$I_\lambda(c,b) = \int_0^\infty \frac{dr}{r^{3/2}}\,\mathrm{e}^{-\lambda r}\,\mathbb{N}_0^{(1)}\left[1 - \mathbf{1}_{\{R\leq br^{-1/4},\ |L|\leq cr^{-1/4}\}}\right]. \tag{8}$$

That is

$$I_\lambda(c,b) = 2\sqrt{2\pi}[\tilde{w}_\lambda(-c,b) - \sqrt{\lambda/2}] = 2\sqrt{2\pi}[w_{\lambda,r_0}(\min(c,b)) - \sqrt{\lambda/2}],$$

where $r_0 = (c+b)/2$. We have proved the following lemma, and thus proposition 4 (with the notation $u_{\lambda,r}(x) = w_{\lambda,r/\sqrt{2}}(x/\sqrt{2})$).

Lemma 8. *Let $\lambda > 0$, $b > 0$, $c > 0$. We have*

$$\int_0^\infty \frac{dr}{r^{3/2}}\,\mathrm{e}^{-\lambda r}\,\mathbb{N}_0^{(1)}\left[1 - \mathbf{1}_{\{R\leq br^{-1/4},\ |L|\leq cr^{-1/4}\}}\right] = 2\sqrt{2\pi}[w_{\lambda,(c+b)/2}(\min(c,b)) - \sqrt{\lambda/2}],$$

where $w_{\lambda,r}$ is the only non-negative solution of (7).

Notice that $I_\lambda(c,b) = I_\lambda(b,c)$, $H_\lambda(b) = \lim_{c\to+\infty} I_\lambda(c,b)$ and that $I_\lambda(c,b) \geq H_\lambda(b)$. Using that

$$1 - \mathbf{1}_{\{R\leq br^{-1/4},\ |L|\leq cr^{-1/4}\}} = \mathbf{1}_{\{R>br^{-1/4}\}} + \mathbf{1}_{\{|L|>cr^{-1/4}\}} - \mathbf{1}_{\{R>br^{-1/4},\ |L|>cr^{-1/4}\}}$$

and that R and $|L|$ have the same distribution under $\mathbb{N}_0^{(1)}$, we deduce

$$(9)\qquad J_\lambda(c,b) = \int_0^\infty \frac{dr}{r^{3/2}}\ \mathrm{e}^{-\lambda r}\,\mathbb{N}_0^{(1)}\left[\mathbf{1}_{\{R>br^{-1/4},\ |L|>cr^{-1/4}\}}\right] = H_\lambda(b) + H_\lambda(c) - I_\lambda(c,b).$$

Proof of the second equality of corollary 5. In particular, taking $c = b$ and letting λ decreases to 0 in (9), we get

$$(10)\qquad J_0(b,b) = \int_0^\infty \frac{dr}{r^{3/2}}\,\mathbb{N}_0^{(1)}\left[\mathbf{1}_{\{\min(R,|L|)\geq br^{-1/4}\}}\right] = 2H_0(b) - I_0(b,b).$$

Notice $I_0(b,b) = 2\sqrt{2\pi}\ w_{0,r_0}(r_0)$, with $r_0 = (c+b)/2 = b$. Let us compute $w_{0,r}(r)$. We set

$$\theta = w_{0,r}(r).$$

Solving the differential equation (7) for $w_{0,r}$, we get for all $t \in (0,r)$,

$$\int_{w_{0,r}(t)}^\infty \sqrt{\frac{3}{8}}\frac{du}{\sqrt{u^3-\theta^3}} = t,$$

that is

$$(11)\qquad \sqrt{\frac{3}{8}}\int_{w_{0,r}(t)/\theta}^\infty \frac{du}{\sqrt{u^3-1}} = \sqrt{\theta}\ t.$$

For $t = r$, we get

$$\sqrt{\frac{3}{8}}\int_1^\infty \frac{du}{\sqrt{u^3-1}} = \sqrt{\theta}\ r.$$

We define

$$\alpha_0 = \int_1^\infty \frac{du}{\sqrt{u^3-1}}.$$

We get that

$$(12)\qquad w_{0,r}(r) = \theta = \frac{3}{8}\left(\frac{\alpha_0}{r}\right)^2,$$

and so $I_0(b,b) = 3\sqrt{2\pi}\ \alpha_0^2/4b^2$. We then deduce from (6) and (10), that

$$J_0(b,b) = \frac{1}{b^2}\left[6\sqrt{2\pi} - \frac{3}{4}\sqrt{2\pi}\ \alpha_0^2\right].$$

If we set $u = r^{-1/4}$ and $b = 1$, we get from (10)

$$4\int_0^\infty du\ u\mathbb{N}_0^{(1)}\left[\mathbf{1}_{\{\min(R,|L|)\geq u\}}\right] = 6\sqrt{2\pi} - \frac{3}{4}\sqrt{2\pi}\ \alpha_0^2.$$

Therefore, we have

$$\mathbb{N}_0^{(1)}[\min(R,|L|)^2] = 3\sqrt{2\pi}[1-\alpha_0^2/8].$$

This prove the second equality of corollary 5. $\square$

Proof of the first equality of corollary 5. We look for a transformation of J_λ which will give the expectation of $R\,|L|$. Notice first that for $\lambda > 0$, J_λ is differentiable in the variable λ and that

$$-\partial_\lambda J_\lambda = \int_0^\infty \frac{dr}{r^{3/2}}\, r\, \mathrm{e}^{-\lambda r}\, \mathbb{N}_0^{(1)} \left[\mathbf{1}_{\{R > br^{-1/4},\, |L| > cr^{-1/4}\}} \right] \geq 0.$$

By Fubini, we have

$$\begin{aligned}
-\int_{(0,\infty)^2} \partial_\lambda J_\lambda(c,b)\; dc\, db &= \int_0^\infty \frac{dr}{r^{1/2}}\; \mathrm{e}^{-\lambda r} \int_{(0,\infty)^2} dc\, db\; \mathbb{N}_0^{(1)} \left[\mathbf{1}_{\{R > br^{-1/4},\, |L| > cr^{-1/4}\}} \right] \\
&= \int_0^\infty \frac{dr}{r^{1/2}}\; \mathrm{e}^{-\lambda r} \sqrt{r}\; \mathbb{N}_0^{(1)} [R\,|L|] \\
&= \frac{1}{\lambda} \mathbb{N}_0^{(1)} [R\,|L|].
\end{aligned}$$

Our next task is to compute $\partial_\lambda J_\lambda(c,b)$. From (8), we deduce the following scaling property: for $\rho > 0$,

$$I_\lambda(c,b) = \frac{1}{\rho^2}\, I_{\lambda\rho^4}(c/\rho, b/\rho).$$

Taking $\rho = \lambda^{-1/4}$, we get

$$I_\lambda(c,b) = \sqrt{\lambda} I_1(c\lambda^{1/4}, b\lambda^{1/4}).$$

Of course, we have a similar scaling property for H. Differentiating with respect to λ, we get

$$\begin{aligned}
\partial_\lambda I_\lambda(c,b) &= \partial_\lambda \sqrt{\lambda} I_1(c\lambda^{1/4}, b\lambda^{1/4}) \\
&= \frac{1}{2\sqrt{\lambda}}\, I_1(c\lambda^{1/4}, b\lambda^{1/4}) + \frac{c}{4\lambda^{1/4}} \partial_c I_1(c\lambda^{1/4}, b\lambda^{1/4}) + \frac{b}{4\lambda^{1/4}} \partial_b I_1(c\lambda^{1/4}, b\lambda^{1/4}) \\
&= \frac{1}{\lambda} \left[\frac{1}{2}\, I_\lambda(c,b) + \frac{1}{4} (c\partial_c I_\lambda(c,b) + b\partial_b I_\lambda(c,b)) \right].
\end{aligned}$$

A similar computation yields

$$\partial_\lambda H_\lambda(b) = \frac{1}{\lambda} \left[\frac{1}{2}\, H_\lambda(b) + \frac{1}{4}\, b\partial_b H_\lambda(b) \right].$$

Therefore, we have

$$\begin{aligned}
&- \lambda \partial_\lambda J_\lambda(c,b) \\
&= \frac{1}{4} [2I_\lambda(c,b) - 2H_\lambda(c) - 2H_\lambda(b) + c\partial_c I_\lambda(c,b) + b\partial_b I_\lambda(c,b) - c\partial_c H_\lambda(c) - b\partial_b H_\lambda(b)].
\end{aligned}$$

Now we will study the limit of $C_{\varepsilon,A} = \int_{[\varepsilon,A]^2} dcdb\; (-\lambda\partial_\lambda J_\lambda(c,b))$ as $A \to \infty$ and $\varepsilon \to 0$, since

$$\lim_{\varepsilon \to 0, A \to \infty} C_{\varepsilon,A} = \mathbb{N}_0^{(1)} [R\,|L|]. \tag{13}$$

An integration by parts gives $4C_{\varepsilon,A} = K_1 + K_2 + K_3 + K_4$, where

$$\begin{aligned}
K_1 &= -2(A-\varepsilon)AH_\lambda(A),\\
K_2 &= 2A\int_\varepsilon^A I_\lambda(A,b)\,db - 2A\int_\varepsilon^A H_\lambda(t)\,dt,\\
K_3 &= 2(A-\varepsilon)\varepsilon H_\lambda(\varepsilon) - 2\int_\varepsilon^A \varepsilon I_\lambda(\varepsilon,b)\,db,\\
K_4 &= 2\varepsilon\int_\varepsilon^A H_\lambda(t)\,dt.
\end{aligned}$$

Study of K_1. We have

$$|K_1| \le 2A^2 H_\lambda(A) = 6\sqrt{2\pi}\left(\frac{(2\lambda)^{1/4}A}{\sinh((2\lambda)^{1/4}A)}\right)^2.$$

In particular, we have

$$\lim_{\varepsilon\to 0, A\to\infty} K_1 = 0. \tag{14}$$

Study of K_2. Since $I_\lambda(A,b) \ge H_\lambda(b)$, and since $J_\lambda(A,b) = H_\lambda(A) + H_\lambda(b) - I_\lambda(A,b)$ is nonnegative, we deduce that

$$0 \le I_\lambda(A,b) - H_\lambda(b) \le H_\lambda(A).$$

This implies

$$0 \le K_2 = 2A\int_\varepsilon^A [I_\lambda(A,b) - H_\lambda(b)]\,db \le 2A\int_\varepsilon^A H_\lambda(A)\,db \le 2A^2 H_\lambda(A).$$

From the study of K_1, we deduce that

$$\lim_{\varepsilon\to 0, A\to\infty} K_2 = 0. \tag{15}$$

Study of K_3. Set $\varepsilon t = b$ and use the scaling property of I and H (with $\rho = \varepsilon$) to get

$$\begin{aligned}
-K_3 &= 2\varepsilon\int_\varepsilon^A [I_\lambda(\varepsilon,b) - H_\lambda(\varepsilon)]\;db\\
&= 2\varepsilon^2\int_1^{A/\varepsilon} [I_\lambda(\varepsilon,\varepsilon t) - H_\lambda(\varepsilon)]\;dt\\
&= 2\int_1^{A/\varepsilon} [I_{\varepsilon^4\lambda}(1,t) - H_{\varepsilon^4\lambda}(1)]\;dt\\
&= 2\int_1^{A/\varepsilon} dt\int_0^\infty \frac{dr}{r^{3/2}}\,\mathrm{e}^{-\varepsilon^4\lambda r}\,\mathbb{N}_0^{(1)}\left[\mathbf{1}_{\{R>tr^{-1/4},\,|L|\le r^{-1/4}\}}\right],
\end{aligned}$$

where we used the definition of I and H for the last equality. By monotone convergence, we get that $-K_3$ increases, as $\varepsilon \downarrow 0$ and $A \uparrow \infty$ to $-\tilde{K}_3$, where

$$\begin{aligned}
-\tilde{K}_3 &= 2\int_1^\infty dt\int_0^\infty \frac{dr}{r^{3/2}}\mathbb{N}_0^{(1)}\left[\mathbf{1}_{\{R>tr^{-1/4},\,|L|\le r^{-1/4}\}}\right]\\
&= 2\int_1^\infty [I_0(1,t) - H_0(1)]\;dt.
\end{aligned}$$

Since $I_0(1,t) \geq H_0(1)$ and $J_0(1,t) = H_0(1) + H_0(t) - I_0(1,t)$ is nonnegative, we deduce that

$$0 \leq I_0(1,t) - H_0(1) \leq H_0(t) = \frac{3\sqrt{2\pi}}{t^2}.$$

Hence, we get that $\tilde{K}_3$ is finite. Let us now compute the value of $\tilde{K}_3$. We have for $t \geq 1$, $I_0(1,t) = 2\sqrt{2\pi}\, w_{0,r_0}(1)$, with $r_0 = (1+t)/2$. We set for $t \geq 1$,

$$G(t) = \int_t^\infty \frac{du}{\sqrt{u^3 - 1}}. \tag{16}$$

In particular, we have $\alpha_0 = G(1)$. From (11) and (12), we get that $G\left(\frac{8r_0^2}{3\alpha_0^2}\, w_{0,r_0}(1)\right) = \frac{\alpha_0}{r_0}$. We deduce that for $t \geq 1$,

$$I_0(1,t) = 2\sqrt{2\pi}\,\frac{3}{8}\left(\frac{\alpha_0}{r_0}\right)^2 G^{-1}\left(\frac{\alpha_0}{r_0}\right) = 2\sqrt{2\pi}\,\frac{3}{8}\left(\frac{2\alpha_0}{1+t}\right)^2 G^{-1}\left(\frac{2\alpha_0}{1+t}\right).$$

Thus, we get with $v = 2\alpha_0/(1+t)$,

$$\begin{aligned}
-\tilde{K}_3 &= 2\int_1^\infty dt\, \left[2\sqrt{2\pi}\,\frac{3}{8}\left(\frac{2\alpha_0}{1+t}\right)^2 G^{-1}\left(\frac{2\alpha_0}{1+t}\right) - 3\sqrt{2\pi}\right] \\
&= 3\sqrt{2\pi}\,\alpha_0 \int_0^{\alpha_0} \left[G^{-1}(v) - 4v^{-2}\right]\, dv.
\end{aligned}$$

From (16), it is easy to check that $G(x) = \frac{2}{\sqrt{x}} + O(x^{-7/2})$ as $x \to \infty$, which implies that

$$G^{-1}(v) = \frac{4}{v^2} + O(v^4), \quad \text{as } v \to 0. \tag{17}$$

This development implies that, with $u = G^{-1}(v)$,

$$\begin{aligned}
-\tilde{K}_3 &= \lim_{\varepsilon \to 0} 3\sqrt{2\pi}\,\alpha_0 \int_\varepsilon^{\alpha_0} G^{-1}(v)\, dv - \frac{12\sqrt{2\pi}\,\alpha_0}{\varepsilon} + 12\sqrt{2\pi} \\
&= \lim_{\varepsilon \to 0} 3\sqrt{2\pi}\,\alpha_0 \int_1^{G^{-1}(\varepsilon)} \frac{u\, du}{\sqrt{u^3-1}} - \frac{12\sqrt{2\pi}\,\alpha_0}{\varepsilon} + 12\sqrt{2\pi} \\
&= 12\sqrt{2\pi} + \lim_{\varepsilon \to 0} 3\sqrt{2\pi}\,\alpha_0 \int_1^{4/\varepsilon^2} \frac{u\, du}{\sqrt{u^3-1}} - \frac{12\sqrt{2\pi}\,\alpha_0}{\varepsilon} \\
&= 12\sqrt{2\pi} + 3\sqrt{2\pi}\,\alpha_0 \lim_{\varepsilon \to 0} \int_1^{4/\varepsilon^2} \left[\frac{u}{\sqrt{u^3-1}} - \frac{1}{\sqrt{u-1}}\right] du \\
&= 12\sqrt{2\pi} - 3\sqrt{2\pi}\,\alpha_0 \int_1^\infty \frac{(u+1)\, du}{\sqrt{u^3-1}(u+\sqrt{u^2+u+1})}.
\end{aligned}$$

Therefore, we deduce that

$$\lim_{\varepsilon \to 0, A \to \infty} K_3 = -12\sqrt{2\pi} + 3\sqrt{2\pi} \int_1^\infty \frac{dt}{\sqrt{t^3-1}} \int_1^\infty \frac{(u+1)\, du}{\sqrt{u^3-1}(u+\sqrt{u^2+u+1})}. \tag{18}$$

Study of K_4. We have for $A \geq 1$,

$$K_4 = 2\varepsilon \int_\varepsilon^A H_\lambda(t)\, dt = 2\varepsilon \int_\varepsilon^A \frac{6\sqrt{\pi\lambda}}{\sinh((2\lambda)^{1/4} t)^2}\, dt$$

$$= 2\varepsilon \left[\frac{-6\sqrt{\pi\lambda}}{(2\lambda)^{1/4}} \coth((2\lambda)^{1/4} t) \right]_\varepsilon^A$$

$$= 6\sqrt{2\pi} + O(\varepsilon).$$

Thus we have

$$\lim_{\varepsilon \to 0, A \to \infty} K_4 = 6\sqrt{2\pi}. \tag{19}$$

Conclusion. Eventually, we deduce from (13), (14), (15), (18) and (19), that

$$\mathbb{N}_0^{(1)}[R\,|L|] = \frac{1}{4}\left[-6\sqrt{2\pi} + 3\sqrt{2\pi} \int_1^\infty \frac{dt}{\sqrt{t^3-1}} \int_1^\infty \frac{(u+1)\, du}{\sqrt{u^3-1}(u+\sqrt{u^2+u+1})} \right].$$

□

6. Proof of lemma 6

We introduce the special Markov property for $X^{(a,b)}$ and we refer to [6] for the complete theory. We set

$$\eta_s = \inf\{t; \int_0^t du\, \mathbf{1}_{\{\zeta_u \leq \tau(W_u)\}} > s\},$$

where $\tau(w) = \inf\{t \in [0, \zeta_w], w(t) \notin (a,b)\}$ is the first exit time of (a,b) for the path $w \in \mathcal{W}$. We define the continuous process W'_s by $W'_s = W_{\eta_s}$. By definition, the σ-field $\mathcal{E}^{(a,b)}$ is generated by $W' = (W'_s, s \geq 0)$ and all the $\mathbb{N}_x$-negligible sets.

From proposition 2.3 in [6], we get that $X^{(a,b)}$ is $\mathcal{E}^{(a,b)}$-measurable. Notice that $\mathbb{N}_x$-a.e., $\{s \geq 0; \zeta_s = \tau(W_s)\}$ is of Lebesgue measure zero and

$$\int_0^\sigma \mathbf{1}_{\{\tau(W_s)=\infty\}}\, ds = \int_0^\sigma \mathbf{1}_{\{\tau(W_s)\geq \zeta_s\}}\, ds = \int_0^\infty \mathbf{1}_{\{W'_s \neq x\}}\, ds.$$

In particular the integral $\int_0^\sigma \mathbf{1}_{\{\tau(W_s)=\infty\}}\, ds$ is $\mathcal{E}^{(a,b)}$-measurable.

The random open set $\{s \in (0,\sigma); \tau(W_s) < \zeta_s\}$ is a countable union of open sets, say $\bigcup_{i\in I}(a_i, b_i)$. Since the set $\{s \in (0,\sigma); \tau(W_s) = \zeta_s\}$ is of Lebesgue measure zero $\mathbb{N}_x$-a.e., we get that $\mathbb{N}_x$-a.e.,

$$\int_0^\sigma \mathbf{1}_{\{\tau(W_s)<\infty\}}\, ds = \int_0^\sigma \mathbf{1}_{\{\tau(W_s)<\zeta_s\}}\, ds = \sum_{i\in I}(b_i - a_i).$$

In particular, we deduce from the special Markov property, theorem 2.4 in [6], that

$$\mathbb{N}_x\left[\mathrm{e}^{-\lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)<\infty\}}\, ds} |\mathcal{E}^{(a,b)} \right] = \mathbb{N}_x\left[\mathrm{e}^{-\lambda \sum_{i\in I}(b_i-a_i)} |\mathcal{E}^{(a,b)} \right] = \mathrm{e}^{-\int X^{(a,b)}(dx)\, \mathbb{N}_x[1-\mathrm{e}^{-\lambda\sigma}]}.$$

From (2) we get that

$$\mathbb{N}_x\left[\mathrm{e}^{-\lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)<\infty\}}\, ds} |\mathcal{E}^{(a,b)} \right] = \mathrm{e}^{-\sqrt{\lambda/2}\, Y^{(a,b)}},$$

where $Y^{(a,b)} = \int X^{(a,b)}(dx)$. This implies that for $a < x < b$, and $\lambda \geq 0, \mu > 0$,

$$\begin{aligned} v_{\mu,\lambda,a,b}(x) &= \mathbb{N}_x\left[1 - \mathrm{e}^{-\mu Y^{(a,b)} - \lambda\sigma}\right] \\ &= \mathbb{N}_x\left[1 - \mathrm{e}^{-\mu Y^{(a,b)} - \lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)=\infty\}}\, ds - \lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)<\infty\}}\, ds}\right] \\ &= \mathbb{N}_x\left[1 - \mathrm{e}^{-\mu Y^{(a,b)} - \lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)=\infty\}}\, ds}\, \mathbb{N}_x\left[\mathrm{e}^{-\lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)<\infty\}}\, ds} | \mathcal{E}^{(a,b)}\right]\right] \\ &= \mathbb{N}_x\left[1 - \mathrm{e}^{-(\mu+\sqrt{\lambda/2}) Y^{(a,b)} - \lambda \int_0^\sigma \mathbf{1}_{\{\tau(W_s)=\infty\}}\, ds}\right]. \end{aligned}$$

We then consider the continuous additive functional of the Brownian snake,

$$dL_s = (\mu + \sqrt{\lambda/2})\; dL_s^{(a,b)} + \lambda \mathbf{1}_{\{\tau(W_s)=\infty\}}\; ds.$$

Of course, we have

$$v_{\mu,\lambda,a,b}(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-L_\sigma}\right] = \mathbb{N}_x\left[\int_0^\sigma dL_s\;\; \mathrm{e}^{-\int_s^\sigma dL_u}\right].$$

Now we replace $\exp(-\int_s^\sigma dL_u)$ by its predictable projection with respect to the filtration of the Brownian snake. Let $\mathbb{E}^*_w$ be the law of the Brownian snake started at the path (w, ζ_w), and whose lifetime is distributed according to a linear Brownian motion started at point ζ_w and stopped as it reaches 0. The predictable projection of $\exp\left(-\int_s^\sigma dL_u\right)$ is given by $\mathbb{E}^*_{W_s}[\mathrm{e}^{-L_\sigma}]$. From proposition 2.1 in [6], we get that

$$\mathbb{E}^*_w\left[\mathrm{e}^{-L_\sigma}\right] = \mathrm{e}^{-2\int_0^{\zeta_w} dt\; \mathbb{N}_{w(t)}\left[1 - \mathrm{e}^{-L_\sigma}\right]}.$$

Therefore, we get that

$$\begin{aligned} v_{\mu,\lambda,a,b}(x) &= \mathbb{N}_x\left[\int_0^\sigma dL_s\;\; \mathrm{e}^{-2\int_0^{\zeta_s} v_{\mu,\lambda,a,b}(W_s(t))\; dt}\right] \\ &= (\mu + \sqrt{\lambda/2})\mathbb{N}_x\left[\int_0^\sigma dL_s^{(a,b)}\;\; \mathrm{e}^{-2\int_0^{\zeta_s} v_{\mu,\lambda,a,b}(W_s(t))\; dt}\right] \\ &\qquad + \lambda \mathbb{N}_x\left[\int_0^\sigma \mathbf{1}_{\{\tau(W_s)=\infty\}} ds\;\; \mathrm{e}^{-2\int_0^{\zeta_s} v_{\mu,\lambda,a,b}(W_s(t))\; dt}\right]. \end{aligned}$$

Let us recall the first moment formula for the Brownian snake: for F a nonnegative measurable function defined on $\mathcal{W}$, we have

$$\mathbb{N}_x\left[\int_0^\sigma F(W_s, \zeta_s) ds\right] = \int_0^\infty dr\; \mathbb{E}_x[F((B_t, t \in [0, r]), r)],$$

and

$$\mathbb{N}_x\left[\int_0^\sigma F(W_s, \zeta_s) dL_s^{(a,b)}\right] = \mathbb{E}_x[F((B_t, t \in [0, \tau]), \tau)],$$

where B is under $\mathbb{E}_x$ a linear Brownian motion started at point x and $\tau = \inf\{t \geq 0; B_t \notin (a,b)\}$.

In particular this implies that $v_{\mu,\lambda,a,b}$ is a nonnegative solution of

$$v(x) = (\mu + \sqrt{\lambda/2})\mathbb{E}_x\left[\mathrm{e}^{-2\int_0^\tau v(B_t) dt}\right] + \lambda \mathbb{E}_x\left[\int_0^\tau dr\;\; \mathrm{e}^{-2\int_0^r v(B_t) dt}\right].$$

From standard arguments on Brownian motion, we deduce lemma 6.

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