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# METASTABILITY OF THE THREE DIMENSIONAL ISING MODEL ON A TORUS AT VERY LOW TEMPERATURES

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# METASTABILITY OF THE THREE DIMENSIONAL ISING MODEL ON A TORUS AT VERY LOW TEMPERATURES

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Abstract. We study the metastability of the stochastic three dimensional Ising model on a finite torus under a small positive magnetic field at very low temperatures.

## 1. Introduction

The theoretical study of metastability has become a field of active research during the last years [2,12,13,16,17,19,20]. The crucial problem is to understand the exit path of a system close to a phase transition relaxing from a metastable state towards a stable state. For the physical motivation of this problem, we refer the reader to [19,20]. The basic situation to investigate the metastability is the stochastic Ising model evolving according to a Glauber or Metropolis dynamics. The first step was accomplished by Neves and Schonmann [16,17,20] who studied the case of the two dimensional Ising model in a finite volume at very low temperatures. They put forward the essential role played by the droplets: small droplets are likely to shrink and disappear whereas big ones tend to grow. The threshold between these opposite behaviours corresponds to the critical droplets whose energy is exactly the energy barrier the system has to overcome to escape from the metastable state. The nucleation occurs through the formation of such a critical droplet. Several more complicated models have also been investigated in the regime of low temperatures, but still in dimension two: an anisotropic case [12] and a case with second nearest

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neighbours interactions [13]. It became progressively clear that all the relevant information to describe the exit path was to be found in the energy landscape associated to the model.

In fact, these Metropolis dynamics at very low temperatures have been thoroughly studied in a more abstract setting in the context of stochastic optimization. Large deviations techniques and the Wentzell–Freidlin theory provide the most powerful tools to analyze these dynamics depending upon a small parameter. The central idea is to decompose the space into a hierarchy of particular subsets called cycles [11]. Roughly speaking, the cycles are the most stable sets we can build for the perturbed process. Their fundamental property is that the process can't escape from a cycle without having forgotten its entrance point. The limiting dynamics is completely determined by the cycle decomposition. For simulated annealing, the best way to cool down the temperature schedule heavily depends upon some quantities associated to the cycle decomposition [3,4,5]. For parallel simulated annealing and generalizations, see also [21,22,23,24]. For genetic algorithms, the cycle decomposition is the key for understanding the behaviour of the process and for controlling the algorithm in order to ensure the convergence towards the global maxima [7,8,9]. For the study of metastability, a general model independent analysis of the exit path for reversible processes was proposed in [18]. In [6] we handled the general case of Markov chains with rare transitions and developed a general method to search efficiently for the exit path. Our technique focuses on the crucial elements of the cycle decomposition necessary to find the exit path. More precisely, we seek only the sequence of cycles the system might visit during the exit. Such a precise and economical method is indispensable to deal with the three dimensional Ising model: the energy landscape is so huge and complex that the least inaccuracy, the least deviation from the metastable path is disastrous and leads to inextricable problems. To apply our general method to a particular model, we need to solve some global and local variational problems. This combinatorial part for the three dimensional Ising model was handled in [1]. Notice that one has to piece together carefully the variational results in order to find the exit path.

Neves has obtained the first important results concerning the $d$–dimensional case in [14]. Using an induction on the dimension, he proves the $d$–dimensional discrete isoperimetric inequality from which he deduces the asymptotic behaviour of the relaxation time. He analyzes also the behaviours of regular droplets in dimension three [15]. However to obtain full information on the exit path one needs more refined variational statements (for instance uniqueness of some minimal shapes) together with a precise investigation of the energy landscape near these minimal shapes. We examine all the configurations which communicate with the metastable state under the global energy barrier (i.e. the greatest cycle containing the metastable state and not the stable state). The configurations of the principal boundary of this cycle which lead to the stable state are the critical droplets, the others are dead–ends: although they are likely to be visited before the relaxation time, they are not likely to be visited during the exit path. This notion of dead–ends was not explicitly apparent in the treatment of the two dimensional case by Neves and Schonmann

[16,17] because they managed in fact to characterize completely the basins of attraction of the metastable state and the stable state. We do not know how to achieve a corresponding result in dimension three and therefore we analyze only the set of configurations visited by the process before relaxation.

This paper is organized as follows. We first describe the model and we recall the main definitions concerning the cycle decomposition. We describe the general method for finding the exit path. We prove some essential facts for the cycles of the Ising model which are valid in any dimension. We then handle the two dimensional case. All the important results of this section have already been proved by Neves and Schonmann [16,17,20]. We reprove them using a technique which will also work in dimension three, for two main reasons: we need to state these results in the language of the cycles in order to understand the dimension three (where the droplets grow by following the two dimensional nucleation mechanism) and moreover we hope that it will make the reading of the three dimensional case easier (especially since there are much more cycles to analyze). We finally handle the three dimensional situation. Synthetic descriptions of the exit path in two and three dimensions are given in theorems 6.32 and 7.36.

## 2. Description of the model

We consider a finite box $\Lambda = \{1 \cdots N\}^d$ of side $N$ in the $d$–dimensional integer lattice $\mathbb{Z}^d$. We will work either in dimension two ($d = 2$) or dimension three ($d = 3$). A point of this box is called a site. Sites will be denoted by the letters $x$, $y$, $z$. We wrap this box into a torus and we define a neighbourhood relation on $\Lambda$ by: $x \sim y$ if all the coordinates of $x$ and $y$ are equal except one which differs by 1 or $N - 1$. At each site $x$ of $\Lambda$ there is a spin taking the values $-1$ or $+1$. The set of all possible spins configurations is $X = \{-1, +1\}^\Lambda$. Configurations of spins will be denoted by the letters $\eta, \sigma, \xi$. The value of the spin at site $x$ for a configuration $\sigma$ is denoted by $\sigma(x)$.

We might see a configuration as a subset of points of the integer lattice $\mathbb{Z}^d$, these points being the sites of the configurations having a positive spin. The inclusion relation is then defined on the configurations by

$$\eta \subset \sigma \quad \Longleftrightarrow \quad \forall x \in \Lambda \quad \eta(x) = +1 \Rightarrow \sigma(x) = +1 \,.$$

The elementary sets operations $\cap$, $\cup$, $\setminus$ are defined by

$$\forall x \in \Lambda \qquad (\sigma \cup \eta)(x) \;=\; \begin{cases} +1 & \text{if } \sigma(x) = +1 \text{ or } \eta(x) = +1 \\ -1 & \text{if } \sigma(x) = \eta(x) = -1 \end{cases}$$

$$\forall x \in \Lambda \qquad (\sigma \cap \eta)(x) \;=\; \begin{cases} +1 & \text{if } \sigma(x) = \eta(x) = +1 \\ -1 & \text{if } \sigma(x) = -1 \text{ or } \eta(x) = -1 \end{cases}$$

$$\forall x \in \Lambda \qquad (\sigma \setminus \eta)(x) \;=\; \begin{cases} +1 & \text{if } \sigma(x) = +1 \text{ and } \eta(x) = -1 \\ -1 & \text{if } \sigma(x) = -1 \text{ or } \eta(x) = +1 \end{cases}$$

There is a natural correspondence between configurations and polyominoes. To a configuration we associate the polyomino which is the union of the unit cubes centered at the sites having a positive spin. The main difference between configurations and polyominoes is that the polyominoes are defined up to translations. We will constantly use the notation introduced in [1] for polyominoes: the relevant definitions are summed up at the beginning of the sections 6 and 7. When we say that a polyomino is included in a configuration, we mean that one of its representative is included in it. For instance, the square $l \times l$ is included in $\sigma$ if $\sigma(x) = +1$ for all $x$ belonging to some square of side $l$ (equivalently, the polyomino associated to $\sigma$ contains $l \times l$).

The energy of a configuration $\sigma$ is

$$E(\sigma) \;=\; -\frac{1}{2} \sum_{\{x,y\}:x\sim y} \sigma(x)\sigma(y) \;-\; \frac{h}{2} \sum_{x\in\Lambda} \sigma(x)$$

where $h > 0$ is the external magnetic field.
The Gibbs probability measure $\mu$ at inverse temperature $\beta$ is

$$\forall \sigma \in X \qquad \mu(\sigma) \;=\; \frac{1}{Z} \exp -\beta E(\sigma)$$

where $Z$ is the partition function.
We build the stochastic Ising model by letting the spins evolve randomly in time. The spin at site $x$ flips with the rate $c(x,\sigma)$ when the system is in the configuration $\sigma$. The standard construction yields a continuous time Markov process $(\sigma_t)_{t\geq 0}$ with state space $X$ and infinitesimal generator $L$ defined by

$$\forall f : \Lambda \to \mathbb{R} \qquad (Lf)(\sigma) \;=\; \sum_{x\in\Lambda} c(x,\sigma)\big(f(\sigma^x) - f(\sigma)\big)$$

where $\sigma^x$ is the configuration $\sigma$ flipped at $x$ i.e.

$$\sigma^x(y) \;=\; \begin{cases} +\sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x \end{cases}$$

We will consider the Metropolis dynamics associated to the rates

$$c(x,\sigma) = \exp -\beta(\Delta_x E(\sigma))^+$$

where $(\quad)^+$ denotes the positive part and

$$\Delta_x E(\sigma) \;=\; E(\sigma^x) - E(\sigma) \;=\; \sigma(x)\big(\sum_{y:y\sim x} \sigma(y) + h\big).$$

Since these rates satisfy the detailed balance condition $\mu(\sigma)c(x,\sigma) = \mu(\sigma^x)c(x,\sigma^x)$, we know that the Gibbs measure $\mu$ is the only invariant measure for the process $(\sigma_t)$.
Here we will deal with the equivalent discrete time version of the Metropolis dynamics. We first define a kernel $q$ on $X$ by $q(\eta,\sigma) = N^{-d}$ if there exists $x$ in $\Lambda$ such that $\eta^x = \sigma$ (i.e $\eta$ and $\sigma$ differ by one spin flip) and $q(\eta,\sigma) = 0$ otherwise. The kernel $q$ is irreducible and symmetric. The transition kernel $p_\beta$ at inverse temperature $\beta$ is the Markov kernel on $X$ defined by

$$\begin{aligned} \forall \eta \neq \sigma \qquad p_\beta(\eta,\sigma) &= q(\eta,\sigma) \exp -\beta\big(E(\sigma) - E(\eta)\big)^+ \\ \forall \eta \qquad p_\beta(\eta,\eta) &= 1 - \sum_{\sigma\neq\eta} p_\beta(\eta,\sigma). \end{aligned}$$

We will study the time homogeneous Markov chain $(\sigma_n)_{n\in\mathbb{N}}$ having for transition probabilities the Markov kernel $p_\beta$.
We denote by $\underline{-1}$ (respectively $\underline{+1}$) the configuration with all spins down (resp. up).
For $D$ an arbitrary subset of $X$ we define the time $\tau(D,m)$ of exit from $D$ after time $m$

$$\tau(D,m) = \min\{\, n \geq m : \sigma_n \notin D \,\}$$

(we make the convention that $\tau(D) = \tau(D,0)$).
For instance, $\tau(\underline{+1}^c)$ is the hitting time of the ground state $\underline{+1}$.
We define also the time $\theta(D,m)$ of the last visit to the set $D$ before time $m$

$$\theta(D,m) = \max\{\, n \leq m : \sigma_n \in D \,\}$$

(if the chain has not visited $D$ before $m$, we take $\theta(D,m) = 0$).
For instance, $\theta(\underline{-1},\tau(\underline{+1}^c))$ is the last visit to the metastable state $\underline{-1}$ before reaching $\underline{+1}$.
We are interested in the laws of $\tau(\underline{+1}^c)$ starting from $\underline{-1}$ (time of the relaxation to equilibrium) and of the way of escaping from the metastable state $\underline{-1}$ to reach $\underline{+1}$, that is the law of the exit path $((\sigma_n),\ \theta(\underline{-1},\tau(\underline{+1}^c)) \leq n \leq \tau(\underline{+1}^c))$, in the limit of vanishing temperature $\beta \to \infty$.

## 3. Decomposition into cycles

The most efficient tool to analyze the dynamical behaviour of a Markov chain with exponentially vanishing transition probabilities is the hierarchical decomposition of the state space into cycles. This fundamental notion is due to Freidlin and Wentzell [10] and was sharpened by Hwang and Sheu for discrete spaces [11]. Here we will deal with an energy landscape $(X,q,E)$ (that is a finite set $X$ equipped with a communication kernel $q$ and the energy $E$), as in the case of the simulated annealing. We recall here briefly the main definitions introduced by Catoni in this context [3].

**Definition 3.1.** (communication at level $\lambda$)
Let $\lambda$ be a real number. The communication relation at level $\lambda$ on the landscape $(X, q, E)$ is the equivalence relation $\mathcal{R}_\lambda$ defined by $\sigma\, \mathcal{R}_\lambda\, \eta$ if and only if either $\sigma = \eta$ or there exists a sequence $\xi_0, \cdots, \xi_r$ of points in $X$ such that

$$\xi_0 = \sigma,\ \xi_r = \eta, \quad \forall i \in \{0 \cdots r-1\} \quad q(\xi_i, \xi_{i+1}) > 0, \quad \forall i \in \{0 \cdots r\} \quad E(\xi_i) \leq \lambda.$$

**Definition 3.2.** (cycles)
A cycle $\pi$ is a subset of $X$ which is an equivalence class of a relation $\mathcal{R}_\lambda$ for some $\lambda$ in $\mathbb{R}$. The smallest such $\lambda$ is denoted by $\lambda(\pi)$ and is called the level of the cycle $\pi$. For $\sigma$ in $X$ and $\lambda$ in $\mathbb{R}$ we denote by $\pi(\sigma, \lambda)$ the equivalence class of $\sigma$ for the relation $\mathcal{R}_\lambda$.

**Proposition 3.3.** *Two cycles are either disjoint or comparable for the inclusion relation.*

**Notation 3.4.** Since $X$ is finite, the set $\{\, E(\sigma) : \sigma \in X \,\}$ is also finite. For $\lambda$ in $\mathbb{R}$ we set

$$\begin{aligned} \operatorname{pred} \lambda &= \max\{\, E(\sigma) : E(\sigma) < \lambda,\ \sigma \in X \,\}, \\ \operatorname{succ} \lambda &= \min\{\, E(\sigma) : E(\sigma) > \lambda,\ \sigma \in X \,\}. \end{aligned}$$

**Definition 3.5.** (maximal partition)
Let $D$ be a subset of $X$. A subcycle of $D$ is a cycle which is included in $D$. The partition of $D$ into its maximal subcycles is denoted by $\mathcal{M}(D)$. For $\sigma$ in $D$ we denote by $\pi(\sigma, D)$ the unique cycle of $\mathcal{M}(D)$ containing $\sigma$. For $\sigma$ in $D^c$ we set by convention that $\pi(\sigma, D) = \{\sigma\}$.

*Remark.* In case $D$ is a cycle, we have $\mathcal{M}(D) = \{D\}$.

*Remark.* There is no ambiguity between the two notations $\pi(\sigma, \lambda)$ and $\pi(\sigma, D)$, since $\lambda$ and $D$ are different objects.

**Definition 3.6.** (communication altitude)
The communication altitude between two points $\sigma$ and $\eta$ of $X$ is the smallest $\lambda$ such that $\sigma\, \mathcal{R}_\lambda\, \eta$. It is denoted by $E(\sigma, \eta)$. Equivalently, it is the level of the smallest cycle containing the points $\sigma$ and $\eta$. The communication altitude between two sets $A$ and $B$ is defined by $E(A, B) = \min\{\, E(a, b) : a \in A,\ b \in B \,\}$.

**Definition 3.7.** Let $\pi$ be a cycle. We define
- its energy $E(\pi) = \min\{\, E(\sigma) : \sigma \in \pi \,\}$,
- its bottom $F(\pi) = \{\, \sigma \in \pi : E(\sigma) = E(\pi) \,\}$,
- its boundary $B(\pi) = \{\, \sigma \notin \pi : \exists\, \eta \in \pi \quad q(\eta, \sigma) > 0 \,\}$,
- its height $H(\pi) = \min\{\, (E(\sigma) - E(\pi))^+ : \sigma \in B(\pi) \,\}$,
- its principal boundary $\widetilde{B}(\pi) = \{\, \sigma \in B(\pi) : E(\sigma) \leq E(\pi) + H(\pi) \,\}$.

*Remark.* If the height $H(\pi)$ is positive, then the principal boundary $\widetilde{B}(\pi)$ is exactly the set $\{\, \sigma \in B(\pi) : E(\sigma) = E(\pi) + H(\pi) \,\}$. The height $H(\pi)$ can vanish only in the case when the cycle $\pi$ is a singleton.

The main part of the work here consists in analyzing the specific energy landscape of the Ising model and some crucial facts concerning its cycle decomposition.

## 4. The exit saddle path

In this section, we make a summary of the general strategy devised in [6] in order to describe the exit path of the process. We take into account the simplifications induced by the reversibility of the process and just state the essential points. A good way to describe the exit path is to look at the points of entrance and exit of the cycles of $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$ the system visits during the exit path. The knowledge of this sequence of points includes the knowledge of the sequence of the cycles visited by the process. Our work will consists in finding the most probable of these sequences i.e. those whose probability does not vanish as $\beta$ goes to infinity.

**Definition 4.1.** (saddle path)
We define recursively a sequence of random times and points:

$$\begin{array}{lll}
\tau_0 = \theta(\underline{-1}, \tau(\underline{+1}^c)) + 1 & s_0 = \sigma_{\tau_0 - 1} & s_1 = \sigma_{\tau_0} \\
\tau_1 = \tau(\pi(s_1, \{\underline{-1},\underline{+1}\}^c), \tau_0) & s_2 = \sigma_{\tau_1 - 1} & s_3 = \sigma_{\tau_1} \\
\quad\vdots & \quad\vdots & \quad\vdots \\
\tau_k = \tau(\pi(s_{2k-1}, \{\underline{-1},\underline{+1}\}^c), \tau_{k-1}) & s_{2k} = \sigma_{\tau_k - 1} & s_{2k+1} = \sigma_{\tau_k} \\
\quad\vdots & \quad\vdots & \quad\vdots \\
\tau_r = \tau(\underline{+1}^c) & s_{2r} = \sigma_{\tau_r - 1} & s_{2r+1} = \sigma_{\tau_r}.
\end{array}$$

Clearly we have $s_0 = \underline{-1}$ and $s_{2r+1} = \underline{+1}$. The sequence $(\underline{-1}, s_1, \cdots, s_{2r}, \underline{+1})$ is called the saddle path of $(\sigma_n)$ relative to $\underline{+1}^c, \underline{-1}$. We define a cost function $V$ on the set of the saddle paths:

$$\begin{aligned}
V(s_0, \cdots, s_{2r+1}) \; = \; & -H(\pi(\underline{-1}, \underline{+1}^c)) \; + \; \big(E(s_1) - E(s_0)\big)^+ + \\
\sum_{k=1}^{r} \Big( & E(s_{2k}) + \big(E(s_{2k+1}) - E(s_{2k})\big)^+ - H(\pi(s_{2k}, \{\underline{-1},\underline{+1}\}^c)) - E(\pi(s_{2k}, \{\underline{-1},\underline{+1}\}^c))\Big).
\end{aligned}$$

We proved in [6] that the law of the saddle path satisfies a large deviation principle with rate function $V$. As a consequence, with a probability converging to one exponentially fast as $\beta$ goes to infinity, the exit saddle path follows a saddle path of null cost. What we have to do is to search for all these saddle paths. Because we are working with the maximal subcycles of $\{\underline{-1},\underline{+1}\}^c$, the saddles of the saddle path are min-max points between the cycles visited and $\{\underline{-1},\underline{+1}\}$ (see [18]). The task of determining the set of all the saddle paths of null $V$–cost can be performed in the following way.

**General strategy to find the exit path.**

$i)$ find the points $\sigma'$ of the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$;

$ii)$ for each such point $\sigma'$ in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$, determine all the sequences of cycles $\pi_0 \cdots \pi_r$ in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ such that $\sigma' \in \pi_0$, $\widetilde{B}(\pi_k) \cap \pi_{k+1} \neq \emptyset$ for $k$ in $\{0 \cdots r-1\}$, $\underline{+1} \in \widetilde{B}(\pi_r)$.

$iii)$ for each such cycle path $\pi_0, \cdots, \pi_r$ determine all the saddle paths $(\sigma', s_1, \cdots, s_{2r}, \underline{+1})$ such that $(s_{2k-1}, s_{2k}) \in \pi_k \times \pi_k$ for $k$ in $\{1 \cdots r\}$, and $(s_{2k}, s_{2k+1})$ is an optimal saddle exiting from $\pi_k$ for all $k$ in $\{0 \cdots r\}$ i.e. $s_{2k} \in \pi_k$, $s_{2k+1} \in \widetilde{B}(\pi_k)$, $q(s_{2k}, s_{2k+1}) > 0$.

$iv)$ for each point $\sigma'$ in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ such that step $ii)$ has succeeded, determine all the sequences of cycles $\pi'_0 \cdots \pi'_{r'}$ in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ such that $\sigma' \in \pi'_0$, $\widetilde{B}(\pi'_k) \cap \pi'_{k+1} \neq \emptyset$ for $k$ in $\{0 \cdots r'-1\}$, $\underline{-1} \in \widetilde{B}(\pi_{r'})$.

$v)$ for each such cycle path $\pi'_0, \cdots, \pi'_{r'}$ determine all the saddle paths $(\sigma', s'_1, \cdots, s'_{2r'}, \underline{-1})$ such that $(s'_{2k-1}, s'_{2k}) \in \pi'_k \times \pi'_k$ for $k$ in $\{1 \cdots r'\}$, and $(s'_{2k}, s'_{2k+1})$ is an optimal saddle exiting from $\pi'_k$ for all $k$ in $\{0 \cdots r'\}$ i.e. $s'_{2k} \in \pi'_k$, $s'_{2k+1} \in \widetilde{B}(\pi'_k)$, $q(s'_{2k}, s'_{2k+1}) > 0$.

$vi)$ the set of all the saddle paths of null $V$–cost passing trough $\sigma'$ is the set of the saddle paths $(s'_{2r'+1}, \cdots, s'_1, \sigma', \sigma', s_1, \cdots, s_{2r+1})$ obtained by glueing together a reversed saddle path obtained at step $v)$ with a saddle path obtained at step $iii)$.

The global variational problem consists in finding the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$. The local variational problems consist in finding the principal boundaries of the cycles appearing in the cycle paths starting at the points of this principal boundary. Notice that we took advantage of the reversibility of the process to complete steps $iv)$ and $vi)$. We used the fact that the portion of the saddle path between $\underline{-1}$ and $\sigma'$ in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ can be obtained by reversing a saddle path starting at $\sigma'$ and reaching $\underline{-1}$ (see last section of [6] as well as [20]). In the general case, the variational condition imposed on this portion of the saddle path is not easy to manipulate.

Remark that step $ii)$ may fail: it might happen that there is no cycle path of null cost in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ starting at $\sigma'$ reaching $\underline{+1}$. In that case, there is no saddle path of null cost passing through $\sigma'$ and it is not necessary to perform the step $v)$: such a point $\sigma'$ is a dead–end.

**Definition 4.2.** A point $\sigma$ is a dead–end to go from $\underline{-1}$ to $\underline{+1}$ if $\sigma$ belongs to $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ and there does not exist a sequence of cycles $\pi_0 \cdots \pi_r$ in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ such that $\sigma \in \pi_0$, $\widetilde{B}(\pi_k) \cap \pi_{k+1} \neq \emptyset$ for $k$ in $\{0 \cdots r-1\}$, $\underline{+1} \in \widetilde{B}(\pi_r)$.

Our strategy to discover the exit path is aimed at analyzing only the relevant features of the energy landscape. However it seems to us that it is impossible to avoid considering the dead–ends, if one wishes to determine completely the points the process is likely to visit during the exit path.

Let us describe more formally this strategy. On the set of all the cycles, we consider the graph $G$ defined by

$$(\pi_1 \to \pi_2) \in G \qquad \Longleftrightarrow \qquad \pi_1 \cap \pi_2 = \emptyset, \quad \widetilde{B}(\pi_1) \cap \pi_2 \neq \emptyset.$$

We denote by $G^+$ the restriction of $G$ to $\pi(\underline{-1}, \underline{+1}^c)^c$ obtained by deleting all the arrows whose starting cycle is included in $\pi(\underline{-1}, \underline{+1}^c)$ i.e.

$$(\pi_1 \to \pi_2) \in G^+ \qquad \Longleftrightarrow \qquad (\pi_1 \to \pi_2) \in G, \quad \pi_1 \cap \pi(\underline{-1}, \underline{+1}^c) = \emptyset.$$

Symmetrically we denote by $G^-$ the restriction of $G$ to $\pi(\underline{+1}, \underline{-1}^c)^c$ obtained by deleting all the arrows whose starting cycle is included in $\pi(\underline{+1}, \underline{-1}^c)$ i.e.

$$(\pi_1 \to \pi_2) \in G^- \qquad \Longleftrightarrow \qquad (\pi_1 \to \pi_2) \in G, \quad \pi_1 \cap \pi(\underline{+1}, \underline{-1}^c) = \emptyset.$$

**Definition 4.3.** Let $I$ be a graph over the set of cycles and let $\pi$ be a vertex of $I$. The orbit $O(\pi, I)$ of $\pi$ in $I$ is the set

$$O(\pi, I) = \{\, \pi' : \exists \pi_1, \cdots, \pi_r \text{ cycles}, \, \pi_1 = \pi, \, \pi_r = \pi', \, (\pi_k \to \pi_{k+1}) \in I, \, 1 \leq k < r \,\}.$$

We denote by $I(\pi)$ the minimal stable subgraph of $I$ containing $\pi$; it is the restriction of $I$ to the orbit of $\pi$ in $I$ i.e.

$$(\pi_1 \to \pi_2) \in I(\pi) \qquad \Longleftrightarrow \qquad (\pi_1 \to \pi_2) \in I, \quad \pi_1 \in O(\pi, I).$$

Step $ii$) consists in determining the graph $G^+(\{\sigma'\})$ for each point $\sigma'$ in the principal boundary of $\pi(\underline{-1}, \underline{+1}^c)$. If $\{\underline{+1}\}$ does not belong to $O(\{\sigma'\}, G^+)$, then $\sigma'$ is a dead–end and the last exit from $\pi(\underline{-1}, \underline{+1}^c)$ before reaching $\underline{+1}$ won't take place through $\sigma'$, although $\sigma'$ is likely to be visited before relaxation. Otherwise, $\sigma'$ is a possible global saddle between $\underline{-1}$ and $\underline{+1}$ and we determine all the paths in $G^+(\{\sigma'\})$ leading from $\{\sigma'\}$ to $\{\underline{+1}\}$. Then we must search for all the cycle paths in $\{\underline{-1}, \underline{+1}\}^c$ of null cost realizing the exit of $\pi(\underline{-1}, \underline{+1}^c)$ at $\sigma'$. More precisely, in step $iv$) we determine all the paths in the graph $G^-(\{\sigma'\})$ starting at $\{\sigma'\}$ and ending at $\{\underline{-1}\}$.

To achieve these goals, we will describe a list of relevant cycles, which are included in $\{\underline{-1}, \underline{+1}\}^c$ and we will precise their principal boundaries. Notice that we don't know *a priori* that these cycles are in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. We will then use the following simple result.

**Lemma 4.4.** *Let $\pi_0, \cdots, \pi_r$ be a sequence of cycles included in $\{\underline{-1}, \underline{+1}\}^c$ such that $(\pi_i \to \pi_{i+1}) \in G$ for $i$ in $\{0 \cdots r-1\}$ and $\widetilde{B}(\pi_r) \cap \{\underline{-1}, \underline{+1}\} \neq \emptyset$. Then all the cycles $\pi_0, \cdots, \pi_r$ are maximal cycles in $\{\underline{-1}, \underline{+1}\}^c$.*

*Proof.* It is enough to prove that $\pi_0$ is a maximal cycle of $\{\underline{-1}, \underline{+1}\}^c$, since $\pi_1, \cdots, \pi_r$ satisfy a similar hypothesis. Let $\pi$ be a cycle containing strictly $\pi_0$. Necessarily, $\pi$ contains $\widetilde{B}(\pi_0)$ (the principal boundary $\widetilde{B}(\pi_0)$ is the set of the points of the boundary of $\pi_0$

which have a minimal energy, see definition 3.7). Since $\widetilde{B}(\pi_0)\cap\pi_1 \neq \emptyset$, then $\pi\cap\pi_1 \neq \emptyset$. By proposition 3.3, we have either $\pi \subset \pi_1$ or $\pi_1 \subset \pi$. However $\pi_0$ and $\pi_1$ are disjoint so that we can't have $\pi \subset \pi_1$. Therefore $\pi_1 \subset \pi$, and the inclusion is strict since $\pi$ contains $\pi_0$. By induction, we obtain that $\pi_r$ is also strictly contained in $\pi$. It follows that $\widetilde{B}(\pi_r) \subset \pi$, implying that $\{\underline{-1}, \underline{+1}\} \cap \pi \neq \emptyset$. Thus $\pi$ is not a subcycle of $\{\underline{-1}, \underline{+1}\}^c$. $\square$

This result will yield "instantaneously" the list of the relevant cycles and will designate those cycles of the list which are in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. Our list of cycles will be large enough to ensure that for each configuration $\sigma'$ in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$, it includes all the vertices of the minimal stable subgraph of $G$ containing $\{\sigma'\}$. We will therefore use lemma 4.4 as a basic tool to find some maximal subcycles of $\{\underline{-1}, \underline{+1}\}^c$. We will not find all the cycles of $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$, which is a formidable task compared to the problem of describing the first passage from $\underline{-1}$ to $\underline{+1}$. For instance we do not investigate the cycles containing several large subcritical droplets close enough to interact instead of disappearing separately.

Step $vi)$ consists in doing the synthesis of all the results gathered in the previous steps $i)$–$v)$. The set of all the cycle paths of null cost is obtained by glueing together an ascending part and a descending part. We define a graph $\mathcal{G}$ over the cycles of $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ by

$$(\pi_1 \to \pi_2) \in \mathcal{G} \iff \exists\, \sigma' \in \widetilde{B}(\pi(\underline{-1}, \underline{+1}^c)),\ (\pi_1 \to \pi_2) \in G^+(\sigma') \text{ or } (\pi_2 \to \pi_1) \in G^-(\sigma')$$

i.e. the graph $\mathcal{G}$ is the union for $\sigma'$ in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ of the graphs $G^+(\sigma')$ and the reversed graphs $G^-(\sigma')$. The set of the cycle paths of null cost is the set of all the paths in the graph $\mathcal{G}$ joining $\{\underline{-1}\}$ to $\{\underline{+1}\}$. Thus the graph $\mathcal{G}$ contains all the information necessary to obtain the set of the cycle paths of null cost between $\underline{-1}$ and $\underline{+1}$.

## 5. The bottom of the cycles

The results of this section are valid in any dimension $d$. They describe some essential facts concerning the geometry of the cycles of the energy landscape of the Ising model. Theorem 5.3 concerns the points of the cycle we might reach starting from a configuration belonging to the bottom of the cycle. Theorem 5.5 gives a condition implying that the bottom of the cycle is reduced to one point. These theorems will be crucial to handle the two and three dimensional cases. Their proofs rely heavily on the following inequality.

**Theorem 5.1.** *(an energy inequality)*
*For any configurations $\eta$, $\sigma$, $\xi$ of $X$ such that $\eta \subset \sigma \subset \xi$, we have (for positive $h$)*

$$E(\sigma \setminus \eta) - E(\sigma) \ \leq\ E(\xi \setminus \eta) - E(\xi).$$

*Proof.* The statement can be checked by a direct computation. $\square$

*Remark.* This inequality might be interpreted as follows. The variation of energy when we turn down a fixed set of spins of a configuration $\sigma$ increases when we enlarge $\sigma$.

**Notation 5.2.** Let $x$ be a site in $\Lambda$. We define the spin flip operator $T(x) : X \to X$ by $T(x)(\sigma) = \sigma^x$ where we recall that

$$\sigma^x(y) = \begin{cases} +\sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x \end{cases}$$

Let $x_1, \cdots, x_r$ be a sequence of sites in $\Lambda$. We define the operator $T(x_1, \cdots, x_r) : X \to X$ by $T(x_1, \cdots, x_r) = T(x_r) \circ \cdots \circ T(x_1)$.
Let $\sigma$ belong to $X$. To $\sigma$ we associate two operators $S_+(\sigma)$ and $S_-(\sigma)$ on the sequences of sites. The sequence $S_-(\sigma)(x_1, \cdots, x_r)$ (respectively $S_+ \cdots$) is the subsequence of $(x_1, \cdots, x_r)$ consisting of the sites $x_i$ such that $\sigma(x_i) = -1$ (resp. $\sigma(x_i) = +1$).

*Remark.* When we speak of a sequence of sites $(x_1, \cdots, x_r)$ without further precision, the sites do not have to be distinct. For any sequence of sites $(x_1, \cdots, x_r)$ and any configuration $\sigma$ we have $T(S_+(\sigma)(x_1, \cdots, x_r))(\sigma) \;\subset\; \sigma \;\subset\; T(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$.

**Theorem 5.3.** *Let $\pi$ be a cycle and let $\sigma$ be a configuration of the bottom $F(\pi)$ of $\pi$. Let $(x_1, \cdots, x_r)$ be a sequence of sites such that $T(x_1, \cdots, x_j)(\sigma)$ belongs to $\pi$ for all $j$ in $\{1 \cdots r\}$. Then $T(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$ and $T(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)$ also belong to $\pi$. Equivalently, if we let $\eta = T(x_1, \cdots, x_r)(\sigma)$, then $\eta \cup \sigma$ and $\eta \cap \sigma$ are in $\pi$.*

*Proof.* We prove the result by induction on the length $r$ of the sequence of sites $(x_1, \cdots, x_r)$. The result is obviously true for $r = 0$ and $r = 1$. Assume it is true at rank $r$ and let $(x_1, \cdots, x_{r+1})$ be a sequence of sites such that $T(x_1, \cdots, x_j)(\sigma)$ belongs to $\pi$ for all $j$ in $\{1 \cdots r+1\}$. Let $\xi = T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$ and let $\eta$ be the unique configuration included in $\sigma$ such that $\sigma \setminus \eta = T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$. We have $\eta \subset \sigma \subset \xi$. By hypothesis, $\xi \setminus \eta = T(x_1, \cdots, x_{r+1})(\sigma) \in \pi$ whence $E(\xi \setminus \eta) \leq \lambda(\pi)$.

- First case: $\sigma(x_{r+1}) = -1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1, \cdots, x_{r+1}) &= (S_-(\sigma)(x_1, \cdots, x_r), x_{r+1}), \\ S_+(\sigma)(x_1, \cdots, x_{r+1}) &= S_+(\sigma)(x_1, \cdots, x_r). \end{aligned}$$

By the induction hypothesis, we know that

$$T(S_-(\sigma)(x_1, \cdots, x_r))(\sigma) \in \pi, \qquad \sigma \setminus \eta = T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma) \in \pi.$$

Now $\xi = T(x_{r+1}) \circ T(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$ i.e. $\xi$ communicates by one spin flip with a configuration of $\pi$. Moreover $\sigma \setminus \eta$ belongs to $\pi$ whence $E(\sigma) \leq E(\sigma \setminus \eta)$ (because $\sigma \in F(\pi)$). By theorem 5.1 we have $E(\xi) \leq E(\xi \setminus \eta) + E(\sigma) - E(\sigma \setminus \eta)$. Therefore $E(\xi)$ is less than the level of $\pi$ so that $\xi$ is in $\pi$.

- Second case: $\sigma(x_{r+1}) = +1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1, \cdots, x_{r+1}) &= S_-(\sigma)(x_1, \cdots, x_r), \\ S_+(\sigma)(x_1, \cdots, x_{r+1}) &= (S_+(\sigma)(x_1, \cdots, x_r), x_{r+1}). \end{aligned}$$

By the induction hypothesis, we know that

$$\xi = T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma) \in \pi, \qquad T(S_+(\sigma)(x_1, \cdots, x_r))(\sigma) \in \pi.$$

By theorem 5.1 we have $E(\sigma \backslash \eta) \leq E(\xi \backslash \eta) + E(\sigma) - E(\xi)$. Therefore $E(\sigma \backslash \eta)$ is less than the level of $\pi$. Since $\sigma \setminus \eta$ differs by one spin flip (at site $x_{r+1}$) from $T(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)$ which is in $\pi$, $\sigma \setminus \eta$ is also in $\pi$.
Thus the induction is completed. $\square$

**Corollary 5.4.** *Let $\pi$ be a cycle an let $\sigma$ be an element of $F(\pi)$. If $\eta$ is a minimal (respectively maximal) element of $\pi$ with respect to the inclusion, then there exists a sequence of sites $(x_1, \cdots, x_r)$ such that $\eta = T(x_1, \cdots, x_r)(\sigma)$, $T(x_1, \cdots, x_j)(\sigma) \in \pi$ and $\sigma(x_j) = +1$ (resp. $\sigma(x_j) = -1$) for all $j$ in $\{1 \cdots r\}$. In particular, $\eta \subset \sigma$ (resp. $\sigma \subset \eta$).*

**Theorem 5.5.** *Let $\pi$ be a cycle and let $\sigma$ be a configuration of $\pi$ satisfying:
for each $\epsilon$ in $\{-1, +1\}$ and each sequence of sites $(x_1, \cdots, x_l)$ in $\Lambda^l$ such that*

$$T(x_1, \cdots, x_l)(\sigma) \neq \sigma, \qquad \forall j \in \{1 \cdots l\} \quad \sigma(x_j) = \epsilon, \quad T(x_1, \cdots, x_j)(\sigma) \in \pi,$$

*we have $E(T(x_1, \cdots, x_l)(\sigma)) > E(\sigma)$.
Then the configuration $\sigma$ also satisfies:
for each sequence of sites $(x_1, \cdots, x_r)$ in $\Lambda^r$ such that $T(x_1, \cdots, x_j)(\sigma)$ belongs to $\pi$ for all $j$ in $\{1 \cdots r\}$, the configurations $T(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$ and $T(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)$ are in $\pi$ and*

$$\begin{aligned}
E(T(x_1, \cdots, x_r)(\sigma)) &\geq E(T(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)) \geq E(\sigma), \\
E(T(x_1, \cdots, x_r)(\sigma)) &\geq E(T(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)) \geq E(\sigma), \\
E(T(x_1, \cdots, x_r)(\sigma)) &= E(\sigma) \implies T(x_1, \cdots, x_r)(\sigma) = \sigma.
\end{aligned}$$

*The bottom $F(\pi)$ of the cycle $\pi$ is then reduced to this single configuration $\sigma$.*

*Proof.* We prove the result by induction on the length $r$ of the sequence of sites $(x_1, \cdots, x_r)$ appearing in the conclusion of the theorem. For $r = 0$ and $r = 1$ it is true. Assume it is true until rank $r$ and let $(x_1, \cdots, x_{r+1})$ be a sequence of sites such that $T(x_1, \cdots, x_j)(\sigma)$ belongs to $\pi$ for all $j$ in $\{1 \cdots r+1\}$. The induction hypothesis yields that for all $j$ in $\{1 \cdots r\}$ the configurations $T(S_-(\sigma)(x_1, \cdots, x_j))(\sigma)$, $T(S_+(\sigma)(x_1, \cdots, x_j))(\sigma)$ are in $\pi$ and moreover

$$\begin{aligned}
E(T(x_1, \cdots, x_j)(\sigma)) &\geq E(T(S_-(\sigma)(x_1, \cdots, x_j))(\sigma)) \geq E(\sigma), \\
E(T(x_1, \cdots, x_j)(\sigma)) &\geq E(T(S_+(\sigma)(x_1, \cdots, x_j))(\sigma)) \geq E(\sigma), \\
E(T(x_1, \cdots, x_j)(\sigma)) &= E(\sigma) \implies T(x_1, \cdots, x_j)(\sigma) = \sigma.
\end{aligned}$$

By hypothesis, $T(x_1, \cdots, x_{r+1})(\sigma)$ is in $\pi$ so that $E(T(x_1, \cdots, x_{r+1})(\sigma)) \leq \lambda(\pi)$.

- First case: $\sigma(x_{r+1}) = -1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1, \cdots, x_{r+1}) &= (S_-(\sigma)(x_1, \cdots, x_r), x_{r+1}), \\ S_+(\sigma)(x_1, \cdots, x_{r+1}) &= S_+(\sigma)(x_1, \cdots, x_r). \end{aligned}$$

By the induction hypothesis, we know that $T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$ is in $\pi$. Since for all $j$ in $\{1 \cdots r+1\}$, $T(S_+(\sigma)(x_1, \cdots, x_j))(\sigma)$ is in $\pi$, the hypothesis on $\sigma$ implies that $E(T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \geq E(\sigma)$. Now theorem 5.1 yields

$$\begin{aligned} E(T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) &\leq E(T(x_1, \cdots, x_{r+1})(\sigma)) + E(\sigma) - \\ &\qquad\qquad E(T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \\ &\leq E(T(x_1, \cdots, x_{r+1})(\sigma)) \leq \lambda(\pi) \end{aligned}$$

and $T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$ is in $\pi$, since its energy is less than the level of $\pi$ and it differs by one spin flip (at site $x_{r+1}$) from a configuration of $\pi$. The hypothesis on $\sigma$ finally implies that $E(T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \geq E(\sigma)$.

- Second case: $\sigma(x_{r+1}) = +1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1, \cdots, x_{r+1}) &= S_-(\sigma)(x_1, \cdots, x_r), \\ S_+(\sigma)(x_1, \cdots, x_{r+1}) &= (S_+(\sigma)(x_1, \cdots, x_r), x_{r+1}). \end{aligned}$$

By the induction hypothesis, we know that $T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$ is in $\pi$. Since for all $j$ in $\{1 \cdots r+1\}$, $T(S_-(\sigma)(x_1, \cdots, x_j))(\sigma)$ is in $\pi$, the hypothesis on $\sigma$ implies that $E(T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \geq E(\sigma)$. Now theorem 5.1 yields

$$\begin{aligned} E(T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) &\leq E(T(x_1, \cdots, x_{r+1})(\sigma)) + E(\sigma) - \\ &\qquad\qquad E(T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \\ &\leq E(T(x_1, \cdots, x_{r+1})(\sigma)) \leq \lambda(\pi) \end{aligned}$$

and $T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$ is in $\pi$, since its energy is less than the level of $\pi$ and it differs by one spin flip (at site $x_{r+1}$) from a configuration of $\pi$. The hypothesis on $\sigma$ finally implies that $E(T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \geq E(\sigma)$.

In both cases, equality can occur only if

$$E(T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) = E(T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) = E(\sigma).$$

The hypothesis on $\sigma$ then implies that

$$T(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma) = T(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma) = \sigma$$

or equivalently $T(x_1, \cdots, x_{r+1})(\sigma) = \sigma$. The induction is completed. $\square$

## 6. Dimension two

A polyomino associated to a configuration is a finite union of unit squares. For $\sigma$ a configuration we will denote by $p(\sigma)$ the perimeter of the associated polyomino and by $a(\sigma)$ its area. We will rely heavily on the notation and results of [1]. Let us recall some essential points.

**Summary of the combinatorial results.**

The rectangle of sides $l_1$ and $l_2$ is denoted by $l_1 \times l_2$. A quasisquare is a rectangle $l_1 \times l_2$ with $|l_1 - l_2| \leq 1$. By $l_1 \times l_2 +_1 k$ (resp. $l_1 \times l_2 +_2 k$) we denote the rectangle $l_1 \times l_2$ plus a vertical (resp. horizontal) bar of length $k$ sticked against the side $l_2$ (resp. $l_1$) starting from the bottom (resp. from the left). When we speak of $l_1 \times l_2 +_1 k$, we assume implicitly that the length of the bar is less than $l_2$. We use the operators $+_1^i, +_2^i$ to specify the position of the bar: $l_1 \times l_2 +_1^i k$ means that the bar starts at the $(i+1)$–th square along the side of $l_1 \times l_2$. By $l_1 \times l_2 \oplus_1 k$ (respectively $l_1 \times l_2 \oplus_2 k$) we denote the set of all the polyominoes obtained by sliding the bar of length $k$ along the side $l_2$ (resp. $l_1$) of the rectangle in such a way that the polyomino is always included in $(l_1+1) \times l_2$ (resp. $l_1 \times (l_2+1)$). Because of the context, there should be no ambiguity between the $+$ operator for integers and polyominoes. Moreover the latter will always have a subscript (i.e. $+_1$, $+_2$).

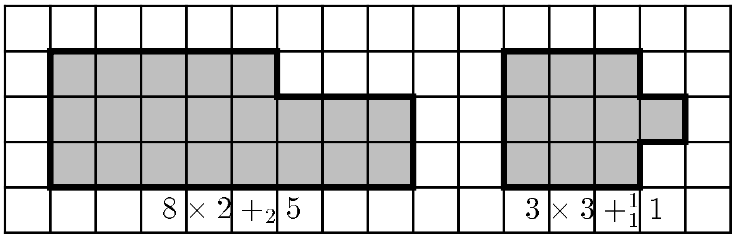


figure 1: the poloyminoes $8 \times 2 +_2 5$, $3 \times 3 +_1^1 1$

For $A$ a set of polyominoes, we denote by $\overline{A}$ its orbit under the action of the planar isometries which leave the integer lattice $\mathbb{Z}^2$ invariant. By $\overline{A}^{12}$ we denote its orbit under the action of the two symmetries with respect to the axis.

**Proposition 6.1.** *For each integer $n$ there exists a unique 3–uple $(l, k, \epsilon)$ such that*

$$\epsilon \in \{0,1\}, \quad 0 \leq k < l + \epsilon \quad \text{and} \quad n = l(l+\epsilon) + k.$$

The set of the polyominoes of area $n$ is $C_n$; the set $M_n$ of the minimal polyominoes of area $n$ is the set of the polyominoes of $C_n$ having minimal perimeter. Let $n = l(l+\epsilon)+k$ be the decomposition of $n$. The canonical polyomino of area $n$ is

$$m_n = \begin{cases} l \times l +_1 k & \text{if } \epsilon = 0 \\ (l+1) \times l +_2 k & \text{if } \epsilon = 1 \end{cases}$$

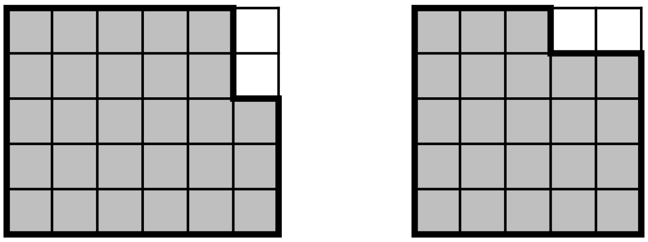

figure 2: the canonical polyominoes $m_{28}, m_{23}$

**Theorem 6.2.** *The canonical polyomino $m_n$ is minimal.*

This theorem is the key for determining the energy barrier the system has to overcome to travel from $\underline{-1}$ to $\underline{+1}$. It simultaneously gives a lower bound for this energy barrier and exhibits a growing sequence of polyominoes realizing this lower bound. This energy barrier gives the constant characterizing the asymptotic behaviour of the relaxation time.

We define several important subsets of $M_n$. The set $S_n$ of the standard polyominoes is

$$S_n = \begin{cases} \overline{l \times l \oplus_1 k} & \text{if } \epsilon = 0 \\ \overline{(l+1) \times l \oplus_2 k} & \text{if } \epsilon = 1 \end{cases}$$

The set $\widetilde{M}_n$ of the principal polyominoes is

$$\widetilde{M}_n = \overline{l \times (l+\epsilon) \oplus_1 k} \bigcup \overline{l \times (l+\epsilon) \oplus_2 k}.$$

The sets $S_n$ and $\widetilde{M}_n$ coincide only when $\epsilon$ is zero. Clearly $\{m_n\} \subset S_n \subset \widetilde{M}_n \subset M_n$. Figure 3 shows that the inclusions may be strict.

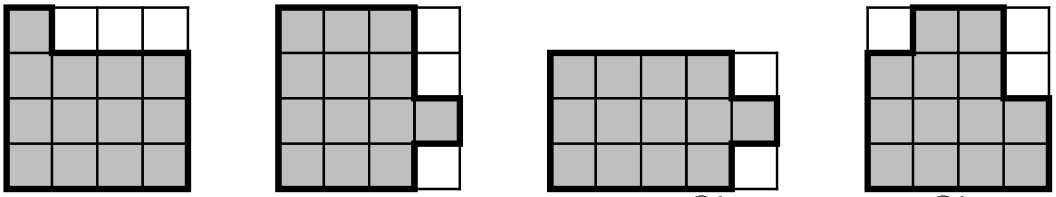

figure 3: elements of $\{m_{13}\}$, $S_{13} \setminus \{m_{13}\}$, $\widetilde{M}_{13} \setminus S_{13}$, $M_{13} \setminus \widetilde{M}_{13}$

In general, the set $M_n$ is much larger than $\widetilde{M}_n$. It turns out that it is not the case for specific values of $n$.

**Theorem 6.3.** *The set $M_n$ is reduced to $\{ m_n \}$ if and only if $n$ is of the form $l^2$. The set $M_n$ coincides with $S_n$ if and only if $n$ is of the form $l^2, l(l+1)-1, l(l+1)$, in which case $S_n = \overline{m}_n$. The set $M_n$ coincide with $\widetilde{M}_n$ if and only if the integer $n$ is of the form $l^2, l(l+1)-1, l(l+1), (l+1)^2-1$.*

This theorem will be crucial to determine precisely the set of the critical droplets. It gives the uniqueness results associated to the discrete isoperimetric inequality. In fact, it will turn out that we do not have uniqueness of the isoperimetric problem for the area of the critical droplet. Luckily enough, there is uniqueness for the area preceding it. We will therefore rely on the following lemma.

**Lemma 6.4.** *For $n$ of the form $l^2, l(l+1)$, we have*

$$\{\, c \in M_{n-1} : q(M_n, c) > 0 \,\} \;=\; S_{n-1},$$
$$\{\, c \in M_{n+1} : q(M_n, c) > 0 \,\} \;=\; \widetilde{M}_{n+1}$$

*(where $q(M_n, c) = \min\{q(d, c) : d \in M_n\}$).*

**Proposition 6.5.** *The principal polyominoes can be completely shrunk through the principal polyominoes: for any integer $n$ and for any principal polyomino $c$ in $\widetilde{M}_n$, there exists an increasing sequence $c_0, \cdots, c_n$ of principal polyominoes such that $c_0 = \emptyset$, $c_n = c$ and $q(c_{i-1}, c_i) > 0$ for $i$ in $\{1 \cdots n\}$.*

A consequence of this proposition is that the set of the principal polyominoes associated to the critical volume is contained in the principal boundary of the greatest cycle containing $\underline{-1}$ and not $\underline{+1}$. Let us remark that it is not possible to grow arbitrarily far through the minimal polyominoes a principal polyomino which is not standard: such a polyomino can only grow until a rectangle $l \times (l+2)$ and is a dead–end.

**Proposition 6.6.** *The standard polyominoes can be grown or shrunk arbitrarily far through the standard polyominoes: for any integers $m \leq n$ and for any standard polyomino $c$ in $S_m$, there exists an increasing sequence $c_0, \cdots, c_n$ of standard polyominoes such that $c_0 = \emptyset$, $c_m = c$ and $q(c_{i-1}, c_i) > 0$ for $i$ in $\{1 \cdots n\}$.*

The statement of proposition 6.6 concerns the set of the standard polyominoes, which (except for specific values of the area) is a strict subset of the set of the principal polyominoes considered in proposition 6.5. The nice feature of the standard polyominoes is that they can be grown arbitrarily far through the minimal polyominoes (proposition 6.6 asserts that a standard polyomino of area $m$ can be grown until any area $n \geq m$).

**Notation 6.7.** We have also some results concerning the best way to shrink or to grow a rectangle. Let $l_1, l_2, k$ be three positive integers. We define

$$M(l_1 \times l_2, -k) \;=\; \{\, c \in C_{l_1 l_2 - k} : c \subset l_1 \times l_2,\; p(c) \text{ minimal} \}.$$

More precisely, a polyomino $c$ belongs to $M(l_1 \times l_2, -k)$ if and only if

$$c \in C_{l_1 l_2 - k}, \quad c \subset l_1 \times l_2, \quad p(c) \;=\; \min\{\, p(d) : d \in C_{l_1 l_2 - k}, d \subset l_1 \times l_2 \,\}.$$

Similarly, we define

$$M(l_1 \times l_2, k) \;=\; \{\, c \in C_{l_1 l_2 + k} : l_1 \times l_2 \subset c,\; p(c) \text{ minimal} \},$$

i.e. a polyomino $c$ belongs to $M(l_1 \times l_2, k)$ if and only if

$$c \in C_{l_1 l_2 + k}, \quad l_1 \times l_2 \subset c, \quad p(c) = \min\{ p(d) : d \in C_{l_1 l_2+k}, l_1 \times l_2 \subset d \}.$$

Notice that the elements of $M(l_1 \times l_2, k)$ (respectively $M(l_1 \times l_2, -k)$) all have the same energy, since they all have the same area and perimeter. We denote by $\mathcal{E}(l_1 \times l_2, k)$ (respectively $\mathcal{E}(l_1 \times l_2, -k)$) the energy of an element of $M(l_1 \times l_2, k)$ (resp. $M(l_1 \times l_2, -k)$). A natural way to remove (add) $k$ squares (for $k < l_1, k < l_2$) is to remove (add) a line on a side of the rectangle; thus we define

$$S(l_1 \times l_2, -k) = \overline{(l_1 - 1) \times l_2 \oplus_1 (l_2 - k)}^{12} \bigcup \overline{l_1 \times (l_2 - 1) \oplus_2 (l_1 - k)}^{12},$$
$$S(l_1 \times l_2, k) = \overline{\{ l_1 \times l_2 \oplus_2 k, \, l_1 \times l_2 \oplus_1 k \}}^{12}.$$

**Proposition 6.8.** *Let $l_1, l_2, k$ be positive integers such that $k < l_1$, $k < l_2$. The set $M(l_1 \times l_2, -k)$ is the set of the polyominoes obtained by removing successively $k$ corner squares from $l_1 \times l_2$. In particular, $S(l_1 \times l_2, -k)$ is included in $M(l_1 \times l_2, -k)$.*

**Proposition 6.9.** *Let $l_1, l_2, k$ be positive integers such that $k < l_1$, $k < l_2$. The set $M(l_1 \times l_2, k)$ is equal to the set $S(l_1 \times l_2, k)$.*

These propositions will be the key to find the principal boundaries of the cycles around the supercritical rectangles, around the subcritical quasisquares and around the dead–ends i.e. the principal non standard polyominoes.

**Application to the two dimensional Ising model.**

We first express the energy of the Ising model with the help of the perimeter and the area of the polyomino associated to the configuration.

**Lemma 6.10.** *For any configuration $\sigma$ in $X$, we have*

$$E(\sigma) = -\frac{1}{2} \sum_{\{x,y\}:x\sim y} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x) = p(\sigma) - ha(\sigma) + N^2(h/2 - 1).$$

We do not change the dynamics nor the cycle decomposition by adding a constant to the energy $E$. In dimension two, we will work with the energy $E(\sigma) = p(\sigma) - ha(\sigma)$. We denote by $\mathcal{E}(n)$ the minimal energy of a configuration of $C_n$ (so that $\mathcal{E}(n) = E(\sigma) = p(\sigma) - hn$ for any $\sigma$ in $M_n$).

*Hypothesis on the magnetic field $h$ and the size $N$.* We suppose that $h$ is small compared to the unity and that for any configurations $\eta, \sigma$, the equality $E(\eta) = E(\sigma)$ implies $a(\eta) = a(\sigma)$ and $p(\eta) = p(\sigma)$. Whenever we take the integral part of a quantity involving $h$, we assume that this quantity is not an integer. For instance $2/h$ is not an integer. Finally, $N$ is large enough to ensure that the combinatorial results proved on the infinite lattice $\mathbb{Z}^2$ remain valid on the torus until the critical area $2/h(2/h+1)$. This is obviously true if $h^2N > 6$.

*Remark.* A careful study would yield the weaker condition $h^2N^2 > K$ for some constant $K$: this would require to push further the techniques used in [1].

We now follow the general strategy outlined in section 4. Our first aim is to determine the communication altitude between $\underline{-1}$ and $\underline{+1}$ (corollary 6.14) and to compute it explicitly (proposition 6.15). We start by giving a lower bound on the communication altitude between the sets of configurations of volume $m$ and $n$ (proposition 6.11) and then exhibit a particular sequence of configurations realizing precisely the value of this lower bound (proposition 6.12).

**Proposition 6.11.** *Let $m \leq n$ be two integers. The communication altitude between the sets $C_m$ and $C_n$ is greater or equal than* $\max\{\,\mathcal{E}(r) : m \leq r \leq n\,\}$.

*Proof.* Let $\sigma_0, \cdots, \sigma_s$ be a sequence of configurations such that $\sigma_0 \in C_m$, $\sigma_s \in C_n$ and $q(\sigma_i, \sigma_{i+1}) > 0$ for $i$ in $\{0 \cdots s-1\}$. Necessarily we have $|a(\sigma_i) - a(\sigma_{i+1})| \leq 1$ so that the sequence of integers $a(\sigma_0), \cdots, a(\sigma_s)$ takes all the values between $m = a(\sigma_0)$ and $n = a(\sigma_s)$. Henceforth

$$\max_{0\leq i\leq s} E(\sigma_i) \;\geq\; \max_{0\leq i\leq s} \mathcal{E}(a(\sigma_i)) \;\geq\; \max_{m\leq r\leq n} \mathcal{E}(r)\,. \quad \square$$

**Proposition 6.12.** *Let $\sigma$ belong to $S_m$ and let $m_1$, $m_2$ in $\mathbb{N}$ be such that $m_1 \leq m \leq m_2$. There exist $\sigma_1$, $\sigma_2$ in $S_{m_1}$, $S_{m_2}$ satisfying*

$$\begin{aligned} E(\sigma_1, \sigma) &= \max\{\,\mathcal{E}(n) : m_1 \leq n \leq m\,\}\,,\\ E(\sigma, \sigma_2) &= \max\{\,\mathcal{E}(n) : m \leq n \leq m_2\,\}\,. \end{aligned}$$

*Remark.* Let us note that the statement of the proposition 6.12 concerns only the standard configurations and does not hold for any configuration $\sigma$ in $C_n$.

*Proof.* We only deal with $\sigma_1$ and $S_{m_1}$, the other case being similar. Proposition 6.6 yields the existence of a sequence $\sigma_{m_1}, \cdots, \sigma_{m-1}, \sigma_m$ such that

$$\sigma_m = \sigma\,, \qquad \forall j \in \{m_1 \cdots m-1\} \quad \sigma_j \in S_j\,, \quad q(\sigma_j, \sigma_{j+1}) > 0\,.$$

Taking $\sigma_1 = \sigma_{m_1}$ we have

$$E(\sigma_1, \sigma) \;\leq\; \max_{m_1\leq n\leq m} E(\sigma_n) \;\leq\; \max_{m_1\leq n\leq m} \mathcal{E}(n).$$

The reverse inequality is a consequence of proposition 6.11. $\square$

**Corollary 6.13.** *Let $m \le n$ be two integers. The communication altitude between the sets $C_m$ and $C_n$ (or $S_m$ and $S_n$) is*

$$E(C_m, C_n) \;=\; E(S_m, S_n) \;=\; \max\{\, \mathcal{E}(r) : m \le r \le n \,\} \;=\; E(m_m, m_n).$$

*Proof.* This is a straightforward consequence of propositions 6.11 and 6.12. $\square$

**Corollary 6.14.** $E(\underline{-1}, \underline{+1}) \;=\; \max\{\, \mathcal{E}(n) : 0 \le n \le N^2 \,\}$.

The figure below shows a graph of $\mathcal{E}(n)$ for $h = 0.21$.

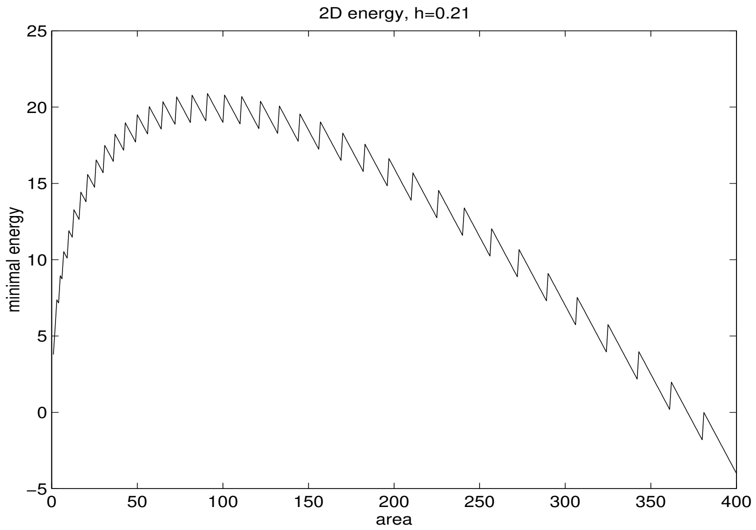


figure 4: the two dimensional minimal energy $\mathcal{E}(n)$

To compute $\mathcal{E}(n)$ we apply theorem 6.2. The minimal perimeter of a polyomino of area $n$ is

$$\min_{\sigma \in C_n} p(\sigma) \;=\; \begin{cases} 2(2l+\epsilon) & \text{if } k = 0 \\ 2(2l+\epsilon)+2 & \text{if } k > 0 \end{cases}$$

where $(l, k, \epsilon)$ is the unique 3–uple in $\mathbb{N}^3$ satisfying $n = l(l+\epsilon)+k$, $\epsilon \in \{0,1\}$, $k < l+\epsilon$. Let $\mathcal{E}_1$ be the corresponding one dimensional energy i.e. $\mathcal{E}_1(k) \;=\; 2 \cdot 1(k > 0) - hk$ (where $1(A)$ denotes the characteristic function of $A$). We have $\mathcal{E}(n) = \mathcal{E}(l(l+\epsilon)) + \mathcal{E}_1(k)$. Starting with $\mathcal{E}_1$ we find

$$\max_{k_1 \le k \le k_2} \mathcal{E}_1(k) \;=\; \begin{cases} 2 - hk_1 & \text{if } k_1 \ge 2 \\ 0 & \text{if } k_2 < 1 \\ 2 - h & \text{if } k_1 \le 1 \le k_2 \end{cases}$$

We next compute the energy barrier between two consecutive quasisquares:

$$\max_{l^2\le n\le l(l+1)} \mathcal{E}(n) \;=\; \mathcal{E}(l^2) + \max_{0\le k\le l} \mathcal{E}_1(k) \;=\; \mathcal{E}(l^2)+2-h\,,$$
$$\max_{l(l+1)\le n\le (l+1)^2} \mathcal{E}(n) \;=\; \mathcal{E}(l(l+1))+2-h\,.$$

Thus

$$\max_{l(l+1)\le n\le (l+1)^2} \mathcal{E}(n) - \max_{l^2\le n\le l(l+1)} \mathcal{E}(n) \;=\; 2-hl\,,$$
$$\max_{l^2\le n\le l(l+1)} \mathcal{E}(n) - \max_{l(l-1)\le n\le l^2} \mathcal{E}(n) \;=\; 2-hl\,.$$

Putting $l_c = \lceil 2/h \rceil$, we see that these energy variations are negative for $l \ge l_c$ and positive for $l < l_c$. From this, we deduce the energy barrier between two remote quasisquares:

$$\max_{l_1(l_1+\epsilon_1)\le n\le l_2(l_2+\epsilon_2)} \mathcal{E}(n) \;=\; \begin{cases} \mathcal{E}(l_2(l_2+\epsilon_2-1)+1) & \text{if } (l_2,\epsilon_2)\le (l_c-1,1)\\ \mathcal{E}((l_c-1)l_c+1) & \text{if } (l_1,\epsilon_1)\le (l_c-1,1)<(l_2,\epsilon_2)\\ \mathcal{E}(l_1(l_1+\epsilon_1)+1) & \text{if } (l_c-1,1)<(l_1,\epsilon_1)\end{cases}$$

(we use the lexicographical order on the pairs $(l,\epsilon)$).
We finally obtain the global energy barrier.

**Proposition 6.15.** *The altitude of communication between $\underline{-1}$ and $\underline{+1}$ is equal to*

$$E(\underline{-1},\underline{+1}) \;=\; \mathcal{E}(n_c) \;=\mathcal{E}((l_c-1)l_c+1) \;=\; 4\lceil 2/h\rceil - h\lceil 2/h\rceil^2 + h\lceil 2/h\rceil - h$$

*where the critical length $l_c$ is $\lceil 2/h \rceil$ and the critical area $n_c$ is $(l_c-1)l_c+1$.*

*Remark.* In particular, $E(\underline{-1},\underline{+1}) \sim 4/h$ when $h$ goes to zero: the energy barrier the system has to overcome goes to infinity like $h^{-1}$.

From this we deduce the level of the greatest cycle containing $\underline{-1}$ and not $\underline{+1}$.

**Corollary 6.16.** *The level of the cycle $\pi(\underline{-1},\underline{+1}^c)$ is pred$\,\mathcal{E}(n_c)$.*

*Remark.* We recall that $\operatorname{pred}\lambda = \max\{\, E(\sigma) : E(\sigma)<\lambda,\ \sigma\in X\,\}$, see notation 3.4.

**Notation 6.17.** If $Y$ is a subset of $X$, its minimal and maximal areas $\underline{a}(Y)$ and $\overline{a}(Y)$ are

$$\underline{a}(Y) \;=\; \min\{\, a(\sigma):\sigma\in Y\,\}\,,\quad \overline{a}(Y) \;=\; \max\{\, a(\sigma):\sigma\in Y\,\}\,.$$

Since $E(\underline{-1},C_n) \;=\; \mathcal{E}(n_c)$ for any $n \ge n_c$ (by corollary 6.13 and proposition 6.15), then all the configurations of the cycle $\pi(\underline{-1},\underline{+1}^c)$ have an area less than $n_c-1$ i.e. $\overline{a}(\pi(\underline{-1},\underline{+1}^c)) \le n_c-1$. To complete step $i)$ of the general strategy, we determine the configurations of the principal boundary of the cycle $\pi(\underline{-1},\underline{+1}^c)$.

**Theorem 6.18.** *The principal boundary $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$ of the cycle $\pi(\underline{-1},\underline{+1}^c)$ is the set $\widetilde{M}_{n_c}$ of the principal configurations of area $n_c$. For $\sigma$ in $\widetilde{M}_{n_c}$ there exists a unique configuration in $\pi(\underline{-1},\underline{+1}^c)$ communicating with $\sigma$, which is the quasisquare $(l_c-1)\times l_c$ or $l_c\times(l_c-1)$ included in $\sigma$.*

*Proof.* Let $\sigma$ belong to $\widetilde{M}_{n_c}$. Then $E(\sigma)=\mathcal{E}(n_c)$ and proposition 6.5 yields the existence of a sequence $\sigma_0,\cdots,\sigma_{n_c}$ such that

$$\sigma_0=\underline{-1},\quad \sigma_{n_c}=\sigma,\quad \forall j\in\{0\cdots n_c-1\}\quad \sigma_j\in M_j,\quad q(\sigma_j,\sigma_{j+1})>0.$$

In particular, we have $\max\{\,E(\sigma_n)\,:\,0\leq n<n_c\,\}<\mathcal{E}(n_c)$ so that $\sigma_{n_c-1}$ belongs to $\pi(\underline{-1},\underline{+1}^c)$. Since $q(\sigma_{n_c-1},\sigma_{n_c})>0$, the configuration $\sigma_{n_c}$ is in the principal boundary of $\pi(\underline{-1},\underline{+1}^c)$. Thus $\widetilde{M}_{n_c}\subset\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$.
Conversely, let $\sigma$ belong to $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$. Necessarily $E(\sigma)=\mathcal{E}(n_c)$ so that $\sigma$ is of area $n_c$ and it is a minimal configuration. In addition there must exist $\eta$ in $\pi(\underline{-1},\underline{+1}^c)$ communicating with $\sigma$. This $\eta$ is in $C_{n_c-1}$ and satisfies $E(\eta)<E(\sigma)$. Thus $p(\eta)\leq p(\sigma)-h=4l_c-h$ and $p(\eta)\geq\min\{\,p(\xi)\,:\,\xi\in C_{n_c-1}\,\}=4l_c-2$. However the perimeter is an even integer. The only possibility is $p(\eta)=4l_c-2$ whence $\eta$ is minimal. Yet $n_c-1=(l_c-1)l_c$ and by theorem 6.3, $M_{n_c-1}=S_{n_c-1}$ so that $\eta$ belongs to $S_{n_c-1}$ and it is a quasisquare $(l_c-1)\times l_c$ or $l_c\times(l_c-1)$. Lemma 6.4 shows that the only points of $M_{n_c}$ which communicate with $M_{n_c-1}$ are the configurations of $\widetilde{M}_{n_c}$. Thus $\sigma$ is a principal configuration of area $n_c$ and $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))\subset\widetilde{M}_{n_c}$. $\square$

We now proceed to steps *ii*) and *iii*) of the general strategy. Moreover, we will handle separately the case of the configurations in $S_{n_c}$ and in $\widetilde{M}_{n_c}\setminus S_{n_c}$: it turns out that step *ii*) succeeds for the standard configurations and fails for the principal non standard configurations. The latter are dead–ends.

**The standard configurations.** We start by describing the relevant list of cycles for determining the minimal stable subgraph of $G^+$ containing the standard configurations of area $n_c$. These are the cycles around the supercritical rectangles $l_1\times l_2$ and the cycles $\{l_1\times l_2+_1^i k\}$, $\{l_1\times l_2+_2^i k\}$, with $l_1\geq l_c, l_2\geq l_c$.

**Theorem 6.19.** *Let $l_1,l_2$ be two integers greater or equal than $l_c$.*
*The cycle $\pi(l_1\times l_2,\mathrm{pred}\,E(l_1\times l_2+_1 1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\begin{aligned}
&\underline{a}(\pi(l_1\times l_2,\mathrm{pred}\,E(l_1\times l_2+_1 1)))\;=\;l_1l_2-l_c+2\,,\\
\forall k\in\{1\cdots l_c-2\}\qquad &M(l_1\times l_2,-k)\;\subset\;\pi(l_1\times l_2,\mathrm{pred}\,E(l_1\times l_2+_1 1))\,,\\
&\overline{a}(\pi(l_1\times l_2,\mathrm{pred}\,E(l_1\times l_2+_1 1)))\;=\;l_1l_2\,.
\end{aligned}$$

*The bottom of this cycle is $\{l_1\times l_2\}$ and its principal boundary is $\overline{l_1\times l_2\oplus 1}^{12}$.*

*Proof.* We check that the rectangle $l_1\times l_2$ and the cycle $\pi(l_1\times l_2,\mathrm{pred}\,E(l_1\times l_2+_1 1))$ satisfy the hypothesis of theorem 5.5. Let $x_1,\cdots,x_r$ be a sequence of sites such that

$T(x_1, \cdots, x_j)(l_1 \times l_2)$ is in $\pi(l_1 \times l_2, \operatorname{pred} E(l_1 \times l_2 +_1 1))$ for $j$ in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\operatorname{pred} E(l_1 \times l_2 +_1 1)$). We put $\eta_j = T(x_1, \cdots, x_j)(l_1 \times l_2)$ for $j$ in $\{0 \cdots r\}$.

- First case: all the sites $(x_1, \cdots, x_r)$ are outside $l_1 \times l_2$. We have that $l_1 \times l_2 \subset \eta_j$ whence $E(\eta_j) \geq \mathcal{E}(l_1 \times l_2, a(\eta_j) - l_1 l_2)$ and (see notation 6.7)

$$\operatorname{pred} E(l_1 \times l_2 +_1 1) \;\geq\; \max_{0 \leq j \leq r} E(\eta_j) \;\geq\; \max_{0 \leq j \leq r} \mathcal{E}(l_1 \times l_2, a(\eta_j) - l_1 l_2).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|a(\eta_{j+1}) - a(\eta_j)| \leq 1$ and $(a(\eta_j),\, 0 \leq j \leq r)$ takes all the values between $l_1 l_2$ and $a(\eta_r)$. Henceforth

$$\max_{0 \leq j \leq r} \mathcal{E}(l_1 \times l_2, a(\eta_j) - l_1 l_2) \;\geq\; \max\{\, \mathcal{E}(l_1 \times l_2, k) : 0 \leq k \leq a(\eta_r) - l_1 l_2 \,\}$$

and the area of $\eta_r$ must satisfy

$$\max\{\, \mathcal{E}(l_1 \times l_2, k) : 0 \leq k \leq a(\eta_r) - l_1 l_2 \,\} \;\leq\; \operatorname{pred} E(l_1 \times l_2 +_1 1).$$

By proposition 6.9, we have $\mathcal{E}(l_1 \times l_2, 1) = E(l_1 \times l_2 +_1 1) > \operatorname{pred} E(l_1 \times l_2 +_1 1)$ so that $a(\eta_r) = l_1 l_2$ and $\eta_r = l_1 \times l_2$.

- Second case: all the sites are inside $l_1 \times l_2$. Now $\eta_j \subset l_1 \times l_2$ so that

$$\begin{aligned} \operatorname{pred} E(l_1 \times l_2 +_1 1) \;&\geq\; \max_{0 \leq j \leq r} E(\eta_j) \;\geq\; \max_{0 \leq j \leq r} \mathcal{E}(l_1 \times l_2, a(\eta_j) - l_1 l_2) \\ &\geq\; \max\{\, \mathcal{E}(l_1 \times l_2, -k) : 0 \leq k \leq l_1 l_2 - a(\eta_r) \,\}. \end{aligned}$$

Proposition 6.8 shows that for $0 \leq k \leq l_1$, $\mathcal{E}(l_1 \times l_2, -k) = E(l_1 \times (l_2 - 1) \oplus_2 (l_1 - k))$ so that $E(l_1 \times l_2 +_1 1) - \mathcal{E}(l_1 \times l_2, -k) = 2 - h(k+1)$. For this quantity to be positive, we must have $k \leq l_c - 2$ whence $a(\eta_r) \geq l_1 l_2 - (l_c - 2)$. In addition for any $k$, $0 \leq k \leq l_c - 2$, $\mathcal{E}(l_1 \times l_2, -k) - E(l_1 \times l_2) = hk > 0$ so that $E(\eta_r) > E(l_1 \times l_2)$ whenever $\eta_r \neq l_1 \times l_2$.
We have proved that $l_1 \times l_2$ and the cycle $\pi(l_1 \times l_2, \operatorname{pred} E(l_1 \times l_2 +_1 1))$ satisfy the hypothesis of theorem 5.5. Thus the bottom of the cycle is $\{l_1 \times l_2\}$. That $M(l_1 \times l_2, -k)$ is included in the cycle for $0 \leq k \leq l_c - 2$ is obvious: each configuration of this set communicates with $l_1 \times l_2$ under the level $\operatorname{pred} E(l_1 \times l_2 +_1 1)$ (by proposition 6.8, these configurations are obtained by deleting successively $k$ corner squares from $l_1 \times l_2$). Finally a configuration $\xi$ of the principal boundary of this cycle is of energy $E(\xi) = E(l_1 \times l_2 +_1 1)$ so that its area is equal to $l_1 l_2 + 1$, and its perimeter to $2(l_1 + l_2) + 2$. Let $\eta$ be a configuration of the cycle such that $q(\eta, \xi) > 0$. Necessarily, the area of $\eta$ is $l_1 l_2$. Thus $\eta$ is a configuration of maximal area of the cycle and as such it is a maximal configuration of the cycle for the inclusion relation. By theorem 5.3 the rectangle $l_1 \times l_2$ is included in $\eta$ so that in fact $\eta = l_1 \times l_2$. Thus $\xi$ belongs to $M(l_1 \times l_2, 1)$, which is equal to $\overline{l_1 \times l_2 \oplus 1}^{12}$ (proposition 6.9). Conversely each configuration of this set communicates with $l_1 \times l_2$ and belongs to the principal boundary of the cycle. $\square$

**Lemma 6.20.** *Let $l_1, l_2, k, i$ be integers with $l_1 > 0$, $l_2 > 0$, $k > 1$, $i \geq 0$, $l_1 \geq k + i$. The cycle $\{l_1 \times l_2 +_2^i k\}$ has one or two configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{l_1 \times l_2 +_2^i (k+1)\}$,*
- *if $i = l_1 - k$ then $\widetilde{B} = \{l_1 \times l_2 +_2^{i-1} (k+1)\}$,*
- *if $0 < i < l_1 - k$ then $\widetilde{B} = \{l_1 \times l_2 +_2^{i-1} (k+1),\ l_1 \times l_2 +_2^i (k+1)\}$.*

**Lemma 6.21.** *Let $l_1, l_2, i$ be integers with $l_1 > 0$, $l_2 > 0$, $i \geq 0$, $l_1 \geq i + 1$. The cycle $\{l_1 \times l_2 +_2^i 1\}$ has two or three configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{l_1 \times l_2,\ l_1 \times l_2 +_2^i 2\}$,*
- *if $i = l_1 - 1$ then $\widetilde{B} = \{l_1 \times l_2,\ l_1 \times l_2 +_2^{i-1} 2\}$,*
- *if $0 < i < l_1 - 1$ then $\widetilde{B} = \{l_1 \times l_2,\ l_1 \times l_2 +_2^{i-1} 2,\ l_1 \times l_2 +_2^i 2\}$.*

*Remark.* Results similar to those stated in lemmas 6.20 and 6.21 hold for the configurations $l_1 \times l_2 +_1^i k$ and also for any configuration in $\overline{l_1 \times l_2 \oplus_1 k} \cup \overline{l_1 \times l_2 \oplus_2 k}$.

**Corollary 6.22.** *The following cycles are maximal cycles of $\{\underline{-1}, \underline{+1}\}^c$: (with $l_1 \geq l_c$ and $l_2 \geq l_c$)*

$$\begin{aligned}
&\{\eta\}\,, &&\eta \in \overline{l_1 \times l_2 \oplus_2 k}, &&0 < k < l_1 - l_c + 2\,,\\
&\{\eta\}\,, &&\eta \in \overline{l_1 \times l_2 \oplus_1 k}, &&0 < k < l_2 - l_c + 2\,,\\
&\pi(l_1 \times l_2, \mathrm{pred}\, E(l_1 \times l_2 +_1 1))\,.
\end{aligned}$$

*Proof.* This corollary is a consequence of lemma 4.4 together with theorem 6.19 and lemmas 6.20 and 6.21. Notice that we have to put together the descriptions of the cycles of theorem 6.19, lemmas 6.20, 6.21 in order to check that for each cycle $\pi$ in the above list, there is a sequence of cycles $\pi_0, \cdots, \pi_r$ such that $\pi_0 = \pi$, $\widetilde{B}(\pi_i) \cap \pi_{i+1} \neq \emptyset$, $0 \leq i < r$ and $\underline{+1} \in \widetilde{B}(\pi_r)$. $\square$

**Corollary 6.23.** *Let $\sigma$ belong to $S_{n_c+1}$. The minimal stable subgraph $G^+(\sigma)$ of $G^+$ containing $\sigma$ is the restriction of $G$ to the vertices listed in corollary 6.22. The arrows of $G^+(\sigma)$ are (in the following list, the rectangles $l_1 \times l_2$ are configurations containing $\sigma$)*

$$\begin{aligned}
&\{l_1 \times l_2 +_2^i k\} \to \{l_1 \times l_2 +_2^i\ \ k+1\}, &&0 \leq i < l_1 - k, \quad 0 < k \leq l_1 - l_c\,,\\
&\{l_1 \times l_2 +_2^i k\} \to \{l_1 \times l_2 +_2^{i-1} k+1\}, &&0 < i \leq l_1 - k, \quad 0 < k \leq l_1 - l_c\,,\\
&\{l_1 \times l_2 +_1^i k\} \to \{l_1 \times l_2 +_1^i\ \ k+1\}, &&0 \leq i < l_2 - k, \quad 0 < k \leq l_2 - l_c\,,\\
&\{l_1 \times l_2 +_1^i k\} \to \{l_1 \times l_2 +_1^{i-1} k+1\}, &&0 < i \leq l_2 - k, \quad 0 < k \leq l_2 - l_c\,,\\
&\{l_1 \times l_2 +_2^i l_1 - l_c + 1\} \to \pi(l_1 \times (l_2+1), \mathrm{pred}\, E(l_1 \times (l_2+1) +_1 1)), &&0 \leq i \leq l_c - 1\,,\\
&\{l_1 \times l_2 +_1^i l_2 - l_c + 1\} \to \pi((l_1+1) \times l_2, \mathrm{pred}\, E((l_1+1) \times l_2 +_1 1)), &&0 \leq i \leq l_c - 1\,,\\
&\pi(l_1 \times l_2, \mathrm{pred}\, E(l_1 \times l_2 +_1 1)) \leftrightarrow \{\eta\}\,, &&\eta \in l_1 \times l_2 \oplus_1 1\,,\\
&\pi(l_1 \times l_2, \mathrm{pred}\, E(l_1 \times l_2 +_1 1)) \leftrightarrow \{\eta\}\,, &&\eta \in l_1 \times l_2 \oplus_2 1\,.
\end{aligned}$$

*The symbol $\leftrightarrow$ means that both arrows $\rightarrow$ and $\leftarrow$ are present. The above list should be completed with all the isometric arrows (obtained by applying the same isometry to both ends of an arrow).*

The only loops in the graph $G^+(\sigma)$ are

$$\pi(l_1\times l_2, \operatorname{pred} E(l_1\times l_2 +_1 1)) \leftrightarrow \{\eta\}\,,\qquad \eta\in l_1\times l_2\oplus_1 1\cup l_1\times l_2\oplus_2 1\,.$$

Any other arrow $\pi_1\to\pi_2$ of $G^+(\sigma)$ satisfies $\overline{a}(\pi_1)<\underline{a}(\pi_2)$. As a consequence a path in $G^+(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{+1}\}$.

We are done with the standard configurations. We have to examine the remaining configurations of the principal boundary of the cycle $\pi(\underline{-1},\underline{+1}^c)$ i.e. the principal non standard configurations.

**The principal non standard configurations.** We now do the same work for the configurations in $\widetilde{M}_{n_c}\setminus S_{n_c}$. The relevant cycles are the cycles around the rectangles $(l_c-1)\times(l_c+1)$ and around the configurations of $\widetilde{M}_{n_c+1}\setminus S_{n_c+1}$.

**Theorem 6.24.** *The cycle $\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))$ is the greatest cycle containing $(l_c-1)\times(l_c+1)$ included in $\{\underline{-1},\underline{+1}\}^c$. Moreover,*

$$\begin{aligned}
&\underline{a}(\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))) \;=\; (l_c-1)l_c+2\,,\\
\forall k\in\{1\cdots l_c-3\}\qquad &M((l_c-1)\times(l_c+1),-k)\;\subset\;\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))\,,\\
&\overline{a}(\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))) \;=\; (l_c-1)(l_c+1)\,.
\end{aligned}$$

*The bottom of this cycle is $\{(l_c-1)\times(l_c+1)\}$; its principal boundary is $M((l_c-1)\times(l_c+1),-(l_c-2))$ and thus contains $\widetilde{M}_{n_c}=\overline{(l_c-1)\times l_c\oplus_2 1}^{12}$.*

*Remark.* Obviously, similar statements are true for the rectangle $(l_c+1)\times(l_c-1)$.

*Proof.* We check that the rectangle $(l_c-1)\times(l_c+1)$ and the cycle $\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))$ satisfy the hypothesis of theorem 5.5. Let $x_1,\cdots,x_r$ be a sequence of sites such that $T(x_1,\cdots,x_j)((l_c-1)\times(l_c+1))$ is in $\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))$ for $j$ in $\{1\cdots r\}$ (i.e. these configurations have an energy less or equal than $\operatorname{pred}\mathcal{E}(n_c)$). We put $\eta_j=T(x_1,\cdots,x_j)((l_c-1)\times(l_c+1))$ for $j$ in $\{0\cdots r\}$.

- First case: all the sites are outside $(l_c-1)\times(l_c+1)$. We have that $(l_c-1)\times(l_c+1)\subset\eta_j$ whence $E(\eta_j)\geq\mathcal{E}((l_c-1)\times(l_c+1),a(\eta_j)-(l_c-1)(l_c+1))$ and

$$\operatorname{pred}\mathcal{E}(n_c)\;\geq\;\max_{0\leq j\leq r}E(\eta_j)\;\geq\;\max_{0\leq j\leq r}\mathcal{E}((l_c-1)\times(l_c+1),a(\eta_j)-(l_c-1)(l_c+1)).$$

Since the sequence $\eta_0,\cdots,\eta_r$ is a sequence of spin flips we have $|a(\eta_{j+1})-a(\eta_j)|\leq 1$ and $(a(\eta_j),\,0\leq j\leq r)$ takes all the values between $(l_c-1)(l_c+1)$ and $a(\eta_r)$. Henceforth

$$\max_{0\leq j\leq r}\mathcal{E}((l_c-1)\times(l_c+1),a(\eta_j)-(l_c-1)(l_c+1))\;\geq\;
\max\{\,\mathcal{E}((l_c-1)\times(l_c+1),k):0\leq k\leq a(\eta_r)-(l_c-1)(l_c+1)\,\}$$

and the area of $\eta_r$ must satisfy

$$\max\{\,\mathcal{E}((l_c-1)\times(l_c+1),k) : 0 \le k \le a(\eta_r)-(l_c-1)(l_c+1)\,\} \;\le\; \operatorname{pred}\mathcal{E}(n_c).$$

By proposition 6.9, we have $\mathcal{E}((l_c-1)\times(l_c+1),1) = E((l_c-1)\times(l_c+1)+_1 1)$ whence $\mathcal{E}((l_c-1)\times(l_c+1),1)-\mathcal{E}(n_c) = 2+h-hl_c$ which is strictly positive since $l_c = \lceil 2/h\rceil$. Thus $a(\eta_r) = (l_c-1)(l_c+1)$ and $\eta_r = (l_c-1)\times(l_c+1)$.

- Second case: all the sites are inside $(l_c-1)\times(l_c+1)$. Now $\eta_j \subset (l_c-1)\times(l_c+1)$ so that

$$\begin{aligned}\operatorname{pred}\mathcal{E}(n_c) \;&\ge\; \max_{0\le j\le r} E(\eta_j) \;\ge\; \max_{0\le j\le r} \mathcal{E}((l_c-1)\times(l_c+1),a(\eta_j)-(l_c-1)(l_c+1))\\ &\ge\; \max\{\,\mathcal{E}((l_c-1)\times(l_c+1),-k) : 0\le k\le (l_c-1)(l_c+1)-a(\eta_r)\,\}.\end{aligned}$$

Proposition 6.8 shows that for $0\le k\le l_c-1$, $\mathcal{E}((l_c-1)\times(l_c+1),-k) \;=\; E((l_c-1)\times l_c \oplus_2 (l_c-1-k))$ so that $\mathcal{E}(n_c)-\mathcal{E}((l_c-1)\times(l_c+1),-k) = h(l_c-k-2)$. For this quantity to be positive, we must have $k<l_c-2$ whence $a(\eta_r)\ge (l_c-1)(l_c+1)-(l_c-3)$. In addition for any $k$, $0\le k\le l_c-3$, $\mathcal{E}((l_c-1)\times(l_c+1),-k)-E((l_c-1)\times(l_c+1)) = hk>0$ so that $E(\eta_r)>E((l_c-1)\times(l_c+1))$ whenever $\eta_r\neq(l_c-1)\times(l_c+1)$.

We have proved that $(l_c-1)\times(l_c+1)$ and the cycle $\pi((l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))$ satisfy the hypothesis of theorem 5.5. It follows that the bottom of the cycle is $\{(l_c-1)\times(l_c+1)\}$. That $M((l_c-1)\times(l_c+1),-k)$ is included in the cycle for $0\le k\le l_c-3$ is obvious: each configuration of this set communicates with $(l_c-1)\times(l_c+1)$ under the level $\operatorname{pred}\mathcal{E}(n_c)$ (proposition 6.8 shows that all these configurations are obtained by deleting successively $k$ corner squares from $(l_c-1)\times(l_c+1)$). Finally a configuration $\xi$ of the principal boundary of this cycle is of energy $E(\xi)=\mathcal{E}(n_c)$ so that its area is equal to $n_c$, and its perimeter to $\min\{\,p(\sigma):\sigma\in C_{n_c}\,\}$. Let $\eta$ be a configuration of the cycle such that $q(\eta,\xi)>0$. Necessarily, the area of $\eta$ is $n_c+1$. Thus $\eta$ is a configuration of minimal area of the cycle and as such it is a minimal configuration of the cycle for the inclusion relation. By theorem 5.3 it is included in the rectangle $(l_c-1)\times(l_c+1)$. Thus $\xi$ belongs to $M((l_c-1)\times(l_c+1),-(l_c-2))$. Conversely each configuration of this set communicates with $M((l_c-1)\times(l_c+1),-(l_c-3))$ and belongs to the principal boundary of the cycle. Finally, proposition 6.8 shows that this set contains $\widetilde{M}_{n_c}$. $\square$

**Corollary 6.25.** *Let $\sigma$ belong to $\widetilde{M}_{n_c+1}\setminus S_{n_c+1}$. Suppose $(l_c-1)\times l_c\subset\sigma\subset(l_c-1)\times(l_c+1)$. Let $G^+(\sigma)$ be the minimal stable subgraph of $G^+$ containing $\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)$. The only arrows of $G^+(\sigma)$ entering $\pi(\underline{-1},\underline{+1}^c)$ are*

$$\{\xi\}\to\pi((l_c-1)\times l_c,\{\underline{-1},\underline{+1}\}^c),\quad \xi\in\widetilde{M}_{n_c}\setminus S_{n_c},\quad (l_c-1)\times l_c\subset\xi.$$

*The remaining arrows of $G^+(\sigma)$ are*

$$\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)\leftrightarrow\{\eta\},\quad \eta\in M((l_c-1)\times(l_c+1),-(l_c-2)).$$

*There is no arrow in $G^+(\sigma)$ ending at $\{\underline{+1}\}$.*

*Remark.* In this statement, the rectangle $(l_c - 1) \times l_c$ should be understood as a configuration and not as a polyomino. That is, $(l_c - 1) \times l_c \subset \xi$ means that the same (fixed) rectangle is included in $\sigma$ and $\xi$.

*Proof.* That these arrows belong to $G^+(\sigma)$ is a straightforward consequence of theorem 6.24 which implies in particular that $\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c) = \pi((l_c - 1) \times (l_c + 1), \{\underline{-1}, \underline{+1}\}^c)$. We have to check that there are no other arrow. Let $\eta$ belong to the principal boundary of $\pi((l_c - 1) \times (l_c + 1), \operatorname{pred} \mathcal{E}(n_c))$. If $\eta$ is in $\widetilde{M}_{n_c}$, then all arrows of $G$ starting at the cycle $\{\eta\}$ are present in the above list (lemma 6.21). If $\eta$ is not in $\widetilde{M}_{n_c}$, we claim that

$$\{\eta\} \to \pi((l_c - 1) \times (l_c + 1), \operatorname{pred} \mathcal{E}(n_c))$$

is the unique arrow of $G$ starting at $\{\eta\}$. Let $\xi$ be a point such that $q(\eta, \xi) > 0$ and $E(\xi) \leq E(\eta)$. Since $a(\eta) = n_c$, then $a(\xi)$ is equal to $n_c - 1$ or $n_c + 1$. Moreover $\eta$ is minimal, and the inequality $E(\xi) \leq E(\eta)$ implies in both cases that $\xi$ is also minimal and that $E(\xi) \leq \operatorname{pred} \mathcal{E}(n_c)$. Thus $\xi$ cannot be of area $n_c - 1$ (by lemma 6.4, the only configurations of $M_{n_c}$ communicating with $M_{n_c - 1}$ are the principal configurations $\widetilde{M}_{n_c}$). Thus $a(\xi) = n_c + 1$. We next show that $\xi$ belongs to $\pi((l_c - 1) \times (l_c + 1), \operatorname{pred} \mathcal{E}(n_c))$. Let $x_1, \cdots, x_r$ be a sequence of sites inside $\sigma$ such that $T(x_1, \cdots, x_j)((l_c - 1) \times (l_c + 1))$ is in $\pi((l_c - 1) \times (l_c + 1), \operatorname{pred} \mathcal{E}(n_c))$ for $j$ in $\{1 \cdots r - 1\}$ and $\eta = T(x_1, \cdots, x_r)((l_c - 1) \times (l_c + 1))$. We put $\eta_j = T(x_1, \cdots, x_j)((l_c - 1) \times (l_c + 1))$, $0 \leq j \leq r$. Since $\eta$ is in the boundary of the cycle, we have $E(\eta_{r-1}) < E(\eta_r)$. Moreover $\eta_{r-1}$ is a minimal configuration of the cycle, of area $n_c + 1$, so that the last spin flip at site $x_r$ has decreased the area. Let $x_{r+1}$ be the unique site such that $\xi = T(x_{r+1})(\eta)$. Suppose $\xi \neq \eta_{r-1}$ so that $x_{r+1} \neq x_r$ (and $\eta(x_{r+1}) = -1$). We have $\xi = T(x_{r+1})(\eta_r) = T(x_{r+1}, x_r)(\eta_{r-1}) = T(x_r, x_{r+1})(\eta_{r-1}) = T(x_r)(\eta'_r)$ where $\eta'_r = T(x_{r+1})(\eta_{r-1})$. The energy inequality 5.1 yields $E(\eta'_r) - E(\eta_{r-1}) \leq E(\xi) - E(\eta)$ whence $E(\eta'_r) \leq E(\eta_{r-1})$. It follows that $\eta'_r$ is in the cycle, as well as $\xi$ (their energies are less or equal than $\operatorname{pred} \mathcal{E}(n_c)$ and they communicate with a configuration of the cycle). $\square$

**Corollary 6.26.** *The principal non standard configurations of area $n_c$ (i.e. the set $\widetilde{M}_{n_c} \setminus S_{n_c}$) are dead–ends: there is no saddle path of null cost between $\underline{-1}$ and $\underline{+1}$ passing through them.*

**Corollary 6.27.** *The set of the global saddle points between $\underline{-1}$ and $\underline{+1}$ is exactly $S_{n_c}$. These configurations are the critical two dimensional configurations.*

Steps $ii)$ and $iii)$ are now completed and we proceed to steps $iv)$ and $v)$.

**The ascending part.** For each configuration $\sigma'$ in $S_{n_c}$, we must determine the minimal stable subgraph of $G^-$ containing $\sigma'$ and all the paths in this graph starting at $\{\sigma'\}$ and ending at $\underline{-1}$. Our exposition is similar as before: we first list the set of the relevant cycles and we use lemma 4.4 to find those belonging to $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. We finally check that we have in hand all the vertices of $G^-$.

**Theorem 6.28.** *Let $l$ be an integer strictly less than $l_c$.*
*The cycle $\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\begin{aligned} \underline{a}(\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1))) \;&=\; l(l-1)+2\,,\\ \forall k\in\{1\cdots l-2\}\qquad M(l\times l,-k)\;&\subset\;\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1))\,,\\ \overline{a}(\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1)))\;&=\;l^2\,. \end{aligned}$$

*The bottom of this cycle is $\{l\times l\}$; its principal boundary is $M(l\times l,-(l-1))$ and thus contains $S_{l(l-1)+1}$.*

*Proof.* We apply corollary 6.13. For any $n$ greater than $l^2$, we have

$$E(l\times l, C_n)\;\geq\;\mathcal{E}(l^2+1)\;\geq\;\mathcal{E}(l(l-1)+1)$$

whence $\overline{a}(\pi(l\times l,\mathrm{pred}\,\mathcal{E}(l(l-1)+1)))=l^2$. Analogously, for any $n$ strictly smaller than $l(l-1)+2$, we have $E(l\times l,C_n)\;\geq\;\mathcal{E}(l(l-1)+1)$; moreover,

$$E(l\times l, C_{l(l-1)+2})\;=\;\mathcal{E}(l(l-1)+2)\;<\;\mathcal{E}(l(l-1)+1)$$

so that $\underline{a}(\pi(l\times l,\mathrm{pred}\,\mathcal{E}(l(l-1)+1)))\;=\;l(l-1)+2$. To prove that the bottom of the cycle is $\{l\times l\}$, we could proceed as before and use theorem 5.5. However, a direct application of our geometrical results (theorem 6.2) yields that the minimum $\min\{\,\mathcal{E}(n):l(l-1)+2\leq n\leq l^2\,\}$ is equal to $\mathcal{E}(l^2)$; the unique configuration of energy $\mathcal{E}(l^2)$ is the square $l\times l$. Theorem 5.3 implies also that the altitude of communication between two different squares $l\times l$ is greater than $\mathcal{E}(l^2+1)$ (one has to make a spin–flip outside the initial square to create another square). The statement concerning the principal boundary is a consequence of proposition 6.8. $\square$

**Theorem 6.29.** *Let $l$ be an integer strictly less than $l_c$.*
*The cycle $\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\begin{aligned} \underline{a}(\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1)))\;&=\;l^2+2\,,\\ \forall k\in\{1\cdots l-2\}\qquad M(l\times(l+1),-k)\;&\subset\;\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1))\,,\\ \overline{a}(\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1)))\;&=\;l(l+1)\,. \end{aligned}$$

*The bottom of this cycle is $\{l\times(l+1)\}$; its principal boundary is $M(l\times(l+1),-(l-1))$ and thus contains $S_{l^2+1}$.*

*Proof.* The proof is similar as the proof of theorem 6.28.

*Remark.* Similar statements hold for the quasisquares $(l+1)\times l$.

**Corollary 6.30.** *Suppose $l < l_c$. The following cycles are maximal cycles of $\{\underline{-1},\underline{+1}\}^c$:*

$$
\begin{aligned}
&\{\eta\}, \qquad \eta \in M(l\times(l+1),-(l-1)),\\
&\{\eta\}, \qquad \eta \in M((l+1)\times l,-(l-1)),\\
&\{\eta\}, \qquad \eta \in M(l\times l,-(l-1)),\\
&\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1)),\\
&\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1)),\\
&\pi((l+1)\times l, pred\,\mathcal{E}(l^2+1)).
\end{aligned}
$$

*Proof.* This corollary is a consequence of lemma 4.4 together with theorems 6.28 and 6.29. Notice that we have to put together the descriptions of the cycles of theorems 6.28, 6.29 in order to check that for each cycle $\pi$ in the above list, there is a sequence of cycles $\pi_0,\cdots,\pi_r$ such that $\pi_0=\pi$, $\widetilde{B}(\pi_i)\cap\pi_{i+1}\neq\emptyset$, $0\leq i<r$ and $\underline{-1}\in\widetilde{B}(\pi_r)$. $\square$

**Corollary 6.31.** *Let $\sigma$ belong to $S_{n_c-1}$. The minimal stable subgraph $G^-(\sigma)$ of $G^-$ containing $\sigma$ is the restriction of $G$ to the vertices listed in corollary 6.30. The arrows of $G^-(\sigma)$ are (in the following list, each square $l\times l$ or quasisquare $l\times(l+1)$ must be included in $\sigma$):*

$$
\begin{aligned}
&\{\eta\}\leftrightarrow\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1)), && \eta\in M(l\times(l+1),-(l-1)),\\
&\{\eta\}\leftrightarrow\pi((l+1)\times l, pred\,\mathcal{E}(l^2+1)), && \eta\in M((l+1)\times l,-(l-1)),\\
&\{\eta\}\leftrightarrow\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1)), && \eta\in M(l\times l,-(l-1)),\\
&\{\eta\}\rightarrow\pi(l\times l, pred\,\mathcal{E}(l(l-1)+1)), && \eta\in \widetilde{M}_{l^2+1},\\
&\{\eta\}\rightarrow\pi(l\times(l+1), pred\,\mathcal{E}(l^2+1)), && \eta\in \overline{l\times(l+1)\oplus_1 1},\\
&\{\eta\}\rightarrow\pi((l+1)\times l, pred\,\mathcal{E}(l^2+1)), && \eta\in \overline{(l+1)\times l\oplus_2 1}.
\end{aligned}
$$

The only loops in the graph $G^-(\sigma)$ are loops around two cycles (corresponding to the arrows described in the first three lines of the list). Any other arrow $\pi_1\to\pi_2$ of $G^-(\sigma)$ satisfies $\overline{a}(\pi_1)>\underline{a}(\pi_2)$. As a consequence a path in $G^-(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{-1}\}$.

**The exit path.** We have finally reached the last step *vi*). We notice at this point that there exists only one optimal saddle between two cycles associated to each arrow of the graph $\mathcal{G}$. Thus the graph $\mathcal{G}$ contains all the information necessary to obtain the set of the saddle paths of null cost between $\underline{-1}$ and $\underline{+1}$. We describe for instance the canonical saddle path, which follows the sequence of the canonical configurations:

$$\underline{-1}\to m_1,\ m_1\to m_2,\ m_2\to m_3,\ m_3\to m_4,\ m_4\to m_5,\ m_5\to m_6,\ m_6\to m_7,$$

$$m_7 \to m_8,\, m_9 \to m_{10},\, m_{10} \to m_{11},\, m_{12} \to m_{13},\, m_{13} \to m_{14},\, m_{16} \to m_{17},\, \cdots,$$
$$m_{k^2} \to m_{k^2+1},\, m_{k^2+1} \to m_{k^2+2},\, m_{k(k+1)} \to m_{k(k+1)+1},\, \cdots,$$
$$m_{(l_c-1)^2} \to m_{(l_c-1)^2+1},\, m_{(l_c-1)^2+1} \to m_{(l_c-1)^2+2},\, m_{(l_c-1)l_c} \to m_{(l_c-1)l_c+1},$$
$$m_{(l_c-1)l_c+1} \to m_{(l_c-1)l_c+2},\, m_{l_c^2} \to m_{l_c^2+1},\, \cdots,\, m_{N^2-1} \to m_{N^2}.$$

We state here some consequences of the information provided by the graph $\mathcal{G}$ which describes the set of all the saddle paths of null cost. Most of them had already been proved by Neves and Schonmann with completely different methods (mainly coupling techniques). We stress that the graph $\mathcal{G}$ provides the most complete information available on the limiting dynamics. Once we know this graph, the results obtained in a general framework [6] may be applied in a systematic fashion to obtain various estimates. We let the process $(\sigma_n)_{n\in\mathbb{N}}$ start from $\underline{-1}$. We recall that $\tau(\underline{+1}^c)$ is the hitting time of the ground state $\underline{+1}$ and $\theta(\underline{-1}, \tau(\underline{+1}^c))$ is the last visit to the metastable state $\underline{-1}$ before reaching $\underline{+1}$.

**Theorem 6.32.** *(the exit path)*
*For any positive $\epsilon$, the following events take place with probability converging to one exponentially fast as $\beta$ goes to infinity:*

- $\exp\beta(\mathcal{E}(n_c)-\epsilon) \,\le\, \tau(\underline{+1}^c) \,\le\, \exp\beta(\mathcal{E}(n_c)+\epsilon)$;
- $\exp\beta(2-h-\epsilon) \,\le\, \tau(\underline{+1}^c) - \theta(\underline{-1}, \tau(\underline{+1}^c)) \,\le\, \exp\beta(2-h+\epsilon)$;
- *during the exit path $(\sigma_n, \theta \le n \le \tau)$, the process crosses the set $S_{n_c}$ of the critical configurations at exactly one point $\sigma_c$; it does not cross $C_{n_c} \setminus S_{n_c}$;*
- *if we let $n_* = \min\{n \ge \theta : a(\sigma_n) = n_c\}$, $n^* = \max\{n \le \tau : a(\sigma_n) = n_c\}$, we have that $\sigma_n = \sigma_{n_*} = \sigma_{n^*}$ for all $n$ in $\{n_* \cdots n^*\}$ and $n^* - n_* \le \exp(\beta\epsilon)$;*
- *all the configurations of the exit path before time $n_*$ are of area less than $n_c$, all the configurations after $n^*$ are of area greater than $n_c$;*
- *during the whole exit path, the configuration stays connected i.e. there is only one cluster of spins in the system; this cluster is a quasisquare $l \times (l+\epsilon)$ minus at most $l-1$ corner squares before $n_*$ and a rectangle $l_1 \times l_2$ minus at most $l_c - 2$ corner squares or plus a line after $n^*$;*
- *during the ascending part, the process goes through an increasing sequence of quasisquares ($l \times (l+\epsilon)$, $l < l_c$), and visits a quasisquare $l \times (l+\epsilon)$ approximately $\exp\beta(l-1)h$ times; when it leaves definitely a quasisquare $l \times (l+1)$, it makes a spin flip along the largest side of the quasisquare in order to create a square $(l+1) \times (l+1)$ and it does not create a rectangle $l \times (l+2)$;*
- *during the descending part, the process goes through an increasing sequence of rectangles ($l_1 \times l_2$, $\min(l_1, l_2) \ge l_c$); he visits each of them approximately $\exp\beta(2-h)$ times; when it leaves definitely a rectangle, he choses randomly the next rectangle it creates, either $(l_1+1) \times l_2$ or $l_1 \times (l_2+1)$;*
- *the process reaches the thermal equilibrium within each cycle of $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ it crosses: the distribution of the process before the exit of a cycle is very close to the Gibbs distribution at inverse temperature $\beta$ restricted to this cycle.*

## 7. Dimension three

A polyomino associated to a configuration is a finite union of unit cubes. For $\sigma$ a configuration we will denote by $a(\sigma)$ the area of the associated polyomino and by $v(\sigma)$ its volume. We will rely heavily on the notation and results of [1]. Let us recall some essential points.

**Summary of the combinatorial results.**

The parallelepiped of sides $j_1, j_2, j_3$ is denoted by $j_1 \times j_2 \times j_3$. A quasicube is a parallelepiped $j_1 \times j_2 \times j_3$ with $|j_1 - j_2| \leq 1, |j_2 - j_3| \leq 1, |j_3 - j_1| \leq 1$. We describe next a simple mechanism for adding a polyomino to a parallelepiped. For the sake of clarity, we need to work here with instances of the polyominoes having a definite position on the lattice $\mathbb{Z}^3$. Let $c$ be a polyomino. By $c(x_1, x_2, x_3)$ we denote the unique polyomino obtained by translating $c$ in such a way that

$$\begin{aligned}
\min\{\, y_1 : \exists\, (y_2, y_3) \quad (y_1, y_2, y_3) \in c(x_1, x_2, x_3)\,\} \;&=\; x_1,\\
\min\{\, y_2 : \exists\, (y_1, y_3) \quad (y_1, y_2, y_3) \in c(x_1, x_2, x_3)\,\} \;&=\; x_2,\\
\min\{\, y_3 : \exists\, (y_1, y_2) \quad (y_1, y_2, y_3) \in c(x_1, x_2, x_3)\,\} \;&=\; x_3.
\end{aligned}$$

Let $e_1, e_2, e_3$ be the standard basis of $\mathbb{Z}^3$. We define an operator $+_1$ which adds a polyomino $c$ to a parallelepiped $j_1 \times j_2 \times j_3$ in the direction of $e_1$ by

$$j_1 \times j_2 \times j_3 +_1 c \;=\; j_1 \times j_2 \times j_3(0,0,0) \cup c(j_1, 0, 0).$$

The operators $+_2$ and $+_3$ are defined similarly, working with the vectors $e_2$ and $e_3$:

$$\begin{aligned}
j_1 \times j_2 \times j_3 +_2 c &= j_1 \times j_2 \times j_3(0,0,0) \cup c(0, j_2, 0)\,,\\
j_1 \times j_2 \times j_3 +_3 c &= j_1 \times j_2 \times j_3(0,0,0) \cup c(0, 0, j_3)\,.
\end{aligned}$$

Let now $c$ be a two dimensional polyomino. We define the three dimensional polyomino $j_1 \times j_2 \times j_3 +_1 c$ as follows. First, we transform $c$ into a planar three dimensional polyomino $c'$ by replacing its squares by unit cubes. We rotate $c'$ so that its normal unit vector becomes $e_1$ (as if the two dimensional polyomino $c$ was initially included in the plane $(e_2, e_3)$). Then we use the previous definition to set $j_1 \times j_2 \times j_3 +_1 c = j_1 \times j_2 \times j_3 +_1 c'$. We make the following convention: when we speak of $j_1 \times j_2 \times j_3 +_3 c$, we assume implicitly that the two dimensional polyomino $c$ is included in the rectangle $j_1 \times j_2$ so that the resulting polyomino is included in $j_1 \times j_2 \times (j_3 + 1)$. Most often, $c$ will be a two dimensional polyomino like $l_1 \times l_2 +_1 k$. Because of the context, there should be no ambiguity between the $+$ operator for integers and polyominoes. Moreover, the latter will always have a subscript (i.e. $+_1$, $+_2$ $+_3$).

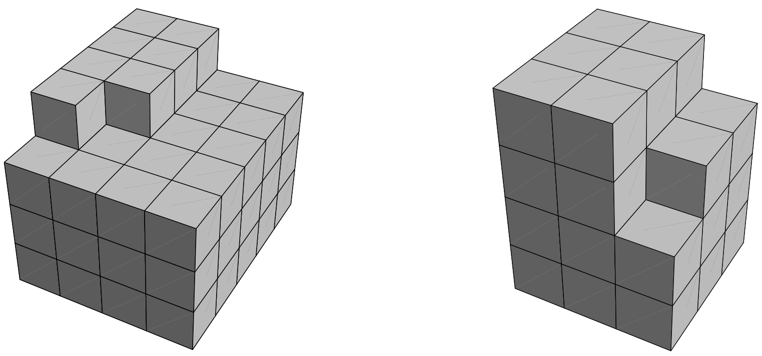

figure 5: the polyominoes $5 \times 4 \times 3 +_3 3 \times 2 +_1 1$, $3 \times 2 \times 4 +_2 3 \times 2 +_2 2$

By $j_1 \times j_2 \times j_3 \oplus_1 c$ (respectively $j_1 \times j_2 \times j_3 \oplus_2 c$, $j_1 \times j_2 \times j_3 \oplus_3 c$), we denote the set of all the polyominoes obtained by translating the polyomino $c$ along the side $j_2 \times j_3$ (resp. $j_1 \times j_3$, $j_1 \times j_2$) in such a way that the polyomino is always included in $(j_1+1) \times j_2 \times j_3$ (resp. $j_1 \times (j_2+1) \times j_3$, $j_1 \times j_2 \times (j_3+1)$). We set also

$$j_1 \times j_2 \times j_3 \oplus c = \bigcup_{i \in \{1,2,3\}} j_1 \times j_2 \times j_3 \oplus_i c.$$

For $A$ a set of polyominoes, we denote by $\overline{A}$ its orbit under the action of the spatial isometries which leave the integer lattice $\mathbb{Z}^3$ invariant. By $\overline{A}^{123}$ we denote its orbit under the action of the three symmetries with respect to the planes $(e_1, e_2), (e_2, e_3), (e_1, e_3)$.

**Proposition 7.1.** *For each integer $n$ there exists a unique 6–uple $(j, l, k, \delta, \gamma, \epsilon)$ such that $\delta, \gamma, \epsilon \in \{0, 1\}$, $\delta \leq \gamma$, $k < l+\epsilon$, $l(l+\epsilon)+k < (j+\delta)(j+\gamma)$ and $n = j(j+\delta)(j+\gamma)+l(l+\epsilon)+k$.*

The set of the polyominoes of volume $n$ is $\mathcal{C}_n$; the set $\mathcal{M}_n$ of the minimal polyominoes of volume $n$ is the set of the polyominoes of $\mathcal{C}_n$ having minimal area. Let $n = j(j+\delta)(j+\gamma)+l(l+\epsilon)+k$ be the decomposition of $n$. We put $r = l(l+\epsilon)+k$. The canonical polyomino $\mathfrak{m}_n$ of volume $n$ is obtained by adding the two dimensional canonical polyomino $m_r$ to the right side of a quasicube of volume $j(j+\delta)(j+\gamma)$. The general formula is

$$\mathfrak{m}_n = (j+\gamma) \times (j+\delta) \times j +_{1+\delta+\gamma} \big((l+\epsilon) \times l +_{1+\epsilon} k\big).$$

**Theorem 7.2.** *The canonical polyomino $\mathfrak{m}_n$ is minimal.*

This theorem is the key for determining the energy barrier the system has to overcome to travel from $\underline{-1}$ to $\underline{+1}$. It simultaneously gives a lower bound for this energy barrier and exhibits a growing sequence of polyominoes realizing this lower bound. This energy barrier gives the constant characterizing the asymptotic behaviour of the relaxation time. Neves has obtained the corresponding result for any dimension $d$ [14].

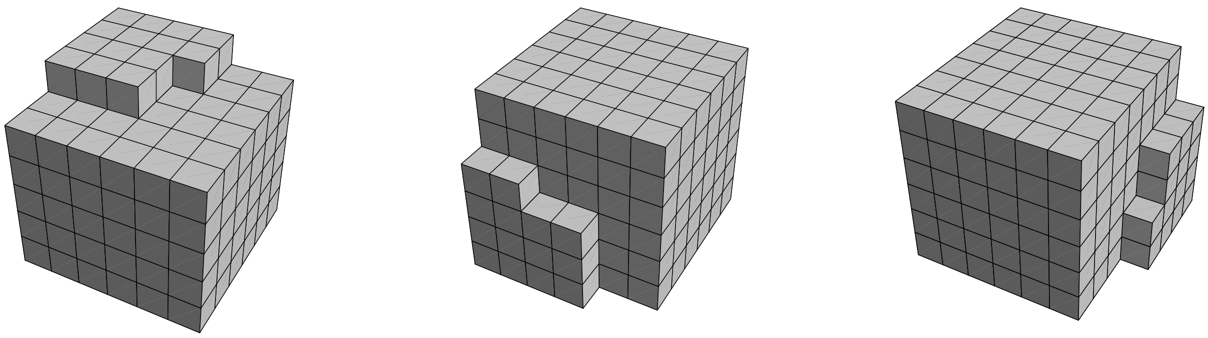

figure 6: the canonical polyominoes $\mathfrak{m}_{194}, \mathfrak{m}_{230}, \mathfrak{m}_{266}$

We define several important subsets of $\mathcal{M}_n$. The set $\mathcal{S}_n$ of the standard polyominoes is

$$\mathcal{S}_n = \overline{(j+\gamma) \times (j+\delta) \times j \oplus_{1+\delta+\gamma} \overline{(l+\epsilon) \times l \oplus_{1+\epsilon} k}}$$

and the set $\widetilde{\mathcal{M}}_n$ of the principal polyominoes is

$$\widetilde{\mathcal{M}}_n = \bigcup_{\substack{t=1,2,3 \\ u=1,2}} \overline{(j+\gamma) \times (j+\delta) \times j \oplus_t \overline{(l+\epsilon) \times l \oplus_u k}}$$

The sets $\mathcal{S}_n$ and $\widetilde{\mathcal{M}}_n$ coincide if $\delta = \gamma = \epsilon = 0$. Moreover we have $\{\mathfrak{m}_n\} \subset \mathcal{S}_n \subset \widetilde{\mathcal{M}}_n \subset \mathcal{M}_n$. However, the inclusions might be strict, as shown by the examples of figures 7, 8.

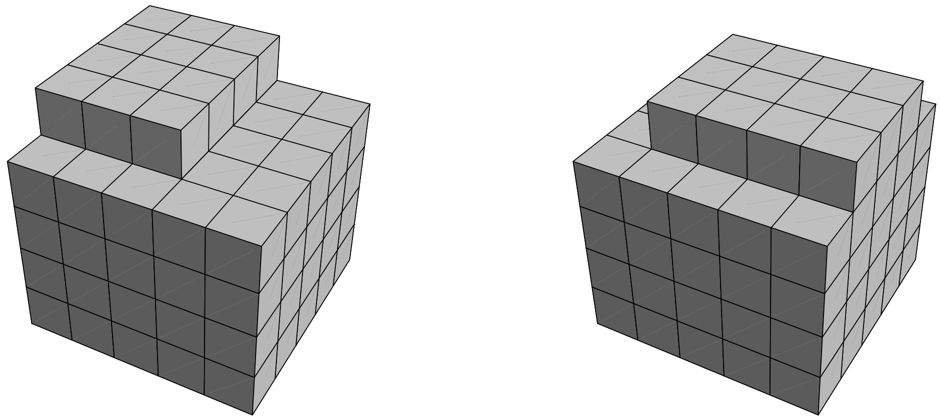

figure 7: polyominoes in $\{m_{112}\}$, $\mathcal{S}_{112} \setminus \{m_{112}\}$.

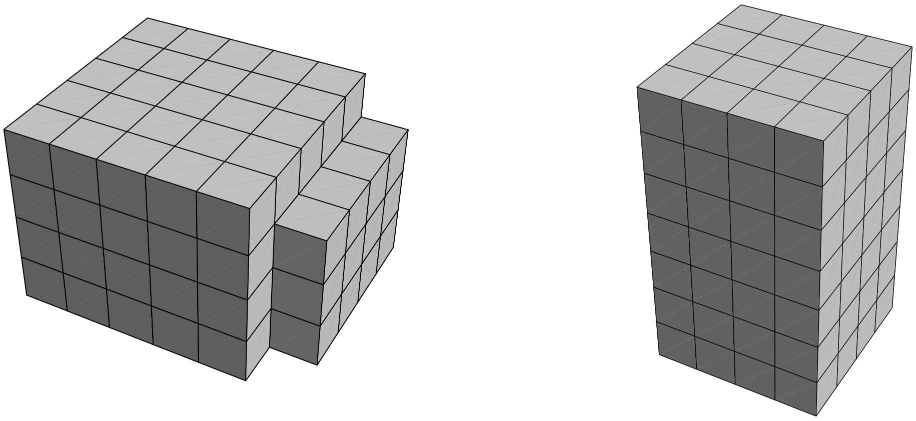

figure 8: polyominoes in $\widetilde{\mathcal{M}}_{112} \setminus \mathcal{S}_{112}$, $\mathcal{M}_{112} \setminus \widetilde{\mathcal{M}}_{112}$.

**Theorem 7.3.** *The set $\mathcal{M}_n$ is reduced to $\{\mathfrak{m}_n\}$ if and only if $n$ is of the form $j^3$.*
*The set $\mathcal{M}_n$ is equal to $\mathcal{S}_n$ if and only if $n$ is of the form $j^3, j^2(j+1), j(j+1)^2$ or $j^3-1, j^2(j+1)-1, j(j+1)^2-1$ (in which case $\mathcal{S}_n = \overline{\mathfrak{m}}_n$), or $j^3+l^2, j^3+l(l+1)$.*
*The set $\mathcal{M}_n$ coincides with $\widetilde{\mathcal{M}}_n$ if and only if $n$ is of the form $j^3-1, j^2(j+1)-1, j(j+1)^2-1$ or*

$$\begin{array}{ccc} j^3 & j^3+l^2 & j^3+l(l+1) \\ j^2(j+1) & j^2(j+1)+l^2 & j^2(j+1)+l(l+1) \\ j(j+1)^2 & j(j+1)^2+l^2 & j(j+1)^2+l(l+1) \end{array}$$

*(where in an expression $j(j+\delta)(j+\gamma)+l(l+\epsilon)$, we have $l(l+\epsilon) < (j+\delta)(j+\gamma)$).*

This theorem will be crucial to determine precisely the set of the critical droplets. It gives the uniqueness results associated to the discrete isoperimetric inequality. In fact, it will turn out that we do not have uniqueness of the isoperimetric problem for the volume of the critical droplet. Luckily enough, there is uniqueness for the volume preceding it. We will therefore rely on the following lemma.

**Lemma 7.4.** *For $n$ the volume of a quasicube plus a quasisquare i.e. of the form $j(j+\delta)(j+\gamma)+l(l+\epsilon)$ (where $0 < l(l+\epsilon) < (j+\delta)(j+\gamma)$) we have*

$$\begin{aligned} \{\, c \in \mathcal{M}_{n-1} : q(\widetilde{\mathcal{M}}_n \setminus \mathcal{S}_n, c) = 1 \,\} &\supset \widetilde{\mathcal{M}}_{n-1} \setminus \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n-1} : q(\mathcal{S}_n, c) > 0 \,\} &\supset \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n+1} : q(\mathcal{M}_n, c) > 0 \,\} &= \widetilde{\mathcal{M}}_{n+1}, \end{aligned}$$

*where $q(A,c) = \min\{q(d,c) : d \in A\}$.*

**Proposition 7.5.** *The principal polyominoes can be completely shrunk through the principal polyominoes: for any integer $n$ and for any principal polyomino $c$ in $\widetilde{\mathcal{M}}_n$, there exists an increasing sequence $c_0, \cdots, c_n$ of principal polyominoes such that $c_0 = \emptyset$, $c_n = c$ and $q(c_{i-1}, c_i) > 0$ for $i$ in $\{1 \cdots n\}$.*

A consequence of this proposition is that the set of the principal polyominoes associated to the critical volume is contained in the principal boundary of the greatest cycle containing $\underline{-1}$ and not $\underline{+1}$. Let us remark that it is not possible to grow arbitrarily far through the minimal polyominoes a principal polyomino which is not standard. The growth will be stopped at either a quasicube plus a rectangle $l \times (l+2)$ or at a parallelepiped $j \times j \times (j+2)$, $j \times (j+1) \times (j+2)$. Such a polyomino is a dead–end.

**Proposition 7.6.** *The standard polyominoes can be grown or shrunk arbitrarily far through the standard polyominoes: for any integers $m \leq n$ and for any standard polyomino $c$ in $\mathcal{S}_m$, there exists an increasing sequence $c_0, \cdots, c_n$ of standard polyominoes such that $c_0 = \emptyset$, $c_m = c$ and $q(c_{i-1}, c_i) > 0$ for $i$ in $\{1 \cdots n\}$.*

The statement of proposition 7.6 concerns the set of the standard polyominoes, which (except for specific values of the volume) is a strict subset of the set of the principal polyominoes considered in proposition 7.5. The nice feature of the standard polyominoes is that they can be grown arbitrarily far through the minimal polyominoes (proposition 7.6 asserts that a standard polyomino of volume $m$ can be grown until any volume $n \geq m$).

**Notation 7.7.** We have also some results concerning the best way to shrink or to grow a parallelepiped plus a rectangle. Let $c$ be either a parallelepiped or a parallelepiped plus a rectangle and let $k$ be a positive integer. We define

$$\mathcal{M}(c, -k) \;=\; \{\, d \in \mathcal{C}_{v(c)-k} : d \subset c,\; a(d) \text{ minimal} \,\}\,,$$

i.e. a polyomino $d$ belongs to $\mathcal{M}(c, -k)$ if and only if

$$d \in \mathcal{C}_{v(c)-k}, \quad d \subset c, \quad a(d) \;=\; \min\{\, a(d') : d' \in \mathcal{C}_{v(c)-k},\; d' \subset c \,\}.$$

Similarly, we define

$$\mathcal{M}(c, k) \;=\; \{\, d \in \mathcal{C}_{v(c)+k} : c \subset d,\; a(d) \text{ minimal} \,\}\,,$$

i.e. a polyomino $d$ belongs to $\mathcal{M}(c, k)$ if and only if

$$d \in \mathcal{C}_{v(c)+k}, \quad c \subset d, \quad a(d) \;=\; \min\{\, a(d') : d' \in \mathcal{C}_{v(c)+k},\; c \subset d' \,\}.$$

Notice that the elements of $\mathcal{M}(c, k)$ (respectively $\mathcal{M}(c, -k)$) all have the same energy, since they all have the same volume and area. We denote by $\mathcal{E}(c, k)$ (respectively $\mathcal{E}(c, -k)$) the energy of an element of $\mathcal{M}(c, k)$ (resp. $\mathcal{M}(c, -k)$).

The next two results are restatements of Proposition 3.25, Corollary 3.26, 3.27 and Proposition 3.28 of [1].

**Proposition 7.8.** *Let $j_1, j_2, j_3, r$ be positive integers such that $r < \min(j_1 j_2, j_2 j_3, j_3 j_1)$. The set $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$ is the set of the polyominoes obtained by removing from $j_1 \times j_2 \times j_3$ as many bars as possible, and then removing a succession of corner cubes until reaching the volume $j_1 j_2 j_3 - r$. In particular, a polyomino obtained from $j_1 \times j_2 \times j_3$ by the successive removal of $r$ cubes in such a way that each cube removal takes place on a bar of minimal length is in $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$.*

**Proposition 7.9.** *Let $j_1, j_2, j_3, r$ be positive integers such that $r < \min(j_1^2, j_2^2, j_3^2)$. The best way to add $r$ cubes to the parallelepiped $j_1 \times j_2 \times j_3$ is to add a minimal two dimensional polyomino of $M_r$ on one side of the parallelepiped. Equivalently, we have*

$$\mathcal{M}(j_1 \times j_2 \times j_3, r) \;=\; \overline{\{\, j_1 \times j_2 \times j_3 \oplus_i d,\ 1 \leq i \leq 3,\ d \in M_r \,\}}^{123}.$$

*In particular, $j_1 \times j_2 \times j_3 \oplus_i m_r \subset \mathcal{M}(j_1 \times j_2 \times j_3, r)$ for any $i$ in $\{1, 2, 3\}$.*

These propositions will be the key to find the principal boundary of the cycles around the supercritical parallelepipeds, around the subcritical quasicubes and around the dead–ends i.e. the principal non standard polyominoes.

**Proposition 7.10.** *Let $j_1, j_2, j_3, l_1, l_2, r$ be integers. We consider a polyomino $c$ of the set $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2$. We suppose that $r < \min(l_1, l_2) \leq \min(j_1, j_2, j_3)$. The set $\mathcal{M}(c, -r)$ is the set of the polyominoes obtained by removing successively $r$ corner cubes from $c$. The set $\mathcal{M}(c, r)$ is equal to the set of the polyominoes obtained by adding a bar of length $r$ against a compatible side of the rectangle $l_1 \times l_2$ (in such a way that $l_1 \times l_2 \oplus r$ fits into the side of the parallelepiped).*

This proposition will be the key to find the principal boundary of the cycles around the configurations which are parallelepipeds plus rectangles.

**Application to the three dimensional Ising model.**

We first express the energy of the Ising model with the help of the area and the volume of the polyomino associated to the configuration.

**Lemma 7.11.** *For any configuration $\sigma$ in $X$, we have*

$$E(\sigma) \;=\; -\frac{1}{2} \sum_{\{x,y\}: x \sim y} \sigma(x)\sigma(y) \;-\; \frac{h}{2} \sum_{x \in \Lambda} \sigma(x) \;=\; a(\sigma) \;-\; hv(\sigma) \;+\; N^3(h-3)/2\,.$$

We do not change the dynamics nor the cycle decomposition by adding a constant to the energy $E$. In dimension three, we will work with the energy $E(\sigma) = a(\sigma) - hv(\sigma)$. We denote by $\mathcal{E}(n)$ the minimal energy of a configuration of $\mathcal{C}_n$ (so that $\mathcal{E}(n) = E(\sigma) = a(\sigma) - hn$ for any $\sigma$ in $\mathcal{M}_n$).

*Hypothesis on the magnetic field h and the size N.* We suppose that $h$ is small compared to the unity and that for any configurations $\eta, \sigma$, the equality $E(\eta) = E(\sigma)$ implies $v(\eta) = v(\sigma)$ and $a(\eta) = a(\sigma)$. Whenever we take the integral part of a quantity involving $h$, we assume that this quantity is not an integer. For instance $4/h$ is not an integer. Finally, $N$ is large enough to ensure that the combinatorial results proved on the infinite lattice $\mathbb{Z}^3$ remain valid on the torus until the critical volume $4/h(4/h+1)^2 + 2/h(2/h+1)$. This is obviously true if $h^3 N > 106$.

*Remark.* We believe that a careful study would yield the weaker condition $h^3N^3 > K$ for some constant $K$. However this seems to require lengthy extensions of the techniques used in [1].

As we did in dimension two, we follow the general strategy outlined in section 4. Our first aim is to determine the communication altitude between $\underline{-1}$ and $\underline{+1}$ (corollary 7.13) and to compute it explicitly (proposition 7.14).

**Proposition 7.12.** *(communication altitude between configurations of different volumes) Let $m \le n$ be two integers. The communication altitude between the sets $\mathcal{C}_m$ and $\mathcal{C}_n$ (or $\mathcal{S}_m$ and $\mathcal{S}_n$) is*

$$E(\mathcal{C}_m, \mathcal{C}_n) \;=\; E(\mathcal{S}_m, \mathcal{S}_n) \;=\; \max\{\, \mathcal{E}(r) : m \le r \le n \,\} \;=\; E(\mathfrak{m}_m, \mathfrak{m}_n).$$

*Proof.* This result might be proven in exactly the same way as in dimension two (see propositions 6.11 and 6.12, corollary 6.13). $\square$

**Corollary 7.13.** $E(\underline{-1}, \underline{+1}) \;=\; \max\{\, \mathcal{E}(n) : 0 \le n \le N^3 \,\}$.

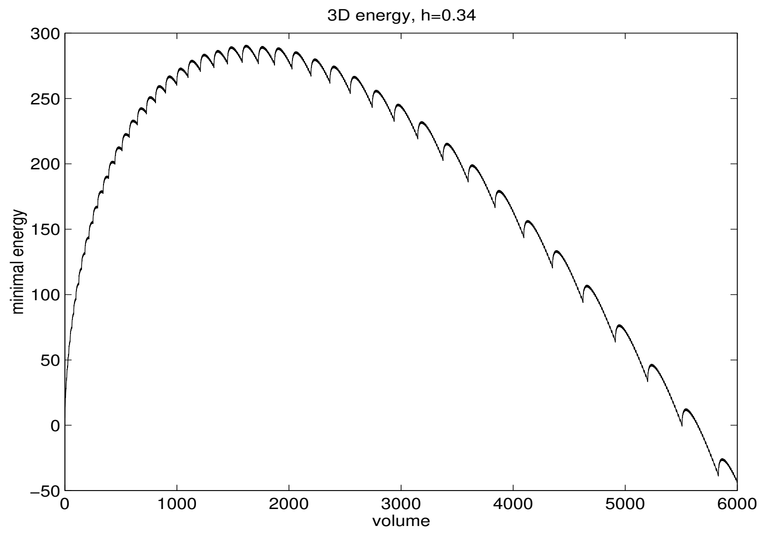


figure 9: the three dimensional minimal energy $\mathcal{E}(n)$

To compute $\mathcal{E}(n)$ we apply theorem 7.2. For each integer $n$ there exists a unique 6–uple $(j,l,k,\delta,\gamma,\epsilon)$ such that $\delta,\gamma,\epsilon\in\{0,1\}$, $\delta\leq\gamma$, $k<l+\epsilon$, $l(l+\epsilon)+k<(j+\delta)(j+\gamma)$ and $n=j(j+\delta)(j+\gamma)+l(l+\epsilon)+k$. The minimal area of a polyomino of volume $n$ is then

$$\min_{\sigma\in\mathcal{C}_n} a(\sigma) \;=\; 2(j(j+\delta)+j(j+\gamma)+(j+\delta)(j+\gamma))+2(2l+\epsilon)+2\times 1_{\{k>0\}}.$$

Figure 9 shows a graph of $\mathcal{E}(n)$ for $h=0.34$. Let us point out that each arch corresponds to a two dimensional energy curve. Figure 10 is a narrow view of the graph around its maximum.

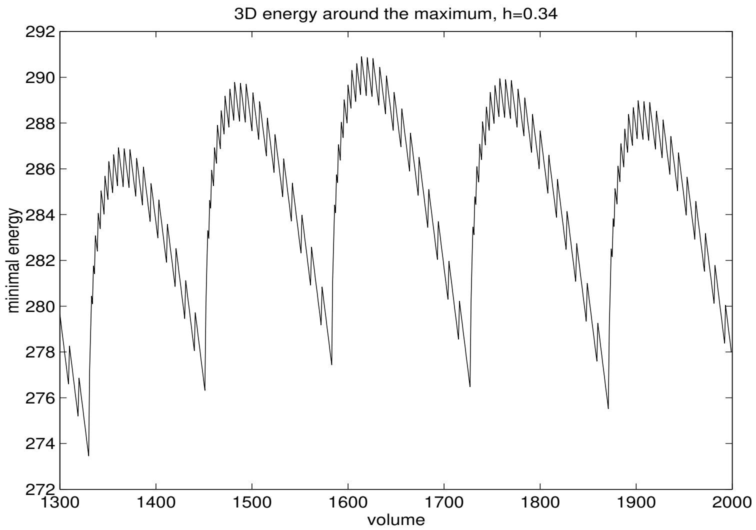


figure 10: $\mathcal{E}(n)$ around the maximum

**Proposition 7.14.** *We set $\delta_c=1$ if $4+\sqrt{16+h^2}\;<\;h(2\lceil 4/h\rceil-1)$ and $\delta_c=0$ otherwise. The communication altitude between $\underline{-1}$ and $\underline{+1}$ is*

$$E(\underline{-1},\underline{+1}) \;=\; \mathcal{E}(n_c) \;=\mathcal{E}((j_c-1)(j_c-\delta_c)j_c)+\mathcal{E}_2((l_c-1)l_c+1)$$

*where $l_c=\lceil 2/h\rceil$ is the two dimensional critical length, $j_c=\lceil 4/h\rceil$ is the three dimensional critical length, $n_c=(j_c-1)(j_c-\delta_c)j_c+(l_c-1)l_c+1$ is the critical volume.*

*Remark.* We have $E(\underline{-1},\underline{+1})\sim 32/h^2$ as $h\to 0$: the global energy barrier goes to infinity like $h^{-2}$ as $h$ goes to zero.

*Proof.* We denote by $\mathcal{E}_2$ the two dimensional energy, so that $\mathcal{E}(n)=\mathcal{E}(j(j+\delta)(j+\gamma))+\mathcal{E}_2(l(l+\epsilon)+k)$. We first compute the energy barrier between two consecutive quasicubes:

$$\max_{j^3\leq n\leq j^2(j+1)}\mathcal{E}(n) \;=\; \mathcal{E}(j^3)+\max_{0\leq n\leq j^2}\mathcal{E}_2(n)\,,$$

$$\max_{j^2(j+1)\leq n\leq j(j+1)^2}\mathcal{E}(n) \;=\; \mathcal{E}(j^2(j+1))+\max_{0\leq n\leq j(j+1)}\mathcal{E}_2(n)\,,$$

$$\max_{j(j+1)^2\leq n\leq (j+1)^3}\mathcal{E}(n) \;=\; \mathcal{E}(j(j+1)^2)+\max_{0\leq n\leq (j+1)^2}\mathcal{E}_2(n)\,.$$

We next compare these quantities. Let

$$\begin{aligned}
\Delta_1 &= \max_{j^3\le n\le j^2(j+1)} \mathcal{E}(n) - \max_{(j-1)j^2\le n\le j^3} \mathcal{E}(n) = j(4-hj)\,,\\
\Delta_2 &= \max_{j^2(j+1)\le n\le j(j+1)^2} \mathcal{E}(n) - \max_{j^3\le n\le j^2(j+1)} \mathcal{E}(n) = j(4-hj) + \max_{0\le n\le j(j+1)} \mathcal{E}_2(n) - \max_{0\le n\le j^2} \mathcal{E}_2(n)\,,\\
\Delta_3 &= \max_{j(j+1)^2\le n\le (j+1)^3} \mathcal{E}(n) - \max_{j^2(j+1)\le n\le j(j+1)^2} \mathcal{E}(n) = j(4-hj) + 2 - hj + \max_{0\le n\le (j+1)^2} \mathcal{E}_2(n)\\
&\qquad - \max_{0\le n\le j(j+1)} \mathcal{E}_2(n)\,.
\end{aligned}$$

The unique positive root of $4j - hj^2 + 2 - hj$ is $j_c' = 2/h(1+\sqrt{1+h^2/16}) - 1/2$ which is less than $j_c'' = 4/h$, but greater than $l_c$. For $j < j_c''$, $\Delta_1$ is positive, and for $j > j_c''$, $\Delta_1$ is negative. For $j < j_c''$, $\Delta_2$ is positive. For $j > j_c''$, we have $j > l_c$ (the two dimensional critical length) so that the two dimensional maxima are equal and $\Delta_2 = j(4-hj)$ which is negative. Finally, $\Delta_3$ is positive for $j < j_c'$ and negative for $j > j_c'$.
We have shown that the quantity

$$\max\{\,\mathcal{E}(n) : j(j+\delta)(j+\gamma) \le n \le (j+1)(j+\delta)(j+\gamma)\,\} - \max\{\,\mathcal{E}(n) : j(j+\delta)(j+\gamma-1) \le n \le j(j+\delta)(j+\gamma)\,\}$$

is positive for

- $\delta = 0$, $\gamma = 0$, $j < j_c''$ (case of $\Delta_1$)
- $\delta = 0$, $\gamma = 1$, $j < j_c''$ (case of $\Delta_2$)
- $\delta = 1$, $\gamma = 1$, $j < j_c'$ (case of $\Delta_3$)

and negative for

- $\delta = 0$, $\gamma = 0$, $j > j_c''$ (case of $\Delta_1$)
- $\delta = 0$, $\gamma = 1$, $j > j_c''$ (case of $\Delta_2$)
- $\delta = 1$, $\gamma = 1$, $j > j_c'$ (case of $\Delta_3$)

We now define the critical length.

- If $\lceil j_c' \rceil = \lceil j_c'' \rceil$ we put $j_c = \lceil j_c' \rceil$ and $\delta_c = 0$.
- If $\lceil j_c' \rceil < \lceil j_c'' \rceil$ we put $j_c = \lceil j_c'' \rceil$ and $\delta_c = 1$.

The volume of the critical quasicube is $(j_c-1)(j_c-\delta_c)j_c$. We can now compute the energy barrier between two remote quasicubes.

$$\max\{\,\mathcal{E}(n) : j_1(j_1+\delta_1)(j_1+\gamma_1) \le n \le j_2(j_2+\delta_2)(j_2+\gamma_2)\,\} =
\begin{cases}
\mathcal{E}((j_c-1)(j_c-\delta_c)j_c) + \mathcal{E}_2((l_c-1)l_c+1) \\
\qquad\text{if} \quad j_1(j_1+\delta_1)(j_1+\gamma_1) \le (j_c-1)(j_c-\delta_c)j_c < j_2(j_2+\delta_2)(j_2+\gamma_2) \\
\mathcal{E}(j_2(j_2+\delta_2)(j_2+\gamma_2-1)) + \mathcal{E}_2((l_c-1)l_c+1) \\
\qquad\text{if} \quad j_2(j_2+\delta_2)(j_2+\gamma_2) \le (j_c-1)(j_c-\delta_c)j_c,\ j_2 \ge l_c \\
\mathcal{E}(j_1(j_1+\delta_1)(j_1+\gamma_1)) + \mathcal{E}_2((l_c-1)l_c+1) \\
\qquad\text{if} \quad (j_c-1)(j_c-\delta_c)j_c < j_1(j_1+\delta_1)(j_1+\gamma_1)
\end{cases}$$

We finally obtain the value of the global energy barrier stated in the proposition by choosing $j_1 = \delta_1 = \gamma_1 = 0$ and $j_2 = N, \delta_2 = \gamma_2 = 0$. $\square$

From this we deduce the level of the greatest cycle containing $\underline{-1}$ and not $\underline{+1}$.

**Corollary 7.15.** *The level of the cycle $\pi(\underline{-1}, \underline{+1}^c)$ is pred $\mathcal{E}(n_c)$.*

**Notation 7.16.** If $Y$ is a subset of $X$, its minimal and maximal volumes $\underline{v}(Y)$ and $\overline{v}(Y)$ are

$$\underline{v}(Y) = \min\{\, v(\sigma) : \sigma \in Y\,\}, \quad \overline{v}(Y) = \max\{\, v(\sigma) : \sigma \in Y\,\}.$$

Since $E(\underline{-1}, \mathcal{C}_n) = \mathcal{E}(n_c)$ for any $n \geq n_c$ (by propositions 7.12 and 7.14), then all the configurations of the cycle $\pi(\underline{-1}, \underline{+1}^c)$ have a volume less than $n_c - 1$ i.e. $\overline{v}(\pi(\underline{-1}, \underline{+1}^c)) \leq n_c - 1$. To complete step $i)$ of the general strategy, we determine the configurations of the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$.

**Theorem 7.17.** *The principal boundary $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ of the cycle $\pi(\underline{-1}, \underline{+1}^c)$ is the set $\widetilde{\mathcal{M}}_{n_c}$ of the principal configurations of volume $n_c$. For $\sigma$ in $\widetilde{\mathcal{M}}_{n_c}$ there exists a unique configuration in $\pi(\underline{-1}, \underline{+1}^c)$ communicating with $\sigma$, which is the quasicube plus a quasisquare of the set $\overline{(j_c - 1) \times (j_c - \delta_c) \times j_c \oplus (l_c - 1) \times l_c}$ included in $\sigma$.*

*Proof.* Let $\sigma$ belong to $\widetilde{\mathcal{M}}_{n_c}$. Then $E(\sigma) = \mathcal{E}(n_c)$ and proposition 7.5 yields the existence of a sequence $\sigma_0, \cdots, \sigma_{n_c}$ such that

$$\sigma_0 = \underline{-1}, \quad \sigma_{n_c} = \sigma, \quad \forall j \in \{0 \cdots n_c - 1\} \quad \sigma_j \in \mathcal{M}_j, \quad q^+(\sigma_j, \sigma_{j+1}) = 1.$$

In particular, we have $\max\{\, E(\sigma_n) : 0 \leq n < n_c\,\} < \mathcal{E}(n_c)$ so that $\sigma_{n_c - 1}$ belongs to $\pi(\underline{-1}, \underline{+1}^c)$. Since $q(\sigma_{n_c-1}, \sigma_{n_c}) > 0$, the configuration $\sigma_{n_c}$ is in the principal boundary of $\pi(\underline{-1}, \underline{+1}^c)$. Thus $\widetilde{\mathcal{M}}_{n_c} \subset \widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$.
Conversely, let $\sigma$ belong to $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$. Necessarily $E(\sigma) = \mathcal{E}(n_c)$ so that $\sigma$ is of volume $n_c$ and it is a minimal configuration. In addition there must exist $\eta$ in $\pi(\underline{-1}, \underline{+1}^c)$ communicating with $\sigma$. This $\eta$ is in $\mathcal{C}_{n_c - 1}$ and satisfies $E(\eta) < E(\sigma)$. Thus

$$\min\{\, a(\xi) : \xi \in \mathcal{C}_{n_c-1}\,\} \leq a(\eta) \leq a(\sigma) - h < a(\sigma) = \min\{\, a(\xi) : \xi \in \mathcal{C}_{n_c-1}\,\} + 2.$$

However the area is an even integer. The only possibility is $a(\eta) = \min\{\, a(\xi) : \xi \in \mathcal{C}_{n_c-1}\,\}$ whence $\eta$ is minimal. Yet $n_c - 1 = (j_c - 1)(j_c - \delta_c)j_c + (l_c - 1)l_c$ is the volume of a quasicube plus a quasisquare; by theorem 7.3, $\mathcal{M}_{n_c-1} = \widetilde{\mathcal{M}}_{n_c-1}$ so that $\eta$ belongs to $\widetilde{\mathcal{M}}_{n_c-1}$ and it is a quasicube plus a quasisquare $j \times (j - \delta_c) \times j \oplus (l_c - 1) \times l_c$. Lemma 7.4 shows that the only points of $\mathcal{M}_{n_c}$ which communicate with $\mathcal{M}_{n_c-1}$ are the configurations of $\widetilde{\mathcal{M}}_{n_c}$. Thus $\sigma$ is a principal configuration of volume $n_c$ and $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c)) \subset \widetilde{\mathcal{M}}_{n_c}$. $\square$

We now proceed to steps $ii)$ and $iii)$ of the general strategy. Moreover, we will handle separately the case of the configurations in $\mathcal{S}_{n_c}$ and in $\widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}$: it turns out that step $ii)$ succeeds for the standard configurations and fails for the principal non standard configurations. The latter are dead–ends.

**The standard configurations.** We start by describing the relevant list of cycles for determining the minimal stable subgraph of $G^+$ containing the standard configurations of volume $n_c$. These are the cycles around a parallelepiped $j_1 \times j_2 \times j_3$, a parallelepiped plus a rectangle $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2$ and finally a parallelepiped plus a rectangle plus a bar $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2 \oplus k$, where each configuration contains strictly a critical configuration.

**Theorem 7.18.** *Let $j_1, j_2, j_3$ be three integers such that the parallelepiped $j_1 \times j_2 \times j_3$ contains a configuration of $\mathcal{S}_{n_c}$. The cycle*

$$\pi = \pi(j_1 \times j_2 \times j_3, \operatorname{pred} E(j_1 \times j_2 \times j_3 \oplus (l_c - 1) \times l_c \oplus 1))$$

*does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\begin{aligned} \underline{v}(\pi) &> j_1 j_2 j_3 - \min(j_1 j_2, j_1 j_3, j_2 j_3) + (l_c - 1) l_c + 1 \,, \\ \overline{v}(\pi) &= j_1 j_2 j_3 + (l_c - 1) l_c \,. \end{aligned}$$

*The bottom of this cycle is $\{j_1 \times j_2 \times j_3\}$ and its principal boundary is*

$$\widetilde{B}(\pi) \;=\; \overline{j_1 \times j_2 \times j_3 \oplus \overline{(l_c - 1) \times l_c \oplus 1}}^{123} \,.$$

*Proof.* We check that the parallelepiped $j_1 \times j_2 \times j_3$ and the cycle $\pi$ satisfy the hypothesis of theorem 5.5. Let $x_1, \cdots, x_r$ be a sequence of sites such that $T(x_1, \cdots, x_s)(j_1 \times j_2 \times j_3)$ is in $\pi$ for $s$ in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\operatorname{pred} E(j_1 \times j_2 \times j_3 \oplus (l_c - 1) \times l_c \oplus 1)$). We put $\eta_s = T(x_1, \cdots, x_s)(j_1 \times j_2 \times j_3)$ for $s$ in $\{0 \cdots r\}$.

- First case: all the sites $(x_1, \cdots, x_r)$ are outside $j_1 \times j_2 \times j_3$. We have that $j_1 \times j_2 \times j_3 \subset \eta_s$ whence $E(\eta_s) \geq \mathcal{E}(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3)$ (see notation 7.7) and

$$\operatorname{pred} E(j_1 \times j_2 \times j_3 \oplus (l_c - 1) \times l_c \oplus 1) \;\geq\; \max_{0 \leq s \leq r} \mathcal{E}(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|v(\eta_{s+1}) - v(\eta_s)| \leq 1$ and $(v(\eta_s),\, 0 \leq s \leq r)$ takes all the values between $j_1 j_2 j_3$ and $v(\eta_r)$. Henceforth

$$\max_{0 \leq s \leq r} \mathcal{E}(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3) \;\geq\; \max\{\, \mathcal{E}(j_1 \times j_2 \times j_3, k) : 0 \leq k \leq v(\eta_r) - j_1 j_2 j_3 \,\}$$

and the volume of $\eta_r$ must satisfy

$$\max\{\,\mathcal{E}(j_1\times j_2\times j_3,k):0\le k\le v(\eta_r)-j_1j_2j_3\,\}\ \le\ \mathrm{pred}\,E(j_1\times j_2\times j_3\oplus(l_c-1)\times l_c\oplus 1).$$

By proposition 7.9, we have

$$\mathcal{E}(j_1\times j_2\times j_3,(l_c-1)l_c+1)=E(j_1\times j_2\times j_3\oplus(l_c-1)\times l_c\oplus 1)$$

so that $v(\eta_r)<j_1j_2j_3+(l_c-1)l_c+1$.

- Second case: all the sites are inside $j_1\times j_2\times j_3$. Now $\eta_s\subset j_1\times j_2\times j_3$ so that

$$\begin{aligned}\mathrm{pred}\,E(j_1\times j_2\times j_3\oplus(l_c-1)\times l_c\oplus 1)\ &\ge\ \max_{0\le s\le r}\mathcal{E}(j_1\times j_2\times j_3,v(\eta_s)-j_1j_2j_3)\\ &\ge\ \max\{\,\mathcal{E}(j_1\times j_2\times j_3,-k):0\le k\le j_1j_2j_3-v(\eta_r)\,\}.\end{aligned}$$

Suppose for instance that $j_1j_2$ is the smallest side of the parallelepiped. Proposition 7.8 shows that for $j_1(j_2-j_1)\le k<j_1j_2$,

$$\mathcal{E}(j_1\times j_2\times j_3,-k)\ =\ E(j_1\times j_2\times(j_3-1)\oplus_3 m_{j_1j_2-k}).$$

However, $E(j_1\times j_2\times(j_3-1)\oplus_3(l_c-1)\times l_c\oplus_1 1)\ >\ E(j_1\times j_2\times j_3\oplus(l_c-1)\times l_c\oplus 1)$ whence necessarily $j_1j_2-k>(l_c-1)l_c+1$ and $v(\eta_r)>j_1j_2j_3-j_1j_2+(l_c-1)l_c+1$. In addition, we have that $\mathcal{E}(j_1\times j_2\times j_3,k)>E(j_1\times j_2\times j_3)$ for all $k$ such that

$$-j_1j_2+(l_c-1)l_c+1<k\le(l_c-1)l_c,\ k\ne 0.$$

We have thus proved that $j_1\times j_2\times j_3$ and the cycle $\pi$ satisfy the hypothesis of theorem 5.5. Thus the bottom of the cycle is $\{j_1\times j_2\times j_3\}$. That $\mathcal{M}(j_1\times j_2\times j_3,k)$ is included in the cycle for $0\le k\le(l_c-1)l_c$ is obvious: each configuration of this set communicates with $j_1\times j_2\times j_3$ under the level $\mathrm{pred}\,E(j_1\times j_2\times j_3\oplus(l_c-1)\times l_c\oplus 1)$ (by proposition 7.9). Finally a configuration $\xi$ of the principal boundary of this cycle is of energy $E(\xi)=E(j_1\times j_2\times j_3\oplus(l_c-1)\times l_c\oplus 1)$ so that its volume is equal to $j_1j_2j_3+(l_c-1)l_c+1$, and its area is $2(j_1j_2+j_2j_3+j_1j_3)+2(2l_c-1)+2$. Let $\eta$ be a configuration of the cycle such that $q(\eta,\xi)>0$. Necessarily, the volume of $\eta$ is $j_1j_2j_3+(l_c-1)l_c$. Thus $\eta$ is a configuration of maximal volume of the cycle $\pi$ and as such it is a maximal configuration of the cycle for the inclusion relation. By theorem 5.3 the parallelepiped $j_1\times j_2\times j_3$ is included in $\eta$. In addition $E(\eta)<E(\xi)$ implies that $a(\eta)\le 2(j_1j_2+j_2j_3+j_1j_3)+2(2l_c-1)$ whence in fact $\eta$ belongs to $\mathcal{M}(j_1\times j_2\times j_3,(l_c-1)l_c)$. Thus $\eta$ is the sum of $j_1\times j_2\times j_3$ and a quasisquare $(l_c-1)\times l_c$ (proposition 7.9). Also $\xi$ belongs to $\mathcal{M}(j_1\times j_2\times j_3,(l_c-1)l_c+1)$. The only configurations of this set which communicate with $\mathcal{M}(j_1\times j_2\times j_3,(l_c-1)l_c)$ are the configurations of

$$\overline{j_1\times j_2\times j_3\oplus\overline{(l_c-1)\times l_c\oplus 1}}^{123}.$$

Conversely, it is clear that all these configurations belong to the principal boundary of the cycle. $\square$

**Theorem 7.19.** *Let $j_1, j_2, j_3, l_1, l_2$ be integers such that the parallelepiped $j_1 \times j_2 \times j_3$ contains a critical quasicube and the rectangle $l_1 \times l_2$ contains a critical two dimensional configuration. The cycle*

$$\pi = \pi(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2, pred\, E(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 \oplus 1))$$

*does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\underline{v}(\pi) \,=\, j_1 j_2 j_3 + l_1 l_2 - l_c + 2\,, \quad \overline{v}(\pi) \,=\, j_1 j_2 j_3 + l_1 l_2\,.$$

*The bottom of this cycle is $\{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2\}$ and its principal boundary is*

$$\widetilde{B}(\pi) \,=\, j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 \oplus 1.$$

*Proof.* This theorem might be proved in the same way as theorem 7.18, using the variational results of proposition 7.10. In fact it is the three dimensional counterpart of the two dimensional theorem 6.19. $\square$

*Remark.* Similar results hold for any parallelepiped plus rectangle $\overline{j_1 \times j_2 \times j_3 \oplus l_1 \times l_2}$ satisfying the requirements of the theorem.

**Theorem 7.20.** *Let $j_1 \times j_2 \times j_3$ be a parallelepiped containing a three dimensional critical configuration. The cycle*

$$\pi = \pi(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times (l_c + 1), pred\, E(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_2 1))$$

*is included in $\{\underline{-1}, \underline{+1}\}^c$. Moreover,*

$$\begin{array}{ll} \underline{v}(\pi) \,=\, j_1 j_2 j_3 + (l_c - 1) l_c + 2\,, & \overline{v}(\pi) \,=\, j_1 j_2 j_3 + (l_c - 1)(l_c + 1)\,. \\ \forall k \in \{1 \cdots l_c - 3\} & \mathcal{M}(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times (l_c + 1), -k) \,\subset\, \pi\,. \end{array}$$

*The bottom of this cycle is $\{j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times (l_c + 1)\}$; its principal boundary is*

$$\widetilde{B}(\pi) \,=\, \mathcal{M}(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times (l_c + 1), -(l_c - 2))$$

*and thus contains $j_1 \times j_2 \times j_3 +_1 \overline{(l_c - 1) \times l_c \oplus_2 1}^{12}$.*

*Proof.* These results are the three dimensional counterpart of the two dimensional theorem 6.24 and can be proved as usual with the help of the variational results of proposition 7.10. $\square$

*Remark.* Similar statements hold for any configuration in $\overline{j_1 \times j_2 \times j_3 \oplus (l_c + 1) \times (l_c - 1)}$.

**Lemma 7.21.** *Let $j_1, j_2, j_3, l_1, l_2$ be positive integers with $l_1 \times l_2 \subset j_2 \times j_3$ and let $k, i$ be such that $k > 1$, $i \geq 0$, $l_1 \geq k + i$. The cycle $\{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i k\}$ has one or two configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i (k+1)\}$,*
- *if $i = l_1 - k$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} (k+1)\}$,*
- *if $0 < i < l_1 - k$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} (k+1),\ l_1 \times l_2 +_2^i (k+1)\}$.*

*Proof.* The configuration $\sigma = j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i k$ communicates with at least one configuration of lower energy, so that the height of the cycle $\{\sigma\}$ is zero, and its principal boundary consists exactly of the configurations communicating with $\sigma$ and having and energy less or equal than $\sigma$. $\square$

**Lemma 7.22.** *Let $j_1, j_2, j_3, l_1, l_2$ be positive integers with $l_1 \times l_2 \subset j_2 \times j_3$ and let $i$ be such that $i \geq 0$, $l_1 \geq i + 1$. The cycle $\{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i 1\}$ has two or three configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2,\ j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i 2\}$,*
- *if $i = l_1 - 1$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2,\ j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} 2\}$,*
- *if $0 < i < l_1 - 1$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2,\ j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} 2,$ $j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i 2\}$.*

*Proof.* The proof is based on the same argument as the proof of lemma 7.21. $\square$

*Remark.* Lemmas 7.21, 7.22 are the three dimensional counterparts of lemmas 6.20, 6.21. Results similar to those stated in lemmas 7.21 and 7.22 hold for the configurations in $\overline{j_1 \times j_2 \times j_3 \oplus l_1 \times l_2 \oplus 1}$.

**Corollary 7.23.** *Suppose $j_1 \times j_2 \times j_3$ contains a configuration of $\mathcal{S}_{n_c}$ (i.e. a critical three dimensional configuration) and $l_1 \times l_2$ contains a critical two dimensional configuration. The following cycles are maximal cycles of $\{\underline{-1}, \underline{+1}\}^c$:*

1) $\pi(j_1 \times j_2 \times j_3, \mathrm{pred}\, E(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_2 1))$,
2) $\pi(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times (l_c + 1), \mathrm{pred}\, E(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 1))$,
3) $\pi(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2, \mathrm{pred}\, E(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 \oplus 1))$,
   *and this cycle is not included in a cycle of type 1),*
4) $\pi((j_c - 1) \times (j_c - \delta_c) \times j_c +_1 l_1 \times l_2, \mathrm{pred}\, E((j_c - 1) \times (j_c - \delta_c) \times j_c +_1 l_1 \times l_2 \oplus 1))$,
   *and this cycle is not included in a cycle of type 1),*
5) $\{\eta\}$, $\quad \eta \in \mathcal{M}(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times (l_c + 1), -(l_c - 2))$,
6) $\{\eta\}$, $\quad \eta \in j_1' \times j_2' \times j_3' \oplus \overline{l_1' \times l_2' \oplus k}$ *where $j_1' \times j_2' \times j_3'$ contains a quasicube $(j_c - 1) \times (j_c - \delta_c) \times j_c$, $l_1' \times l_2'$ contains a quasisquare $(l_c - 1) \times l_c$, $k$ is positive, and this cycle is not included in a cycle of type 1), 2), 3), 4), 5).*

*The list should be completed with all the isometric cycles (obtained by applying an isometry) as well as all the cycles whose bottom is obtained by translating the rectangle $l_1 \times l_2$ along the side of the parallelepiped (i.e. the configurations in $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2$).*

*Proof.* This corollary is a consequence of lemma 4.4 together with theorems 7.18, 7.19, 7.20 and lemmas 7.21, 7.22. Notice that we have to put together the descriptions of the cycles of theorems 7.18, 7.19, 7.20, lemmas 7.21, 7.22 in order to check that for each cycle $\pi$ in the above list, there is a sequence of cycles $\pi_0, \cdots, \pi_r$ such that $\pi_0 = \pi$, $\widetilde{B}(\pi_i) \cap \pi_{i+1} \neq \emptyset$, $0 \leq i < r$, and $\underline{+1} \in \widetilde{B}(\pi_r)$. Formally, this would require a tedious induction. For instance, the following arrows are in the graph $G$: $\pi(j_1 \times j_2 \times j_3, \operatorname{pred} E(j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 1)) \rightarrow \{\, j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 1 \,\} \rightarrow \{\, j_1 \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 2 \,\} \rightarrow \cdots \rightarrow \pi(j_1 \times j_2 \times j_3 +_1 l_c \times l_c, \operatorname{pred} E(j_1 \times j_2 \times j_3 +_1 l_c \times l_c +_1 1)) \rightarrow \cdots \rightarrow \pi(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2, \operatorname{pred} E(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_1 1)) \rightarrow \cdots \rightarrow \pi((j_1+1) \times j_2 \times j_3, \operatorname{pred} E((j_1 + 1) \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 1)) \rightarrow \cdots \rightarrow \{\underline{+1}\}$. $\square$

**Corollary 7.24.** *Let $\sigma$ belong to $\mathcal{S}_{n_c+1}$. The minimal stable subgraph $G^+(\sigma)$ of $G^+$ containing $\sigma$ is the restriction of $G$ to the vertices listed in corollary 7.23. The arrows of $G^+(\sigma)$ are (we denote by $\eta$ the parallelepiped $j_1 \times j_2 \times j_3$ and the cycles which are the starting point of the arrows belong to the list of cycles of corollary 7.23; the rectangles $l_1 \times l_2$ contain a critical two dimensional configuration; the rectangles $l_1' \times l_2'$ are very large):*

*1) around the parallelepipeds:*

$$\pi(\eta, pred\, E(\eta \oplus (l_c - 1) \times l_c \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \overline{\eta \oplus \overline{\overline{(l_c - 1) \times l_c \oplus 1}}}^{123}\ ;$$

*2) growing in the direction of $e_1$:*

$$\begin{aligned}
&\pi(\eta +_1 (l_c - 1) \times (l_c + 1), pred\, E(\eta +_1 (l_c - 1) \times l_c \oplus 1)) \leftrightarrow \{\xi\},\\
&\qquad \xi \in \mathcal{M}(\eta +_1 (l_c - 1) \times (l_c + 1), -(l_c - 2)),\\
&\pi(\eta +_1 l_1 \times l_2, pred\, E(\eta +_1 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_1 l_1 \times l_2 \oplus 1,\\
&\{\eta +_1 l_1 \times l_2 +_1 k\} \rightarrow \{\eta +_1 l_1 \times l_2 +_1 (k + 1)\},\ 0 < k \leq l_2 - l_c,\\
&\{\eta +_1 l_1 \times l_2 +_2 k\} \rightarrow \{\eta +_1 l_1 \times l_2 +_2 (k + 1)\},\ 0 < k \leq l_1 - l_c,\\
&\{\eta +_1 l_1 \times l_2 +_2 l_1 - l_c + 1\} \rightarrow \pi(\eta +_1 l_1 \times (l_2 + 1), pred\, E(\eta +_1 l_1 \times (l_2 + 1) + 1)),\\
&\{\eta +_1 l_1 \times l_2 +_1 l_2 - l_c + 1\} \rightarrow \pi(\eta +_1 (l_1 + 1) \times l_2, pred\, E(\eta +_1 (l_1 + 1) \times l_2 + 1)),\\
&\{\eta +_1 l_1' \times l_2' +_2 l_1' - l_c + 1\} \rightarrow \pi((j_1 + 1) \times j_2 \times j_3, pred\, E((j_1 + 1) \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 1)),\\
&\{\eta +_1 l_1' \times l_2' +_1 l_2' - l_c + 1\} \rightarrow \pi((j_1 + 1) \times j_2 \times j_3, pred\, E((j_1 + 1) \times j_2 \times j_3 +_1 (l_c - 1) \times l_c +_1 1));
\end{aligned}$$

*3) growing in the direction of $e_2$:*

$$\begin{aligned}
&\pi(\eta +_2 (l_c - 1) \times (l_c + 1), pred\, E(\eta +_2 (l_c - 1) \times l_c \oplus 1)) \leftrightarrow \{\xi\},\\
&\qquad \xi \in \mathcal{M}(\eta +_2 (l_c - 1) \times (l_c + 1), -(l_c - 2)),\\
&\pi(\eta +_2 l_1 \times l_2, pred\, E(\eta +_2 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_2 l_1 \times l_2 \oplus 1,\\
&\{\eta +_2 l_1 \times l_2 +_1 k\} \rightarrow \{\eta +_2 l_1 \times l_2 +_1 (k + 1)\},\ 0 < k \leq l_2 - l_c,\\
&\{\eta +_2 l_1 \times l_2 +_2 k\} \rightarrow \{\eta +_2 l_1 \times l_2 +_2 (k + 1)\},\ 0 < k \leq l_1 - l_c,\\
&\{\eta +_2 l_1 \times l_2 +_2 l_1 - l_c + 1\} \rightarrow \pi(\eta +_2 l_1 \times (l_2 + 1), pred\, E(\eta +_2 l_1 \times (l_2 + 1) + 1)),
\end{aligned}$$

$\{\eta +_2 l_1 \times l_2 +_1 l_2 - l_c + 1\} \to \pi(\eta +_2 (l_1+1) \times l_2, \mathit{pred}\, E(\eta +_2 (l_1+1) \times l_2 + 1))$,
$\{\eta +_2 l'_1 \times l'_2 +_2 l'_1 - l_c + 1\} \to \pi(j_1 \times (j_2+1) \times j_3, \mathit{pred}\, E(j_1 \times (j_2+1) \times j_3 +_2 (l_c-1) \times l_c +_1 1))$,
$\{\eta +_2 l'_1 \times l'_2 +_1 l'_2 - l_c + 1\} \to \pi(j_1 \times (j_2+1) \times j_3, \mathit{pred}\, E(j_1 \times (j_2+1) \times j_3 +_2 (l_c-1) \times l_c +_1 1))$;

*4) growing in the direction of $e_3$:*

$\pi(\eta +_3 (l_c - 1) \times (l_c + 1), \mathit{pred}\, E(\eta +_3 (l_c-1) \times l_c \oplus 1)) \leftrightarrow \{\xi\}$,
  $\xi \in \mathcal{M}(\eta +_3 (l_c-1) \times (l_c+1), -(l_c-2))$,
$\pi(\eta +_3 l_1 \times l_2, \mathit{pred}\, E(\eta +_3 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\}$, $\xi \in \eta +_3 l_1 \times l_2 \oplus 1$,
$\{\eta +_3 l_1 \times l_2 +_1 k\} \to \{\eta +_3 l_1 \times l_2 +_1 (k+1)\}$, $0 < k \le l_2 - l_c$,
$\{\eta +_3 l_1 \times l_2 +_2 k\} \to \{\eta +_3 l_1 \times l_2 +_2 (k+1)\}$, $0 < k \le l_1 - l_c$,
$\{\eta +_3 l_1 \times l_2 +_2 l_1 - l_c + 1\} \to \pi(\eta +_3 l_1 \times (l_2+1), \mathit{pred}\, E(\eta +_3 l_1 \times (l_2+1) + 1))$,
$\{\eta +_3 l_1 \times l_2 +_1 l_2 - l_c + 1\} \to \pi(\eta +_3 (l_1+1) \times l_2, \mathit{pred}\, E(\eta +_3 (l_1+1) \times l_2 + 1))$,
$\{\eta +_3 l'_1 \times l'_2 +_2 l'_1 - l_c + 1\} \to \pi(j_1 \times j_2 \times (j_3+1), \mathit{pred}\, E(j_1 \times j_2 \times (j_3+1) +_3 (l_c-1) \times l_c +_1 1))$,
$\{\eta +_3 l'_1 \times l'_2 +_1 l'_2 - l_c + 1\} \to \pi(j_1 \times j_2 \times (j_3+1), \mathit{pred}\, E(j_1 \times j_2 \times (j_3+1) +_3 (l_c-1) \times l_c +_1 1))$.

*The symbol $\leftrightarrow$ means that both arrows $\to$ and $\leftarrow$ are present. The above list should be completed with all the isometric arrows (obtained by applying the same isometry to both ends of an arrow), as well as all the arrows obtained by sliding either the bar along the rectangle (i.e. the configurations in $l_1 \times l_2 \oplus_1 k$) or by translating the rectangle $l_1 \times l_2$ on the side of the parallelepiped $\eta$ (i.e. the configurations in $\eta \oplus l_1 \times l_2$).*

The loops in the graph $G^+(\sigma)$ are

$$\pi(\eta, \mathrm{pred}\, E(\eta \oplus (l_c-1) \times l_c \oplus 1)) \leftrightarrow \{\xi\}, \qquad \xi \in \overline{\eta \oplus \overline{\overline{(l_c-1) \times l_c \oplus 1}}}^{123},$$
$$\pi(\eta +_1 (l_c-1) \times (l_c+1), \mathrm{pred}\, E(\eta +_1 (l_c-1) \times l_c \oplus 1)) \leftrightarrow \{\xi\},$$
$$\xi \in \mathcal{M}(\eta +_1 (l_c-1) \times (l_c+1), -(l_c-2)),$$
$$\pi(\eta +_1 l_1 \times l_2, \mathrm{pred}\, E(\eta +_1 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\}, \qquad \xi \in \eta +_1 l_1 \times l_2 \oplus 1,$$
$$\pi(\eta +_2 (l_c-1) \times (l_c+1), \mathrm{pred}\, E(\eta +_2 (l_c-1) \times l_c \oplus 1)) \leftrightarrow \{\xi\},$$
$$\xi \in \mathcal{M}(\eta +_2 (l_c-1) \times (l_c+1), -(l_c-2)),$$
$$\pi(\eta +_2 l_1 \times l_2, \mathrm{pred}\, E(\eta +_2 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\}, \qquad \xi \in \eta +_2 l_1 \times l_2 \oplus 1,$$
$$\pi(\eta +_3 (l_c-1) \times (l_c+1), \mathrm{pred}\, E(\eta +_3 (l_c-1) \times l_c \oplus 1)) \leftrightarrow \{\xi\},$$
$$\xi \in \mathcal{M}(\eta +_3 (l_c-1) \times (l_c+1), -(l_c-2)),$$
$$\pi(\eta +_3 l_1 \times l_2, \mathrm{pred}\, E(\eta +_3 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\}, \qquad \xi \in \eta +_3 l_1 \times l_2 \oplus 1.$$

All these loops are of length two. Any other arrow $\pi_1 \to \pi_2$ of $G^+(\sigma)$ satisfies $\overline{v}(\pi_1) < \underline{v}(\pi_2)$. As a consequence a path in $G^+(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{+1}\}$.

We are done with the standard configurations. We have to examine the remaining configurations of the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$ i.e. the principal non standard configurations.

**The principal non standard configurations.** We now do the same work for the configurations in $\widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}$. The relevant cycles are the cycles around the parallelepipeds $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)$, around the quasicubes plus rectangles $(j_c - 1) \times (j_c - \delta_c) \times j_c +_1 (l_c - 1) \times (l_c + 1)$, and around the configurations of $\widetilde{\mathcal{M}}_{n_c+1} \setminus \mathcal{S}_{n_c+1}$. We handle first the case of the configurations of $\widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}$ which lead to a parallelepiped $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)$ and then those which lead to a quasicube plus a rectangle $(j_c - 1) \times (j_c - \delta_c) \times j_c +_1 (l_c - 1) \times (l_c + 1)$.

**Theorem 7.25.** *The cycle $\pi = \pi((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), \mathrm{pred}\,\mathcal{E}(n_c))$ is the greatest cycle containing $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)$ included in $\{\underline{-1}, \underline{+1}\}^c$. Moreover,*

$$\underline{v}(\pi) \,=\, n_c + 1\,, \quad \overline{v}(\pi) \,<\, (j_c - 1)(j_c - \delta_c)(j_c + 1) + (l_c - 1)l_c + 1\,.$$

*The bottom of this cycle is $\{(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)\}$; its principal boundary contains*

$$\{\, \xi : \xi \subset (j_c - 1) \times (j_c - \delta_c) \times (j_c + 1),\, \xi \in \widetilde{\mathcal{M}}_{n_c} \,\}$$

*and is included in $\mathcal{M}((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), n_c - (j_c - 1)(j_c - \delta_c)(j_c + 1))$.*

*Remark.* Obviously, similar statements are true for the parallelepipeds isometric to $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)$.

*Proof.* We check that the parallelepiped $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)$ and the cycle $\pi$ satisfy the hypothesis of theorem 5.5. Let $x_1, \cdots, x_r$ be a sequence of sites such that $T(x_1, \cdots, x_s)((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1))$ is in $\pi$ for $s$ in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\mathrm{pred}\,\mathcal{E}(n_c)$). We put $\eta_s = T(x_1, \cdots, x_s)((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1))$ for $s$ in $\{0 \cdots r\}$.

- First case: all the sites are outside $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1)$. We have that $(j_c - 1) \times (j_c - \delta_c) \times (j_c + 1) \subset \eta_s$ whence $E(\eta_s) \geq \mathcal{E}((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), v(\eta_s) - (j_c - 1)(j_c - \delta_c)(j_c + 1))$ and

$$\mathrm{pred}\,\mathcal{E}(n_c) \,\geq\, \max_{0 \leq s \leq r} \mathcal{E}((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), v(\eta_s) - (j_c - 1)(j_c - \delta_c)(j_c + 1)).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|v(\eta_{s+1}) - v(\eta_s)| \leq 1$ and $(v(\eta_s),\, 0 \leq s \leq r)$ takes all the values between $(j_c - 1)(j_c - \delta_c)(j_c + 1)$ and $v(\eta_r)$. Henceforth

$$\max_{0 \leq s \leq r} \mathcal{E}((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), v(\eta_s) - (j_c - 1)(j_c - \delta_c)(j_c + 1)) \,\geq\, \max\{\, \mathcal{E}((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), k) : 0 \leq k \leq v(\eta_r) - (j_c - 1)(j_c - \delta_c)(j_c + 1) \,\}$$

and the volume of $\eta_r$ must satisfy

$$\max\{\, \mathcal{E}((j_c - 1) \times (j_c - \delta_c) \times (j_c + 1), k) : 0 \leq k \leq v(\eta_r) - (j_c - 1)(j_c - \delta_c)(j_c + 1) \,\} \,\leq\, \mathrm{pred}\,\mathcal{E}(n_c).$$

By proposition 7.9, we have $\mathcal{E}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),(l_c-1)l_c+1)=E((j_c-1)\times(j_c-\delta_c)\times(j_c+1)+_1(l_c-1)\times l_c+_1 1)$ which is strictly greater than $\operatorname{pred}\mathcal{E}(n_c)$. Thus $v(\eta_r)$ is strictly less than $(j_c-1)\times(j_c-\delta_c)\times(j_c+1)+(l_c-1)l_c+1$.

- Second case: all the sites are inside $(j_c-1)\times(j_c-\delta_c)\times(j_c+1)$. Now $\eta_s\subset(j_c-1)\times(j_c-\delta_c)\times(j_c+1)$ so that

$$\begin{aligned}\operatorname{pred}\mathcal{E}(n_c) &\geq \max_{0\leq s\leq r}\mathcal{E}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),v(\eta_s)-(j_c-1)(j_c-\delta_c)(j_c+1))\\ &\geq \max\{\,\mathcal{E}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),-k):0\leq k\leq(j_c-1)(j_c-\delta_c)(j_c+1)-v(\eta_r)\,\}.\end{aligned}$$

Proposition 7.8 shows that $\mathcal{E}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),-(j_c-1)(j_c-\delta_c)+(l_c-1)l_c+1)$ is equal to $\mathcal{E}(n_c)$. Necessarily, $v(\eta_r)$ is strictly greater than $n_c$.
In addition, we have that $\mathcal{E}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),k)>E((j_c-1)\times(j_c-\delta_c)\times(j_c+1))$ for all $k$ such that

$$\underline{v}(\pi)-(j_c-1)(j_c-\delta_c)(j_c+1)\leq k\leq\overline{v}(\pi)-(j_c-1)(j_c-\delta_c)(j_c+1),\ k\neq 0.$$

We have thus proved that $(j_c-1)\times(j_c-\delta_c)\times(j_c+1)$ and the cycle $\pi$ satisfy the hypothesis of theorem 5.5. It follows that the bottom of the cycle is $\{(j_c-1)\times(j_c-\delta_c)\times(j_c+1)\}$. Moreover the set $\{\,\xi:\xi\subset(j_c-1)\times(j_c-\delta_c)\times(j_c+1),\ \xi\in\widetilde{\mathcal{M}}_{n_c+1}\,\}$ is included in the cycle: each configuration of this set communicates with $(j_c-1)\times(j_c-\delta_c)\times(j_c+1)$ under the level $\operatorname{pred}\mathcal{E}(n_c)$. Thus $\{\,\xi:\xi\subset(j_c-1)\times(j_c-\delta_c)\times(j_c+1),\ \xi\in\widetilde{\mathcal{M}}_{n_c}\,\}$ is included in the principal boundary of $\pi$. Finally a configuration $\xi$ of the principal boundary of this cycle is of energy $E(\xi)=\mathcal{E}(n_c)$ so that its volume is equal to $n_c$, and its area to $\min\{\,a(\sigma):\sigma\in\mathcal{C}_{n_c}\,\}$. Let $\eta$ be a configuration of the cycle such that $q(\eta,\xi)>0$. Necessarily, the volume of $\eta$ is $n_c+1$. Thus $\eta$ is a configuration of minimal volume of the cycle and as such it is a minimal configuration of the cycle for the inclusion relation. By theorem 5.3 it is included in the parallelepiped $(j_c-1)\times(j_c-\delta_c)\times(j_c+1)$. Thus $\xi$ belongs to $\mathcal{M}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),n_c-(j_c-1)(j_c-\delta_c)(j_c+1))$. $\square$

**Corollary 7.26.** *Let $\sigma$ belong to $\widetilde{\mathcal{M}}_{n_c+1}\setminus\mathcal{S}_{n_c+1}$. Suppose $(j_c-1)\times(j_c-\delta_c)\times j_c+_3(l_c-1)\times l_c\subset\sigma$. Let $G^+(\sigma)$ be the minimal stable subgraph of $G^+$ containing $\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)$. The only arrows of $G^+(\sigma)$ entering $\pi(\underline{-1},\underline{+1}^c)$ are*

$$\begin{gathered}\{\xi\}\to\pi((j_c-1)\times(j_c-\delta_c)\times j_c+_3(l_c-1)\times l_c,\{\underline{-1},\underline{+1}\}^c),\\ \xi\in\widetilde{\mathcal{M}}_{n_c}\setminus\mathcal{S}_{n_c},\quad(j_c-1)\times(j_c-\delta_c)\times j_c+_3(l_c-1)\times l_c\subset\xi.\end{gathered}$$

*The remaining arrows of $G^+(\sigma)$ are*

$$\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)\leftrightarrow\{\eta\},\quad\eta\in\widetilde{B}(\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)).$$

*There is no arrow in $G^+(\sigma)$ ending at $\{\underline{+1}\}$.*

*Remark.* We have proved in theorem 7.25 that

$$\widetilde{B}(\pi(\sigma,\{\underline{-1},\underline{+1}\}^c))\subset\mathcal{M}((j_c-1)\times(j_c-\delta_c)\times(j_c+1),n_c-(j_c-1)(j_c-\delta_c)(j_c+1)).$$

*Proof.* That these arrows belong to $G^+(\sigma)$ is a straightforward consequence of theorem 7.25 which implies in particular that $\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)=\pi((j_c-1)\times(j_c-\delta_c)\times(j_c+1),\{\underline{-1},\underline{+1}\}^c)$. We have to check that there is no other arrow. Let $\eta$ belong to the principal boundary of $\pi((j_c-1)\times(j_c-\delta_c)\times(j_c+1),\operatorname{pred}\mathcal{E}(n_c))$. If $\eta$ is in $\widetilde{\mathcal{M}}_{n_c}$, then all arrows of $G$ starting at the cycle $\{\eta\}$ are present in the above list (lemma 7.22). If $\eta$ is not in $\widetilde{\mathcal{M}}_{n_c}$, we claim that

$$\{\eta\}\to\pi((j_c-1)\times(j_c-\delta_c)\times(j_c+1),\operatorname{pred}\mathcal{E}(n_c))$$

is the unique arrow of $G$ starting at $\{\eta\}$. Let $\xi$ be a point such that $q(\eta,\xi)>0$ and $E(\xi)\leq E(\eta)$. Necessarily $E(\xi)\leq\operatorname{pred}\mathcal{E}(n_c)$. Since $v(\eta)=n_c$, then $v(\xi)$ is equal to $n_c-1$ or $n_c+1$. Moreover $\eta$ is minimal, and the inequality $E(\xi)\leq E(\eta)$ implies in both cases that $\xi$ is also minimal. Thus $\xi$ cannot be of volume $n_c-1$ (by lemma 7.4, the only configurations of $\mathcal{M}_{n_c}$ communicating with $\mathcal{M}_{n_c-1}$ are the principal configurations $\widetilde{\mathcal{M}}_{n_c}$). Thus $v(\xi)=n_c+1$. We next show that $\xi$ belongs to $\pi((j_c-1)\times(j_c-\delta_c)\times(j_c+1),\operatorname{pred}\mathcal{E}(n_c))$. Let $x_1,\cdots,x_r$ be a sequence of sites inside $\sigma$ such that $T(x_1,\cdots,x_s)((j_c-1)\times(j_c-\delta_c)\times(j_c+1))$ is in $\pi((j_c-1)\times(j_c-\delta_c)\times(j_c+1),\operatorname{pred}\mathcal{E}(n_c))$ for $s$ in $\{1\cdots r-1\}$ and $\eta=T(x_1,\cdots,x_r)((j_c-1)\times(j_c-\delta_c)\times(j_c+1))$. We put $\eta_s=T(x_1,\cdots,x_j)((j_c-1)\times(j_c-\delta_c)\times(j_c+1))$, $0\leq s\leq r$. Since $\eta$ is in the boundary of the cycle, we have $E(\eta_{r-1})<E(\eta_r)$. Moreover $\eta_{r-1}$ is a minimal configuration of the cycle, of volume $n_c+1$, so that the last spin flip at site $x_r$ has decreased the volume. Let $x_{r+1}$ be the unique site such that $\xi=T(x_{r+1})(\eta)$. If $x_r=x_{r+1}$ then $\xi=\eta_{r-1}$. Suppose $x_r\neq x_{r+1}$ (so that $\eta_{r-1}(x_{r+1})=-1$). We have $\xi=T(x_{r+1})(\eta_r)=T(x_{r+1},x_r)(\eta_{r-1})=T(x_r,x_{r+1})(\eta_{r-1})=T(x_r)(\eta'_r)$ where $\eta'_r=T(x_{r+1})(\eta_{r-1})$. The energy inequality 5.1 yields $E(\eta'_r)-E(\eta_{r-1})\leq E(\xi)-E(\eta)$ whence $E(\eta'_r)\leq E(\eta_{r-1})$. It follows that $\eta'_r$ is in the cycle, as well as $\xi$ (their energies are less or equal than $\operatorname{pred}\mathcal{E}(n_c)$ and they both communicate with a configuration of the cycle). $\square$

We deal now with the principal non standard configurations which lead to a quasicube plus a rectangle $(j_c-1)\times(j_c-\delta_c)\times j_c+_1(l_c-1)\times(l_c+1)$. This situation corresponds to the presence of a two dimensional principal non standard configuration on the side of the critical quasicube.

**Theorem 7.27.** *The cycle $\pi((j_c-1)\times(j_c-\delta_c)\times j_c+_1(l_c-1)\times(l_c+1),\operatorname{pred}\mathcal{E}(n_c))$ is the greatest cycle containing $(j_c-1)\times(j_c-\delta_c)\times j_c+_1(l_c-1)\times(l_c+1)$ included in $\{\underline{-1},\underline{+1}\}^c$. Moreover,*

$$\begin{aligned}\underline{v}(\pi)&=(j_c-1)(j_c-\delta_c)j_c+(l_c-1)l_c+2\,,\\ \overline{v}(\pi)&=(j_c-1)(j_c-\delta_c)j_c+(l_c-1)(l_c+1)\,,\\ \forall k\in\{1\cdots l_c-3\}\qquad \mathcal{M}((j_c-1)&\times(j_c-\delta_c)\times j_c+_1(l_c-1)\times(l_c+1),-k)\subset\pi\,.\end{aligned}$$

*The bottom of this cycle is* $\{(j_c-1)\times(j_c-\delta_c)\times j_c +_1 (l_c-1)\times(l_c+1)\}$*; its principal boundary is*

$$\widetilde{B}(\pi) \;=\; \mathcal{M}((j_c-1)\times(j_c-\delta_c)\times j_c +_1 (l_c-1)\times(l_c+1), -(l_c-2))$$

*and thus contains* $(j_c-1)\times(j_c-\delta_c)\times j_c +_1 \overline{(l_c-1)\times l_c \oplus_2 1}^{12}$ *(which belong to* $\widetilde{\mathcal{M}}_{n_c}$*).*

*Proof.* The proof is similar as for theorem 6.24. $\square$

*Remark.* Obviously, similar statements are true for the configurations in the set

$$\overline{(j_c-1)\times(j_c-\delta_c)\times j_c \oplus_1 \overline{\overline{(l_c-1)\times(l_c+1)}}}.$$

**Corollary 7.28.** *Let* $\sigma$ *belong to* $\widetilde{\mathcal{M}}_{n_c+1}\setminus\mathcal{S}_{n_c+1}$*. Suppose* $(j_c-1)\times(j_c-\delta_c)\times j_c +_1 (l_c-1)\times l_c \subset \sigma \subset (j_c-1)\times(j_c-\delta_c)\times j_c +_1 (l_c-1)\times(l_c+1)$*. Let* $G^+(\sigma)$ *be the minimal stable subgraph of* $G^+$ *containing* $\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)$*. The only arrows of* $G^+(\sigma)$ *entering* $\pi(\underline{-1},\underline{+1}^c)$ *are*

$$\begin{gathered}\{\xi\}\to\pi((j_c-1)\times(j_c-\delta_c)\times j_c +_1 (l_c-1)\times l_c,\{\underline{-1},\underline{+1}\}^c),\\ \xi\in\widetilde{\mathcal{M}}_{n_c}\setminus\mathcal{S}_{n_c},\quad (j_c-1)\times(j_c-\delta_c)\times j_c +_1 (l_c-1)\times l_c\subset\xi.\end{gathered}$$

*The remaining arrows of* $G^+(\sigma)$ *are*

$$\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)\leftrightarrow\{\eta\},\quad \eta\in\widetilde{B}(\pi(\sigma,\{\underline{-1},\underline{+1}\}^c)).$$

*There is no arrow in* $G^+(\sigma)$ *ending at* $\{\underline{+1}\}$*.*

*Proof.* The proof is similar as for corollary 6.25. $\square$

We sum up the consequences of the previous results in the next corollaries.

**Corollary 7.29.** *The principal non standard configurations of volume* $n_c$ *(i.e. the set* $\widetilde{\mathcal{M}}_{n_c}\setminus\mathcal{S}_{n_c}$*) are dead–ends: there is no saddle path of null cost between* $\underline{-1}$ *and* $\underline{+1}$ *passing through them.*

**Corollary 7.30.** *The set of the global saddle points between* $\underline{-1}$ *and* $\underline{+1}$ *is exactly* $\mathcal{S}_{n_c}$*. These configurations are the critical three dimensional configurations.*

Steps $ii)$ and $iii)$ are now completed and we proceed to steps $iv)$ and $v)$.

**The ascending part.** We now proceed to the last part of the program: for each configuration $\sigma'$ in $\mathcal{S}_{n_c}$, we must determine the minimal stable subgraph of $G^-$ containing $\sigma'$ and all the paths in this graph starting at $\{\sigma'\}$ and ending at $\underline{-1}$. Our exposition is similar as before: we first list the set of the relevant cycles and we use lemma 4.4 to find those belonging to $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$. We finally check that we have in hand all the vertices of $G^-$.

**Theorem 7.31.** *Let $j$ be an integer strictly less than $j_c$ and greater or equal than $l_c$. Let $\delta \leq \gamma$ belong to $\{0,1\}$. The cycle $\pi = \pi(j \times (j+\delta) \times (j+\gamma), pred\, \mathcal{E}(j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\underline{v}(\pi) \,=\, j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+2\,, \quad \overline{v}(\pi) \,<\, j(j+\delta)(j+\gamma)+(l_c-1)l_c+1\,.$$

*The bottom of this cycle is $\{j \times (j+\delta) \times (j+\gamma)\}$; its principal boundary contains the set*

$$\{\,\xi : \xi \subset j \times (j+\delta) \times (j+\gamma),\, \xi \in \mathcal{S}_{j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1}\,\}$$

*and is included in $\mathcal{M}(j \times (j+\delta) \times (j+\gamma), (l_c-1)l_c+1-j(j+\delta))$.*

*Proof.* We apply proposition 7.12. For any $n$ greater than $j(j+\delta)(j+\gamma)+(l_c-1)l_c+1$,

$$E(j \times (j+\delta) \times (j+\gamma), \mathcal{C}_n) \,\geq\, \mathcal{E}(j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1)$$

whence $\overline{v}(\pi) < j(j+\delta)(j+\gamma)+(l_c-1)l_c+1$. Analogously, for any $n$ strictly smaller than $j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+2$ we have $E(j(j+\delta)(j+\gamma), \mathcal{C}_n) \geq \mathcal{E}(j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1)$; moreover,

$$E(j(j+\delta)(j+\gamma), \mathcal{C}_{j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+2}) \,<\, \mathcal{E}(j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1)$$

so that $\underline{v}(\pi) = j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+2$. To prove that the bottom of the cycle is $\{j \times (j+\delta) \times (j+\gamma)\}$, we could proceed as before and use theorem 5.5. However, a direct application of our geometrical results (theorem 7.2) yields that the minimum $\min\{\,\mathcal{E}(n) : j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+2 \leq n < j(j+\delta)(j+\gamma)+(l_c-1)l_c+1\,\}$ is equal to $\mathcal{E}(j(j+\delta)(j+\gamma))$; the unique configurations of energy $\mathcal{E}(j(j+\delta)(j+\gamma))$ are the quasicubes isometric to $j \times (j+\delta) \times (j+\gamma)$. Theorem 5.3 implies also that the altitude of communication between two different quasicubes of this set is greater than $\mathcal{E}(j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1)$ (one has to fill a face of the initial quasicube to create another quasicube). The statement concerning the principal boundary is a consequence of proposition 7.8. $\square$

**Theorem 7.32.** *Let $j$ be an integer strictly less than $l_c$. Let $\delta \leq \gamma$ belong to $\{0,1\}$. The cycle*

$$\pi \,=\, \pi(j \times (j+\delta) \times (j+\gamma), pred\, \mathcal{E}(j(j+\delta)(j+\gamma)-j+1))$$

*does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\underline{v}(\pi) \,=\, j(j+\delta)(j+\gamma)-j+2\,, \quad \overline{v}(\pi) \,=\, j(j+\delta)(j+\gamma)\,,$$
$$\forall k \in \{1 \cdots j-2\} \qquad \mathcal{M}(j \times (j+\delta) \times (j+\gamma), -k) \,\subset\, \pi\,.$$

*The bottom of this cycle is $\{j \times (j+\delta) \times (j+\gamma)\}$; its principal boundary is*

$$\widetilde{B}(\pi) \,=\, \mathcal{M}(j \times (j+\delta) \times (j+\gamma), -j+1)$$

*and thus contains $\{\,\xi : \xi \subset j \times (j+\delta) \times (j+\gamma),\, \xi \in \mathcal{S}_{j(j+\delta)(j+\gamma)-j+1}\,\}$.*

*Proof.* The proof is similar as for theorem 7.31. $\square$

**Theorem 7.33.** *Let $j$ be an integer strictly less than $j_c$. Let $\delta \leq \gamma$ belong to $\{0,1\}$. Let $l \times (l+\epsilon)$ be a quasisquare strictly included in $j \times (j+\delta)$, where $l < l_c$. The cycle*

$$\pi \;=\; \pi(j \times (j+\delta) \times (j+\gamma) +_3 l \times (l+\epsilon), pred\,\mathcal{E}(j(j+\delta)(j+\gamma) + l(l+\epsilon) - l + 1))$$

*does not contain $\underline{-1}$ and $\underline{+1}$. Moreover*

$$\underline{v}(\pi) \;=\; j(j+\delta)(j+\gamma) + l(l+\epsilon) - l + 2\,, \quad \overline{v}(\pi) \;=\; j(j+\delta)(j+\gamma) + l(l+\epsilon)\,,$$
$$\forall k \in \{1 \cdots l-2\} \qquad \mathcal{M}(j \times (j+\delta) \times (j+\gamma) +_3 l \times (l+\epsilon), -k) \;\subset\; \pi\,.$$

*The bottom of this cycle is $\{j \times (j+\delta) \times (j+\gamma) +_3 l \times (l+\epsilon)\}$; its principal boundary is*

$$\widetilde{B}(\pi) \;=\; \mathcal{M}(j \times (j+\delta) \times (j+\gamma) +_3 l \times (l+\epsilon), -l+1)$$

*and thus contains $\{\, \xi : \xi \subset j \times (j+\delta) \times (j+\gamma),\, \xi \in \mathcal{S}_{j(j+\delta)(j+\gamma)+l(l+\epsilon)-l+1} \,\}$.*

*Remark.* This theorem cover the cases $j \leq l_c$ and $j > l_c$, which are of slightly different natures. When $j \leq l_c$, the system does not have to create a two dimensional critical droplet in order to fill a face of the quasicube.

**Corollary 7.34.** *Suppose $j < j_c$. The following cycles are maximal cycles of $\{\underline{-1}, \underline{+1}\}^c$:*

$\pi(j \times (j+\delta) \times (j+\gamma), pred\,\mathcal{E}(j(j+\delta)(j+\gamma-1) + (l_c-1)l_c + 1)),\ l_c \leq j\,,$

$\{\eta\}, \quad \eta \in \widetilde{B}(\pi(j \times (j+\delta) \times (j+\gamma), pred\,\mathcal{E}(j(j+\delta)(j+\gamma-1) + (l_c-1)l_c + 1))),\ l_c \leq j,$

$\pi(j \times (j+\delta) \times (j+\gamma), pred\,\mathcal{E}(j(j+\delta)(j+\gamma) - j + 1)),\ j < l_c\,,$

$\{\eta\}, \quad \eta \in \mathcal{M}(j \times (j+\delta) \times (j+\gamma), -j+1),\ j < l_c\,,$

$\pi(j \times (j+\delta) \times (j+\gamma) +_3 l \times (l+\epsilon), pred\,\mathcal{E}(j(j+\delta)(j+\gamma) + l(l+\epsilon) - l + 1)),\ l < l_c,$

$\{\eta\}, \quad \eta \in \mathcal{M}(j \times (j+\delta) \times (j+\gamma) +_3 l \times (l+\epsilon), -l+1),\ l < l_c\,.$

*Proof.* This corollary is a consequence of lemma 4.4 together with theorems 7.31, 7.32, 7.33. Notice that we have to put together the descriptions of the cycles of these theorems in order to check that for each cycle $\pi$ in the above list, there is a sequence of cycles $\pi_0, \cdots, \pi_r$ such that $\pi_0 = \pi$, $\widetilde{B}(\pi_i) \cap \pi_{i+1} \neq \emptyset$, $0 \leq i < r$, and $\underline{-1} \in \widetilde{B}(\pi_r)$. $\square$

**Corollary 7.35.** *Suppose $\sigma$ is a configuration of $\mathcal{S}_{n_c-1}$. The minimal stable subgraph $G^-(\sigma)$ of $G^-$ containing $\sigma$ is the restriction of $G$ to the vertices listed in corollary 7.34. The arrows of $G^-(\sigma)$ are (in the following list, the starting cycles belong to the list given in corollary 7.34):*

$$
\begin{aligned}
&\pi(j\times(j+\delta)\times(j+\gamma), pred\,\mathcal{E}(j(j+\delta)(j+\gamma-1)+(l_c-1)l_c+1)) \leftrightarrow \{\eta\}\,,\\
&\hfill \eta\in\widetilde{B}(\pi)\,,\quad l_c\le j<j_c\,,\\
&\{j\times(j+\delta)\times(j+\gamma-1)+_3(l_c-1)\times l_c+_1 1\}\to\\
&\hfill \pi(j\times(j+\delta)\times(j+\gamma-1)+_3(l_c-1)\times l_c,\{\underline{-1},\underline{+1}\}^c)\,,l_c\le j<j_c\,,\\
&\pi(j\times(j+\delta)\times(j+\gamma), pred\,\mathcal{E}(j(j+\delta)(j+\gamma)-j+1)) \leftrightarrow \{\eta\}\,,\\
&\hfill \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\gamma),-j+1)\,,\quad j<l_c\,,\\
&\{j\times(j+\delta)\times(j+\gamma-1)+_3 j\times(j+\delta-1)+_1 1\}\to\\
&\hfill \pi(j\times(j+\delta)\times(j+\gamma-1)+_3 j\times(j+\delta-1),\{\underline{-1},\underline{+1}\}^c)\,,j<l_c\,,\\
&\pi(j\times(j+\delta)\times(j+\gamma)+_3 l\times(l+\epsilon), pred\,\mathcal{E}(j(j+\delta)(j+\gamma)+l(l+\epsilon)-l+1)) \leftrightarrow \{\eta\}\,,\\
&\hfill \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\gamma)+_3 l\times(l+\epsilon),-l+1)\,,\quad l<l_c,\\
&\{j\times(j+\delta)\times(j+\gamma)+_3 l\times(l+\epsilon-1)+_1 1\}\to\\
&\hfill \pi(j\times(j+\delta)\times(j+\gamma)+_3 l\times(l+\epsilon-1),\{\underline{-1},\underline{+1}\}^c)\,,l<l_c\,.
\end{aligned}
$$

*As usual, this list should be completed with all the isometric arrows as well as with the arrows between configurations where the quasisquares $l\times(l+\epsilon)$ are slid against the side of the quasicubes, and the unit cube 1 is slid against the side of the quasisquare (i.e. the configurations in $j\times(j+\delta)\times(j+\gamma)\oplus_3 l\times(l+\epsilon)\oplus 1$).*

The only loops in the graph $G^-(\sigma)$ are loops around two cycles. Any other arrow $\pi_1\to\pi_2$ of $G^-(\sigma)$ satisfies $\underline{v}(\pi_1)>\overline{v}(\pi_2)$. As a consequence a path in $G^-(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{-1}\}$.

**The exit path.** We have finally reached the last step $vi)$. We notice at this point that there exists only one optimal saddle between two cycles associated to each arrow of the graph $\mathcal{G}$. Thus the graph $\mathcal{G}$ contains all the information necessary to obtain the set of the saddle paths of null cost between $\underline{-1}$ and $\underline{+1}$. We describe for instance the canonical saddle path, which follows the sequence of the canonical configurations:

$$
\begin{gathered}
\underline{-1}\to\mathfrak{m}_1,\,\mathfrak{m}_1\to\mathfrak{m}_2,\,\mathfrak{m}_2\to\mathfrak{m}_3,\,\mathfrak{m}_3\to\mathfrak{m}_4,\,\mathfrak{m}_4\to\mathfrak{m}_5,\,\mathfrak{m}_5\to\mathfrak{m}_6,\,\mathfrak{m}_6\to\mathfrak{m}_7,\,\mathfrak{m}_7\to\mathfrak{m}_8,\\
\mathfrak{m}_8\to\mathfrak{m}_9,\,\mathfrak{m}_9\to\mathfrak{m}_{10},\,\mathfrak{m}_{10}\to\mathfrak{m}_{11},\,\mathfrak{m}_{11}\to\mathfrak{m}_{12},\,\mathfrak{m}_{12}\to\mathfrak{m}_{13},\,\mathfrak{m}_{13}\to\mathfrak{m}_{14},\,\mathfrak{m}_{14}\to\mathfrak{m}_{15},\\
\mathfrak{m}_{15}\to\mathfrak{m}_{16},\,\mathfrak{m}_{16}\to\mathfrak{m}_{17},\,\mathfrak{m}_{18}\to\mathfrak{m}_{19},\,\mathfrak{m}_{19}\to\mathfrak{m}_{20},\,\mathfrak{m}_{20}\to\mathfrak{m}_{21},\,\mathfrak{m}_{22}\to\mathfrak{m}_{23},\,\mathfrak{m}_{24}\to\mathfrak{m}_{25},\\
\mathfrak{m}_{25}\to\mathfrak{m}_{26},\,\mathfrak{m}_{27}\to\mathfrak{m}_{28},\,\cdots,\,\mathfrak{m}_{j^3+(l_c-1)l_c}\to\mathfrak{m}_{j^3+(l_c-1)l_c+1},\,\mathfrak{m}_{j^3+(l_c-1)l_c+1}\to\mathfrak{m}_{j^3+(l_c-1)l_c+2},\\
\mathfrak{m}_{j^2(j+1)+(l_c-1)l_c}\to\mathfrak{m}_{j^2(j+1)+(l_c-1)l_c+1},\,\mathfrak{m}_{j^2(j+1)+(l_c-1)l_c+1}\to\mathfrak{m}_{j^2(j+1)+(l_c-1)l_c+2},\\
\mathfrak{m}_{j(j+1)^2+(l_c-1)l_c}\to\mathfrak{m}_{j(j+1)^2+(l_c-1)l_c+1},\,\mathfrak{m}_{j(j+1)^2+(l_c-1)l_c+1}\to\mathfrak{m}_{j(j+1)^2+(l_c-1)l_c+2},\\
\mathfrak{m}_{(j+1)^3+(l_c-1)l_c}\to\mathfrak{m}_{(j+1)^3+(l_c-1)l_c+1},\,\cdots,\,\mathfrak{m}_{(j_c-1)(j_c-\delta_c)j_c+(l_c-1)l_c}\to\mathfrak{m}_{(j_c-1)(j_c-\delta_c)j_c+(l_c-1)l_c+1},\\
\mathfrak{m}_{(j_c-1)(j_c-\delta_c)j_c+(l_c-1)l_c+1}\to\mathfrak{m}_{(j_c-1)(j_c-\delta_c)j_c+(l_c-1)l_c+2},\,\cdots,\,\mathfrak{m}_{N^3-1}\to\mathfrak{m}_{N^3}.
\end{gathered}
$$

We state here some consequences of the information provided by the graph $\mathcal{G}$ which describes the set of all the saddle paths of null cost. We stress once more that the graph $\mathcal{G}$ provides the most complete information available on the limiting dynamics. Once we know this graph, the results obtained in a general framework [6] may be applied in a systematic fashion to obtain various estimates. We let the process $(\sigma_n)_{n\in\mathbb{N}}$ start from $\underline{-1}$. We recall that $\tau(\underline{+1}^c)$ is the hitting time of the ground state $\underline{+1}$ and $\theta(\underline{-1},\tau(\underline{+1}^c))$ is the last visit to the metastable state $\underline{-1}$ before reaching $\underline{+1}$.

**Theorem 7.36.** *(the exit path)*
*For any positive $\epsilon$, the following events take place with probability converging to one exponentially fast as $\beta$ goes to infinity:*

- $\exp\beta(\mathcal{E}(n_c)-\epsilon) \leq \tau(\underline{+1}^c) \leq \exp\beta(\mathcal{E}(n_c)+\epsilon)$;
- $\exp\beta(E_{2c}-\epsilon) \leq \tau(\underline{+1}^c)-\theta(\underline{-1},\tau(\underline{+1}^c)) \leq \exp\beta(E_{2c}+\epsilon)$, *where $E_{2c}$ is the two dimensional global energy barrier given in proposition* 6.15*;*
- *during the exit path $(\sigma_n,\theta\leq n\leq\tau)$, the process crosses the set $\mathcal{S}_{n_c}$ of the critical configurations at exactly one point $\sigma_c$; it does not cross $\mathcal{C}_{n_c}\setminus\mathcal{S}_{n_c}$;*
- *if we let $n_*=\min\{n\geq\theta : v(\sigma_n)=n_c\}$, $n^*=\max\{n\leq\tau : v(\sigma_n)=n_c\}$, we have that $\sigma_n=\sigma_{n_*}=\sigma_{n^*}$ for all $n$ in $\{n_*\cdots n^*\}$ and $n^*-n_*\leq\exp(\beta\epsilon)$;*
- *all the configurations of the exit path before time $n_*$ are of volume less than $n_c$, all the configurations after $n^*$ are of volume greater than $n_c$;*
- *during the ascending part, the process goes through an increasing sequence of quasicubes $(j\times(j+\delta)\times(j+\gamma), j<j_c)$. It visits a quasicube $j\times(j+\delta)\times(j+\gamma)$ approximately $\exp\beta(j-1)h$ times if $j<l_c$ and $\exp\beta(\mathcal{E}_2((l_c-1)l_c+1)-\mathcal{E}_2(j(j+\delta)))$ times if $j\geq l_c$. When it definitely leaves a quasicube, it fills the right side of the quasicube in order to obtain the next larger quasicube, this filling occurs according to the two dimensional nucleation mechanism; in particular, it does not make a parallelepiped which is not a quasicube;*
- *during the descending part, the process goes through an increasing sequence of parallelepipeds $(j_1\times j_2\times j_3,\ \min(j_1,j_2,j_3)\geq j_c)$. It visits a parallelepiped $j_1\times j_2\times j_3$ approximately $\exp\beta E_{2c}$ times. When it definitely leaves a parallelepiped it fills a side of the parallelepiped chosen at random in order to obtain a larger parallelepiped, this filling occurs by following the two dimensional nucleation mechanism;*
- *the process reaches the thermal equilibrium within each cycle of $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$ it crosses: the distribution of the process before the exit of a cycle is very close to the Gibbs distribution at inverse temperature $\beta$ restricted to this cycle.*

Let us notice a new phenomenon occurring in dimension three. Before growing a quasicube or a parallelepiped whose sides are larger than $l_c$, the system will build every configuration reachable below the energy barrier $E_{2c}$. This energy barrier goes to infinity like $4/h$ and can thus be terribly high. The system will form small clusters very far from the main cluster. There will be small sparks appearing everywhere on the torus. It will try to build very strange shapes around the main cluster, like small antennae and wires.

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