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The Exit Place of Brownian Motion in an Unbounded Domain

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Abstract

For Brownian motion in an unbounded domain we study the influence of the "far away" behavior of the domain on the probability that the modulus of the Brownian motion is large when it exits the domain. Roughly speaking, if the domain expands at a sublinear rate, then the chance of a large exit place decays in a subexponential fashion. The decay rate can be explicitly given in terms of the sublinear expansion rate. Our results encompass and extend some known special cases.

Key words: Exit place of Brownian motion, parabolic-type domain, horn-shaped domain, *h*-transform, Green function, harmonic measure.

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1 Introduction

In this article we use a new approach to study the effect of the "far away" behavior of an unbounded domain on the probability that the modulus of the Brownian motion is large when it exits the domain. We study domains of the form

$$D=\left\{(x_1,\tilde{x})\in\mathbb{R}\times\mathbb{R}^{d-1}:\ x_1>\frac{1}{2},|\tilde{x}|<a(x_1)\right\},\quad d\geq 3 \tag{1}$$

and

$$\Omega=\left\{(\rho,z)\in\mathbb{R}^2:\ \rho>\frac{1}{2},|z|<a(\rho)\right\}\times S^{n-1},\quad n\geq 2 \tag{2}$$

where $a\colon\ \left[\frac{1}{2},\infty\right)\to[0,\infty)$ is continuous and positive on $\left(\frac{1}{2},\infty\right)$. In the case of Ω, we understand it to be a subset of $\mathbb{R}^{n+1}$ where the triple (ρ,z,θ) denotes the cylindrical coordinates of a point $(\tilde{x},x_{n+1})\in\mathbb{R}^n\times\mathbb{R}$ with $\tilde{x}\neq 0$:

$$\rho=|\tilde{x}|,\quad z=x_{n+1},\quad \theta=\frac{\tilde{x}}{|\tilde{x}|}.$$

One way to understand the difference between the domains D and Ω is to look at them in three dimensions. There D is obtained by revolving the set $\{(x,y):\ x>\frac{1}{2},|y|<a(x)\}$ about the x-axis, while Ω is obtained by revolving about the y-axis.

Roughly speaking, we will show that if the growth of a is sublinear and the oscillations far away are not too severe, then the probability the modulus of Brownian motion is large when it exits the domain is subexponentially small and we can explicitly identify the decay rate.

An equivalent analytic formulation of this problem is to ask how the harmonic measure behaves outside a compact set. A related problem is to determine the sharp order of integrability of the exit position.

Several authors have studied these questions in various domains, using different tools. For a cone of angle θ (which corresponds to D above with $a(x)=cx$), Burkholder (1977) used his L^p-inequalities for Brownian motion to explicitly find $p(\theta)>0$ for which the $p>0$ moment of the exit place is finite iff $p<p(\theta)$. By classical estimates for harmonic measure (Haliste (1984) and Essén and Haliste (1984)), there are positive C_1 and C_2 such that

$$C_1r^{-p(\theta)}\leq P_x(|B(\tau)|>r)\leq C_2r^{-p(\theta)},$$

where B is Brownian motion and τ is the exit time from the cone. The analogue is also true for more general cones. Using the explicit form of the heat kernel for a cone (due to Bañuelos and Smits (1997)), Bañuelos and DeBlassie (2006) obtained a series expansion for $\frac{d}{dr}P_x(|B(\tau)|>r)$ which implies the behavior

$$P_x(|B(\tau)|>r)\sim Cr^{-p(\theta)}\quad\text{as}\quad r\to\infty,$$

and C was explicitly identified. Again, there is an analogous result for more general cones.

Bañuelos and Carroll (2005) studied the domain D above with $a(x) = Ax^\alpha$, where $A > 0$ and $0 < \alpha < 1$. Denoting the exit time of Brownian motion from D by τ_D, those authors showed that

$$\lim_{t\to\infty} t^{\alpha-1} \log P_z(|B(\tau_D)| > t) = -\frac{\sqrt{\lambda_1}}{A(1-\alpha)}, \tag{3}$$

where $\lambda_1 > 0$ is the smallest eigenvalue for the Dirichlet Laplacian in the unit ball of $\mathbb{R}^{d-1}$ (note when $d = 2$, $\sqrt{\lambda_1} = \frac{\pi}{2}$). They also showed that for $d = 2$,

$$E_z[\exp(b|B(\tau_D)|^{1-\alpha})] < \infty \tag{4}$$

iff $b < \frac{\pi}{2A(1-\alpha)}$. In dimension $d \geq 3$, they proved the expectation is finite for $b < \frac{\sqrt{\lambda_1}}{A(1-\alpha)}$ and infinite for $b > \frac{\sqrt{\lambda_1}}{A(1-\alpha)}$. The critical case was left open. Their method was to use a conformal mapping and a technique of Carleman to estimate harmonic measure.

For the domain Ω above with $a(x) = Ax^\alpha$, $0 < \alpha < 1$, DeBlassie (2008b) reduced the computation of the probability the modulus of Brownian motion is large upon exiting the domain to the two-dimensional case studied by Bañuelos and Carroll (2005). The method used conformal maps coupled with the Feynman–Kac formula and the Comparison Theorem for stochastic differential equations. The main result obtained was that

$$\lim_{N\to\infty} N^{\alpha-1} \log P_x(|B(\tau_\Omega)| > N) = -\frac{\pi}{2A(1-\alpha)}. \tag{5}$$

It is interesting to note that in contrast with the domains considered by Bañuelos and Carroll, the limit is independent of the dimension. This is counter-intuitive; see DeBlassie (2008a) for more explanation.

It does not seem possible to use the conformal method mentioned above for boundary functions other than $a(x) = Ax^\alpha$ for $\alpha \in (0, 1)$. This is because the method relies very much on delicate estimates, due to Carroll and Hayman (2004), of the derivative of a certain conformal map. The specific power law growth x^α is crucial to their argument and it is not at all clear how to extend their estimates to more general functions.

Instead, we take a new approach that will permit extension of (3) and (5) to much more general functions $a(x)$, and it will also resolve the critical case in dimension $d \geq 3$ for finiteness of $E_x[\exp(b|B(\tau_D)|^{1-\alpha}]$ left open by Bañuelos and Carroll. The basic idea is to represent the density (with respect to surface measure on the boundary) of harmonic measure as the normal derivative of the Green function. Then we can use estimates of harmonic functions due to Cranston and Li (1997) to estimate the normal derivative. The representation of the density of harmonic measure as the normal derivative of the Green function is a classical fact for bounded smooth domains (Miranda (1970), Garabedian (1986), Gilbarg and Trudinger (1983)). But here our domains are unbounded and the classical proof must be modified. Basically, trouble arises because the proof requires the Divergence Theorem to hold, and since the domain is unbounded, integrability issues become significant.

Before stating our results, we present the basic assumptions on D from (1).

Blanket Assumptions

- D is C^3
- $\lim_{t\to\infty} [|a'(t)| + a(t)|a''(t)|] = 0$
- $\limsup_{M\to\infty} \sup\{|a''(M + ta(M))|a(M): \ -1 \le t \le 1\} < \infty$
- $\limsup_{M\to\infty} \sup\{|a^{(3)}(M + ta(M))|a^2(M): \ -1 \le t \le 1\} < \infty$

Remark 1.1. i) The lim sup conditions as well as the condition $a(t)a''(t) \to 0$ as $t \to \infty$ quantify the statement that the oscillations of $a(\cdot)$ far away are not too severe.
ii) By the Mean Value Theorem, the requirement that $a'(t) \to 0$ as $t \to \infty$ implies $a(t)/t \to 0$ as $t \to \infty$, which in turn implies $a(t)$ is sublinear and

$$\lim_{x\to\infty} \int_1^x \frac{dt}{a(t)} = \infty. \tag{6}$$

iii) If $a(\cdot)$ is C^3, then the domain D satisfies the blanket assumptions in the following cases:

- $a(t) = At^\alpha$ for large t, where $A > 0$ and $\alpha < 1$ with $\alpha \neq 0$; note if $\alpha = 1$, then D is a cone.
- $a(t) = Ae^{-\gamma t^p}$ for large t, where A, γ and p are positive;
- $a(t) = At(\log t)^{-p}$ for large t, where A and p are positive. □

For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, write

$$B_\varepsilon(x) = \{y \in \mathbb{R}^d : \ |x - y| < \varepsilon\}.$$

For any process Z_t in $\mathbb{R}^d$, we will write

$$\tau_D(Z) = \inf\{t > 0 : \ Z_t \notin D\}$$

for the exit time of Z from D.
Letting

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{z}{2})^{2k}}{k!\Gamma(\nu + k + 1)}$$

be the Bessel function of the first kind, we denote its first positive zero by j_ν.

Theorem 1.2. *Let D from* (1) *satisfy the Blanket Assumptions. For dimension $d \ge 3$, if B_t is d-dimensional Brownian motion and if $\tau_D = \tau_D(B)$, then for $\nu = \frac{d-3}{2}$,*

$$\lim_{N\to\infty} \left[\int_1^N \frac{dt}{a(t)}\right]^{-1} \log P_x(|B(\tau_D)| > N) = -j_\nu.$$ □

Remark 1.3. i) When $d = 2$, the problem is easily handled with conformal methods and the limiting behavior given holds upon replacing ν by $|\nu|$. Note too that $j_\nu = \pi/2$ in this case.
ii) Note that the first Dirichlet eigenvalue of $\Delta_{\mathbb{R}^{d-1}}$ on the unit ball in $\mathbb{R}^{d-1}$ is in fact j_ν^2.
iii) If $a(t) = At^\alpha$, where $A > 0$ and $0 \neq \alpha < 1$, then $\int_1^N \frac{dt}{a(t)} \sim \frac{1}{A(1-\alpha)} N^{1-\alpha}$ as $N \to \infty$. Thus we recover the result (3) of Bañuelos and Carroll for $0 < \alpha < 1$ and extend it to $\alpha < 0$. □

With some additional conditions on $a(\cdot)$, Theorem 1.2 can be extended with no further effort:

Theorem 1.4. *Let $d \geq 3$. In addition to the Blanket Assumptions on D from* (1)*, suppose*

$$(a')^2/a, a'', aa^{(3)}, a^2a^{(4)} \in L_1\left(\left[\tfrac{1}{2}, \infty\right)\right)$$

and

$$\lim_{t\to\infty} [a^2(t)|a^{(3)}(t)| + a^3(t)|a^{(4)}(t)|] = 0.$$

Then for some positive C_1 and C_2, for all large N,

$$C_1 a(N)^{d-1} \leq P_x(|B(\tau_D)| > N) \exp\left(j_\nu \int_1^N \frac{dt}{a(t)}\right) \leq C_2 a(N)^{d-1}.$$ □

Remark 1.5. If $A > 0$ and $0 \neq \alpha < 1$, then $a(t) = At^\alpha$ satisfies the hypotheses of Theorem 1.4. By Remark 1.3 iii), we can mimic the argument of Bañuelos and Carroll giving (4) for $d = 2$ and resolve the critical case for $d \geq 3$ they left open:

$$E_x[\exp(b|B(\tau_D)|^{1-\alpha})] < \infty$$

iff $b < \frac{\sqrt{\lambda_1}}{A(1-\alpha)}$. □

Next consider the domain Ω from (2). Then we can write

$$\Omega = D \times S^{n-1},$$

where D is now given by (1) with $d = 2$, and we continue to make the Blanket Assumptions on D.

Theorem 1.6. *Let $n \geq 2$ and suppose B_t is $(n+1)$-dimensional Brownian motion. Then*

$$\lim_{N\to\infty} \left[\int_1^N \frac{dt}{a(t)}\right]^{-1} \log P_x(|B(\tau_\Omega)| > N) = -\frac{\pi}{2}.$$ □

Remark 1.7. i) Analogous to Remark 1.3 iii), we recover our earlier result (5).
ii) Theorem 1.6 can be sharpened much like Theorem 1.2 was sharpened by Theorem 1.4.
iii) Note that in contrast with Theorem 1.2, there is dimensional independence in the limit. The explanation for this counter-intuitive result is the same as that given in DeBlassie (2008a) for the special case of $a(x) = Ax^\alpha$.

In addition to the results of the authors mentioned above, there are other studies involving the domains D and Ω. Ioffe and Pinsky (1994) identified the Martin boundary of Ω. This result was extended by Aikawa and Murata (1996) and Murata (2002), (2005) to asymmetric versions of Ω

and they also found a series expansion for the Martin kernel. Related results were announced in Maz'ya (1977) and Kesten (1979). The growth of the Martin kernel at infinity for Ω and D was determined in DeBlassie (2008b) and (2009), respectively.

Collet et al. (2006) proved a ratio limit theorem for the Dirichlet heat kernel in Ω for $a(t) = \sqrt{t}$ in two dimensions. They used their theorem to determine the probability that Brownian motion remains in this particular domain for a long time. Using different methods, DeBlassie (2007) extended the latter result to all dimensions for the functions $a(t) = t^\alpha$, where $0 < \alpha < 1$.

Pinsky (2009) determined spectral properties of the Neumann Laplacian in Ω (and more general domains) as well as a transience/recurrence dichotomy for Brownian motion in Ω with normal reflection at the boundary.

In Bañuelos et al. (2001), Li (2003) and Lifshits and Shi (2002), the probability that Brownian motion remains in D for a long time was derived in the case when $a(t) = t^\alpha$, $0 < \alpha < 1$. For this particular domain, van den Berg (2003) found long-time asymptotics for the corresponding Dirichlet heat kernel.

In the case of the domain D with $a(t) \to 0$ as $t \to \infty$, bounds on the Dirichlet eigenfunctions of the Laplacian in D were obtained by Bañuelos and Davis (1992) and (1994), Bañuelos and van den Berg (1996), Cranston and Li (1997) and Lindemann et al. (1997).

Here is the organization of the article. In section 2 we study the domain D from (1). A representation theorem is stated, giving the density of harmonic measure as the normal derivative of the Green function. In subsection 2.1, the representation theorem is used in conjunction with a result of Cranston and Li on the asymptotics of harmonic functions to prove Theorems 1.2 and 1.4. Then in subsections 2.2–2.4, the proof of the representation theorem is given.

In section 3 we shift attention to the domain Ω from (2). The problem is reduced to two-dimensions, where now the relevant operator is the Laplacian plus a first order term, with corresponding process X. In subsection 3.1, by suitably conditioning the process, we eliminate the drift and state a representation theorem for the exit place density of X in terms of the conditioned process. Then we proceed analogously to the case of D considered in section 2.

Please note that throughout the article, c will be a scalar whose exact value can change from line to line.

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2 The Domain *D* from (1)

In this section we will prove Theorems 1.2 and 1.4. Since the domain D has a non-polar component, the Brownian motion—killed upon exiting D—is transient in D. Hence it has a Green function we denote by $G_D(x, y)$. Note that analytically one says $(\frac{1}{2}\Delta_{\mathbb{R}^d}, D)$ is subcritical and in fact $G_D(x, y)$ is the (minimal) Green function for $(\frac{1}{2}\Delta_{\mathbb{R}^d}, D)$. As indicated in the introduction, the next result describes the normal derivative of $G_D(x, \cdot)$ as more or less being the density of the harmonic measure based at x.

Theorem 2.1. *For any Borel set $A \subseteq \partial D$,*

$$P_x(B(\tau_D) \in A) = \frac{1}{2} \int_A \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] \sigma(dy), \qquad x \in D,$$

where $\frac{\partial}{\partial n_y}$ is the inward normal derivative at $y \in \partial D$ and $\sigma(dy)$ is the surface measure on ∂D induced by the usual Riemannian structure on $\mathbb{R}^d$. □

The unboundedness of D complicates the proof of Theorem 2.1. We will break up the proof into several pieces, but before that, we now use it to prove Theorems 1.2 and 1.4.

2.1 Proof of Theorems 1.2 and 1.4

For $d \geq 3$, let Ω_{d-1} be the unit ball in $\mathbb{R}^{d-1}$ and set $\nu = \frac{d-3}{2}$. As pointed out earlier, the first Dirichlet eigenvalue of $\Delta_{\mathbb{R}^{d-1}}$ on Ω_{d-1} is j_ν^2, where j_ν is the first positive zero of the Bessel function J_ν. Furthermore, the corresponding eigenfunction is

$$w(\tilde{x}) = C_0 |\tilde{x}|^{-\nu} J_\nu(j_\nu |\tilde{x}|), \qquad \tilde{x} \in \Omega_{d-1},$$

where C_0 is chosen so that $w(0) = 1$.

The next theorem is due to Cranston and Li (1997)—see the two paragraphs just after the proof of their Theorem 2.1. Note that although they make the blanket assumption $a(t) \to 0$ as $t \to \infty$, this is not used to prove the version of their theorem that we use.

Theorem 2.2. *Let $a\colon [M,\infty) \to (0,\infty)$ be continuous and, for sufficiently large t, twice differentiable with*

$$\lim_{t\to\infty} [|a'(t)| + a(t)|a''(t)|] = 0.$$

Set

$$D_M = \{(x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}\colon\ x_1 > M, |\tilde{x}| < a(x_1)\},$$

and suppose H is bounded and Hölder continuous on D_M with

$$\lim_{t\to\infty} a(t)^2 \sup_{|z|<a(t)} |H(t,z)| = 0.$$

If u satisfies

$$\begin{aligned} &u > 0 \quad on \quad D_M \\ &(\Delta_{\mathbb{R}^d} + H)u = 0 \quad on \quad D_M \\ &u = 0 \quad on \quad \{x_1 > M\} \cap \partial D_M \\ &\lim_{x_1\to\infty} u(x_1,\tilde{x}) = 0, \end{aligned}$$

then for each $\delta \in (0, j_\nu)$ there exist $M_1 > M$ and $C > 0$ such that

$$C^{-1} w\left(\frac{\tilde{x}}{a(x_1)}\right) \exp\left(-(j_\nu + \delta)\int_M^{x_1} \frac{dt}{a(t)}\right) \leq u(x_1,\tilde{x}) \leq C w\left(\frac{\tilde{x}}{a(x_1)}\right) \exp\left(-(j_\nu - \delta)\int_M^{x_1} \frac{dt}{a(t)}\right)$$

for all $x = (x_1, \tilde{x}) \in D_{M_1}$. □

For $y \in \partial D$, recall n_y is the inward unit normal to ∂D at y.

Lemma 2.3. *Let $y = (y_1, \tilde{y}) \in \partial D$ with $y_1 > \frac{1}{2}$. Then for $z = (z_1, \tilde{z}) = y + hn_y$,*

$$\lim_{h \to 0^+} \frac{w(\tilde{z}/a(z_1))}{h} = -C_0 j_\nu J_\nu'(j_\nu) \frac{\sqrt{1 + [a'(y_1)]^2}}{a(y_1)}.$$

Remark 2.4. It is known that $J_\nu'(j_\nu) < 0$.

Proof of Lemma 2.3. For $y = (y_1, \tilde{y}) \in \partial D$ with $y_1 > \frac{1}{2}$, it is a simple matter to show that

$$n_y = [1 + [a'(y_1)]^2]^{-1/2} \left(a'(y_1), -\frac{\tilde{y}}{a(y_1)} \right).$$

Since $|\tilde{y}| = a(y_1)$, as $h \to 0^+$ we get

$$\begin{aligned} \left| \frac{\tilde{z}}{a(z_1)} \right| &= \left| \frac{\tilde{y} \left[1 - \frac{h}{a(y_1)\sqrt{1+[a'(y_1)]^2}} \right]}{a\left(y_1 + ha'(y_1)/\sqrt{1 + [a'(y_1)]^2} \right)} \right| \\ &= \frac{a(y_1)}{a\left(y_1 + ha'(y_1)/\sqrt{1 + [a'(y_1)]^2} \right)} \left| 1 - \frac{h}{a(y_1)\sqrt{1 + [a'(y_1)]^2}} \right| \qquad (7) \\ &\to 1. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{h \to 0^+} \frac{w(\tilde{z}/a(z_1))}{h} &= \lim_{h \to 0^+} \frac{C_0}{h} \left| \frac{\tilde{z}}{a(z_1)} \right|^{-\nu} J_\nu \left(j_\nu \left| \frac{\tilde{z}}{a(z_1)} \right| \right) \\ &= C_0 \lim_{h \to 0^+} \frac{1}{h} J_\nu \left(j_\nu \left| \frac{\tilde{z}}{a(z_1)} \right| \right) \\ &= C_0 \lim_{h \to 0^+} J_\nu' \left(j_\nu \left| \frac{\tilde{z}}{a(z_1)} \right| \right) \frac{\partial}{\partial h} \left| \frac{\tilde{z}}{a(z_1)} \right| j_\nu \\ &= -C_0 j_\nu J_\nu'(j_\nu) \frac{\sqrt{1 + [a'(y_1)]^2}}{a(y_1)}. \qquad \square \end{aligned}$$

Proof of Theorem 1.2. Define $x_1(N)$ to be the first coordinate of the intersection of the circle $\rho^2 + z^2 = N^2$ with the curve $z = a(\rho)$ in the ρz-plane:

$$x_1(N)^2 + a(x_1(N))^2 = N^2. \qquad (8)$$

Fix $x \in D$ and suppose $M > |x|$ and $\delta \in (0, j_\nu)$. Combined with the fact that $G_D(x, y)$ goes to 0 as the modulus of $y \in D$ goes to infinity, Theorem 2.2 applied to $u(\cdot) = G_D(x, \cdot)$ and $H \equiv 0$ on D_M shows that we can choose $M_1 > M$ and $C > 0$ such that

$$C^{-1}w\left(\frac{\tilde{y}}{a(y_1)}\right)\exp\left(-(j_\nu+\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)\le G_D(x,y)\le Cw\left(\frac{\tilde{y}}{a(y_1)}\right)\exp\left(-(j_\nu-\delta)\int_M^{y_1}\frac{dt}{a(t)}\right) \tag{9}$$

for all $y=(y_1,\tilde{y})\in D_{M_1}$. Since $G_D(x,\cdot)=0$ on ∂D, for $y=(y_1,\tilde{y})\in\partial D$ with $y_1>M$,

$$\frac{\partial}{\partial n_y}G_D(x,y)=\lim_{h\to 0^+}\frac{G_D(x,y+hn_y)}{h}.$$

Using this and Lemma 2.3 in (9), we get that for some positive C_1 and C_2

$$\begin{aligned}C_1\frac{\sqrt{1+[a'(y_1)]^2}}{a(y_1)}\exp\left(-(j_\nu+\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)&\le\frac{\partial}{\partial n_y}G_D(x,y)\\ &\le C_2\frac{\sqrt{1+[a'(y_1)]^2}}{a(y_1)}\exp\left(-(j_\nu-\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)\end{aligned}$$

for $y=(y_1,\tilde{y})\in\partial D$ with $y_1>M_1$. Then by Theorem 2.1, also using that

$$P_x(|B(\tau_D)|>N)=P_x(B_1(\tau_D)>x_1(N))$$

(recall (8)), we get that for $N>M_1$,

$$\begin{aligned}\frac{1}{2}C_1\int\limits_{y_1>x_1(N)}\frac{\sqrt{1+[a'(y_1)]^2}}{a(y_1)}\exp\left(-(j_\nu+\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)dy&\le P_x(|B(\tau_D)|>N)\\ \le\frac{1}{2}C_2\int\limits_{y_1>x_1(N)}\frac{\sqrt{1+[a'(y_1)]^2}}{a(y_1)}\exp\left(-(j_\nu-\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)dy.&\end{aligned}$$

The integrands depend only on y_1, so this reduces to

$$\begin{aligned}\frac{1}{2}C_1\int_{x_1(N)}^{\infty}\frac{\sqrt{1+[a'(y_1)]^2}}{a(y_1)}\exp\left(-(j_\nu+\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)a(y_1)^{d-1}dy_1&\le P_x(|B(\tau_D)|>N)\\ \le\frac{1}{2}C_2\int_{x_1(N)}^{\infty}\frac{\sqrt{1+[a'(y_1)]^2}}{a(y_1)}\exp\left(-(j_\nu-\delta)\int_M^{y_1}\frac{dt}{a(t)}\right)a(y_1)^{d-1}dy_1.&\end{aligned} \tag{10}$$

Since $a'(t)\to 0$ as $t\to\infty$, the identity

$$\log a(t)=\int_1^t\frac{a'(u)}{a(u)}du+\log a(1)$$

implies

$$\left[\int_M^t\frac{du}{a(u)}\right]^{-1}\log a(t)\to 0\quad\text{as}\quad t\to\infty. \tag{11}$$

By l'Hôpital's rule and the fact that $a'(x) \to 0$ as $x \to \infty$, for any $\gamma > 0$ that is close to j_ν,

$$\lim_{N\to\infty} \frac{\int_N^\infty \sqrt{1+[a'(y_1)]^2}\exp\left(-\gamma\int_M^{y_1}\frac{dt}{a(t)}\right)a(y_1)^{d-2}dy_1}{a(N)^{d-1}\exp\left(-\gamma\int_M^N\frac{dt}{a(t)}\right)} = \frac{1}{\gamma}.$$

Combining this with (10), we get that for some positive C_3 and C_4, for large N,

$$\begin{aligned}\frac{C_3}{j_\nu+\delta}a(x_1(N))^{d-1}\exp\left(-(j_\nu+\delta)\int_M^{x_1(N)}\frac{dt}{a(t)}\right) &\le P_x(|B(\tau_D)|>N)\\ \le \frac{C_4}{j_\nu-\delta}a(x_1(N))^{d-1}\exp\left(-(j_\nu-\delta)\int_M^{x_1(N)}\frac{dt}{a(t)}\right).&\end{aligned} \tag{12}$$

Take the natural logarithm, multiply by $[\int_M^{x_1(N)}\frac{dt}{a(t)}]^{-1}$, let $N\to\infty$ and use (11) to get

$$\begin{aligned}-(j_\nu+\delta) &\le \liminf_{N\to\infty}\left[\int_M^{x_1(N)}\frac{dt}{a(t)}\right]^{-1}\log P_x(|B(\tau_D)|>N)\\ &\le \limsup_{N\to\infty}\left[\int_M^{x_1(N)}\frac{dt}{a(t)}\right]^{-1}\log P_x(|B(\tau_D)|>N)\\ &\le -(j_\nu-\delta).\end{aligned}$$

Let $\delta\to 0$ and use the fact that

$$\int_M^{x_1(N)}\frac{dt}{a(t)} \sim \int_1^{x_1(N)}\frac{dt}{a(t)} \quad \text{as} \quad N\to\infty$$

to get

$$\lim_{N\to\infty}\left[\int_1^{x_1(N)}\frac{dt}{a(t)}\right]^{-1}\log P_x(|B(\tau_D)|>N) = -j_\nu.$$

To complete the proof of Theorem 1.2, we show

$$\int_1^{x_1(N)}\frac{dt}{a(t)} \sim \int_1^N \frac{dt}{a(t)} \quad \text{as} \quad N\to\infty.$$

Indeed, recalling the definition of $x_1(N)$ from (8) and writing x_1 for $x_1(N)$, we have

$$N = x_1\sqrt{1+\left(\frac{a(x_1)}{x_1}\right)^2}.$$

Then by the mean value theorem, there is $\tilde{x} \in (x_1, N)$ such that

$$
\begin{aligned}
\frac{a(N)}{a(x_1)} - 1 &= \frac{a(N) - a(x_1)}{a(x_1)} \\
&= \frac{a'(\tilde{x})x_1 \left[\sqrt{1 + \left(\frac{a(x_1)}{x_1}\right)^2} - 1 \right]}{a(x_1)} \\
&= \frac{a'(\tilde{x})x_1}{a(x_1)} \left[1 + \frac{1}{2}\left(\frac{a(x_1)}{x_1}\right)^2 + o\left(\left(\frac{a(x_1)}{x_1}\right)^2\right) - 1 \right] \\
&= a'(\tilde{x}) \left[\frac{1}{2}\frac{a(x_1)}{x_1} + o(1)\frac{a(x_1)}{x_1} \right] \\
&\to 0
\end{aligned}
$$

as $N \to \infty$, since $\frac{a(u)}{u} \to 0$ and $a'(u) \to 0$ as $u \to \infty$. Thus

$$
\frac{a(N)}{a(x_1)} \to 1 \quad \text{as} \quad N \to \infty.
$$

Upon differentiating (8) with respect to N,

$$
\begin{aligned}
x_1' &= \frac{N}{x_1\left(1 + \frac{a(x_1)}{x_1}a'(x_1)\right)} \\
&= \frac{\sqrt{1 + \left(\frac{a(x_1)}{x_1}\right)^2}}{1 + \frac{a(x_1)}{x_1}a'(x_1)} \\
&\to 1
\end{aligned}
$$

as $N \to \infty$.

To finish, use these limits and l'Hôpital's rule to get

$$
\begin{aligned}
\lim_{N\to\infty} \frac{\int_M^{x_1} \frac{dt}{a(t)}}{\int_M^N \frac{dt}{a(t)}} &= \lim_{N\to\infty} \frac{x_1' a(N)}{a(x_1)} \\
&= 1,
\end{aligned}
$$

as desired. □

Proof of Theorem 1.4. Under the additional hypotheses of Theorem 1.2, the conclusion of Theorem 2.2 holds with $\delta = 0$. Thus the argument leading to (12) goes through with $\delta = 0$ there and we get the conclusion of Theorem 1.4. □

2.2 Proof of Theorem 2.1: Preliminaries and a Reduction

To prove Theorem 2.1, it suffices to show that for each $x \in D$, for any nonnegative $f \in C^3(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d \backslash \{x\}$, we have

$$E_x[f(B_{\tau_D})] = \frac{1}{2} \int_{\partial D} f(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] \sigma(dy). \tag{13}$$

To this end, write

$$u(z) = E_z[f(B_{\tau_D})], \qquad z \in D.$$

By the strong Markov property, u is harmonic in D, hence C^∞ there. Given $\varepsilon > 0$, by uniform continuity of f, choose $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Then for $x \in D$ and $y \in \partial D$ with $|x - y| < \frac{\delta}{2}$,

$$\begin{aligned} |u(x) - f(y)| &\le E_x[\,|f(B_{\tau_D}) - f(y)|\,] \\ &= E_x[\,|f(B_{\tau_D}) - f(y)|\,[\,I(|B_{\tau_D} - y| < \delta) + I(|B_{\tau_D} - y| \ge \delta)\,]\,] \\ &\le \varepsilon + 2[\,\sup|f|\,]P_x(|B_{\tau_D} - y| \ge \delta). \end{aligned}$$

But since $|x - y| < \frac{\delta}{2}$, for any $t > 0$ we can apply Theorem 2.2.2 (ii) in Pinsky (1995) to get

$$\begin{aligned} P_x(|B_{\tau_D} - y| \ge \delta) &\le P_x\left(|B_{\tau_D} - x| \ge \frac{\delta}{2}\right) \\ &\le P_x\left(|B_{\tau_D} - x| \ge \frac{\delta}{2}, \tau_D \le t\right) + P_x(\tau_D > t) \\ &\le P_x\left(\sup_{s \le t}|B_s - x| \ge \frac{\delta}{2}\right) + P_x(\tau_D > t) \\ &\le 2d \exp\left(-\frac{\delta^2}{8dt}\right) + P_x(\tau_D > t). \end{aligned}$$

Choosing $t > 0$ so large that the first term is less than ε, we get

$$|u(x) - f(y)| < \varepsilon + 2[\,\sup|f|\,][\,\varepsilon + P_x(\tau_D > t)\,].$$

Now let $x \to y \in \partial D$ and apply Corollary 2.3.4 in Pinsky (1995) to get

$$\limsup_{x \to y} |u(x) - f(y)| \le \varepsilon(1 + 2\sup|f|).$$

Since $\varepsilon > 0$ was arbitrary, we get that $u(x) \to f(y)$ as $x \to y \in \partial D$. Thus

$$\begin{cases} u \in C^{2,\alpha}(D) \\ \frac{1}{2}\Delta u = 0 \quad \text{in} \quad D \\ u(x) \to f(y) \quad \text{as} \quad x \to y \in \partial D. \end{cases} \tag{14}$$

For an unbounded set $G \subseteq \mathbb{R}^d$, we define $C^{2,\alpha}(G)$ to be the set of all functions in $C^2(G)$ whose second order partials are uniformly Hölder continuous on any compact subset of G. Note we do not require G to be open.

Since $f \in C^3(\overline{D})$, by the Elliptic Regularity Theorem (Lemma 6.18 in Gilbarg and Trudinger (1983)), $u \in C^{2,\alpha}(\overline{D})$. Thus (14) becomes

$$\begin{cases} u \in C^{2,\alpha}(\overline{D}) \\ \frac{1}{2}\Delta u = 0 \quad \text{in} \quad D \\ u|_{\partial D} = f \end{cases} \tag{15}$$

and so (13) amounts to the classical Green Representation of the solution to (15). As indicated in the introduction, since D is unbounded, complications arise.

We will write

$$x_0 = (1,0) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Now we fix $x = (x_1, \tilde{x}) \in D$ and prove (13). For simplicity we will assume $x \neq x_0$. Notice $\frac{1}{2}\Delta G_D(x,\cdot) = 0$ on $D \backslash \{x\}$ and $G_D(x,\cdot)$ is continuous on $\overline{D} \backslash \{x\}$ with boundary value 0 (Pinsky (1995), Theorem 7.3.2). Then by another application of the Elliptic Regularity Theorem,

$$G_D(x,\cdot) \in C^{2,\alpha}(\overline{D} \backslash \{x\}).$$

Let $\varepsilon > 0$ be so small that

$$\overline{B_{2\varepsilon}(x)} \subseteq D \backslash \{x_0\}$$

and

$$\text{supp}(f) \subseteq \overline{B_\varepsilon(x)}^c,$$

where the over bar denotes Euclidean closure. Given $M > 0$ so large that

$$\text{supp}(f) \subseteq B_M(0),$$

define

$$E = E(M,\varepsilon) = D \cap \{(x_1,\tilde{x}): \; x_1 < M\} \cap \overline{B_\varepsilon(x)}^c. \tag{16}$$

Then by Green's Second Identity,

$$\begin{aligned} &-\int_{\partial E} \left\{ u(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] - G_D(x,y) \frac{\partial u}{\partial n_y}(y) \right\} \sigma(dy) \\ &\quad = \int_E [u(y)\Delta_y G_D(x,y) - G_D(x,y)\Delta u(y)]\,dy \\ &\quad = 0. \end{aligned} \tag{17}$$

Writing

$$\pi_M = \{(x_1,\tilde{x}): \; x_1 = M\},$$

break up the surface integral over ∂E into the pieces $\partial B_\varepsilon(x)$, $\{x_1 < M\} \cap \partial D$ and $\overline{D} \cap \pi_M$ to end up with

$$\int\limits_{\{x_1<M\}\cap\partial D} = \left[- \int\limits_{\overline{D}\cap\pi_M} + \int\limits_{\partial B_\varepsilon(x)} \right] \left\{ u(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] - G_D(x,y) \frac{\partial u}{\partial n_y}(y) \right\} \sigma(dy) \tag{18}$$

where now the $\frac{\partial}{\partial n_y}$ in the $\partial B_\varepsilon(x)$ integral is inward normal differentiation for $B_\varepsilon(x)$. Now $u = f$ on ∂D, $\operatorname{supp}(f) \subseteq B_M(0) \cap \overline{B_\varepsilon(x)}^c$ and $G_D(x, \cdot) = 0$ on ∂D, so we have

$$\int\limits_{\partial D} f(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] \sigma(dy) = \int\limits_{\{x_1<M\}\cap\partial D} \left\{ u(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] - G_D(x,y) \frac{\partial u}{\partial n_y}(y) \right\} \sigma(dy)$$

$$= \left[- \int\limits_{\overline{D}\cap\pi_M} + \int\limits_{\partial B_\varepsilon(x)} \right] \{''\}, \text{ by (18).} \tag{19}$$

Once we prove

$$\lim_{M\to\infty} \int\limits_{\overline{D}\cap\pi_M} \left\{ u(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] - G_D(x,y) \frac{\partial u}{\partial n_y}(y) \right\} \sigma(dy) = 0 \tag{20}$$

and

$$\lim_{\varepsilon\to 0} \int\limits_{\partial B_\varepsilon(x)} \left\{ u(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] - G_D(x,y) \frac{\partial u}{\partial n_y}(y) \right\} \sigma(dy) = 2u(x), \tag{21}$$

we can let $M \to \infty$ and $\varepsilon \to 0$ in (19) to get

$$u(x) = \frac{1}{2} \int\limits_{\partial D} f(y) \left[\frac{\partial}{\partial n_y} G_D(x,y) \right] \sigma(dy),$$

which is exactly (13), as desired.

Formulas (20)–(21) will be proved in the next two subsections.

2.3 Proof of (20)

The following result is a consequence of the proof of Lemma 6.5 in Gilbarg and Trudinger (1983) combined with the comments subsequent to the proof.

Lemma 2.5. *Let Ω be a domain in $\mathbb{R}^d$ with a $C^{2,\alpha}$ boundary portion T. Suppose $u \in C^{2,\alpha}(\Omega \cup T)$ is a solution of*

$$(\Delta_{\mathbb{R}^d} + H)u = 0 \quad on \quad \Omega$$
$$u = 0 \quad on \quad T,$$

where

$$\Lambda := |H|_{0,\alpha;\Omega} = \sup_{\Omega} |H| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|H(x)-H(y)|}{|x-y|^\alpha} < \infty.$$

Then for any $z \in T$ there is $\delta > 0$ such that for

$$B(z) = B_\delta(z) \cap \Omega \quad \textit{and} \quad T(z) = B_\delta(z) \cap T$$

we have

$$\sup\{d(x, \partial B(z) - T(z))|\nabla u(x)| :\ x \in B(z) \cup T(z)\} \le C \sup_{B(z)} |u|.$$

Here δ depends only on the diameter of the domain of the $C^{2,\alpha}$ diffeomorphism ψ that straightens the boundary near z and C depends only on d, α, Λ and the $C^{2,\alpha}$ bounds on ψ. □

We now apply Lemma 2.5.

Lemma 2.6. *Suppose $b :\ \overline{D} \to \mathbb{R}$ is bounded with*

$$\sup_{M\ge 1} a(M)^{\alpha+2} \sup\left\{ \frac{|b(z)-b(y)|}{|z-y|^\alpha} :\ z, y \in \overline{D}; z \neq y; z_1, y_1 \in (M - a(M), M + a(M)) \right\} < \infty,$$

$$\sup_{M\ge 1} a(M)^2 \sup\{|b(z)| :\ z \in \overline{D}, z_1 \in (M - a(M), M + a(M))\} < \infty.$$

If $v \in C^{2,\alpha}(\overline{D}\backslash\{x\})$ is a solution of

$$(\Delta + b)v = 0 \quad \textit{on} \quad D\backslash\{x\}$$

with $v = 0$ on ∂D outside a compact set, then for large M,

$$|\nabla v(z)| \le C a(M)^{-1} \sup\{|v(y)| :\ y \in D, y_1 \in (M - a(M), M + a(M))\}, \qquad z \in D \cap \pi_M.$$

Proof. Define

$$\gamma_M(t) = \frac{a(ta(M) + M)}{a(M)}, \qquad t \ge \frac{\frac{1}{2} - M}{a(M)}.$$

Then for

$$H_M = \left\{ (z_1, \tilde{z}) :\ z_1 > \frac{\frac{1}{2} - M}{a(M)}, |\tilde{z}| < \gamma_M(z_1) \right\}$$

we have

$$\begin{cases} z \in H_M \Leftrightarrow a(M)z + Mx_0 \in D \\ z \in H_M \cap \{-1 < z_1 < 1\} \Leftrightarrow a(M)z + Mx_0 \in D \cap \{(y_1, \tilde{y}) :\ M - a(M) < y_1 < M + a(M)\} \\ z \in \partial H_M \cap \{-1 < z_1 < 1\} \Leftrightarrow a(M)z + Mx_0 \in \partial D \cap \{(y_1, \tilde{y}) :\ M - a(M) < y_1 < M + a(M)\}. \end{cases} \tag{22}$$

Notice H_M is obtained from D via translating by $-Mx_0$ and then scaling by $1/a(M)$. For any function g on $\overline{D}$, we define

$$g_M(z) = g(a(M)z + Mx_0), \quad z \in \overline{H}_M.$$

Then for

$$L_M = \Delta + a(M)^2 b_M$$

we have

$$L_M v_M = 0 \quad \text{on} \quad H_M \setminus \left\{ \frac{x - Mx_0}{a(M)} \right\}.$$

Since $\frac{a(M)}{M} \to 0$ as $M \to \infty$, for large M we have

$$x_1 < M - a(M).$$

Thus for large M,

$$L_M v_M = 0 \quad \text{on} \quad H_M \cap \{-1 < z_1 < 1\}.$$

Since $v \in C^{2,\alpha}(\overline{D} \setminus \{x\})$ and $v = 0$ on ∂D outside a compact set, by making M larger if necessary, we have that

$$\begin{aligned} v_M &\in C^{2,\alpha}(\overline{H}_M \cap \{-1 \le z_1 \le 1\}) \\ v_M &= 0 \quad \text{on} \quad \partial H_M \cap \{-1 \le z_1 \le 1\}. \end{aligned}$$

We are going to apply Lemma 2.5 to L_M and v_M on

$$\begin{aligned} \Omega &= H_M \cap \{-1 < z_1 < 1\} \\ T &= \partial H_M \cap \{-1 < z_1 < 1\}. \end{aligned}$$

This is legitimate because by our hypotheses on b, there exists $\Lambda > 0$ such that for large M,

$$|a(M)^2 b_M|_{0,\alpha;\Omega} \le \Lambda. \tag{23}$$

By symmetry, compactness and our Blanket Assumptions on $a(\cdot)$, for each $z \in \partial H_M \cap \{-\frac{1}{2} \le z_1 \le \frac{1}{2}\}$, the $C^{2,\alpha}$ bounds on the diffeomorphism straightening the boundary near z are independent of z and large M. (Roughly speaking, for large M, the set $H_M \cap \{-1 < z_1 < 1\}$ looks like the set $\{(z_1, \tilde{z}): \ -1 < z_1 < 1, |\tilde{z}| < 1\}$). Combined with (23), it follows that the constant C appearing in Lemma 2.5 is independent of such z and large M. Likewise, the δ in the lemma is also independent of z and large M. The net effect is that for some $\delta > 0$ and $C > 0$, for

$$\begin{aligned} B(z) &= B_\delta(z) \cap \Omega \\ T(z) &= B_\delta(z) \cap \partial\Omega, \end{aligned}$$

we have

$$\begin{aligned} \sup_{y \in B(z) \cup T(z)} d(y, \partial B(z) - T(z)) |\nabla v_M(y)| &\le C \sup_{B(z)} |v_M| \\ &\le C \sup_{\Omega} |v_M| \end{aligned} \tag{24}$$

whenever $z \in \partial H_M \cap \{-\frac{1}{2} \le z_1 \le \frac{1}{2}\}$ and M is large.

In particular, given $y \in \pi_0 \cap H_M \cap \{|\tilde{y}| > 1 - \frac{\delta}{2}\}$, choose $z \in \pi_0 \cap \partial H_M$ such that $d(y, \pi_0 \cap \partial H_M) = d(y,z)$. Then $y \in B_\delta(z)$ and by (24), for M large,

$$|\nabla v_M(y)| \le C d(y, \partial B(z) - T(z))^{-1} \sup_\Omega |v_M|.$$

Since $y \in \pi_0$, $d(y, \partial B(z) - T(z)) \ge \frac{\delta}{2}$ and we get that for some $C_1 > 0$, for large M,

$$|\nabla v_M(y)| \le C_1 \delta^{-1} \sup_\Omega |v_M|, \qquad y \in \pi_0 \cap H_M \cap \left\{ |\tilde{y}| \ge 1 - \frac{\delta}{2} \right\}. \tag{25}$$

On the other hand, by the Schauder interior estimates (Gilbarg and Trudinger (1983), Theorem 6.2),

$$\sup_\Omega d(y, \partial\Omega) |\nabla v_M(y)| \le C_2 \sup_\Omega |v_M|,$$

where $C_2 > 0$ is independent of large M. Combined with (25), we get that for some $C > 0$, for all large M,

$$|\nabla v_M(y)| \le C \delta^{-1} \sup_\Omega |v_M|, \qquad y \in \pi_0 \cap H_M.$$

Converting back to v and using (22), for all large M we have

$$|a(M)(\nabla v)(a(M)y + Mx_0)| \le C\delta^{-1} \sup_{z \in \Omega} |v(a(M)z + Mx_0)|, \quad a(M)y + Mx_0 \in D \cap \pi_M,$$

which is to say

$$|\nabla v(z)| \le C\delta^{-1} a(M)^{-1} \sup\{|v(w)| : \ w \in D, w_1 \in (M - a(M), M + a(M))\}, \qquad z \in D \cap \pi_M,$$

as desired. □

Now we can prove (20). Taking $b = 0$ in Lemma 2.6, $v = u$ or $G_D(x, \cdot)$ satisfies the required hypotheses, so for some $C > 0$, for all large M,

$$\left| \frac{\partial u}{\partial n_y}(y) \right| \le C a(M)^{-1} \sup |u|, \qquad y \in \pi_M \cap D$$

$$\left| \frac{\partial}{\partial n_y} G_D(x,y) \right| \le C a(M)^{-1} \sup\{G_D(x,z) : \ z \in D, M - a(M) < z_1 < M + a(M)\}, \qquad y \in \pi_M \cap D,$$

where $\frac{\partial}{\partial n_y}$ is inward normal differentiation at the boundary part $\pi_M \cap D$ of $D \cap \{z_1 < M\}$. But it is well-known that $|G_D(x,y)| \le C|x-y|^{2-d}$, and since $\frac{a(M)}{M} \to 0$ as $M \to \infty$, it follows that for large M,

$$\left| \frac{\partial}{\partial n_y} G_D(x,y) \right| \le C a(M)^{-1} M^{2-d}, \qquad y \in \pi_M \cap D.$$

Since u is bounded, we end up with

$$\left| u(y) \frac{\partial}{\partial n_y} G_D(x,y) - G_D(x,y) \frac{\partial u}{\partial n_y}(y) \right| \le C a(M)^{-1} M^{2-d}, \qquad y \in \pi_M \cap D,$$

for all large M. Hence as $M \to \infty$

$$\left| \int_{\overline{D}\cap\pi_M} \left\{ u(y)\frac{\partial}{\partial n_y} G_D(x,y) - G_D(x,y)\frac{\partial u}{\partial n_y}(y) \right\} \sigma(dy) \right| \leq C a(M)^{-1} M^{2-d} a(M)^{d-1}$$
$$= C\left(\frac{a(M)}{M}\right)^{d-2}$$
$$\to 0,$$

as desired. □

2.4 Proof of (21)

Since $u \in C^{2,\alpha}(\overline{D})$, $\frac{\partial u}{\partial n_y}$ is bounded in a neighborhood of x, hence

$$\left| \int_{\partial B_\varepsilon(x)} G_D(x,y)\frac{\partial u}{\partial n_y}\sigma(dy) \right| \leq C\varepsilon^{2-d}\sigma(\partial B_\varepsilon(x))$$
$$= C\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus to prove (21), we need only check

$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(x)} u(y)\left[\frac{\partial}{\partial n_y} G_D(x,y)\right]\sigma(dy) = 2u(x). \tag{26}$$

Now for

$$p(t,y,z) = \frac{1}{(2\pi t)^{d/2}} e^{-|y-z|^2/2t},$$

the transition density of Brownian motion killed upon exiting D is given by

$$p_D(t,y,z) = p(t,y,z) - E_y[I_{\tau_D<t}\, p(t-\tau_D, B_{\tau_D}, z)].$$

Analytically, p_D is the Dirichlet heat kernel for $\frac{1}{2}\Delta_{\mathbb{R}^d}$ on D. Thus

$$G_D(y,z) = \int_0^\infty p_D(t,y,z)dt$$
$$= K_d|y-z|^{2-d} - K_d E_y[|B_{\tau_D} - z|^{2-d}],$$

where

$$K_d = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}}.$$

Consequently, recalling that $\frac{\partial}{\partial n_y}$ is differentiation along the inward unit normal to $\partial B_\varepsilon(x)$,

$$\frac{\partial}{\partial n_y} G_D(x,y) = K_d(d-2)\left[|x-y|^{1-d} - E_y\left[|B_{\tau_D} - y|^{-d}\frac{(B_{\tau_D}-y)\cdot(x-y)}{\varepsilon}\right]\right].$$

To justify the differentiation under the expectation, bound the difference quotients using the Mean Value Theorem. Then dominated convergence applies because

$$\sup\{\nabla_z |w-z|^{2-d} :\ z \in B_{2\varepsilon}(x), w \in \partial D\} < \infty.$$

We have, as $\varepsilon \to 0$,

$$\begin{aligned}
&\left|\int_{\partial B_\varepsilon(x)} u(y) E_x\left[|B_{\tau_D} - y|^{-d}\frac{(B_{\tau_D}-y)\cdot(x-y)}{\varepsilon}\right]\sigma(dy)\right| \\
&\leq C\int_{\partial B_\varepsilon(x)} E_x[|B_{\tau_D}-y|^{1-d}]\sigma(dy) \\
&\leq C d(\partial B_\varepsilon(x), \partial D)^{1-d}\sigma(\partial B_\varepsilon(x)) \\
&\leq C\varepsilon^{d-1} \to 0.
\end{aligned}$$

Thus to prove (26), we need only show

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} K_d(d-2)|x-y|^{1-d} u(y)\sigma(dy) = 2u(x).$$

But this is an immediate consequence of the continuity and boundedness of u, combined with the identities $K_d(d-2)\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} = 2$ and $\sigma(\partial B_\varepsilon(x)) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$. □

3 The Domain Ω from (2)

Recall $n \geq 2$ and the horn Ω in $\mathbb{R}^{n+1}$ is represented in cylindrical coordinates (ρ, z, θ) by $\Omega = D \times S^{n-1}$, where

$$D = \left\{(\rho, z) :\ \rho > \frac{1}{2}, -a(\rho) < z < a(\rho)\right\}.$$

The Laplacian expressed in the coordinates (ρ, z, θ) is

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial\rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\Delta_{S^{n-1}},$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on S^{n-1}. Write L for the nonangular part of $\frac{1}{2}\Delta_{\mathbb{R}^{n+1}}$:

$$L = \frac{1}{2}\left[\frac{\partial^2}{\partial\rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right],$$

and let X_t be the diffusion associated with L in $\{(\rho,z)\colon\ \rho>0\}$. Then by symmetry, for $x=(\rho,z,\theta)$,

$$P_x(|B(\tau_\Omega)|>N)=P_y(|X(\tau_D)|>N), \quad y=(\rho,z).$$

Thus to prove Theorem 1.6, it suffices to show

$$\lim_{N\to\infty}\left[\int_1^N \frac{dt}{a(t)}\right]^{-1}\log P_y(|X(\tau_D)|>N)=-\frac{\pi}{2}.$$

Using the relation

$$\int_1^{x_1(N)}\frac{dt}{a(t)}\sim\int_1^N\frac{dt}{a(t)} \quad \text{as} \quad N\to\infty$$

derived in the proof of Theorem 1.2, we see the proof of Theorem 1.6 comes down to showing

$$\lim_{N\to\infty}\left[\int_1^{x_1(N)} \frac{dt}{a(t)}\right]^{-1}\log P_y(|X(\tau_D)|>N)=-\frac{\pi}{2}. \tag{27}$$

3.1 The Analogue of Theorem 2.1 for X

Since $n\geq 2$, starting at (ρ,z) with $\rho>0$, the process X_t stays in $\{(\rho,z)\colon\ \rho>0\}$ forever. In fact, the first component of X_t is an n-dimensional Bessel process and the second component is an independent one-dimensional Brownian motion. Thus the transition density $p(t,y,w)$ of X_t (with respect to Lebesgue measure) is the product of the transition densities of the components: for $y=(y_1,y_2)$ and $w=(w_1,w_2)$,

$$p(t,y,w)=\frac{e^{-(w_1^2+y_1^2)/2t}}{t(y_1w_1)^{\frac{n}{2}-1}}w_1^{n-1}I_{\frac{n}{2}-1}\left(\frac{y_1w_1}{t}\right)\frac{1}{\sqrt{2\pi t}}e^{-(y_2-w_2)^2/2t}, \tag{28}$$

where

$$I_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^\infty\frac{(\frac{z}{2})^{2k}}{k!\Gamma(\nu+k+1)}$$

is the modified Bessel function (see Ikeda and Watanabe (1981) for the transition density of the Bessel process).

Lemma 3.1. *The operator L is subcritical on $\{(\rho,z)\colon\ \rho>0\}$; equivalently, X_t is transient in $\{(\rho,z)\colon\ \rho>0\}$ and has a Green function G there. In fact, for $y=(y_1,y_2)$ and $w=(w_1,w_2)$,*

$$G(y,w)=K_nw_1^{n-1}[y_1^2+w_1^2+(y_2-w_2)^2]^{-(n-1)/2}F\left(\frac{n-1}{4},\frac{n+1}{4};\frac{n}{2};\left(\frac{2y_1w_1}{y_1^2+w_1^2+(y_2-w_2)^2}\right)^2\right),$$

where

$$K_n=\pi^{-1}2^{(n-3)/2}\frac{\Gamma(\frac{n-1}{4})\Gamma(\frac{n+1}{4})}{\Gamma(\frac{n}{2})}$$

and

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^\infty\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!}$$

is the hypergeometric function with interval of convergence $(-1,1)$.

Proof. Writing

$$\beta = \frac{y_1^2 + w_1^2 + (y_2 - w_2)^2}{2}$$

and changing variables $u = \beta/t$, we have

$$\begin{aligned}
G(y,w) &= \int_0^\infty p(t,y,w)dt \\
&= \frac{w_1^{n/2} y_1^{1-n/2}}{\sqrt{2\pi}} \int_0^\infty t^{-3/2} e^{-\beta/t} I_{\frac{n}{2}-1}\left(\frac{y_1 w_1}{t}\right) dt \\
&= \frac{w_1^{n/2} y_1^{1-n/2}}{\sqrt{2\pi}} \beta^{-1/2} \int_0^\infty u^{-1/2} e^{-u} I_{\frac{n}{2}-1}\left(\frac{y_1 w_1}{\beta} u\right) du \\
&= \frac{w_1^{n/2} y_1^{1-n/2}}{\sqrt{2\pi}} \beta^{-1/2} \sum_{k=0}^\infty \int_0^\infty \frac{u^{-1/2} e^{-u}}{k!\Gamma(\frac{n}{2}+k)} \left(\frac{y_1 w_1}{2\beta} u\right)^{\frac{n}{2}-1+2k} du \\
&= \frac{w_1^{n-1} 2^{(1-n)/2}}{\sqrt{\pi}} \beta^{(1-n)/2} \sum_{k=0}^\infty \frac{\Gamma(\frac{n}{2}+2k-\frac{1}{2})}{\Gamma(\frac{n}{2}+k)} \frac{1}{k!} \left(\frac{y_1 w_1}{2\beta}\right)^{2k}.
\end{aligned}$$

Using the identity

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} 2^{2z-\frac{1}{2}} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

(Abramowitz and Stegun (1972), 6.1.18), we get

$$\begin{aligned}
G(y,w) &= \frac{1}{2\pi} w_1^{n-1} \beta^{-(n-1)/2} \sum_{k=0}^\infty \frac{\Gamma(\frac{n}{4}+k-\frac{1}{4})\Gamma(\frac{n}{4}+k+\frac{1}{4})}{k!\Gamma(\frac{n}{2}+k)} \left(\frac{y_1 w_1}{\beta}\right)^{2k} \\
&= \frac{1}{2\pi} w_1^{n-1} \beta^{-(n-1)/2} \frac{\Gamma(\frac{n-1}{4})\Gamma(\frac{n+1}{4})}{\Gamma(\frac{n}{2})} F\left(\frac{n-1}{4}, \frac{n+1}{4}; \frac{n}{2}; \left(\frac{y_1 w_1}{\beta}\right)^2\right).
\end{aligned}$$

Upon substituting for β, we get the desired expression. □

Since $D \subseteq \{(\rho,z)\colon\ \rho > 0\}$, by Lemma 3.1 (L,D) is subcritical and the corresponding Green function G_D is associated with X_t killed upon exiting D. Because L is not self-adjoint with respect to Lebesgue measure, the analogue of Theorem 2.1 takes on a slightly different form. We h-transform L, converting it into a self-adjoint operator that is easier to analyze. Here, if $h \in C^{2,\alpha}(D)$ is positive, then the h-transform of L is the operator L^h given by

$$L^h f = \frac{1}{h} L(hf).$$

We will take

$$h(\rho,z) = \rho^{-p}, \qquad p = \frac{n-1}{2}. \tag{29}$$

Then

$$L^h = \frac{1}{2}\left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} - \frac{p(p-1)}{\rho^2}\right].$$

Since (L,D) is subcritical, so is (L^h,D) (Pinsky (1995) Proposition 4.2.2) and its Green function is

$$G_D^h(y,w) = G_D(y,w)h(w)/h(y). \tag{30}$$

Now we can state the analogue of Theorem 2.1.

Theorem 3.2. *For any Borel set $A \subseteq \partial D$*

$$P_y(X(\tau_D) \in A) = \frac{1}{2}\int_A \frac{h(y)}{h(w)}\left[\frac{\partial}{\partial n_w} G_D^h(y,w)\right]\sigma(dw), \qquad y \in D. \qquad \square$$

Before proving this theorem, we show how it yields (27), hence Theorem 1.6. Indeed, Theorem 3.2 implies

$$P_y(|X_{\tau_D}| > N) = \frac{1}{2}h(y)\int_{|w|\geq N} \frac{1}{h(w)}\left[\frac{\partial}{\partial n_w} G_D^h(y,w)\right]\sigma(dw).$$

Fix $y \in D$ and let $M > |y|$, $\delta \in (0, j_\nu)$. Since $\frac{a(t)}{t} \to 0$ as $t \to \infty$, the function $H(\rho,z) = -\frac{p(p-1)}{\rho^2}$ satisfies the hypotheses of Theorem 2.2 on D_M; below in (42) we show that $G_D^h(y,w) \to 0$ as $w_1 \to \infty$. Thus for $u(w) = G_D^h(y,w)$, $w \in D_M$,

$$(\Delta_{\mathbb{R}^2} + H)u = 2L^h G_D^h(y,\cdot) = 0 \quad \text{on} \quad D_M$$

and so we can apply Theorem 2.2. Then we can repeat the proof of Theorem 1.2 almost word-for-word to end up with the analogue of (12), except that now $d = 2$ and the upper and lower bounds have an extra factor of $x_1(N)^{-p}$—this is due to the extra factor $\frac{1}{h(w)} = w_1^{-p}$ in the integrand of the expression above for $P_y(|X_{\tau_D}| > N)$. The rest of the argument after (12) still goes through because

$$\lim_{N\to\infty} \frac{\log x_1(N)}{\int_M^{x_1(N)} \frac{dt}{a(t)}} = \lim_{K\to\infty} \frac{\log K}{\int_M^K \frac{dt}{a(t)}} = \lim_{K\to\infty} \frac{a(K)}{K} = 0.$$

3.2 Proof of Theorem 3.2

Now L^h is formally self-adjoint and it satisfies Hypothesis $\widetilde{H}_{\text{loc}}$ in Pinsky (1995). Then by his Theorems 4.2.5, 4.2.8, and 8.1.1,

- $G_D^h(\cdot,y), G_D^h(y,\cdot) \in C^{2,\alpha}(D \backslash \{y\})$;
- G_D^h is positive and jointly continuous off the diagonal;
- $L^h G_D^h(y,\cdot) = 0$ on $D \backslash \{y\}$;

• for each $y \in D$, there exist positive C_1 and C_2 along with $r_0 \in (0,1)$ such that

$$-C_1 \log|y-w| \leq G_D^h(y,w) \leq -C_2 \log|y-w| \quad \text{for} \quad 0 < |y-w| < r_0; \tag{31}$$

- $G_D^h(y,\cdot)$ is continuous on $\overline{D}\backslash\{y\}$ with boundary value 0.

In particular, by the Elliptic Regularity Theorem,

$$G^h_D(y,\cdot) \in C^{2,\alpha}(\overline{D}\backslash\{y\}).$$

Fix $y \in D$. Exactly as in the proof of Theorem 2.1, it suffices to show for any nonnegative $f \in C^3(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2\backslash\{y\}$,

$$E_y[f(X_{\tau_D})] = \frac{1}{2}\int_{\partial D} f(w)\frac{h(y)}{h(w)}\left[\frac{\partial}{\partial n_w}G^h_D(y,w)\right]\sigma(dw). \tag{32}$$

The argument giving (15) also gives that

$$u(w) = E_w[f(X_{\tau_D})]$$

solves

$$\begin{cases} u \in C^{2,\alpha}(\overline{D}) \\ Lu = 0 \quad \text{in} \quad D \\ u|_{\partial D} = f. \end{cases} \tag{33}$$

So if we define

$$u^h(w) = u(w)/h(w),$$

then

$$\begin{cases} u^h \in C^{2,\alpha}(\overline{D}) \\ L^h u^h = 0 \quad \text{in} \quad D \\ u^h|_{\partial D} = f/h. \end{cases} \tag{34}$$

Lemma 3.3. *We have*

$$\lim_{M\to\infty} \sup\{u(w):\ w \in D, w_1 \geq M\} = 0.$$

Proof. It is expedient to convert back to Brownian motion B in Ω. For $x = (\tilde{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, define

$$g(x) = f(|\tilde{x}|, x_{n+1}).$$

Then for $v(x) = E_x[g(B(\tau_\Omega))]$, it suffices to prove

$$\lim_{M\to\infty} \sup\{v(x):\ x \in \Omega, x_{n+1} \geq M\} = 0.$$

To this end, choose K so large that $\text{supp}(g) \subseteq B_K(0)$. For a Borel set E, we will use η_E to denote the first hitting time of E by B_t. If we set $G = \text{supp}(g) \cap \partial\Omega$ and take M large, then for $x \in \Omega$ with $x_{n+1} \geq M$,

$$\begin{aligned} v(x) &\leq (\sup|g|)P_x(B_{\tau_\Omega} \in G) \\ &\leq (\sup|g|)P_x(\eta_G < \infty) \\ &\leq (\sup|g|)P_x(\eta_{B_K(0)} < \infty) \\ &\leq (\sup|g|)\sup\{P_z(\eta_{B_K(0)} < \infty) : |z|^2 \geq M^2 + a(M)^2\}. \end{aligned}$$

It is well-known that

$$P_z(\eta_{B_K(0)} < \infty) = \left[\frac{|x|}{K}\right]^{2-d}$$

(see Example 7.4.2 in Oksendal (2007)), and so we have

$$\begin{aligned}
&\limsup_{M\to\infty} \sup\{v(x):\ x \in \Omega, x_{n+1} \geq M\} \\
&\qquad\qquad \leq (\sup|g|) \limsup_{M\to\infty} \left[\frac{M^2 + a(M)^2}{K^2}\right]^{(2-d)/2} \\
&\qquad\qquad = 0,
\end{aligned}$$

as desired. □

Let Δ_w denote the two-dimensional Laplacian in the variable (w_1, w_2). Now since

$$\frac{1}{2}\Delta_w G_D^h(y,w) = \left(L^h + \frac{p(p-1)}{2w_1^2}\right) G_D^h(y,w) = \frac{p(p-1)}{2w_1^2} G_D^h(y,w)$$

and

$$\frac{1}{2}\Delta_w u^h(w) = \left(L^h + \frac{p(p-1)}{2w_1^2}\right) u^h(w) = \frac{p(p-1)}{2w_1^2} u^h(w),$$

by Green's second identity,

$$\begin{aligned}
&-\int_{\partial E} \left[u^h(w)\left[\frac{\partial}{\partial n_w} G_D^h(y,w)\right] - G_D^h(y,w)\frac{\partial u^h}{\partial n_w}(w)\right]\sigma(dw) \\
&\quad = \int_E [u^h(w)\Delta_w G_D^h(y,w) - G_D^h(y,w)\Delta u^h(w)]dw \\
&\quad = 0, \qquad (35)
\end{aligned}$$

where, analogous to (16),

$$E = E(M,\varepsilon) = D \cap \{(\rho,z):\ \rho < M\} \cap \overline{B_\varepsilon(y)}^c. \qquad (36)$$

Then exactly as in §2 (cf. (17) and what follows), we will have that

$$u^h(y) = \frac{1}{2}\int_{\partial D} f^h(w)\left[\frac{\partial}{\partial n_w} G_D^h(y,w)\right]\sigma(dy) \qquad (37)$$

once we prove the analogues of (20)–(21):

$$\lim_{M\to\infty} \int_{\overline{D}\cap\pi_M} \left[u^h(w)\left[\frac{\partial}{\partial n_w} G_D^h(y,w)\right] - G_D^h(y,w)\frac{\partial u^h}{\partial n_w}(w)\right]\sigma(dw) = 0 \qquad (38)$$

and

$$\lim_{\varepsilon\to 0}\int\limits_{\partial B_\varepsilon(y)}\left[u^h(w)\left[\frac{\partial}{\partial n_w}G_D^h(y,w)\right]-G_D^h(y,w)\frac{\partial u^h}{\partial n_w}(w)\right]\sigma(dw)=2u^h(y). \tag{39}$$

Translating (37), we would then have

$$\frac{u(y)}{h(y)}=\frac{1}{2}\int\limits_{\partial D}\frac{f(w)}{h(w)}\left[\frac{\partial}{\partial n_w}G_D^h(y,w)\right]\sigma(dw),$$

which is equivalent to (32), since $u(y)=E_y[f(X_{\tau_D})]$.
It remains to verify (38) and (39).

Proof of (38). All the hard work was done in section 2. Taking $b(\rho,z)=-\frac{p(p-1)}{\rho^2}$, we have that b is bounded on $\overline{D}$ and for $z=(z_1,z_2)\in\overline{D}$ with $z_1\in(M-a(M),M+a(M))$,

$$|b(z)|\le\frac{C}{z_1^2}\le\frac{C}{[M-a(M)]^2}.$$

If also $w=(w_1,w_2)\in\overline{D}$ with $w_1\in(M-a(M),M+a(M))$ and $w\neq z$,

$$\begin{aligned}\frac{|b(w)-b(z)|}{|w-z|^\alpha}&\le C\frac{|w_1^2-z_1^2|}{w_1^2z_1^2|w-z|^\alpha}\\&\le C\frac{(w_1+z_1)|w_1-z_1|^{1-\alpha}}{w_1^2z_1^2}\\&\le C\frac{[a(M)+M][a(M)+M]^{1-\alpha}}{M^4[1-\frac{a(M)}{M}]^4}.\end{aligned}$$

Since $\frac{a(M)}{M}\to 0$ as $M\to\infty$, it follows that

$$\sup_{M\ge 1}a(M)^2\sup\{|b(z)|:\ z=(z_1,z_2),z_1\in(M-a(M),M+a(M))\}<\infty$$

and

$$\sup_{M\ge 1}a(M)^{\alpha+2}\sup\left\{\frac{|b(w)-b(z)|}{|w-z|^\alpha}:\ w,z\in\overline{D};w\neq z;w_1z_2\in(M-a(M),M+a(M))\right\}<\infty.$$

Now we can apply Lemma 2.6 to $\Delta+b=2L^h$ and $v=u^h$ or $G^h(y,\cdot)$ to get that for some $C>0$, for all large M, for $w\in\pi_M\cap\overline{D}$,

$$\begin{aligned}\left|\frac{\partial u^h}{\partial n_w}(w)\right|&\le Ca(M)^{-1}\sup\{|u^h(z)|:\ z\in\overline{D},M-a(M)<z_1<M+a(M)\} \qquad (40)\\&\le Ca(M)^{-1}[M+a(M)]^p\sup\{|u(z)|:\ z\in\overline{D},z_1>M-a(M)\}\end{aligned}$$

(since $u^h = u/h$)

$$\leq Ca(M)^{-1}M^p \sup_{z_1 > M - a(M)} |u(z)|$$

(since $a(M)/M \to 0$ as $M \to \infty$)

and

$$\left|\frac{\partial}{\partial n_w} G_D^h(y,w)\right| \leq Ca(M)^{-1} \sup\{G_D^h(y,z):\ z \in \overline{D}, M - a(M) < z_1 < M + a(M)\}, \qquad w \in \pi_M \cap D. \tag{41}$$

By Lemma 3.1, for large w_1, $G(y,w)$ is bounded (recall y is fixed), and so

$$\begin{aligned} G_D^h(y,w) &= G_D(y,w)h(w)/h(y) \\ &\leq G(y,w)h(w)/h(y) \\ &\leq Cw_1^{-p}, \qquad w_1 \text{ large.} \end{aligned}$$

Then using that $\frac{a(M)}{M} \to 0$ as $M \to \infty$, we have for large M,

$$\sup\{G_D^h(y,w):\ w \in \overline{D}, M - a(M) < w_1 < M + a(M)\} \leq CM^{-p}. \tag{42}$$

Combining this with (41), for large M, we get

$$\left|\frac{\partial}{\partial n_w} G_D^h(y,w)\right| \leq Ca(M)^{-1}M^{-p}, \qquad w \in \pi_M \cap D. \tag{43}$$

Since $|u^h(w)| \leq M^p \sup_{z_1 \geq M} |u(z)|$ for $w \in \pi_M \cap \overline{D}$, the inequalities (40), (42) and (43) yield that for all large M,

$$\left|u^h(w)\frac{\partial}{\partial n_w} G_D^h(y,w) - G_D^h(y,w)\frac{\partial u^h}{\partial n_w}(w)\right| \leq Ca(M)^{-1} \sup_{z_1 \geq M - a(M)} |u(z)|, \qquad w \in \pi_M \cap \overline{D}.$$

Thus as $M \to \infty$,

$$\begin{aligned} &\left|\int_{\overline{D}\cap\pi_M} \left[u^h(w)\frac{\partial}{\partial n_w} G_D^h(y,w) - G_D^h(y,w)\frac{\partial u^h}{\partial n_w}(w)\right] \sigma(dw)\right| \leq Ca(M)^{-1} \sup_{z_1 \geq M - a(M)} |u(z)| a(M) \\ &\qquad = C \sup_{z_1 \geq M - a(M)} |u(z)| \to 0, \end{aligned}$$

by Lemma 3.3. This completes the proof of (38). □

Proof of (39). Recall the $\frac{\partial}{\partial n_w}$ appearing in (39) is the inward normal derivative at $\partial B_\varepsilon(y)$. Since $u^h \in C^{2,\alpha}(\overline{D})$, $\frac{\partial u^h}{\partial n_w}$ is bounded on a neighborhood $\partial B_\varepsilon(y)$; then by (31),

$$\begin{aligned}\left|\int_{\partial B_\varepsilon(y)} G_D^h(y,w)\frac{\partial u^h}{\partial n_w}(w)\sigma(dw)\right| &\le C\log\frac{1}{\varepsilon}\sigma(\partial B_\varepsilon(y))\\ &= C\varepsilon\log\frac{1}{\varepsilon}\\ &\to 0 \quad \text{as} \quad \varepsilon\to 0.\end{aligned}$$

Thus to prove (39), it suffices to show

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(y)} u^h(w)\left[\frac{\partial}{\partial n_w}G_D^h(y,w)\right]\sigma(dw) = 2u^h(y). \tag{44}$$

The transition density $p_D(t,w,z)$ of X_t killed upon exiting D is given by

$$p_D(t,w,z) = p(t,w,z) - E_w[I_{\tau_D<t}p(t-\tau_D, X_{\tau_D}, z)],$$

where $p(t,w,z)$ is from (28). Then

$$\begin{aligned}G_D(w,z) &= \int_0^\infty p_D(t,w,z)dt\\ &= G(w,z) - E_w[G(X_{\tau_D},z)],\end{aligned}$$

where G is from Lemma 3.1. Writing

$$G^h(w,z) = G(w,z)h(z)/h(w),$$

we get

$$\begin{aligned}G_D^h(w,z) &= G_D(w,z)h(z)/h(w)\\ &= G^h(w,z) - E_w[G^h(X_{\tau_D},z)h(X_{\tau_D})]/h(w).\end{aligned} \tag{45}$$

Lemma 3.4. *a) For $\varepsilon > 0$ so small that $\overline{B_{2\varepsilon}(y)} \subseteq D$,*

$$\sup\{|\nabla_w G^h(z,w)|:\ w\in B_{2\varepsilon}(y), z\in\partial D\} < \infty.$$

b) For $\frac{\partial}{\partial n_w}$ denoting inward normal differentiation at the boundary of $B_\varepsilon(y)$,

$$\sup_{w\in\partial B_\varepsilon(y)}\left|\frac{\partial}{\partial n_w}G^h(y,w)\right| \le C\varepsilon^{-1}, \quad \varepsilon \text{ small}$$

and

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(y)}\frac{\partial}{\partial n_w}G^h(y,w)\sigma(dw) = 2 \qquad \square$$

We defer the technical proof to the next subsection. Differentiating (45) and using Lemma 3.4a to justify differentiation inside the expectation,

$$\frac{\partial}{\partial n_w} G_D^h(y,w) = \frac{\partial}{\partial n_w} G^h(y,w) - E_y \left[\left[\frac{\partial}{\partial n_w} G^h(X_{\tau_D}, w) \right] h(X_{\tau_D})/h(y) \right].$$

Moreover, since u^h is bounded near y and since h is bounded on $\overline{D}$, part a) of Lemma 3.4 also implies that as $\varepsilon \to 0$,

$$\begin{aligned} \int_{\partial B_\varepsilon(y)} & u^h(w) E_y \left[\left[\frac{\partial}{\partial n_w} G^h(X_{\tau_D}, w) \right] h(X_{\tau_D})/h(y) \right] \sigma(dy) \\ & \leq \frac{C}{h(y)} \sigma(\partial B_\varepsilon(y)) \\ & \to 0. \end{aligned}$$

Thus by (45), (44) comes down to showing

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(y)} u^h(w) \left[\frac{\partial}{\partial n_w} G^h(y,w) \right] \sigma(dw) = 2u^h(y).$$

Rewriting the integral as

$$\int_{\partial B_\varepsilon(y)} [u^h(w) - u^h(y)] \left[\frac{\partial}{\partial n_w} G^h(y,w) \right] \sigma(dw) + u^h(y) \int_{\partial B_\varepsilon(y)} \left[\frac{\partial}{\partial n_w} G^h(y,w) \right] \sigma(dw),$$

by part b) of Lemma 3.4, the first integral is bounded by $C \sup_{w \in \partial B_\varepsilon(y)} |u^h(w) - u^h(y)|$, which converges to 0 as $\varepsilon \to 0$. Moreover, the limit in that part of the lemma also implies that the second integral converges to $2u^h(y)$ as $\varepsilon \to 0$. This completes the proof of (39). □

3.3 Proof of Lemma 3.4

Recall $G^h(z,w) = G(z,w)h(w)/h(z)$. By Lemma 3.1, writing

$$\gamma = \gamma(z,w) = \frac{2w_1 z_1}{w_1^2 + z_1^2 + (w_2 - z_2)^2}$$

and

$$H(x) = \frac{1}{2\pi} \frac{\Gamma(\frac{n-1}{4})\Gamma(\frac{n+1}{4})}{\Gamma(\frac{n}{2})} F\left(\frac{n-1}{4}, \frac{n+1}{4}; \frac{n}{2}; x \right),$$

we have

$$G^h(z,w) = \gamma^{\frac{n-1}{2}} H(\gamma^2). \tag{46}$$

Taking the gradient of this gives

$$\nabla_w G^h(z,w) = \left[\frac{n-1}{2} \gamma^{\frac{n-1}{2}} H(\gamma^2) + 2\gamma^{\frac{n-1}{2}+2} H'(\gamma^2) \right] \frac{\nabla_w \gamma}{\gamma}. \tag{47}$$

Notice

$$\frac{\partial\gamma}{\partial w_1} = \frac{\gamma}{w_1}\frac{|w-z|^2 - 2w_1(w_1-z_1)}{w_1^2+z_1^2+(w_2-z_2)^2} \tag{48}$$

$$\frac{\partial\gamma}{\partial w_2} = -2\gamma\frac{w_2-z_2}{w_1^2+z_1^2+(w_2-z_2)^2}. \tag{49}$$

Proof of Lemma 3.4 a). Since $\overline{B_{2\varepsilon}(y)} \subseteq D$,

$$\gamma_1 := \inf\{w_1 :\ w \in \overline{B_{2\varepsilon}(y)}\} \geq \frac{1}{2} > 0$$

and

$$\gamma_2 := \inf\{|z-w| :\ z \in \partial D, w \in B_{2\varepsilon}(y)\} > 0.$$

Using the identity

$$w_1^2 + z_1^2 + (w_2-z_2)^2 = |w-z|^2 + 2z_1 w_1,$$

for $z \in \partial D$ and $w \in B_{2\varepsilon}(y)$ we have from (48)–(49)

$$\begin{aligned}\left|\frac{1}{\gamma}\frac{\partial\gamma}{\partial w_1}\right| &\leq \frac{1}{w_1}\left[\frac{|w-z|^2 + 2w_1|w_1-z_1|}{|w-z|^2+2z_1w_1}\right]\\ &\leq \frac{1}{w_1} + \frac{2|w_1-z_1|}{|w-z|^2}\\ &\leq \frac{1}{\gamma_1} + \frac{2}{\gamma_2}\end{aligned}$$

and

$$\left|\frac{1}{\gamma}\frac{\partial\gamma}{\partial w_2}\right| \leq 2\frac{|w_2-z_2|}{|w-z|^2+2z_1w_1} \leq \frac{2}{\gamma_2}.$$

Thus

$$\sup\left\{\left|\frac{\nabla_w\gamma}{\gamma}(z,w)\right| :\ z \in \partial D, w \in B_{2\varepsilon}(y)\right\} < \infty. \tag{50}$$

Also, for $w \in \overline{B_{2\varepsilon}(y)}$ and $z \in \partial D$ with $z_1 \geq 4(y_1 + 2\varepsilon)$,

$$\begin{aligned}\gamma(z,w) &\leq \frac{2w_1}{z_1}\\ &\leq \frac{2(y_1+2\varepsilon)}{z_1}\\ &\leq \frac{1}{2}.\end{aligned}$$

Since $\gamma(z,w) = 1$ iff $z = w$ and since $\overline{B_{2\varepsilon}(y)} \subseteq D$, we see that

$$\sup\{\gamma(z,w) :\ w \in B_{2\varepsilon}(y), z \in \partial D, z_1 \leq 4(y_1+2\varepsilon)\} < 1.$$

Combining the last two bounds yields

$$\sup\{(\gamma(z,w):\ w \in B_{2\varepsilon}(y), z \in \partial D\} < 1.$$

Then by the series representation for the hypergeometric function F from Lemma 3.1, it follows that $H(\gamma^2)$ and $H'(\gamma^2)$ are bounded for $z \in \partial D$ and $w \in B_{2\varepsilon}(y)$. Together with (50) and (47), this implies that $\nabla_w G^h(z,w)$ is bounded for $z \in \partial D$ and $w \in B_{2\varepsilon}(y)$, as desired. □

Proof of Lemma 3.4 b). By formula 15.3.10 on page 559 in Abramowitz and Stegun (1972),

$$H(x) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{n-1}{4}+k)\Gamma(\frac{n+1}{4}+k)}{\Gamma(\frac{n-1}{4})\Gamma(\frac{n+1}{4})(k!)^2}\left[2\psi(k+1) - \psi\left(\frac{n-1}{4}+k\right) - \psi\left(\frac{n+1}{4}+k\right)\right.$$
$$\left. - \ln(1-x)\right](1-x)^k,$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the Psi function and the interval of convergence is $0 < x < 1$. Using the asymptotic relation

$$\psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to \infty$$

(Abramowitz and Stegun (19792), 6.3.18 on page 259), one can see that $2\psi(k+1) - \psi(\frac{n-1}{4}+k) - \psi(\frac{n+1}{4}+k)$ is bounded in k. By Stirling's formula, for some positive A and C,

$$\frac{\Gamma(\frac{n-1}{4}+k)\Gamma(\frac{n+1}{4}+k)}{\Gamma(\frac{n-1}{4})\Gamma(\frac{n+1}{4})(k!)^2} \le Ck^A, \quad k \text{ large.}$$

Thus we have as $x \to 1^+$,

$$H(x) = -\frac{1}{2\pi}[1+o(1)]\ln(1-x)$$

and

$$H'(x) = \frac{1}{2\pi}[1+o(1)]\frac{1}{1-x}.$$

Together these imply that as $x \to 1^+$,

$$\begin{aligned}
&\frac{n-1}{2}x^{\frac{n-1}{2}}H(x^2) + 2x^{\frac{n-1}{2}+2}H'(x^2) \\
&\quad = \frac{1}{2\pi(1-x)}\left[-\frac{n-1}{2}x^{\frac{n-1}{2}}[1+o(1)](1-x)\ln(1-x^2) + \frac{2}{1+x}x^{\frac{n-1}{2}+2}[1+o(1)]\right] \\
&\quad = \frac{1}{2\pi(1-x)}[1+o(1)]. \qquad (51)
\end{aligned}$$

Note that for $w \in \partial B_\varepsilon(y)$,

$$\begin{aligned}
\gamma(y,w) &= \frac{2y_1w_1}{w_1^2 + y_1^2 + (w_2-y_2)^2} \\
&= \frac{2y_1w_1}{|w-y|^2 + 2y_1w_1} \\
&= \frac{2y_1w_1}{\varepsilon^2 + 2y_1w_1} \\
&= 1 - |o(1)| \qquad (52)
\end{aligned}$$

as $\varepsilon \to 0$, uniformly for $w \in \partial B_\varepsilon(y)$.

From (48)–(49), for $w \in \partial B_\varepsilon(y)$,

$$\nabla_w \gamma(y,w) = \frac{\gamma(y,w)}{[w_1^2 + y_1^2 + (w_2-y_2)^2]} \left(\frac{\varepsilon^2 - 2w_1(w_1-y_1)}{w_1}, -2(w_2-y_2) \right)$$
$$= \frac{\gamma(y,w)}{\varepsilon^2 + 2y_1 w_1} \left(\frac{\varepsilon^2 - 2w_1(w_1-y_1)}{w_1}, -2(w_2-y_2) \right).$$

As a consequence, also using that $|w-z|^2 = \varepsilon^2$,

$$\begin{aligned} \frac{\partial}{\partial n_w}\gamma(y,w) &= \frac{y-w}{\varepsilon} \cdot \nabla_w \gamma(y,w) \\ &= \frac{\gamma(y,w)}{\varepsilon^2 + 2y_1 w_1} \frac{1}{\varepsilon} \left[\varepsilon^2 \frac{y_1 - w_1}{w_1} + 2(w_1-y_1)^2 + 2(w_2-y_2)^2 \right] \\ &= \frac{\gamma(y,w)}{\varepsilon^2 + 2y_1 w_1} \varepsilon \left[\frac{y_1 - w_1}{w_1} + 2 \right]. \end{aligned} \tag{53}$$

Using (52) in (51), as $\varepsilon \to 0$, uniformly for $w \in \partial B_\varepsilon(y)$, for $\gamma = \gamma(y,w)$ we have

$$\begin{aligned} \frac{n-1}{2}\gamma^{\frac{n-1}{2}} H(\gamma^2) + 2\gamma^{\frac{n-1}{2}+2} H'(\gamma^2) &= \frac{1}{2\pi(1-\gamma)}[1+o(1)] \\ &= \frac{(w_1^2 + y_1^2 + (w_2-y_2)^2)}{2\pi\varepsilon^2}[1+o(1)] \end{aligned}$$

(since $|w-y|^2 = \varepsilon^2$)

$$= \frac{\varepsilon^2 + 2y_1 w_1}{2\pi\varepsilon^2}[1+o(1)].$$

Together with (53) we use this in (47) to get that as $\varepsilon \to 0$, uniformly for $w \in \partial B_\varepsilon(y)$,

$$\begin{aligned} \frac{\partial}{\partial n_w} G^h(y,w) &= \left[\frac{n-1}{2}\gamma^{\frac{n-1}{2}} H(\gamma^2) + 2\gamma^{\frac{n-1}{2}+2} H'(\gamma^2) \right] \frac{1}{\gamma} \frac{\partial \gamma}{\partial n_w} \\ &= \frac{\varepsilon^2 + 2y_1 w_1}{2\pi\varepsilon^2} \frac{\varepsilon}{\varepsilon^2 + 2y_1 w_1} \left[\frac{y_1 - w_1}{w_1} + 2 \right] [1+o(1)] \\ &= \frac{1}{2\pi\varepsilon} \left[\frac{y_1 - w_1}{w_1} + 2 \right] [1+o(1)] \\ &= \frac{1}{2\pi\varepsilon}[2+o(1)]. \end{aligned}$$

This immediately yields the bound given in part b) of Lemma 3.4; moreover,

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(y)} \frac{\partial}{\partial n_w} G^h(y,w)\sigma(dw) &= \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(y)} \frac{1}{2\pi\varepsilon}[2+o(1)]\sigma(dw) \\ &= 2, \end{aligned}$$

as desired. □

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