Electronic Journal of Probability

Vol. 12 (2007), Paper no. 18, pages 516–573.

Journal URL
http://www.math.washington.edu/~ejpecp/

Intermittency on catalysts: symmetric exclusion*

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Abstract

We continue our study of intermittency for the parabolic Anderson equation $\partial u/\partial t = \kappa\Delta u + \xi u$, where $u\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, κ is the diffusion constant, Δ is the discrete Laplacian, and $\xi\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is a space-time random medium. The solution of the equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ.

In this paper we focus on the case where ξ is exclusion with a symmetric random walk transition kernel, starting from equilibrium with density $\rho \in (0,1)$. We consider the annealed

*The research in this paper was partially supported by the DFG Research Group 718 "Analysis and Stochastics in Complex Physical Systems" and the ESF Scientific Programme "Random Dynamics in Spatially Extended Systems". GM was supported by a postdoctoral fellowship from the Netherlands Organization for Scientific Research (grant 613.000.307) during his stay at EURANDOM. FdH and GM are grateful to the Pacific Institute for the Mathematical Sciences and the Mathematics Department of the University of British Columbia, Vancouver, Canada, for hospitality: FdH January to August 2006, GM mid-January to mid-February 2006 when part of this paper was completed.

Lyapunov exponents, i.e., the exponential growth rates of the successive moments of u. We show that these exponents are trivial when the random walk is recurrent, but display an interesting dependence on the diffusion constant κ when the random walk is transient, with qualitatively different behavior in different dimensions. Special attention is given to the asymptotics of the exponents for $\kappa \to \infty$, which is controlled by moderate deviations of ξ requiring a delicate expansion argument.
In Gärtner and den Hollander (10) the case where ξ is a Poisson field of independent (simple) random walks was studied. The two cases show interesting differences and similarities. Throughout the paper, a comparison of the two cases plays a crucial role.

Key words: Parabolic Anderson model, catalytic random medium, exclusion process, Lyapunov exponents, intermittency, large deviations, graphical representation.

AMS 2000 Subject Classification: Primary 60H25, 82C44; Secondary: 60F10, 35B40.

Submitted to EJP on September 4 2006, final version accepted February 14 2007.

1 Introduction and main results

1.1 Model

The parabolic Anderson equation is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{1.1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.1.2}$$

($\|\cdot\|$ is the Euclidian norm), while

$$\xi = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d,\ t \geq 0\} \tag{1.1.3}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation. As initial condition for (1.1.1) we take

$$u(\cdot,0) \equiv 1. \tag{1.1.4}$$

Equation (1.1.1) is a discrete heat equation with the ξ-field playing the role of a source. What makes (1.1.1) particularly interesting is that the two terms in the right-hand side *compete with each other*: the diffusion induced by Δ tends to make u flat, while the branching induced by ξ tends to make u irregular. *Intermittency* means that for large t the branching dominates, i.e., the u-field develops sparse high peaks in such a way that u and its moments are each dominated by their own collection of peaks (see Gärtner and König (11), Section 1.3, and den Hollander (10), Section 1.2). In the quenched situation this geometric picture of intermittency is well understood for several classes of *time-independent* random potentials ξ (see Sznitman (21) for Poisson clouds and Gärtner, König and Molchanov (12) for i.i.d. potentials with double-exponential and heavier upper tails). For *time-dependent* random potentials ξ, however, such results are not yet available. Instead one restricts attention to understanding the phenomenon of intermittency indirectly by comparing the successive annealed Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log\langle u(0,t)^p\rangle^{1/p}, \qquad p = 1,2,\ldots \tag{1.1.5}$$

One says that the solution u is *p-intermittent* if the *strict* inequality

$$\lambda_p > \lambda_{p-1} \tag{1.1.6}$$

holds. For a geometric interpretation of this definition, see (11), Section 1.3.

In their fundamental paper (3), Carmona and Molchanov succeeded to investigate the annealed Lyapunov exponents and to draw the qualitative picture of intermittency for potentials of the form

$$\xi(x,t) = \dot{W}_x(t), \tag{1.1.7}$$

where $\{W_x,\ x \in \mathbb{Z}^d\}$ denotes a collection of independent Brownian motions. (In this important case, equation (1.1.1) corresponds to an infinite system of coupled Îto diffusions.) They showed that for $d = 1, 2$ intermittency of all orders is present for all κ, whereas for $d \geq 3$ p-intermittency holds if and only if the diffusion constant κ is smaller than a critical threshold κ_p^* tending to infinity as $p \to \infty$. They also studied the asymptotics of the quenched Lyapunov exponent in the limit as $\kappa \downarrow 0$, which turns out to be singular. Subsequently, the latter was more thoroughly investigated in papers by Carmona, Molchanov and Viens (4), Carmona, Koralov and Molchanov (2), and Cranston, Mountford and Shiga (6), (7).

In the present paper we study a different model, describing the spatial evolution of moving *reactants* under the influence of moving *catalysts*. In this model, the potential has the form

$$\xi(x,t) = \sum_k \delta_{Y_k(t)}(x) \tag{1.1.8}$$

with $\{Y_k,\, k \in \mathbb{N}\}$ a collection of catalyst particles performing a space-time homogeneous reversible particle dynamics with hard core repulsion, and $u(x,t)$ describes the concentration of the reactant particles given the motion of the catalyst particles. We will see later that the study of the annealed Lyapunov exponents leads to *different* dimension effects and requires the development of *different* techniques than in the white noise case (1.1.7). Indeed, because of the *non-Gaussian* nature and the *non-independent* spatial structure of the potential, it is far from obvious how to tackle the computation of Lyapunov exponents.

Let us describe our model in more detail. We consider the case where ξ is *Symmetric Exclusion* (SE), i.e., ξ takes values in $\{0,1\}^{\mathbb{Z}^d} \times [0,\infty)$, where $\xi(x,t) = 1$ means that there is a particle at x at time t and $\xi(x,t) = 0$ means that there is none, and particles move around according to a symmetric random walk transition kernel. We choose $\xi(\cdot, 0)$ according to the Bernoulli product measure with density $\rho \in (0,1)$, i.e., initially each site has a particle with probability ρ and no particle with probability $1-\rho$, independently for different sites. For this choice, the ξ-field is stationary in time.

One interpretation of (1.1.1) and (1.1.4) comes from population dynamics. Consider a spatially homogeneous system of two types of particles, A (catalyst) and B (reactant), subject to:

(i) A-particles behave autonomously, according to a prescribed stationary dynamics, with density ρ;

(ii) B-particles perform independent random walks with diffusion constant κ and split into two at a rate that is equal to the number of A-particles present at the same location;

(iii) the initial density of B-particles is 1.

Then

$$u(x,t) \;=\; \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t\\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{1.1.9}$$

It is possible to add that B-particles die at rate $\delta \in (0,\infty)$. This amounts to the trivial transformation $u(x,t) \to u(x,t)e^{-\delta t}$.

In Kesten and Sidoravicius (16) and in Gärtner and den Hollander (10), the case was considered where ξ is given by a Poisson field of independent simple random walks. The survival versus extinction pattern (in (16) for $\delta > 0$) and the annealed Lyapunov exponents (in (10) for $\delta = 0$) were studied, in particular, their dependence on d, κ and the parameters controlling ξ.

1.2 SE, Lyapunov exponents and comparison with IRW

Throughout the paper, we abbreviate $\Omega = \{0,1\}^{\mathbb{Z}^d}$ (endowed with the product topology), and we let $p\colon \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ be the transition kernel of an irreducible random walk,

$$\begin{aligned} &p(x,y) = p(0,y-x) \geq 0 \ \ \forall\, x,y \in \mathbb{Z}^d, \quad \sum_{y\in\mathbb{Z}^d} p(x,y) = 1 \ \ \forall\, x \in \mathbb{Z}^d, \\ &p(x,x) = 0 \ \ \forall\, x \in \mathbb{Z}^d, \quad p(\cdot,\cdot) \text{ generates } \mathbb{Z}^d, \end{aligned} \tag{1.2.1}$$

that is assumed to be *symmetric*,

$$p(x,y) = p(y,x) \ \ \forall\, x,y \in \mathbb{Z}^d. \tag{1.2.2}$$

A special case is *simple* random walk

$$p(x,y) = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.3}$$

The exclusion process ξ is the Markov process on Ω whose generator L acts on cylindrical functions f as (see Liggett (19), Chapter VIII)

$$(Lf)(\eta) = \sum_{x,y\in\mathbb{Z}^d} p(x,y)\,\eta(x)[1-\eta(y)]\left[f\left(\eta^{x,y}\right) - f(\eta)\right] = \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)\,\left[f\left(\eta^{x,y}\right) - f(\eta)\right], \tag{1.2.4}$$

where the latter sum runs over unoriented bonds $\{x,y\}$ between any pair of sites $x,y \in \mathbb{Z}^d$, and

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases} \tag{1.2.5}$$

The first equality in (1.2.4) says that a particle at site x jumps to a vacancy at site y at rate $p(x,y)$, the second equality says that the states of x and y are interchanged along the bond $\{x,y\}$ at rate $p(x,y)$. For $\rho \in [0,1]$, let ν_ρ be the Bernoulli product measure on Ω with density ρ. This is an invariant measure for SE. Under (1.2.1–1.2.2), $(\nu_\rho)_{\rho\in[0,1]}$ are the only extremal equilibria (see Liggett (19), Chapter VIII, Theorem 1.44). We denote by $\mathbb{P}_\eta$ the law of ξ starting from $\eta \in \Omega$ and write $\mathbb{P}_{\nu_\rho} = \int_\Omega \nu_\rho(d\eta)\,\mathbb{P}_\eta$.

In the *graphical representation* of SE, space is drawn sidewards, time is drawn upwards, and for each pair of sites $x,y \in \mathbb{Z}^d$ links are drawn between x and y at Poisson rate $p(x,y)$. The configuration at time t is obtained from the one at time 0 by transporting the local states along paths that move upwards with time and sidewards along links (see Fig. 1).

We will frequently use the following property, which is immediate from the graphical representation:

$$\mathbb{E}_\eta\left(\xi(y,t)\right) = \sum_{x\in\mathbb{Z}^d} \eta(x)\, p_t(x,y), \qquad \eta \in \Omega,\ y \in \mathbb{Z}^d,\ t \geq 0. \tag{1.2.6}$$

Similar expressions hold for higher order correlations. Here, $p_t(x,y)$ is the probability that the random walk with transition kernel $p(\cdot,\cdot)$ and step rate 1 moves from x to y in time t.

The graphical representation shows that the evolution is invariant under time reversal and, in particular, the equilibria $(\nu_\rho)_{\rho\in[0,1]}$ are *reversible.* This fact will turn out to be very important later on.

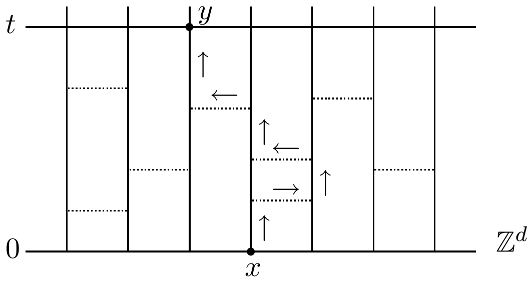

Fig. 1: Graphical representation. The dashed lines are links.
The arrows represent a path from $(x,0)$ to (y,t).

By the Feynman-Kac formula, the solution of (1.1.1) and (1.1.4) reads

$$u(x,t) = E_x\left(\exp\left[\int_0^t ds\ \xi\left(X^\kappa(s), t-s\right)\right]\right), \tag{1.2.7}$$

where X^κ is simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$ and E_x denotes expectation with respect to X^κ given $X^\kappa(0)=x$. We will often write $\xi_t(x)$ and X_t^κ instead of $\xi(x,t)$ and $X^\kappa(t)$, respectively.

For $p\in\mathbb{N}$ and $t>0$, define

$$\Lambda_p(t) = \frac{1}{pt}\log \mathbb{E}_{\nu_\rho}\left(u(0,t)^p\right). \tag{1.2.8}$$

Then

$$\Lambda_p(t) = \frac{1}{pt}\log \mathbb{E}_{\nu_\rho}\left(E_{0,\ldots,0}\left(\exp\left[\int_0^t ds \sum_{q=1}^p \xi\big(X_q^\kappa(s), s\big)\right]\right)\right), \tag{1.2.9}$$

where X_q^κ, $q=1,\ldots,p$, are p independent copies of X^κ, $E_{0,\ldots,0}$ denotes expectation w.r.t. $X_q^\kappa, q=1,\ldots,p$, given $X_1^\kappa(0)=\cdots=X_p^\kappa(0)=0$, and the time argument $t-s$ in (1.2.7) is replaced by s in (1.2.9) via the reversibility of ξ starting from ν_ρ. If the last quantity admits a limit as $t\to\infty$, then we define

$$\lambda_p = \lim_{t\to\infty}\Lambda_p(t) \tag{1.2.10}$$

to be the p-th *annealed Lyapunov exponent.*

From Hölder's inequality applied to (1.2.8) it follows that $\Lambda_p(t)\geq\Lambda_{p-1}(t)$ for all $t>0$ and $p\in\mathbb{N}\setminus\{1\}$. Hence $\lambda_p\geq\lambda_{p-1}$ for all $p\in\mathbb{N}\setminus\{1\}$. As before, we say that the system is p-*intermittent* if $\lambda_p>\lambda_{p-1}$. In the latter case the system is q-intermittent for all $q>p$ as well (cf. Gärtner and Molchanov (13), Section 1.1). We say that the system is *intermittent* if it is p-intermittent for all $p\in\mathbb{N}\setminus\{1\}$.

Let $(\tilde\xi_t)_{t\geq 0}$ be the process of *Independent Random Walks* (IRW) with step rate 1, transition kernel $p(\cdot,\cdot)$ and state space $\widetilde\Omega=\mathbb{N}_0^{\mathbb{Z}^d}$ with $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Let $\mathbb{E}_\eta^{\mathrm{IRW}}$ denote expectation w.r.t. $(\tilde\xi_t)_{t\geq0}$ starting from $\tilde\xi_0=\eta\in\Omega$, and write $\mathbb{E}_{\nu_\rho}^{\mathrm{IRW}}=\int_\Omega \nu_\rho(d\eta)\,\mathbb{E}_\eta^{\mathrm{IRW}}$. Throughout the paper we will make use of the following inequality comparing SE and IRW. The proof of this inequality is given in Appendix A and uses a lemma due to Landim (18).

Proposition 1.2.1. *For any* $K\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ *such that either* $K \geq 0$ *or* $K \leq 0$*, any* $t \geq 0$ *such that* $\sum_{z\in\mathbb{Z}^d}\int_0^t ds\,|K(z,s)| < \infty$ *and any* $\eta \in \Omega$*,*

$$\mathbb{E}_\eta\Bigg(\exp\Bigg[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\;K(z,s)\,\xi_s(z)\Bigg]\Bigg) \leq \mathbb{E}_\eta^{\mathrm{IRW}}\Bigg(\exp\Bigg[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\;K(z,s)\,\tilde{\xi}_s(z)\Bigg]\Bigg). \tag{1.2.11}$$

This powerful inequality will allow us to obtain bounds that are more easily computable.

1.3 Main theorems

Our first result is standard and states that the Lyapunov exponents exist and behave nicely as a function of κ. We write $\lambda_p(\kappa)$ to exhibit the dependence on κ, suppressing d and ρ.

Theorem 1.3.1. *Let* $d \geq 1$*,* $\rho \in (0,1)$ *and* $p \in \mathbb{N}$*.*
(i) For all $\kappa \in [0,\infty)$*, the limit in* (1.2.10) *exists and is finite.*
(ii) On $[0,\infty)$*,* $\kappa \to \lambda_p(\kappa)$ *is continuous, non-increasing and convex.*

Our second result states that the Lyapunov exponents are trivial for recurrent random walk but are non-trivial for transient random walk (see Fig. 2), without any further restriction on $p(\cdot,\cdot)$.

Theorem 1.3.2. *Let* $d \geq 1$*,* $\rho \in (0,1)$ *and* $p \in \mathbb{N}$*.*
(i) If $p(\cdot,\cdot)$ *is recurrent, then* $\lambda_p(\kappa) = 1$ *for all* $\kappa \in [0,\infty)$*.*
(ii) If $p(\cdot,\cdot)$ *is transient, then* $\rho < \lambda_p(\kappa) < 1$ *for all* $\kappa \in [0,\infty)$*. Moreover,* $\kappa \mapsto \lambda_p(\kappa)$ *is strictly decreasing with* $\lim_{\kappa\to\infty}\lambda_p(\kappa) = \rho$*.*

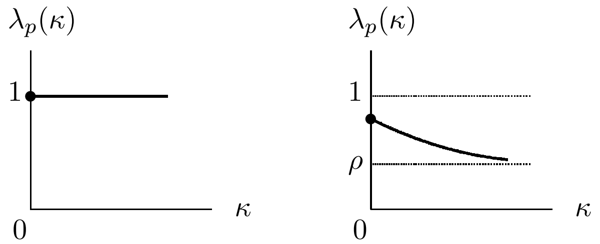

Fig. 2: Qualitative picture of $\kappa \mapsto \lambda_p(\kappa)$ for recurrent, respectively, transient random walk.

Our third result shows that for transient random walk the system is intermittent for *small* κ.

Theorem 1.3.3. *Let* $d \geq 1$ *and* $\rho \in (0,1)$*. If* $p(\cdot,\cdot)$ *is transient, then there exists* $\kappa_0 \in (0,\infty]$ *such that* $p \mapsto \lambda_p(\kappa)$ *is strictly increasing for* $\kappa \in [0,\kappa_0)$*.*

Our fourth and final result identifies the behavior of the Lyapunov exponents for *large* κ when $d \geq 4$ and $p(\cdot,\cdot)$ is simple random walk (see Fig. 3).

Theorem 1.3.4. *Assume* (1.2.3)*. Let* $d \geq 4$*,* $\rho \in (0,1)$ *and* $p \in \mathbb{N}$ *. Then*

$$\lim_{\kappa\to\infty}\kappa[\lambda_p(\kappa) - \rho] = \frac{1}{2d}\rho(1-\rho)G_d \tag{1.3.1}$$

with G_d *the Green function at the origin of simple random walk on* $\mathbb{Z}^d$*.*

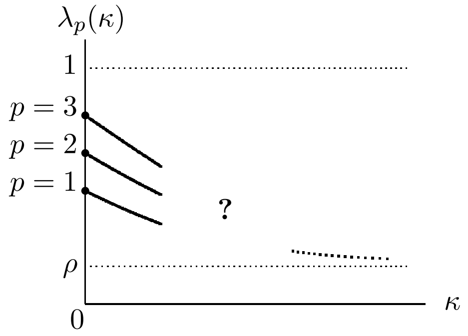

Fig. 3: Qualitative picture of $\kappa \mapsto \lambda_p(\kappa)$ for $p = 1, 2, 3$ for simple random walk in $d \geq 4$. The dotted line moving down represents the asymptotics given by the r.h.s. of (1.3.1).

1.4 Discussion

Theorem 1.3.1 gives general properties that need no further comment. We will see that they in fact hold for *any stationary, reversible and bounded* ξ.

The intuition behind Theorem 1.3.2 is the following. If the catalyst is driven by a recurrent random walk, then it suffers from "traffic jams", i.e., with not too small a probability there is a large region around the origin that the catalyst fully occupies for a long time. Since with not too small a probability the simple random walk (driving the reactant) can stay inside this large region for the same amount of time, the average growth rate of the reactant at the origin is maximal. This phenomenon may be expressed by saying that *for recurrent random walk clumping of the catalyst dominates the growth of the moments*. For transient random walk, on the other hand, clumping of the catalyst is present (the growth rate of the reactant is $> \rho$), but it is *not* dominant (the growth rate of the reactant is < 1). As the diffusion constant κ of the reactant increases, the effect of the clumping of the catalyst gradually diminishes and the growth rate of the reactant gradually decreases to the density of the catalyst.

Theorem 1.3.3 shows that if the reactant stands still and the catalyst is driven by a transient random walk, then the system is intermittent. Apparently, the successive moments of the reactant, which are equal to the exponential moments of the occupation time of the origin by the catalyst (take (1.2.7) with $\kappa = 0$), are sensitive to *successive degrees of clumping*. By continuity, *intermittency persists for small* κ.

Theorem 1.3.4 shows that, when the catalyst is driven by simple random walk, *all* Lyapunov exponents decay to ρ as $\kappa \to \infty$ *in the same manner* when $d \geq 4$. The case $d = 3$ remains open. We conjecture:

Conjecture 1.4.1. *Assume* (1.2.3). *Let* $d = 3$, $\rho \in (0, 1)$ *and* $p \in \mathbb{N}$. *Then*

$$\lim_{\kappa \to \infty} \kappa[\lambda_p(\kappa) - \rho] = \frac{1}{2d}\rho(1-\rho)G_d + [2d\rho(1-\rho)p]^2\mathcal{P} \tag{1.4.1}$$

with

$$\mathcal{P} = \sup_{\substack{f \in H^1(\mathbb{R}^3) \\ \|f\|_2 = 1}} \left[\left\| (-\Delta_{\mathbb{R}^3})^{-1/2} \ f^2 \right\|_2^2 - \|\nabla_{\mathbb{R}^3} f\|_2^2 \right] \in (0, \infty), \tag{1.4.2}$$

where $\nabla_{\mathbb{R}^3}$ and $\Delta_{\mathbb{R}^3}$ are the continuous gradient and Laplacian, $\|\cdot\|_2$ is the $L^2(\mathbb{R}^3)$-norm, $H^1(\mathbb{R}^3) = \{f\colon \mathbb{R}^3 \to \mathbb{R}\colon f, \nabla_{\mathbb{R}^3} f \in L^2(\mathbb{R}^3)\}$, and

$$\left\|(-\Delta_{\mathbb{R}^3})^{-1/2} f^2\right\|_2^2 = \int_{\mathbb{R}^3} dx\, f^2(x) \int_{\mathbb{R}^3} dy\, f^2(y)\, \frac{1}{4\pi\|x-y\|}. \tag{1.4.3}$$

In Section 1.5 we will explain how this conjecture arises in analogy with the case of IRW studied in Gärtner and den Hollander (10). If Conjecture 1.4.1 holds true, then in $d = 3$ *intermittency persists for large κ*. It would still remain open whether the same is true for $d \geq 4$. To decide the latter, we need a finer asymptotics for $d \geq 4$. A large diffusion constant of the reactant hampers the localization of u around the regions where the catalyst clumps, but it is not a priori clear whether this is able to destroy intermittency for $d \geq 4$.

We further conjecture:

Conjecture 1.4.2. *In $d = 3$, the system is intermittent for all $\kappa \in [0, \infty)$.*

Conjecture 1.4.3. *In $d \geq 4$, there exists a strictly increasing sequence $0 < \kappa_2 < \kappa_3 < \dots$ such that for $p = 2, 3, \dots$ the system is p-intermittent if and only if $\kappa \in [0, \kappa_p)$.*

In words, we conjecture that in $d = 3$ the curves in Fig. 3 never merge, whereas for $d \geq 4$ the curves merge successively.

Let us briefly compare our results for the simple symmetric exclusion dynamics with those of the IRW dynamics studied in (10). If the catalysts are moving freely, then they can accumulate with a not too small probability at single lattice sites. This leads to a double-exponential growth of the moments for $d = 1, 2$. The same is true for $d \geq 3$ for certain choices of the model parameters ('strongly catalytic regime'). Otherwise the annealed Lyapunov exponents are finite ('weakly catalytic regime'). For our exclusion dynamics, there can be at most one catalytic particle per site, leading to the degenerate behavior for $d = 1, 2$ (i.e., the recurrent case) as stated in Theorem 1.3.2(i). For $d \geq 3$, the large κ behavior of the annealed Lyapunov exponents turns out to be the same as in the weakly catalytic regime for IRW. The proof of Theorem 1.3.4 will be carried out in Section 4 essentially by 'reducing' its assertion to the corresponding statement in (10), as will be explained in Section 1.5. The reduction is highly technical, but seems to indicate a degree of 'universality' in the behavior of a larger class of models.

Finally, let us explain why we cannot proceed directly along the lines of (10). In that paper, the key is a Feynman-Kac representation of the moments. For the first moment, for instance, we have

$$\langle u(0,t)\rangle = e^{\nu t} E_0\left(\exp\left[\nu \int_0^t w(X(s), s)\, ds\right]\right), \tag{1.4.4}$$

where X is simple random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ starting from the origin, ν is the density of the catalysts, and w denotes the solution of the random Cauchy problem

$$\frac{\partial}{\partial t} w(x,t) = \varrho \Delta w(x,t) + \delta_{X(t)}(x)\{w(x,t) + 1\}, \qquad w(\cdot, 0) \equiv 0, \tag{1.4.5}$$

with ϱ the diffusion constant of the catalysts. In the weakly catalytic regime, for large κ, we may combine (1.4.4) with the approximation

$$w(X(s), s) \approx \int_0^s p_{s-u}(X(u), X(s))\, du, \tag{1.4.6}$$

where $p_t(x,y)$ is the transition kernel of the catalysts. Observe that $w(X(s),s)$ depends on the full past of X up to time s. The *entire proof* in (10) is based on formula (1.4.4). But for our exclusion dynamics there is *no* such formula for the moments.

1.5 Heuristics behind Theorem 1.3.4 and Conjecture 1.4.1

The *heuristics* behind Theorem 1.3.4 and Conjecture 1.4.1 is the following. Consider the case $p=1$. Scaling time by κ in (1.2.9), we have $\lambda_1(\kappa)=\kappa\lambda_1^*(\kappa)$ with

$$\lambda_1^*(\kappa)=\lim_{t\to\infty}\Lambda_1^*(\kappa;t) \quad \text{and} \quad \Lambda_1^*(\kappa;t)=\frac{1}{t}\log\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^t ds\,\xi\Big(X(s),\frac{s}{\kappa}\Big)\right]\right), \tag{1.5.1}$$

where $X=X^1$ and we abbreviate

$$\mathbb{E}_{\nu_\rho,0}=\mathbb{E}_{\nu_\rho}E_0. \tag{1.5.2}$$

For large κ, the ξ-field in (1.5.1) evolves slowly and therefore does not manage to cooperate with the X-process in determining the growth rate. Also, the prefactor $1/\kappa$ in the exponent is small. As a result, the expectation over the ξ-field can be computed via a *Gaussian approximation* that becomes sharp in the limit as $\kappa\to\infty$, i.e.,

$$\begin{aligned}\Lambda_1^*(\kappa;t)-\frac{\rho}{\kappa}&=\frac{1}{t}\log\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^t ds\,\Big[\xi\Big(X(s),\frac{s}{\kappa}\Big)-\rho\Big]\right]\right)\\ &\approx\frac{1}{t}\log E_0\left(\exp\left[\frac{1}{2\kappa^2}\int_0^t ds\int_0^t du\,\mathbb{E}_{\nu_\rho}\left(\Big[\xi\Big(X(s),\frac{s}{\kappa}\Big)-\rho\Big]\Big[\xi\Big(X(u),\frac{u}{\kappa}\Big)-\rho\Big]\right)\right]\right).\end{aligned} \tag{1.5.3}$$

(In essence, what happens here is that the asymptotics for $\kappa\to\infty$ is driven by moderate deviations of the ξ-field, which fall in the Gaussian regime.) The exponent in the r.h.s. of (1.5.3) equals

$$\frac{1}{\kappa^2}\int_0^t ds\int_s^t du\,\mathbb{E}_{\nu_\rho}\left(\Big[\xi\Big(X(s),\frac{s}{\kappa}\Big)-\rho\Big]\Big[\xi\Big(X(u),\frac{u}{\kappa}\Big)-\rho\Big]\right). \tag{1.5.4}$$

Now, for $x,y\in\mathbb{Z}^d$ and $b\ge a\ge 0$ we have

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big([\xi(x,a)-\rho]\,[\xi(y,b)-\rho]\Big)&=\mathbb{E}_{\nu_\rho}\Big([\xi(x,0)-\rho]\,[\xi(y,b-a)-\rho]\Big)\\ &=\int_\Omega\nu_\rho(d\eta)\,[\eta(x)-\rho]\,\mathbb{E}_\eta\Big([\xi(y,b-a)-\rho]\Big)\\ &=\sum_{z\in\mathbb{Z}^d}p_{b-a}(z,y)\int_\Omega\nu_\rho(d\eta)\,[\eta(x)-\rho]\,[\eta(z)-\rho]\\ &=\rho(1-\rho)\,p_{b-a}(x,y),\end{aligned} \tag{1.5.5}$$

where the first equality uses the stationarity of ξ, the third equality uses (1.2.6) from the graphical representation, and the fourth equality uses that ν_ρ is Bernoulli. Substituting (1.5.5) into (1.5.4), we get that the r.h.s. of (1.5.3) equals

$$\frac{1}{t}\log E_0\left(\exp\left[\frac{\rho(1-\rho)}{\kappa^2}\int_0^t ds\int_s^t du\,p_{\frac{u-s}{\kappa}}(X(s),X(u))\right]\right). \tag{1.5.6}$$

This is precisely the integral that was investigated in Gärtner and den Hollander (10) (see Sections 5–8 and equations (1.5.4–1.5.11) of that paper and 1.4.4-1.4.5). Therefore the limit

$$\lim_{\kappa\to\infty}\kappa[\lambda_1(\kappa)-\rho]=\lim_{\kappa\to\infty}\kappa^2\lim_{t\to\infty}\Big[\Lambda_1^*(\kappa;t)-\frac{\rho}{\kappa}\Big]=\lim_{\kappa\to\infty}\kappa^2\lim_{t\to\infty}(1.5.6) \tag{1.5.7}$$

can be read off from (10) and yields (1.3.1) for $d\geq 4$ and (1.4.1) for $d=3$. A similar heuristics applies for $p>1$.

The r.h.s. of (1.3.1), which is valid for $d\geq 4$, is obtained from the above computations by moving the expectation in (1.5.6) into the exponent. Indeed,

$$E_0\Big(p_{\frac{u-s}{\kappa}}(X(s),X(u))\Big)=\sum_{x,y\in\mathbb{Z}^d}p_{2ds}(0,x)p_{2d(u-s)}(x,y)p_{\frac{u-s}{\kappa}}(x,y)=p_{2d(u-s)(1+\frac{1}{2d\kappa})}(0,0) \tag{1.5.8}$$

and hence

$$\int_0^t ds\int_s^t du\;E_0\Big(p_{\frac{u-s}{\kappa}}(X(s),X(u))\Big)=\int_0^t ds\int_0^{t-s}dv\,p_{2dv(1+\frac{1}{2d\kappa})}(0,0)\sim t\,\frac{1}{2d(1+\frac{1}{2d\kappa})}G_d. \tag{1.5.9}$$

Thus we see that the result in Theorem 1.3.4 comes from a *second order* asymptotics on ξ and a *first order* asymptotics on X in the limit as $\kappa\to\infty$. Despite this simple fact, it turns out to be hard to make the above heuristics rigorous. For $d=3$, on the other hand, we expect the first order asymptotics on X to fail, leading to the more complicated behavior in (1.4.1).

Remark 1: In (1.1.1), the ξ-field may be multiplied by a coupling constant $\gamma\in(0,\infty)$. This produces no change in Theorems 1.3.1, 1.3.2(i) and 1.3.3. In Theorem 1.3.2(ii), $(\rho,1)$ becomes $(\gamma\rho,\gamma)$, while in the r.h.s. of Theorem 1.3.4 and Conjecture 1.4.1, $\rho(1-\rho)$ gets multiplied by γ^2. Similarly, if the simple random walk in Theorem 1.3.4 is replaced by a random walk with transition kernel $p(\cdot,\cdot)$ satisfying (1.2.1–1.2.2), then we expect that in (1.3.1) and (1.4.1) G_d becomes the Green function at the origin of this random walk and a factor $1/\sigma^4$ appears in front of the last term in the r.h.s. of (1.4.1) with σ^2 the variance of $p(\cdot,\cdot)$.

Remark 2: In Gärtner and den Hollander (10) the catalyst was γ times a Poisson field with density ρ of independent simple random walks stepping at rate $2d\theta$, where $\gamma,\rho,\theta\in(0,\infty)$ are parameters. It was found that the Lyapunov exponents are infinite in $d=1,2$ for all p and in $d\geq 3$ for $p\geq 2d\theta/\gamma G_d$, irrespective of κ and ρ. In $d\geq 3$ for $p<2d\theta/\gamma G_d$, on the other hand, the Lyapunov exponents are finite for all κ, and exhibit a dichotomy similar to the one expressed by Theorem 1.3.4 and Conjecture 1.4.1. Apparently, in this regime the two types of catalyst are qualitatively similar. Remarkably, the *same* asymptotic behavior for large κ was found (with $\rho\gamma^2$ replacing $\rho(1-\rho)$ in (1.3.1)), and the *same* variational formula as in (1.4.2) was seen to play a central role in $d=3$. [Note: In (10) the symbols ν,ρ,G_d were used instead of $\rho,\theta,G_d/2d$.]

1.6 Outline

In Section 2 we derive a variational formula for λ_p from which Theorem 1.3.1 follows immediately. The arguments that will be used to derive this variational formula apply to an arbitrary bounded, stationary and *reversible* catalyst. Thus, the properties in Theorem 1.3.1 are quite general. In Section 3 we do a range of estimates, either directly on (1.2.9) or on the variational formula for

λ_p derived in Section 2, to prove Theorems 1.3.2 and 1.3.3. Here, the special properties of SE, in particular, its space-time correlation structure expressed through the graphical representation (see Fig. 1), are crucial. These results hold for an arbitrary random walk subject to (1.2.1–1.2.2). Finally, in Section 4 we prove Theorem 1.3.4, which is restricted to simple random walk. The analysis consists of a long series of estimates, taking up more than half of the paper and, in essence, showing that the problem reduces to understanding the asymptotic behavior of (1.5.6). This reduction is important, because it explains why there is some degree of *universality* in the behavior for $\kappa \to \infty$ under different types of catalysts: apparently, the Gaussian approximation and the two-point correlation function in space and time determine the asymptotics (recall the heuristic argument in Section 1.5). The main steps of this long proof are outlined in Section 4.2.

2 Lyapunov exponents: general properties

In this section we prove Theorem 1.3.1. In Section 2.1 we formulate a large deviation principle for the occupation time of the origin in SE due to Landim (18), which will be needed in Section 3.2. In Section 2.2 we extend the line of thought in (18) and derive a variational formula for λ_p from which Theorem 1.3.1 will follow immediately.

2.1 Large deviations for the occupation time of the origin

Kipnis (17), building on techniques developed by Arratia (1), proved that the *occupation time of the origin up to time* t,

$$T_t = \int_0^t \xi(0, s)\, ds, \tag{2.1.1}$$

satisfies a strong law of large numbers and a central limit theorem. Landim (18) subsequently proved that T_t satisfies a *large deviation principle*, i.e.,

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho}\left(T_t/t \in F\right) &\le -\inf_{\alpha\in F} \Psi_d(\alpha), \qquad F \subseteq [0,1] \text{ closed},\\ \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho}\left(T_t/t \in G\right) &\ge -\inf_{\alpha\in G} \Psi_d(\alpha), \qquad G \subseteq [0,1] \text{ open}, \end{aligned} \tag{2.1.2}$$

with the rate function $\Psi_d\colon [0,1] \to [0,\infty)$ given by an associated Dirichlet form. This rate function is continuous, for transient random walk kernels $p(\cdot,\cdot)$ it has a unique zero at ρ, whereas for recurrent random walk kernels it vanishes identically.

2.2 Variational formula for $\lambda_p(\kappa)$: proof of Theorem 1.3.1

Return to (1.2.9). In this section we show that, by considering ξ and $X_1^\kappa,\dots,X_p^\kappa$ as a joint random process and exploiting the reversibility of ξ, we can use the spectral theorem to express the Lyapunov exponents in terms of a variational formula. From the latter it will follow that $\kappa \mapsto \lambda_p(\kappa)$ is continuous, non-increasing and convex on $[0,\infty)$.

Define

$$Y(t) = \left(\xi(t), X_1^\kappa(t), \dots, X_p^\kappa(t)\right), \quad t \ge 0, \tag{2.2.1}$$

and

$$V(\eta, x_1, \ldots, x_p) = \sum_{i=1}^{p} \eta(x_i), \quad \eta \in \Omega,\, x_1, \ldots, x_p \in \mathbb{Z}^d. \tag{2.2.2}$$

Then we may write (1.2.9) as

$$\Lambda_p(t) = \frac{1}{pt} \log \mathbb{E}_{\nu_\rho,0,\ldots,0} \left(\exp \left[\int_0^t V(Y(s))ds \right] \right). \tag{2.2.3}$$

The random process $Y = (Y(t))_{t\geq 0}$ takes values in $\Omega \times (\mathbb{Z}^d)^p$ and has generator

$$G^\kappa = L + \kappa \sum_{i=1}^{p} \Delta_i \tag{2.2.4}$$

in $L^2(\nu_\rho \otimes m^p)$ (endowed with the inner product $(\cdot,\cdot)$), with L given by (1.2.4), Δ_i the discrete Laplacian acting on the i-th spatial coordinate, and m the counting measure on $\mathbb{Z}^d$. Let

$$G_V^\kappa = G^\kappa + V. \tag{2.2.5}$$

By (1.2.2), this is a *self-adjoint* operator. Our claim is that λ_p equals $\frac{1}{p}$ times the upper boundary of the spectrum of G_V^κ.

Proposition 2.2.1. *$\lambda_p = \frac{1}{p}\mu_p$ with $\mu_p = \sup \mathrm{Sp}\,(G_V^\kappa)$.*

Although this is a general fact, the proofs known to us (e.g. Carmona and Molchanov (3), Lemma III.1.1) do not work in our situation.

Proof. Let $(\mathcal{P}_t)_{t\geq 0}$ denote the semigroup generated by G_V^κ.

Upper bound: Let $Q_{t\log t} = [-t\log t, t\log t]^d \cap \mathbb{Z}^d$. By a standard large deviation estimate for simple random walk, we have

$$\begin{aligned} &\mathbb{E}_{\nu_\rho,0,\ldots,0} \left(\exp \left[\int_0^t V(Y(s))ds \right] \right) \\ &\quad = \mathbb{E}_{\nu_\rho,0,\ldots,0} \left(\exp \left[\int_0^t V(Y(s))ds \right] \mathbb{1}\left\{ X_i^\kappa(t) \in Q_{t\log t} \text{ for } i = 1, \ldots, p \right\} \right) + R_t \end{aligned} \tag{2.2.6}$$

with $\lim_{t\to\infty} \frac{1}{t} \log R_t = -\infty$. Thus it suffices to focus on the term with the indicator. Estimate, with the help of the spectral theorem (Kato (15), Section VI.5),

$$\begin{aligned} &\mathbb{E}_{\nu_\rho,0,\ldots,0} \left(\exp \left[\int_0^t V(Y(s))ds \right] \mathbb{1}\left\{ X_i^\kappa(t) \in Q_{t\log t} \text{ for } i = 1, \ldots, p \right\} \right) \\ &\quad \leq \left(\mathbb{1}_{(Q_{t\log t})^p}, \mathcal{P}_t \mathbb{1}_{(Q_{t\log t})^p} \right) = \int_{(-\infty,\mu_p]} e^{\mu t}\, d\|E_\mu \mathbb{1}_{(Q_{t\log t})^p}\|^2_{L^2(\nu_\rho\otimes m^p)} \\ &\quad \leq e^{\mu_p t}\, \|\mathbb{1}_{(Q_{t\log t})^p}\|^2_{L^2(\nu_\rho\otimes m^p)}, \end{aligned} \tag{2.2.7}$$

where $\mathbb{1}_{(Q_{t\log t})^p}$ is the indicator function of $(Q_{t\log t})^p \subset (\mathbb{Z}^d)^p$ and $(E_\mu)_{\mu\in\mathbb{R}}$ denotes the spectral family of orthogonal projection operators associated with G_V^κ. Since $\|\mathbb{1}_{(Q_{t\log t})^p}\|^2_{L^2(\nu_\rho\otimes m^p)} =$

$|Q_{t\log t}|^p$ does not increase exponentially fast, it follows from (1.2.10), (2.2.3) and (2.2.6–2.2.7) that $\lambda_p \leq \frac{1}{p}\mu_p$.

Lower bound: For every $\delta > 0$ there exists an $f_\delta \in L^2(\nu_\rho \otimes m^p)$ such that

$$(E_{\mu_p} - E_{\mu_p - \delta}) f_\delta \neq 0 \tag{2.2.8}$$

(see Kato (15), Section VI.2; the spectrum of G_V^κ coincides with the set of μ's for which $E_{\mu+\delta} - E_{\mu-\delta} \neq 0$ for all $\delta > 0$). Approximating f_δ by bounded functions, we may without loss of generality assume that $0 \leq f_\delta \leq 1$. Similarly, approximating f_δ by bounded functions with finite support in the spatial variables, we may assume without loss of generality that there exists a finite $K_\delta \subset \mathbb{Z}^d$ such that

$$0 \leq f_\delta \leq 1\!\!1_{(K_\delta)^p}. \tag{2.2.9}$$

First estimate

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho, 0, \ldots, 0}\left(\exp\left[\int_0^t V(Y(s))ds\right]\right)\\
&\geq \sum_{x_1,\ldots,x_p \in K_\delta} \mathbb{E}_{\nu_\rho,0,\ldots,0}\left(1\!\!1\{X_1^\kappa(1) = x_1, \ldots, X_p^\kappa(1) = x_p\} \exp\left[\int_1^t V(Y(s))ds\right]\right)\\
&= \sum_{x_1,\ldots,x_p \in K_\delta} p_1^\kappa(0,x_1)\ldots p_1^\kappa(0,x_p)\, \mathbb{E}_{\nu_\rho, x_1,\ldots,x_p}\left(\exp\left[\int_0^{t-1} V(Y(s))ds\right]\right)\\
&\geq C_\delta^p \sum_{x_1,\ldots,x_p \in K_\delta} \mathbb{E}_{\nu_\rho,x_1,\ldots,x_p}\left(\exp\left[\int_0^{t-1} V(Y(s))ds\right]\right),
\end{aligned} \tag{2.2.10}$$

where $p_t^\kappa(x,y) = \mathbb{P}_x(X^\kappa(t) = y)$ and $C_\delta = \min_{x\in K_\delta} p_1^\kappa(0,x) > 0$. The equality in (2.2.10) uses the Markov property and the fact that ν_ρ is invariant for the SE-dynamics. Next estimate

$$\begin{aligned}
\text{r.h.s. (2.2.10)} &\geq C_\delta^p \int_\Omega \nu_\rho(d\eta) \sum_{x_1,\ldots,x_p \in \mathbb{Z}^d} f_\delta(\eta, x_1, \ldots, x_p)\\
&\qquad\qquad \times \mathbb{E}_{\eta,x_1,\ldots,x_p}\left(\exp\left[\int_0^{t-1} V(Y(s))ds\right]\right) f_\delta(Y(t-1))\\
&= C_\delta^p\,(f_\delta, \mathcal{P}_{t-1} f_\delta) \geq \frac{C_\delta^p}{|K_\delta|^p} \int_{(\mu_p-\delta,\mu_p]} e^{\mu(t-1)}\, d\|E_\mu f_\delta\|^2_{L^2(\nu_\rho\otimes m^p)}\\
&\geq C_\delta^p\, e^{(\mu_p-\delta)(t-1)}\, \|(E_{\mu_p} - E_{\mu_p-\delta}) f_\delta\|^2_{L^2(\nu_\rho \otimes m^p)},
\end{aligned} \tag{2.2.11}$$

where the first inequality uses (2.2.9). Combine (2.2.10–2.2.11) with (2.2.8), and recall (2.2.3), to get $\lambda_p \geq \frac{1}{p}(\mu_p - \delta)$. Let $\delta \downarrow 0$, to obtain $\lambda_p \geq \frac{1}{p}\mu_p$. ∎

The Rayleigh-Ritz formula for μ_p applied to Proposition 2.2.1 gives (recall (1.2.4), (2.2.2) and (2.2.4–2.2.5)):

Proposition 2.2.2. *For all $p \in \mathbb{N}$,*

$$\lambda_p = \frac{1}{p}\mu_p = \frac{1}{p} \sup_{\|f\|_{L^2(\nu_\rho\otimes m^p)}=1} (G_V^\kappa f, f) \tag{2.2.12}$$

with

$$(G_V^\kappa f, f) = A_1(f) - A_2(f) - \kappa A_3(f), \tag{2.2.13}$$

where

$$\begin{aligned} A_1(f) &= \int_\Omega \nu_\rho(d\eta) \sum_{z_1,\ldots,z_p\in\mathbb{Z}^d} V(\eta; z_1,\ldots,z_p)\, f(\eta, z_1,\ldots,z_p)^2,\\ A_2(f) &= \int_\Omega \nu_\rho(d\eta) \sum_{z_1,\ldots,z_p\in\mathbb{Z}^d} \frac{1}{2} \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)\,[f(\eta^{x,y}, z_1,\ldots,z_p) - f(\eta, z_1,\ldots,z_p)]^2,\\ A_3(f) &= \int_\Omega \nu_\rho(d\eta) \sum_{z_1,\ldots,z_p\in\mathbb{Z}^d} \frac{1}{2} \sum_{i=1}^{p} \sum_{\substack{y_i\in\mathbb{Z}^d\\ \|y_i-z_i\|=1}} [f(\eta, z_1,\ldots,z_p)|_{z_i\to y_i} - f(\eta, z_1,\ldots,z_p)]^2, \end{aligned} \tag{2.2.14}$$

and $z_i \to y_i$ *means that the argument* z_i *is replaced by* y_i.

Remark 2.2.3. *Propositions* 2.2.1–2.2.2 *are valid for general bounded measurable potentials* V *instead of* (2.2.2). *The proof also works for modifications of the random walk* Y *for which a lower bound similar to that in the last two lines of* (2.2.10) *can be obtained. Such modifications will be used later in Sections* 4.5–4.6.

We are now ready to give the proof of Theorem 1.3.1.

Proof. The existence of λ_p was established in Proposition 2.2.1. By (2.2.13–2.2.14), the r.h.s. of (2.2.12) is a supremum over functions that are linear and non-increasing in κ. Consequently, $\kappa \mapsto \lambda_p(\kappa)$ is lower semi-continuous, convex and non-increasing on $[0,\infty)$ (and, hence, also continuous). ∎

The variational formula in Proposition 2.2.2 is useful to deduce qualitative properties of λ_p, as demonstrated above. Unfortunately, it is not clear how to deduce from it more detailed information about the Lyapunov exponents. To achieve the latter, we resort in Sections 3 and 4 to different techniques, only occasionally making use of Proposition 2.2.2.

3 Lyapunov exponents: recurrent vs. transient random walk

In this section we prove Theorems 1.3.2 and 1.3.3. In Section 3.1 we consider recurrent random walk, in Section 3.2 transient random walk.

3.1 Recurrent random walk: proof of Theorem 1.3.2(i)

The key to the proof of Theorem 1.3.2(i) is the following.

Lemma 3.1.1. *If* $p(\cdot,\cdot)$ *is recurrent, then for any finite box* $Q \subset \mathbb{Z}^d$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho}\Big(\xi(x,s) = 1\ \forall\, s \in [0,t]\ \forall\, x \in Q\Big) = 0. \tag{3.1.1}$$

Proof. In the spirit of Arratia (1), Section 3, we argue as follows. Let

$$H_t^Q = \Big\{x \in \mathbb{Z}^d\colon \text{ there is a path from } (x,0) \text{ to } Q \times [0,t] \text{ in the graphical representation}\Big\}. \tag{3.1.2}$$

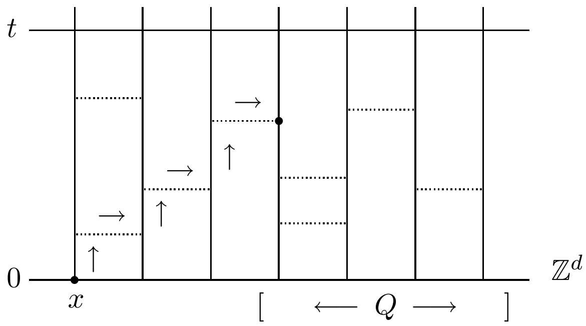

Fig. 4: A path from $(x,0)$ to $Q \times [0,t]$ (recall Fig. 1).

Note that $H_0^Q = Q$ and that $t \mapsto H_t^Q$ is non-decreasing. Denote by $\mathcal{P}$ and $\mathcal{E}$, respectively, probability and expectation associated with the graphical representation. Then

$$\mathbb{P}_{\nu_\rho}\Big(\xi(x,s) = 1\ \forall\, s \in [0,t]\ \forall\, x \in Q\Big) = (\mathcal{P} \otimes \nu_\rho)\Big(H_t^Q \subseteq \xi(0)\Big), \tag{3.1.3}$$

where $\xi(0) = \{x \in \mathbb{Z}^d\colon \xi(x,0) = 1\}$ is the set of initial locations of the particles. Indeed, (3.1.3) holds because if $\xi(x,0) = 0$ for some $x \in H_t^Q$, then this 0 will propagate into Q prior to time t (see Fig. 4).

By Jensen's inequality,

$$(\mathcal{P} \otimes \nu_\rho)\Big(H_t^Q \subseteq \xi(0)\Big) = \mathcal{E}\left(\rho^{|H_t^Q|}\right) \geq \rho^{\mathcal{E}|H_t^Q|}. \tag{3.1.4}$$

Moreover, $H_t^Q \subseteq \cup_{y \in Q} H_t^{\{y\}}$, and hence

$$\mathcal{E}|H_t^Q| \leq |Q|\, \mathcal{E}|H_t^{\{0\}}|. \tag{3.1.5}$$

Furthermore, we have

$$\mathcal{E}|H_t^{\{0\}}| = \mathbb{E}_0^{p(\cdot,\cdot)} R_t, \tag{3.1.6}$$

where R_t is the range after time t of the random walk with transition kernel $p(\cdot,\cdot)$ driving ξ and $\mathbb{E}_0^{p(\cdot,\cdot)}$ denotes expectation w.r.t. this random walk starting from 0. Indeed, by time reversal, the probability that there is a path from $(x,0)$ to $\{0\} \times [0,t]$ in the graphical representation is equal to the probability that the random walk starting from 0 hits x prior to time t. It follows from (3.1.3–3.1.6) that

$$\frac{1}{t} \log \mathbb{P}_{\nu_\rho}\Big(\xi(x,s) = 1\ \forall\, s \in [0,t]\ \forall\, x \in Q\Big) \geq -|Q| \log\left(\frac{1}{\rho}\right) \left\{\frac{1}{t}\mathbb{E}_0^{p(\cdot,\cdot)} R_t\right\}. \tag{3.1.7}$$

Finally, since $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0^{p(\cdot,\cdot)} R_t = 0$ when $p(\cdot,\cdot)$ is recurrent (see Spitzer (20), Chapter 1, Section 4), we get (3.1.1). ∎

We are now ready to give the proof of Theorem 1.3.2(i).

Proof. Since $p \mapsto \lambda_p$ is non-decreasing and $\lambda_p \le 1$ for all $p \in \mathbb{N}$, it suffices to give the proof for $p = 1$. For $p = 1$, (1.2.9) gives

$$\Lambda_1(t) = \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t \xi\left(X^\kappa(s), s\right) ds\right]\right). \tag{3.1.8}$$

By restricting X^κ to stay inside a finite box $Q \subset \mathbb{Z}^d$ up to time t and requiring ξ to be 1 throughout this box up to time t, we obtain

$$\begin{aligned}&\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t \xi(X^\kappa(s), s)\, ds\right]\right)\\ &\ge e^t\, \mathbb{P}_{\nu_\rho}\Big(\xi(x,s) = 1\ \forall\, s \in [0,t]\ \forall\, x \in Q\Big)\, P_0\Big(X^\kappa(s) \in Q\ \forall\, s \in [0,t]\Big).\end{aligned} \tag{3.1.9}$$

For the second factor, we apply (3.1.1). For the third factor, we have

$$\lim_{t\to\infty} \frac{1}{t} \log P_0\Big(X^\kappa(s) \in Q\ \forall\, s \in [0,t]\Big) = -\lambda^\kappa(Q) \tag{3.1.10}$$

with $\lambda^\kappa(Q) > 0$ the principal Dirichlet eigenvalue on Q of $-\kappa\Delta$, the generator of the simple random walk X^κ. Combining (3.1.1) and (3.1.8–3.1.10), we arrive at

$$\lambda_1 = \lim_{t\to\infty} \Lambda_1(t) \ge 1 - \lambda^\kappa(Q). \tag{3.1.11}$$

Finally, let $Q \to \mathbb{Z}^d$ and use that $\lim_{Q\to\mathbb{Z}^d} \lambda^\kappa(Q) = 0$ for any κ, to arrive at $\lambda_1 \ge 1$. Since, trivially, $\lambda_1 \le 1$, we get $\lambda_1 = 1$. ∎

3.2 Transient random walk: proof of Theorems 1.3.2(ii) and 1.3.3

Theorem 1.3.2(ii) is proved in Sections 3.2.1 and 3.2.3–3.2.5, Theorem 1.3.3 in Section 3.2.2. Throughout the present section we assume that the random walk kernel $p(\cdot,\cdot)$ is transient.

3.2.1 Proof of the lower bound in Theorem 1.3.2(ii)

Proposition 3.2.1. *$\lambda_p(\kappa) > \rho$ for all $\kappa \in [0,\infty)$ and $p \in \mathbb{N}$.*

Proof. Since $p \mapsto \lambda_p(\kappa)$ is non-decreasing for all κ, it suffices to give the proof for $p = 1$. For every $\epsilon > 0$ there exists a function $\phi_\epsilon \colon \mathbb{Z}^d \to \mathbb{R}$ such that

$$\sum_{x\in\mathbb{Z}^d} \phi_\epsilon(x)^2 = 1 \quad \text{and} \quad \sum_{\substack{x,y\in\mathbb{Z}^d\\ \|x-y\|=1}} [\phi_\epsilon(x) - \phi_\epsilon(y)]^2 \le \epsilon^2. \tag{3.2.1}$$

Let

$$f_\epsilon(\eta, x) = \frac{1 + \epsilon\eta(x)}{[1 + (2\epsilon + \epsilon^2)\rho]^{1/2}}\, \phi_\epsilon(x), \qquad \eta \in \Omega,\ x \in \mathbb{Z}^d. \tag{3.2.2}$$

Then

$$\|f_\epsilon\|^2_{L^2(\nu_\rho\otimes m)} = \int_\Omega \nu_\rho(d\eta) \sum_{x\in\mathbb{Z}^d} \frac{[1+\epsilon\eta(x)]^2}{1+(2\epsilon+\epsilon^2)\rho}\, \phi_\epsilon(x)^2 = \sum_{x\in\mathbb{Z}^d} \phi_\epsilon(x)^2 = 1. \tag{3.2.3}$$

Therefore we may use f_ϵ as a test function in (2.2.12) in Proposition 2.2.2. This gives

$$\lambda_1 = \mu_1 \geq \frac{1}{1+(2\epsilon+\epsilon^2)\rho}\,(I - II - \kappa\, III) \tag{3.2.4}$$

with

$$I = \int_\Omega \nu_\rho(d\eta) \sum_{z\in\mathbb{Z}^d} \eta(z)\,[1+\epsilon\eta(z)]^2\,\phi_\epsilon(z)^2 = (1+2\epsilon+\epsilon^2)\rho \sum_{z\in\mathbb{Z}^d} \phi_\epsilon(z)^2 = (1+2\epsilon+\epsilon^2)\rho \tag{3.2.5}$$

and

$$\begin{aligned} II &= \int_\Omega \nu_\rho(d\eta) \sum_{z\in\mathbb{Z}^d} \frac{1}{4} \sum_{x,y\in\mathbb{Z}^d} p(x,y)\,\epsilon^2[\eta^{x,y}(z)-\eta(z)]^2\phi_\epsilon(z)^2 \\ &= \frac{1}{2}\int_\Omega \nu_\rho(d\eta) \sum_{x,y\in\mathbb{Z}^d} p(x,y)\,\epsilon^2[\eta(x)-\eta(y)]^2\,\phi_\epsilon(x)^2 \\ &= \epsilon^2\rho(1-\rho) \sum_{\substack{x,y\in\mathbb{Z}^d\\ x\neq y}} p(x,y)\,\phi_\epsilon(x)^2 \leq \epsilon^2\rho(1-\rho) \end{aligned} \tag{3.2.6}$$

and

$$\begin{aligned} III &= \frac{1}{2}\int_\Omega \nu_\rho(d\eta) \sum_{\substack{x,y\in\mathbb{Z}^d\\ \|x-y\|=1}} \big\{[1+\epsilon\eta(x)]\phi_\epsilon(x) - [1+\epsilon\eta(y)]\phi_\epsilon(y)\big\}^2 \\ &= \frac{1}{2} \sum_{\substack{x,y\in\mathbb{Z}^d\\ \|x-y\|=1}} \big\{[1+(2\epsilon+\epsilon^2)\rho][\phi_\epsilon(x)^2+\phi_\epsilon(y)^2] - 2(1+\epsilon\rho)^2\phi_\epsilon(x)\phi_\epsilon(y)\big\} \\ &= \frac{1}{2}[1+(2\epsilon+\epsilon^2)\rho] \sum_{\substack{x,y\in\mathbb{Z}^d\\ \|x-y\|=1}} [\phi_\epsilon(x)-\phi_\epsilon(y)]^2 + \epsilon^2\rho(1-\rho) \sum_{\substack{x,y\in\mathbb{Z}^d\\ \|x-y\|=1}} \phi_\epsilon(x)\phi_\epsilon(y) \\ &\leq \frac{1}{2}[1+(2\epsilon+\epsilon^2)\rho]\epsilon^2 + 2d\epsilon^2\rho(1-\rho). \end{aligned} \tag{3.2.7}$$

In the last line we use that $\phi_\epsilon(x)\phi_\epsilon(y) \leq \frac{1}{2}\phi_\epsilon(x)^2 + \frac{1}{2}\phi_\epsilon(y)^2$. Combining (3.2.4–3.2.7), we find

$$\lambda_1 = \mu_1 \geq \rho\,\frac{1+2\epsilon+O(\epsilon^2)}{1+2\epsilon\rho+O(\epsilon^2)}. \tag{3.2.8}$$

Because $\rho\in(0,1)$, it follows that for ϵ small enough the r.h.s. is strictly larger than ρ. ∎

3.2.2 Proof of Theorem 1.3.3

Proof. It is enough to show that $\lambda_2(0) > \lambda_1(0)$. Then, by continuity (recall Theorem 1.3.1(ii)), there exists $\kappa_0 \in (0, \infty]$ such that $\lambda_2(\kappa) > \lambda_1(\kappa)$ for all $\kappa \in [0, \kappa_0)$, after which the inequality $\lambda_{p+1}(\kappa) > \lambda_p(\kappa)$ for $\kappa \in [0, \kappa_0)$ and arbitrary p follows from general convexity arguments (see Gärtner and Heydenreich (9), Lemma 3.1).

For $\kappa = 0$, (1.2.9) reduces to

$$\Lambda_p(t) = \frac{1}{pt} \log \mathbb{E}_{\nu_\rho} \left(\exp \left[p \int_0^t \xi(0, s) ds \right] \right) = \frac{1}{pt} \log \mathbb{E}_{\nu_\rho} \left(\exp \left[p T_t \right] \right) \tag{3.2.9}$$

(recall (2.1.1)). In order to compute $\lambda_p(0) = \lim_{t\to\infty} \Lambda_p(t)$, we may use the large deviation principle for $(T_t)_{t\geq 0}$ cited in Section 2.1 due to Landim (18). Indeed, by applying Varadhan's Lemma (see e.g. den Hollander (14), Theorem III.13) to (3.2.9), we get

$$\lambda_p(0) = \frac{1}{p} \max_{\alpha \in [0,1]} \left[p\alpha - \Psi_d(\alpha) \right] \tag{3.2.10}$$

with Ψ_d the rate function introduced in (2.1.2). Since Ψ_d is continuous, (3.2.10) has at least one maximizer α_p:

$$\lambda_p(0) = \alpha_p - \frac{1}{p} \Psi_d(\alpha_p). \tag{3.2.11}$$

By Proposition 3.2.1 for $\kappa = 0$, we have $\lambda_p(0) > \rho$. Hence $\alpha_p > \rho$ (because $\Psi_d(\rho) = 0$). Since $p(\cdot, \cdot)$ is transient, it follows that $\Psi_d(\alpha_p) > 0$. Therefore we get from (3.2.10–3.2.11) that

$$\lambda_{p+1}(0) \geq \frac{1}{p+1} \left[\alpha_p(p+1) - \Psi_d(\alpha_p) \right] = \alpha_p - \frac{1}{p+1} \Psi_d(\alpha_p) > \alpha_p - \frac{1}{p} \Psi_d(\alpha_p) = \lambda_p(0). \tag{3.2.12}$$

In particular $\lambda_2(0) > \lambda_1(0)$, and so we are done. ∎

3.2.3 Proof of the upper bound in Theorem 1.3.2(ii)

Proposition 3.2.2. $\lambda_p(\kappa) < 1$ *for all* $\kappa \in [0, \infty)$ *and* $p \in \mathbb{N}$.

Proof. By Theorem 1.3.3, which was proved in Section 3.2.2, we know that $p \mapsto \lambda_p(0)$ is strictly increasing. Since $\lambda_p(0) \leq 1$ for all $p \in \mathbb{N}$, it therefore follows that $\lambda_p(0) < 1$ for all $p \in \mathbb{N}$. Moreover, by Theorem 1.3.1(ii), which was proved in Section 2.2, we know that $\kappa \mapsto \lambda_p(\kappa)$ is non-increasing. It therefore follows that $\lambda_p(\kappa) < 1$ for all $\kappa \in [0, \infty)$ and $p \in \mathbb{N}$. ∎

3.2.4 Proof of the asymptotics in Theorem 1.3.2(ii)

The proof of the next proposition is somewhat delicate.

Proposition 3.2.3. $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \rho$ *for all* $p \in \mathbb{N}$.

Proof. We give the proof for $p = 1$. The generalization to arbitrary p is straightforward and will be explained at the end. We need a cube $Q = [-R, R]^d \cap \mathbb{Z}^d$ of length $2R$, centered at the origin and $\delta \in (0, 1)$. Limits are taken in the order

$$t \to \infty, \ \kappa \to \infty, \ \delta \downarrow 0, \ Q \uparrow \mathbb{Z}^d. \tag{3.2.13}$$

The proof proceeds in 4 steps, each containing a lemma.

Step 1: Let $X^{\kappa,Q}$ be simple random walk on Q obtained from X^κ by suppressing jumps outside of Q. Then $(\xi_t, X_t^{\kappa,Q})_{t\geq 0}$ is a Markov process on $\Omega \times Q$ with self-adjoint generator in $L^2(\nu_\rho \otimes m_Q)$, where m_Q is the counting measure on Q.

Lemma 3.2.4. *For all Q finite (centered and cubic) and $\kappa \in [0, \infty)$,*

$$\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\, \xi(X_s^\kappa, s)\right]\right) \leq e^{o(t)}\, \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\, \xi\big(X_s^{\kappa,Q}, s\big)\right]\right), \ t \to \infty. \tag{3.2.14}$$

Proof. We consider the partition of $\mathbb{Z}^d$ into cubes $Q_z = 2Rz + Q$, $z \in \mathbb{Z}^d$. The Lyapunov exponent $\lambda_1(\kappa)$ associated with X^κ is given by the variational formula (2.2.12–2.2.14) for $p = 1$. It can be estimated from above by splitting the sums over $\mathbb{Z}^d$ in (2.2.14) into separate sums over the individual cubes Q_z and suppressing in $A_3(f)$ the summands on pairs of lattice sites belonging to different cubes. The resulting expression is easily seen to coincide with the original variational expression (2.2.12), except that the supremum is restricted in addition to functions f with spatial support contained in Q. But this is precisely the Lyapunov exponent $\lambda_1^Q(\kappa)$ associated with $X^{\kappa,Q}$. Hence, $\lambda_1(\kappa) \leq \lambda_1^Q(\kappa)$, and this implies (3.2.14). ∎

Step 2: For large κ the random walk $X^{\kappa,Q}$ moves fast through the finite box Q and therefore samples it in a way that is close to the uniform distribution.

Lemma 3.2.5. *For all Q finite and $\delta \in (0, 1)$, there exist $\varepsilon = \varepsilon(\kappa, \delta, Q)$ and $N_0 = N_0(\delta, \varepsilon)$, satisfying* $\lim_{\kappa\to\infty} \varepsilon(\kappa, \delta, Q) = 0$ *and* $\lim_{\delta,\varepsilon\downarrow 0} N_0(\delta, \varepsilon) = N_0 > 1$, *such that*

$$\begin{aligned}\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\, \xi\big(X_s^{\kappa,Q}, s\big)\right]\right) &\leq o(1) + \exp\left[\left(\left(1 + \frac{1+\varepsilon}{1-\delta}\right)\delta N_0 |Q| + \frac{\delta+\varepsilon}{1-\delta}\right)(t+\delta)\right] \\ &\quad\times \mathbb{E}_{\nu_\rho}\left(\exp\left[\int_0^{t+\delta} ds\, \frac{1}{|Q|}\sum_{y\in Q} \xi(y, s)\right]\right), \ t \to \infty.\end{aligned} \tag{3.2.15}$$

Proof. We split time into intervals of length $\delta > 0$. Let I_k be the indicator of the event that ξ has a jump time in Q during the k-th time interval. If $I_k = 0$, then $\xi_s = \xi_{(k-1)\delta}$ for all $s \in [(k-1)\delta, k\delta)$. Hence,

$$\int_{(k-1)\delta}^{k\delta} ds\, \xi_s\big(X_s^{\kappa,Q}\big) \leq \int_{(k-1)\delta}^{k\delta} ds\, \xi_{(k-1)\delta}\big(X_s^{\kappa,Q}\big) + \delta I_k \tag{3.2.16}$$

and, consequently, we have for all $x \in \mathbb{Z}^d$ and $k = 1, \ldots, \lceil t/\delta \rceil$,

$$E_x\left(\exp\left[\int_0^\delta ds\, \xi_{(k-1)\delta+s}\big(X_s^{\kappa,Q}\big)\right]\right) \leq e^{\delta I_k}\, E_x\left(\exp\left[\int_0^\delta ds\, \eta\big(X_s^{\kappa,Q}\big)\right]\right), \tag{3.2.17}$$

where we abbreviate $\xi_{(k-1)\delta} = \eta$. Next, we do a Taylor expansion and use the Markov property of $X^{\kappa,Q}$, to obtain ($s_0 = 0$)

$$
\begin{aligned}
&E_x\left(\exp\left[\int_0^\delta ds\,\eta\big(X_s^{\kappa,Q}\big)\right]\right) = \sum_{n=0}^\infty \left(\prod_{l=1}^n \int_{s_{l-1}}^\delta ds_l\right) E_x\left(\prod_{m=1}^n \eta\big(X_{s_m}^{\kappa,Q}\big)\right)\\
&\leq \sum_{n=0}^\infty \left(\prod_{l=1}^n \int_{s_{l-1}}^\delta ds_l\right)\left(\prod_{m=1}^n \max_{x\in Q} E_x\left(\eta\big(X_{s_m - s_{m-1}}^{\kappa,Q}\big)\right)\right)\\
&\leq \sum_{n=0}^\infty \left\{\int_0^\delta ds\,\max_{x\in Q} E_x\left(\eta\big(X_s^{\kappa,Q}\big)\right)\right\}^n \leq \exp\left[\frac{1}{1-\delta}\int_0^\delta ds\,\max_{x\in Q} E_x\Big(\eta\big(X_s^{\kappa,Q}\big)\Big)\right]\\
&\leq \exp\left[\frac{1}{1-\delta}\sum_{y\in Q}\eta(y)\int_0^\delta ds\,\max_{x\in Q} E_x\Big(\delta_y\big(X_s^{\kappa,Q}\big)\Big)\right],
\end{aligned}
\tag{3.2.18}
$$

where we use that $\max_{x\in Q} E_x\left(\int_0^\delta ds\,\eta\big(X_s^{\kappa,Q}\big)\right) \leq \delta$. Now, let $p_s^{\kappa,Q}(\cdot,\cdot)$ denote the transition kernel of $X^{\kappa,Q}$. Note that

$$
\lim_{\kappa\to\infty} p_s^{\kappa,Q}(x,y) = \frac{1}{|Q|} \quad \text{for all } s>0,\ Q \text{ finite and } x,y\in Q. \tag{3.2.19}
$$

Hence

$$
\lim_{\kappa\to\infty} E_x\Big(\delta_y\big(X_s^{\kappa,Q}\big)\Big) = \frac{1}{|Q|} \quad \text{for all } s>0,\ Q \text{ finite and } x,y\in Q. \tag{3.2.20}
$$

Therefore, by the Lebesgue dominated convergence theorem, we have

$$
\lim_{\kappa\to\infty}\int_0^\delta ds\,\max_{x\in Q} E_x\Big(\delta_y\big(X_s^{\kappa,Q}\big)\Big) = \delta\,\frac{1}{|Q|} \quad \text{for all } \delta>0,\ Q \text{ finite and } y\in Q. \tag{3.2.21}
$$

This implies that the expression in the exponent in the r.h.s. of (3.2.18) converges to

$$
\frac{\delta}{1-\delta}\frac{1}{|Q|}\sum_{y\in Q}\eta(y), \tag{3.2.22}
$$

uniformly in $\eta\in\Omega$. Combining the latter with (3.2.18), we see that there exists some $\varepsilon = \varepsilon(\kappa,\delta,Q)$, satisfying $\lim_{\kappa\to\infty}\varepsilon(\kappa,\delta,Q) = 0$, such that for all $x\in Q$,

$$
E_x\bigg(\exp\left[\int_0^\delta ds\,\eta\big(X_s^{\kappa,Q}\big)\right]\bigg) \leq \exp\left[\frac{1+\varepsilon}{1-\delta}\,\delta\,\frac{1}{|Q|}\sum_{y\in Q}\eta(y)\right] \quad \text{for all } \delta\in(0,1) \text{ and } Q \text{ finite.} \tag{3.2.23}
$$

Next, similarly as in (3.2.16), we have

$$
\delta\,\frac{1}{|Q|}\sum_{y\in Q}\xi_{(k-1)\delta}(y) \leq \int_{(k-1)\delta}^{k\delta} ds\,\frac{1}{|Q|}\sum_{y\in Q}\xi_s(y) + \delta I_k. \tag{3.2.24}
$$

Applying the Markov property to $X^{\kappa,Q}$, and using (3.2.16) and (3.2.23-3.2.24), we find that

$$
\begin{aligned}
\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\,\xi\big(X_s^{\kappa,Q},s\big)\right]\right) \leq \mathbb{E}_{\nu_\rho}\Bigg(&\exp\left[\Big(1+\frac{1+\varepsilon}{1-\delta}\Big)\delta\,N_{t+\delta} + \frac{\delta+\epsilon}{1-\delta}(t+\delta)\right]\\
&\times\exp\left[\int_0^{t+\delta} ds\,\frac{1}{|Q|}\sum_{y\in Q}\xi_s(y)\right]\Bigg),
\end{aligned}
\tag{3.2.25}
$$

where $N_{t+\delta}$ is the total number of jumps that ξ makes inside Q up to time $t+\delta$. The second term in the r.h.s. of (3.2.25) equals the second term in the r.h.s. of (3.2.15). The first term will be negligible on an exponential scale for $\delta \downarrow 0$, because, as can be seen from the graphical representation, $N_{t+\delta}$ is stochastically smaller that the total number of jumps up to time $t+\delta$ of a Poisson process with rate $|Q \cup \partial Q|$. Indeed, abbreviating

$$a = \left(1 + \frac{1+\varepsilon}{1-\delta}\right)\delta, \quad b = \frac{\delta+\varepsilon}{1-\delta}, \quad M_{t+\delta} = \int_0^{t+\delta} ds\, \frac{1}{|Q|} \sum_{y \in Q} \xi_s(y), \tag{3.2.26}$$

we estimate, for each N,

$$\begin{aligned} \text{r.h.s. } (3.2.25) &= \mathbb{E}_{\nu_\rho}\left(e^{aN_{t+\delta}+b(t+\delta)+M_{t+\delta}}\right) \\ &\leq e^{(b+1)(t+\delta)}\, \mathbb{E}_{\nu_\rho}\Big(e^{aN_{t+\delta}}\, 1\{N_{t+\delta} \geq N|Q|(t+\delta)\}\Big) + e^{(aN|Q|+b)(t+\delta)}\, \mathbb{E}_{\nu_\rho}\left(e^{M_{t+\delta}}\right). \end{aligned} \tag{3.2.27}$$

For $N \geq N_0 = N_0(a,b)$, the first term tends to zero as $t \to \infty$ and can be discarded. Hence

$$\text{r.h.s. } (3.2.25) \leq e^{(aN_0|Q|+b)(t+\delta)}\, \mathbb{E}_{\nu_\rho}\left(e^{bM_{t+\delta}}\right), \tag{3.2.28}$$

which is the desired bound in (3.2.15). Note that $a \downarrow 0$, $b \downarrow 1$ as $\delta, \varepsilon \downarrow 0$ and hence $N_0(a,b) \downarrow N_0 > 1$. ∎

Step 3: By combining Lemmas 3.2.4–3.2.5, we now know that for any Q finite,

$$\lim_{\kappa \to \infty} \lambda_1(\kappa) \leq \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|} \sum_{y \in Q} \xi_s(y)\right]\right), \tag{3.2.29}$$

where we have taken the limits $\kappa \to \infty$ and $\delta \downarrow 0$. According to Proposition 1.2.1 (with $K(z,s) = (1/|Q|)1_Q(z)$),

$$\mathbb{E}_{\nu_\rho}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|} \sum_{y \in Q} \xi_s(y)\right]\right) \leq \mathbb{E}^{\mathrm{IRW}}_{\nu_\rho}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|} \sum_{y \in Q} \tilde{\xi}_s(y)\right]\right), \tag{3.2.30}$$

where $(\tilde{\xi}_t)_{t \geq 0}$ is the process of Independent Random Walks on $\mathbb{Z}^d$ with step rate 1 and transition kernel $p(\cdot,\cdot)$, and $\mathbb{E}^{\mathrm{IRW}}_{\nu_\rho} = \int_\Omega \nu_\rho(d\eta)\, \mathbb{E}^{\mathrm{IRW}}_\eta$. The r.h.s. can be computed and estimated as follows. Write

$$(\Delta^{(p)} f)(x) = \sum_{y \in \mathbb{Z}^d} p(x,y)[f(y) - f(x)], \qquad x \in \mathbb{Z}^d, \tag{3.2.31}$$

to denote the generator of the random walk with step rate 1 and transition kernel $p(\cdot,\cdot)$.

Lemma 3.2.6. *For all Q finite,*

$$\text{r.h.s. } (3.2.30) \leq e^{\rho t} \exp\left[\int_0^t ds\, \frac{1}{|Q|} \sum_{x \in Q} w^Q(x,s)\right], \tag{3.2.32}$$

where $w^Q\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial w^Q}{\partial t}(x,t) = \Delta^{(p)} w^Q(x,t) + \left\{\frac{1}{|Q|} 1_Q(x)\right\} [w^Q(x,t)+1], \qquad w^Q(\cdot,0) \equiv 0, \tag{3.2.33}$$

which has the representation

$$w^Q(x,t) = E_x^{\mathrm{RW}}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|} 1_Q(Y_s)\right]\right) - 1 \geq 0, \tag{3.2.34}$$

where $Y = (Y_t)_{t\geq 0}$ is the single random walk with step rate 1 and transition kernel $p(\cdot,\cdot)$, and E_x^{RW} denotes the expectation w.r.t. to Y starting from $Y_0 = x$.

Proof. Let

$$A_\eta = \{x \in \mathbb{Z}^d\colon\ \eta(x) = 1\}, \qquad \eta \in \Omega. \tag{3.2.35}$$

Then

$$\begin{aligned}\text{r.h.s. } (3.2.30) &= \int_\Omega \nu_\rho(d\eta)\, \mathbb{E}_\eta^{\mathrm{IRW}}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|} \sum_{x\in A_\eta} \sum_{y\in Q} 1_y(\tilde{\xi}_{s,x})\right]\right) \\ &= \int_\Omega \nu_\rho(d\eta) \prod_{x\in A_\eta} E_x^{\mathrm{RW}}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|}\, 1_Q(Y_s)\right]\right),\end{aligned} \tag{3.2.36}$$

where $\tilde{\xi}_{s,x}$ is the position at time s of the random walk starting from $\tilde{\xi}_{0,x} = x$ (in the process of Independent Random Walks $\tilde{\xi} = (\tilde{\xi}_t)_{t\geq 0}$). Let

$$v^Q(x,t) = E_x^{\mathrm{RW}}\left(\exp\left[\int_0^t ds\, \frac{1}{|Q|}\, 1_Q(Y_s)\right]\right). \tag{3.2.37}$$

By the Feynman-Kac formula, $v^Q(x,t)$ is the solution of the Cauchy problem

$$\frac{\partial v^Q}{\partial t}(x,t) = \Delta^{(p)} v^Q(x,t) + \left\{\frac{1}{|Q|} 1_Q(x)\right\} v^Q(x,t), \qquad v^Q(\cdot,0) \equiv 1. \tag{3.2.38}$$

Now put

$$w^Q(x,t) = v^Q(x,t) - 1. \tag{3.2.39}$$

Then (3.2.38) can be rewritten as (3.2.33). Combining (3.2.36–3.2.37) and (3.2.39), we get

$$\begin{aligned}\text{r.h.s. } (3.2.30) &= \int_\Omega \nu_\rho(d\eta) \prod_{x\in A_\eta} \left(1 + w^Q(x,t)\right) = \int_\Omega \nu_\rho(d\eta) \prod_{x\in\mathbb{Z}^d} \left(1 + \eta(x)\, w^Q(x,t)\right) \\ &= \prod_{x\in\mathbb{Z}^d} \left(1 + \rho\, w^Q(x,t)\right) \leq \exp\left[\rho \sum_{x\in\mathbb{Z}^d} w^Q(x,t)\right],\end{aligned} \tag{3.2.40}$$

where we use that ν_ρ is the Bernoulli product measure with density ρ. Summing (3.2.33) over $\mathbb{Z}^d$, we have

$$\frac{\partial}{\partial t}\left(\sum_{x\in\mathbb{Z}^d} w^Q(x,t)\right) = \sum_{x\in Q} \frac{1}{|Q|}\, w^Q(x,t) + 1. \tag{3.2.41}$$

Integrating (3.2.41) w.r.t. time, we get

$$\sum_{x\in\mathbb{Z}^d} w^Q(x,t) = \int_0^t ds \sum_{x\in Q} \frac{1}{|Q|}\, w^Q(x,s) + t. \tag{3.2.42}$$

Combining (3.2.40) and(3.2.42), we get the claim. ∎

Step 4: The proof is completed by showing the following:

Lemma 3.2.7.

$$\lim_{Q\uparrow\mathbb{Z}^d} \lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, \frac{1}{|Q|}\sum_{x\in Q} w^Q(x,s) = 0. \tag{3.2.43}$$

Proof. Let $\mathcal{G}$ denote the Green operator acting on functions $V\colon \mathbb{Z}^d \to [0,\infty)$ as

$$(\mathcal{G}V)(x) = \sum_{y\in\mathbb{Z}^d} G(x,y)V(y), \qquad x\in\mathbb{Z}^d, \tag{3.2.44}$$

where $G(x,y) = \int_0^\infty dt\, p_t(x,y)$ denotes the Green kernel on $\mathbb{Z}^d$. We have

$$\left\|\mathcal{G}\Big(\frac{1}{|Q|}1_{|Q|}\Big)\right\|_\infty = \sup_{x\in\mathbb{Z}^d} \sum_{y\in Q} G(x,y)\frac{1}{|Q|}. \tag{3.2.45}$$

The r.h.s. tends to zero as $Q\uparrow\mathbb{Z}^d$, because $G(x,y)$ tends to zero as $\|x-y\|\to\infty$. Hence Lemma 8.2.1 in Gärtner and den Hollander (10) can be applied to (3.2.34) for Q large enough, to yield

$$\sup_{\substack{x\in\mathbb{Z}^d\\ s\geq 0}} w^Q(x,s) \leq \varepsilon(Q)\downarrow 0 \quad \text{as} \quad Q\uparrow\mathbb{Z}^d, \tag{3.2.46}$$

which proves (3.2.43). ∎

Combine (3.2.29–3.2.30), (3.2.32) and (3.2.43) to get the claim in Proposition 3.2.3.
This completes the proof of Proposition 3.2.3 for $p=1$. The generalization to arbitrary p is straightforward and runs as follows. Return to (1.2.9). Separate the p terms under the sum with the help of Hölder's inequality with weights $1/p$. Next, use (3.2.14) for each of the p factors, leading to $\frac{1}{p}\log$ of the r.h.s. of (3.2.14) with an extra factor p in the exponent. Then proceed as before, which leads to Lemma 3.2.6 but with w^Q the solution of (3.2.33) with $\frac{p}{|Q|}1_Q(x)$ between braces. Then again proceed as before, which leads to (3.2.40) but with an extra factor p in the r.h.s. of (3.2.42). The latter gives a factor $e^{p\rho t}$ replacing $e^{\rho t}$ in (3.2.32). Now use Lemma 3.2.7 to get the claim. ∎

3.2.5 Proof of the strict monotonicity in Theorem 1.3.2(ii)

By Theorem 1.3.1(ii), $\kappa\mapsto\lambda_p(\kappa)$ is convex. Because of Proposition 3.2.1 and Proposition 3.2.3, it must be strictly decreasing. This completes the proof of Theorem 1.3.2(ii).

4 Lyapunov exponents: transient simple random walk

This section is devoted to the proof of Theorem 1.3.4, where $d \geq 4$ and $p(\cdot,\cdot)$ is simple random walk given by (1.2.3), i.e., ξ is simple symmetric exclusion (SSE). *The proof is long and technical*, taking up more than half of the present paper. After a time scaling in Section 4.1, an outline of the proof will be given in Section 4.2. The proof for $p = 1$ will then be carried out in Sections 4.3–4.7. In Section 4.8, we will indicate how to extend the proof to arbitrary p.

4.1 Scaling

As before, we write $X^\kappa_s, \xi_s(x)$ instead of $X^\kappa(s), \xi(x,s)$. We abbreviate

$$1[\kappa] = 1 + \frac{1}{2d\kappa}, \tag{4.1.1}$$

and write $\{a,b\}$ to denote the unoriented bond between nearest-neighbor sites $a, b \in \mathbb{Z}^d$ (recall (1.2.3)–(1.2.4)). Three parameters will be important: t, κ and T. We will take limits in the following order:

$$t \to \infty, \quad \kappa \to \infty, \quad T \to \infty. \tag{4.1.2}$$

For $t \geq 0$, let

$$Z_t = (\xi_{\frac{t}{\kappa}}, X_t) \tag{4.1.3}$$

and denote by $\mathbb{P}_{\eta,x}$ the law of Z starting from $Z_0 = (\eta, x)$. Then $Z = (Z_t)_{t\geq 0}$ is a Markov process on $\Omega \times \mathbb{Z}^d$ with generator

$$\mathcal{A} = \frac{1}{\kappa} L + \Delta \tag{4.1.4}$$

(acting on the Banach space of bounded continuous functions on $\Omega \times \mathbb{Z}^d$, equipped with the supremum norm). Abbreviate $X^\kappa_t = X_{\kappa t}$, $t \geq 0$, where $X = (X_t)_{t\geq 0}$ is simple random walk with step rate $2d$, being independent of $(\xi_t)_{t\geq 0}$. We therefore have

$$\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\ \xi_s\left(X^\kappa_s\right)\right]\right) = \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^{\kappa t} ds\ \xi_{\frac{s}{\kappa}}\left(X_s\right)\right]\right). \tag{4.1.5}$$

Define the scaled Lyapunov exponent (recall (1.2.9–1.2.10))

$$\lambda^*_1(\kappa) = \lim_{t\to\infty} \Lambda^*_1(\kappa;t) \quad \text{with} \quad \Lambda^*_1(\kappa;t) = \frac{1}{t}\log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^t ds\ \xi_{\frac{s}{\kappa}}\left(X_s\right)\right]\right). \tag{4.1.6}$$

Then $\lambda_1(\kappa) = \kappa\lambda^*_1(\kappa)$. Therefore, in what follows we will focus on the quantity

$$\lambda^*_1(\kappa) - \frac{\rho}{\kappa} = \lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^t ds\ \left(\xi_{\frac{s}{\kappa}}(X_s) - \rho\right)\right]\right) \tag{4.1.7}$$

and compute its asymptotic behavior for large κ. We must show that

$$\lim_{\kappa\to\infty} 2d\kappa^2\left[\lambda^*_1(\kappa) - \frac{\rho}{\kappa}\right] = \rho(1-\rho)G_d. \tag{4.1.8}$$

4.2 Outline

To prove (4.1.8), we have to study the asymptotics of the expectation on the r.h.s. of (4.1.7) as $t \to \infty$ and $\kappa \to \infty$ (in this order). This expectation has the form

$$\mathbb{E}_{\nu_\rho,0}\Big(\exp\Big[\frac{1}{\kappa}\int_0^t ds\, \phi(Z_s)\Big]\Big), \tag{4.2.1}$$

where $\phi(\eta, x) = \eta(x) - \rho$. Let ψ be the bounded solution of the equation

$$-\mathcal{A}\psi = \phi. \tag{4.2.2}$$

(In fact, such a solution exists only after an appropriate regularization, which turns out to be asymptotically correct for $d \geq 4$ but not for $d = 3$.) Then the term in the exponent of (4.2.1) is a martingale M_t modulo a remainder that stays bounded as $t \to \infty$:

$$\frac{1}{\kappa}\int_0^t ds\, \phi(Z_s) = M_t + \frac{1}{\kappa}[\psi(Z_0) - \psi(Z_t)] \tag{4.2.3}$$

(Lemma 4.3.1(i) below). Hence, the asymptotic investigation of (4.2.1) reduces to the study of

$$\mathbb{E}_{\nu_\rho,0}\Big(e^{M_t}\Big) = \mathbb{E}_{\nu_\rho,0}\Big(\big(N_t^r\big)^{1/r} \exp\Big[\frac{1}{r}\int_0^t ds\, \Big[\Big(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\Big) - \mathcal{A}\Big(\frac{r}{\kappa}\psi\Big)\Big](Z_s)\Big]\Big), \tag{4.2.4}$$

where

$$N_t^r = \exp\Big[rM_t - \int_0^t ds\Big[\Big(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\Big) - \mathcal{A}\Big(\frac{r}{\kappa}\psi\Big)\Big](Z_s)\Big] \tag{4.2.5}$$

is an exponential martingale (Lemma 4.3.1(iii) below) and r is close to 1. Hence, applying Hölder's inequality, we may bound the expectation in the r.h.s. of (4.2.4) from above by

$$\Big(\mathbb{E}_{\nu_\rho,0}\Big(\exp\Big[\frac{q}{r}\int_0^t ds\, \Big[\Big(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\Big) - \mathcal{A}\Big(\frac{r}{\kappa}\psi\Big)\Big](Z_s)\Big]\Big)\Big)^{1/q} \tag{4.2.6}$$

with $1/r + 1/q = 1$ (and q large). A reverse Hölder inequality shows that this is a lower bound for large negative q. Because of the structure of the expected result (coming from a linear approximation of the exponential), the choice of a large $|q|$ does not hurt. (This is not true for the result in Conjecture 1.4.1 pertaining to $d = 3$.) Hence, the whole proof essentially reduces to the derivation of an appropriate upper bound for

$$\mathbb{E}_{\nu_\rho,0}\Big(\exp\Big[\frac{\alpha}{r}\int_0^t ds\, \Big[\Big(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\Big) - \mathcal{A}\Big(\frac{r}{\kappa}\psi\Big)\Big](Z_s)\Big]\Big) \tag{4.2.7}$$

with arbitrary $\alpha \in \mathbb{R}$ (c.f. Proposition 4.4.1 below). A Taylor expansion up to second order shows that

$$\Big[\Big(e^{-\frac{r}{\kappa}\psi}\mathcal{A}\, e^{\frac{r}{\kappa}\psi}\Big) - \mathcal{A}\,\Big(\frac{r}{\kappa}\psi\Big)\Big](\eta, x) = \frac{r^2}{2\kappa^2}\sum_{e:\ \|e\|=1}\Big(\psi(\eta, x+e) - \psi(\eta, x)\Big)^2 + O\Big(\Big(\frac{1}{\kappa}\Big)^3\Big) \tag{4.2.8}$$

as $\kappa \to \infty$ (Lemma 4.6.1 below). The expression under the integral in (4.2.7) depends on the process $(X_s)_{s\geq 0}$. A combination of the spectral representation of the associated semigroup with the Rayleigh-Ritz formula shows that, asymptotically as $t \to \infty$, the expectation in (4.2.7) gets larger when we replace X_s by 0. Using an explicit representation of ψ, we see that

$$\begin{aligned}\sum_{e\colon\ \|e\|=1} \big(\psi(\eta,e)-\psi(\eta,0)\big)^2 = &\sum_{z\in\mathbb{Z}^d} K_{\rm diag}(z)\big(\eta(z)-\rho\big)^2\\ &+\sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}} K_{\rm off}(z_1,z_2)\big(\eta(z_1)-\rho\big)\big(\eta(z_2)-\rho\big)\end{aligned} \tag{4.2.9}$$

for certain kernels $K_{\rm diag}$ and $K_{\rm off}$ (Lemma 4.6.2 below). Substituting this into the previous formulas and separating the “diagonal” term from the “off-diagonal” term by use of the Cauchy-Schwarz inequality, we finally see that the whole proof reduces to showing that

$$\limsup_{\kappa\to\infty}\limsup_{t\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds\sum_{z\in\mathbb{Z}^d}K_{\rm diag}(z)\Big(\xi_{\frac{s}{\kappa}}(z)-\rho\Big)^2\right]\right)\leq \alpha r\,\rho(1-\rho)\frac{1}{d}G_d \tag{4.2.10}$$

and

$$\limsup_{\kappa\to\infty}\limsup_{t\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds\sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}}K_{\rm off}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\right)\leq 0 \tag{4.2.11}$$

(Lemmas 4.6.3 and 4.6.4 below). To prove the latter statements, we use Jensen's inequality to move the kernels $K_{\rm diag}$ and $K_{\rm off}$ out of the exponents. Then we are left with the derivation of upper bounds for terms of the form

$$\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\beta}{\kappa^2}\int_0^t ds\,\Big(\xi_{\frac{s}{\kappa}}(z)-\rho\Big)^2\right]\right),\qquad z\in\mathbb{Z}^d, \tag{4.2.12}$$

and

$$\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\beta}{\kappa^2}\int_0^t ds\,\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\right),\qquad z_1,z_2\in\mathbb{Z}^d,\ z_1\neq z_2 \tag{4.2.13}$$

(Lemmas 4.6.8 and 4.6.10 below). The first expectation can be handled with the help of the IRW approximation (Proposition 1.2.1). The handling of the second expectation is more involved and requires, in addition, spectral methods.

4.3 SSE+RW generator and an auxiliary exponential martingale

Recall (4.1.3–4.1.4). Let $(\mathcal{P}_t)_{t\geq 0}$ be the semigroup generated by $\mathcal{A}$. The following lemma will be crucial to rewrite the expectation in the r.h.s. of (4.1.7) in a more manageable form.

Lemma 4.3.1. *Fix $\kappa>0$ and $r>0$. For all $t\geq 0$ and all bounded continuous functions $\psi\colon\ \Omega\times\mathbb{Z}^d\to\mathbb{R}$ such that ψ and $\exp[(r/\kappa)\psi]$ belong to the domain of $\mathcal{A}$, define*

$$\begin{aligned} M_t^r &= \frac{r}{\kappa}\left[\psi(Z_t)-\psi(Z_0)-\int_0^t ds\,\mathcal{A}\psi(Z_s)\right],\\ N_t^r &= \exp\left[M_t^r-\int_0^t ds\,\left[\left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\right)-\mathcal{A}\left(\frac{r}{\kappa}\psi\right)\right](Z_s)\right].\end{aligned} \tag{4.3.1}$$

Then:
(i) $M^r = (M_t^r)_{t\geq 0}$ *is a* $\mathbb{P}_{\eta,x}$*-martingale for all* (η, x)*.*
(ii) For $t \geq 0$*, let* $\mathcal{P}_t^{\mathrm{new}}$ *be the operator defined by*

$$(\mathcal{P}_t^{\mathrm{new}} f)(\eta, x) = e^{-\frac{r}{\kappa}\psi(\eta,x)}\,\mathbb{E}_{\eta,x}\left(\exp\left[-\int_0^t ds\,\left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\right)(Z_s)\right]\left(e^{\frac{r}{\kappa}\psi} f\right)(Z_t)\right) \tag{4.3.2}$$

for bounded continuous $f\colon \Omega \times \mathbb{Z}^d \to \mathbb{R}$*. Then* $(\mathcal{P}_t^{\mathrm{new}})_{t\geq 0}$ *is a strongly continuous semigroup with generator*

$$(\mathcal{A}^{\mathrm{new}} f)(\eta, x) = \left[e^{-\frac{r}{\kappa}\psi}\mathcal{A}\left(e^{\frac{r}{\kappa}\psi} f\right) - \left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\right) f\right](\eta, x). \tag{4.3.3}$$

(iii) $N^r = (N_t^r)_{t\geq 0}$ *is a* $\mathbb{P}_{\eta,x}$*-martingale for all* (η, x)*.*
(iv) Define a new path measure $\mathbb{P}_{\eta,x}^{\mathrm{new}}$ *by putting*

$$\frac{d\,\mathbb{P}_{\eta,x}^{\mathrm{new}}}{d\,\mathbb{P}_{\eta,x}}\left((Z_s)_{0\leq s\leq t}\right) = N_t^r, \qquad t \geq 0. \tag{4.3.4}$$

Then, under $\mathbb{P}_{\eta,x}^{\mathrm{new}}$*,* $(Z_t)_{t\geq 0}$ *is a Markov process with semigroup* $(\mathcal{P}_t^{\mathrm{new}})_{t\geq 0}$*.*

Proof. The proof is standard.
(i) This follows from the fact that $\mathcal{A}$ is a Markov generator and ψ belongs to its domain (see Liggett (19), Chapter I, Section 5).
(ii) Let $\eta \in \Omega$, $x \in \mathbb{Z}^d$ and $f\colon \Omega \times \mathbb{Z}^d \to \mathbb{R}$ bounded measurable. Rewrite (4.3.2) as

$$\begin{aligned}(\mathcal{P}_t^{\mathrm{new}} f)(\eta, x) &= \mathbb{E}_{\eta,x}\left(\exp\left[\frac{r}{\kappa}\psi(Z_t) - \frac{r}{\kappa}\psi(Z_0) - \int_0^t ds\,\left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\right)(Z_s)\right] f(Z_t)\right)\\ &= \mathbb{E}_{\eta,x}\left(N_t^r f(Z_t)\right).\end{aligned} \tag{4.3.5}$$

This gives

$$(\mathcal{P}_0^{\mathrm{new}} f)(\eta, x) = f(\eta, x) \tag{4.3.6}$$

and

$$\begin{aligned}(\mathcal{P}_{t_1+t_2}^{\mathrm{new}} f)(\eta, x) &= \mathbb{E}_{\eta,x}\left(N_{t_1+t_2}^r f(Z_{t_1+t_2})\right) = \mathbb{E}_{\eta,x}\left(N_{t_1}^r \frac{N_{t_1+t_2}^r}{N_{t_1}^r} f(Z_{t_1+t_2})\right)\\ &= \mathbb{E}_{\eta,x}\Big(N_{t_1}^r \mathbb{E}_{Z_{t_1}}\left(N_{t_2}^r f(Z_{t_2})\right)\Big) = \Big(\mathcal{P}_{t_1}^{\mathrm{new}}(\mathcal{P}_{t_2}^{\mathrm{new}} f)\Big)(\eta, x),\end{aligned} \tag{4.3.7}$$

where we use the Markov property of Z at time t_1 (under $\mathbb{P}_{\eta,x}$) together with the fact that $N_{t_1+t_2}^r / N_{t_1}^r$ only depends on Z_t for $t \in [t_1, t_1 + t_2]$. Equations (4.3.6–4.3.7) show that $(\mathcal{P}_t^{\mathrm{new}})_{t\geq 0}$ is a semigroup which is easily seen to be strongly continuous.
Taking the derivative of (4.3.2) in the norm w.r.t. t at $t = 0$, we get (4.3.3). Next, if $f \equiv 1$, then (4.3.3) gives $\mathcal{A}^{\mathrm{new}}1 = 0$. This last equality implies that

$$\frac{1}{\lambda}\left(\lambda Id - \mathcal{A}^{\mathrm{new}}\right)1 = 1 \quad \forall\,\lambda > 0. \tag{4.3.8}$$

Since $\lambda Id - \mathcal{A}^{\mathrm{new}}$ is invertible, we get

$$\left(\lambda Id - \mathcal{A}^{\mathrm{new}}\right)^{-1} 1 = \frac{1}{\lambda} \quad \forall\,\lambda > 0, \tag{4.3.9}$$

i.e.,

$$\int_0^\infty dt\, e^{-\lambda t}\, \mathcal{P}_t^{\mathrm{new}} 1 = \frac{1}{\lambda} \quad \forall\, \lambda > 0. \tag{4.3.10}$$

Inverting this Laplace transform, we see that

$$\mathcal{P}_t^{\mathrm{new}} 1 = 1 \quad \forall\, t \geq 0. \tag{4.3.11}$$

(iii) Fix $t \geq 0$ and $h > 0$. Since N_t^r is $\mathcal{F}_t$-measurable, with $\mathcal{F}_t$ the σ-algebra generated by $(Z_s)_{0 \leq s \leq t}$, we have

$$\begin{aligned} &\mathbb{E}_{\eta,x}\left(N_{t+h}^r \middle| \mathcal{F}_t\right) \\ &= N_t^r\, \mathbb{E}_{\eta,x}\left(\exp\left[M_{t+h}^r - M_t^r - \int_t^{t+h} ds\, \left[\left(e^{-\frac{r}{\kappa}\psi} \mathcal{A} e^{\frac{r}{\kappa}\psi}\right) - \mathcal{A}\left(\frac{r}{\kappa}\psi\right)\right](Z_s)\right] \middle| \mathcal{F}_t\right). \end{aligned} \tag{4.3.12}$$

Applying the Markov property of Z at time t, we get

$$\begin{aligned} \mathbb{E}_{\eta,x}\left(N_{t+h}^r \mid \mathcal{F}_t\right) &= N_t^r\, \mathbb{E}_{Z_t}\left(\exp\left[\frac{r}{\kappa}\psi(Z_h) - \frac{r}{\kappa}\psi(Z_0) - \int_0^h ds\, \left(e^{-\frac{r}{\kappa}\psi} \mathcal{A} e^{\frac{r}{\kappa}\psi}\right)(Z_s)\right]\right) \\ &= N_t^r\, \left(\mathcal{P}_h^{\mathrm{new}} 1\right)(Z_t) = N_t^r, \end{aligned} \tag{4.3.13}$$

where the third equality uses (4.3.11).

(iv) This follows from (iii) via a calculation similar to (4.3.7). ∎

4.4 Proof of Theorem 1.3.4

In this section we compute upper and lower bounds for the r.h.s. of (4.1.7) in terms of certain key quantities (Proposition 4.4.1 below). We then state two propositions for these quantities (Propositions 4.4.2–4.4.3 below), from which Theorem 1.3.4 will follow. The proof of these two propositions is given in Sections 4.6–4.7.

For $T > 0$, let $\psi\colon\ \Omega \times \mathbb{Z}^d$ be defined by

$$\psi(\eta, x) = \int_0^T ds\, (\mathcal{P}_s \phi)(\eta, x) \quad \text{with} \quad \phi(\eta, x) = \eta(x) - \rho, \tag{4.4.1}$$

where $(\mathcal{P}_t)_{t \geq 0}$ is the semigroup generated by $\mathcal{A}$ (recall (4.1.4)). We have

$$\psi(\eta, x) = \int_0^T ds\, \mathbb{E}_{\eta,x}\left(\phi(Z_s)\right) = \int_0^T ds\, \mathbb{E}_\eta \sum_{y \in \mathbb{Z}^d} p_{2ds}(y, x)\Big(\xi_{\frac{s}{\kappa}}(y) - \rho\Big), \tag{4.4.2}$$

where $p_t(x, y)$ is the probability that simple random walk with step rate 1 moves from x to y in time t (recall that we assume (1.2.3)). Using (1.2.6), we obtain the representation

$$\psi(\eta, x) = \int_0^T ds \sum_{z \in \mathbb{Z}^d} p_{2ds1[\kappa]}(z, x)\big[\eta(z) - \rho\big], \tag{4.4.3}$$

where $1[\kappa]$ is given by (4.1.1). Note that ψ depends on κ and T. We suppress this dependence. Similarly,

$$-\mathcal{A}\psi = \int_0^T ds\,(-\mathcal{A}\mathcal{P}_s\phi) = \phi - \mathcal{P}_T\phi, \tag{4.4.4}$$

with

$$(\mathcal{P}_T\phi)(\eta,x) = \mathbb{E}_{\eta,x}\left(\phi(Z_T)\right) = \mathbb{E}_{\eta,x}\left(\xi_{\frac{T}{\kappa}}(X_T)-\rho\right) = \sum_{z\in\mathbb{Z}^d} p_{2dT1[\kappa]}(z,x)\,[\eta(z)-\rho]. \tag{4.4.5}$$

The auxiliary function ψ will play a key role throughout the remaining sections. The integral in (4.4.1) is a regularization that is useful when dealing with central limit type behavior of Markov processes (see e.g. Kipnis (17)). Heuristically, $T=\infty$ corresponds to $-\mathcal{A}\psi=\phi$. Later we will let $T\to\infty$.

The following proposition serves as the starting point of our asymptotic analysis.

Proposition 4.4.1. *For any $\kappa, T>0$,*

$$\lambda_1^*(\kappa) - \frac{\rho}{\kappa} \begin{array}{c}\le\\ \ge\end{array} I_1^{r,q}(\kappa,T) + I_2^{r,q}(\kappa,T), \tag{4.4.6}$$

where

$$\begin{aligned} I_1^{r,q}(\kappa,T) &= \frac{1}{2q}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{2q}{r}\int_0^t ds\,\left[\left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\right)-\mathcal{A}\left(\frac{r}{\kappa}\psi\right)\right](Z_s)\right]\right),\\ I_2^{r,q}(\kappa,T) &= \frac{1}{2q}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{2q}{\kappa}\int_0^t ds\,\left(\mathcal{P}_T\phi\right)(Z_s)\right]\right), \end{aligned} \tag{4.4.7}$$

and $1/r+1/q=1$ for any $r,q>1$ in the first inequality and any $q<0<r<1$ in the second inequality.

Proof. Recall (4.1.7). From the first line of (4.3.1) and (4.4.4) it follows that

$$\frac{1}{r}M_t^r + \frac{1}{\kappa}\psi(Z_0) - \frac{1}{\kappa}\psi(Z_t) = \frac{1}{\kappa}\int_0^t ds\,[(-\mathcal{A})\psi](Z_s) = \frac{1}{\kappa}\int_0^t ds\,\phi(Z_s) - \frac{1}{\kappa}\int_0^t ds\,\left(\mathcal{P}_T\phi\right)(Z_s). \tag{4.4.8}$$

Hence

$$\begin{aligned} &\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^t ds\,\phi(Z_s)\right]\right)\\ &\qquad = \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{r}M_t^r + \frac{1}{\kappa}\psi(Z_0) - \frac{1}{\kappa}\psi(Z_t) + \frac{1}{\kappa}\int_0^t ds\,(\mathcal{P}_T\phi)(Z_s)\right]\right)\\ &\qquad = \mathbb{E}_{\nu_\rho,0}\left(\exp\left[U_t^r + \frac{1}{r}V_t^r\right]\right) \end{aligned} \tag{4.4.9}$$

with

$$U_t^r = \frac{1}{r}\int_0^t ds\,\left[\left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}e^{\frac{r}{\kappa}\psi}\right) - \mathcal{A}\left(\frac{r}{\kappa}\psi\right)\right](Z_s) + \frac{1}{\kappa}\Big(\psi(Z_0)-\psi(Z_t)\Big) + \frac{1}{\kappa}\int_0^t ds\,\left(\mathcal{P}_T\phi\right)(Z_s) \tag{4.4.10}$$

and

$$V_t^r = M_t^r - \int_0^t ds\, \left[\left(e^{-\frac{r}{\kappa}\psi} \mathcal{A} e^{\frac{r}{\kappa}\psi} \right) - \mathcal{A}\left(\frac{r}{\kappa}\psi \right) \right] (Z_s). \tag{4.4.11}$$

By Hölder's inequality, with $r, q > 1$ such that $1/r + 1/q = 1$, it follows from (4.4.9) that

$$\begin{aligned} \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{1}{\kappa} \int_0^t ds\, \phi(Z_s) \right] \right) &\leq \Big(\mathbb{E}_{\nu_\rho,0}\big(\exp\left[V_t^r \right] \big) \Big)^{1/r} \Big(\mathbb{E}_{\nu_\rho,0}\big(\exp\left[q U_t^r \right] \big) \Big)^{1/q} \\ &= \Big(\mathbb{E}_{\nu_\rho,0}\big(\exp\left[q U_t^r \right] \big) \Big)^{1/q}, \end{aligned} \tag{4.4.12}$$

where the second line of (4.4.12) comes from the fact that $N_t^r = \exp[V_t^r]$ is a martingale, by Lemma 4.3.1(iii). Similarly, by the reverse of Hölder's inequality, with $q < 0 < r < 1$ such that $1/r + 1/q = 1$, it follows from (4.4.9) that

$$\begin{aligned} \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{1}{\kappa} \int_0^t ds\, \phi(Z_s) \right] \right) &\geq \Big(\mathbb{E}_{\nu_\rho,0}\big(\exp\left[V_t^r \right] \big) \Big)^{1/r} \Big(\mathbb{E}_{\nu_\rho,0}\big(\exp\left[q U_t^r \right] \big) \Big)^{1/q} \\ &= \Big(\mathbb{E}_{\nu_\rho,0}\big(\exp\left[q U_t^r \right] \big) \Big)^{1/q}. \end{aligned} \tag{4.4.13}$$

The middle term in the r.h.s. of (4.4.10) can be discarded, because (4.4.3) shows that $-\rho T \leq \psi \leq (1-\rho)T$. Apply the Cauchy-Schwarz inequality to the r.h.s. of (4.4.12–4.4.13) to separate the other two terms in the r.h.s. of (4.4.10). ∎

Note that in the r.h.s. of (4.4.7) the prefactors of the logarithms and the prefactors in the exponents are *both positive for the upper bound and both negative for the lower bound*. This will be important later on.

The following two propositions will be proved in Sections 4.6-4.7, respectively. Abbreviate

$$\limsup_{t,\kappa,T\to\infty} = \limsup_{T\to\infty} \limsup_{\kappa\to\infty} \limsup_{t\to\infty}. \tag{4.4.14}$$

Proposition 4.4.2. *If $d \geq 3$, then for any $\alpha \in \mathbb{R}$ and $r > 0$,*

$$\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{r} \int_0^t ds\, \left[e^{-\frac{r}{\kappa}\psi} \mathcal{A} e^{\frac{r}{\kappa}\psi} - \mathcal{A}\Big(\frac{r}{\kappa}\psi \Big) \right] (Z_s) \right] \right) \leq \alpha r\, \rho(1-\rho) \frac{1}{2d} G_d. \tag{4.4.15}$$

Proposition 4.4.3. *If $d \geq 4$, then for any $\alpha \in \mathbb{R}$,*

$$\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathbb{E}_{\nu_\rho,0}\bigg(\exp\left[\frac{\alpha}{\kappa} \int_0^t ds\, \left(\mathcal{P}_T \phi \right)(Z_s) \right] \bigg) \leq 0. \tag{4.4.16}$$

Picking $\alpha = 2q$ in Proposition 4.4.2, we see that the first term in the r.h.s. of (4.4.6) satisfies the bounds

$$\begin{aligned} \limsup_{T\to\infty} \limsup_{\kappa\to\infty} \kappa^2 I_1^{r,q}(\kappa, T) &\leq r\, \rho(1-\rho) \frac{1}{2d} G_d \quad \text{if } r > 1, \\ \liminf_{T\to\infty} \liminf_{\kappa\to\infty} \kappa^2 I_1^{r,q}(\kappa, T) &\geq r\, \rho(1-\rho) \frac{1}{2d} G_d \quad \text{if } r < 1. \end{aligned} \tag{4.4.17}$$

Letting r tend to 1, we obtain

$$\lim_{T\to\infty}\lim_{\kappa\to\infty}\kappa^2 I_1^{r,q}(\kappa,T)=\rho(1-\rho)\frac{1}{2d}G_d. \tag{4.4.18}$$

Picking $\alpha=2q$ in Proposition 4.4.3, we see that the second term in the r.h.s. of (4.4.6) satisfies

$$\limsup_{T\to\infty}\limsup_{\kappa\to\infty}\kappa^2 I_2^{r,q}(\kappa,T)=0 \qquad \text{if } d\geq 4. \tag{4.4.19}$$

Combining (4.4.18–4.4.19), we see that we have completed the proof of Theorem 1.3.4 for $d\geq 4$. In order to prove Conjecture 1.4.1, we would have to extend Proposition 4.4.3 to $d=3$ and show that it contributes the second term in the r.h.s. of (4.4.16) rather than being negligible.

4.5 Preparatory facts and notation

In order to estimate $I_1^{r,q}(\kappa,T)$ and $I_2^{r,q}(\kappa,T)$, we need a number of preparatory facts. These are listed in Lemmas 4.5.1–4.5.4 below.

It follows from (4.4.3) that

$$\psi(\eta,b)-\psi(\eta,a)=\int_0^T ds\sum_{z\in\mathbb{Z}^d}\big(p_{2ds1[\kappa]}(z,b)-p_{2ds1[\kappa]}(z,a)\big)[\eta(z)-\rho] \tag{4.5.1}$$

and

$$\begin{aligned}\psi(\eta^{a,b},x)-\psi(\eta,x)&=\int_0^T ds\sum_{z\in\mathbb{Z}^d}p_{2ds1[\kappa]}(z,x)\left[\eta^{a,b}(z)-\eta(z)\right]\\&=\int_0^T ds\,\big(p_{2ds1[\kappa]}(b,x)-p_{2ds1[\kappa]}(a,x)\big)\,[\eta(a)-\eta(b)]\,,\end{aligned} \tag{4.5.2}$$

where we recall the definitions of $1[\kappa]$ and $\eta^{a,b}$ in (4.1.1) and (1.2.5), respectively. We need bounds on both these differences.

Lemma 4.5.1. *For any $\eta\in\Omega$, $a,b,x\in\mathbb{Z}^d$ and $\kappa,T>0$,*

$$\big|\psi(\eta,b)-\psi(\eta,a)\big|\leq 2T, \tag{4.5.3}$$

$$\Big|\psi\big(\eta^{a,b},x\big)-\psi(\eta,x)\Big|\leq 2G_d<\infty, \tag{4.5.4}$$

and

$$\sum_{\{a,b\}}\Big(\psi\big(\eta^{a,b},x\big)-\psi(\eta,x)\Big)^2\leq\frac{1}{2d}G_d<\infty, \tag{4.5.5}$$

where G_d is the Green function at the origin of simple random walk.

Proof. The bound in (4.5.3) is immediate from (4.5.1). By (4.5.2), we have

$$\Big|\psi\big(\eta^{a,b},x\big)-\psi(\eta,x)\Big|\leq\int_0^T ds\,\big|p_{2ds1[\kappa]}(b,x)-p_{2ds1[\kappa]}(a,x)\big|\,. \tag{4.5.6}$$

Using the bound $p_t(x,y) \le p_t(0,0)$ (which is immediate from the Fourier representation of the transition kernel), we get

$$\Big|\psi(\eta^{a,b},x) - \psi(\eta,x)\Big| \le 2\int_0^\infty ds\, p_{2ds1[\kappa]}(0,0) \le 2G_d. \tag{4.5.7}$$

Again by (4.5.2), we have

$$\begin{aligned}
\sum_{\{a,b\}} \Big(\psi(\eta^{a,b},x) - \psi(\eta,x)\Big)^2 &= \sum_{\{a,b\}} [\eta(a)-\eta(b)]^2 \left(\int_0^T ds\, \Big(p_{2ds1[\kappa]}(b,x) - p_{2ds1[\kappa]}(a,x)\Big)\right)^2 \\
&\le 2\int_0^T du \int_u^T dv \sum_{\{a,b\}} \Big(p_{2du1[\kappa]}(b,x) - p_{2du1[\kappa]}(a,x)\Big)\Big(p_{2dv1[\kappa]}(b,x) - p_{2dv1[\kappa]}(a,x)\Big) \\
&= -2\int_0^T du \int_u^T dv \sum_{a\in\mathbb{Z}^d} p_{2du1[\kappa]}(a,x)\Big[\Delta_1 p_{2dv1[\kappa]}(a,x)\Big] \\
&= -\frac{2}{1[\kappa]}\int_0^T du \int_u^T dv \sum_{a\in\mathbb{Z}^d} p_{2du1[\kappa]}(a,x)\left[\frac{\partial}{\partial v} p_{2dv1[\kappa]}(a,x)\right] \\
&= -\frac{2}{1[\kappa]}\int_0^T du \sum_{a\in\mathbb{Z}^d} p_{2du1[\kappa]}(a,x)\big(p_{2dT1[\kappa]}(a,x) - p_{2du1[\kappa]}(a,x)\big) \\
&\le \frac{2}{1[\kappa]}\int_0^T du \sum_{a\in\mathbb{Z}^d} p^2_{2du1[\kappa]}(a,x) \\
&\le \frac{2}{1[\kappa]}\int_0^\infty du\, p_{4du1[\kappa]}(0,0) = \frac{1}{2d(1[\kappa])^2} G_d(0) \le \frac{1}{2d} G_d,
\end{aligned} \tag{4.5.8}$$

where Δ_1 denotes the discrete Laplacian acting on the first coordinate, and in the fifth line we use that $(\partial/\partial t)p_t = (1/2d)\Delta_1 p_t$. ∎

For $x \in \mathbb{Z}^d$, let $\tau_x\colon \Omega \to \Omega$ be the x-shift on Ω defined by

$$\big(\tau_x\eta\big)(z) = \eta(z+x), \qquad \eta\in\Omega,\ z\in\mathbb{Z}^d. \tag{4.5.9}$$

Lemma 4.5.2. *For any bounded measurable $W\colon \Omega\times\mathbb{Z}^d \to \mathbb{R}$,*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\bigg(\exp\bigg[\int_0^t ds\ W\big(\xi_{\frac{s}{\kappa}}, X_s\big)\bigg]\bigg) \\
&\qquad \le \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho}\bigg(\exp\bigg[\int_0^t ds\ W\big(\xi_{\frac{s}{\kappa}}, 0\big)\bigg]\bigg),
\end{aligned} \tag{4.5.10}$$

provided

$$W(\eta,x) = W(\tau_x\eta, 0) \quad \forall\, \eta\in\Omega,\ x\in\mathbb{Z}^d. \tag{4.5.11}$$

Proof. The proof uses arguments similar to those in Section 2.2. Recall (4.1.3). Proposition 2.2.2 with $p=1$ and Remark 2.2.3, applied to the self-adjoint operator $G^\kappa_W = \frac{1}{\kappa}L + \Delta + W$

(instead of G_V^κ in (2.2.4–2.2.5)), gives

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\, W(Z_s)\right]\right) = \sup_{\|f\|_{L^2(\nu_\rho\otimes m)}=1} \left(B_1(f) - \frac{1}{\kappa}B_2(f) - B_3(f)\right) \tag{4.5.12}$$

with

$$\begin{aligned} B_1(f) &= \int_\Omega \nu_\rho(d\eta) \sum_{z\in\mathbb{Z}^d} W(\eta,z)\, f(\eta,z)^2, \\ B_2(f) &= \int_\Omega \nu_\rho(d\eta) \sum_{z\in\mathbb{Z}^d} \frac{1}{2} \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)[f(\eta^{x,y},z) - f(\eta,z)]^2, \\ B_3(f) &= \int_\Omega \nu_\rho(d\eta) \sum_{z\in\mathbb{Z}^d} \frac{1}{2} \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-z\|=1}} [f(\eta,y) - f(\eta,z)]^2. \end{aligned} \tag{4.5.13}$$

An upper bound is obtained by dropping $B_3(f)$, i.e., the part associated with the simple random walk X. After that, split the supremum into two parts,

$$\begin{aligned} &\sup_{\|f\|_{L^2(\nu_\rho\otimes m)}=1} \Big(B_1(f) - B_2(f)\Big) \\ &= \sup_{\|g\|_{L^2(m)}=1} \; \sup_{\|f_z\|_{L^2(\nu_\rho)}=1\,\forall\, z\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} g(z)^2 \int_\Omega \nu_\rho(d\eta) \\ &\qquad \times \left(W(\eta,z)\, f_z(\eta)^2 - \frac{1}{2} \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)[f_z(\eta^{x,y}) - f_z(\eta)]^2\right), \end{aligned} \tag{4.5.14}$$

where $f_z(\eta) = f(\eta,z)/g(z)$ with $g(z)^2 = \int_\Omega \nu_\rho(d\eta) f(\eta,z)^2$. The second supremum in (4.5.14), which runs over a family of functions indexed by z, can be brought under the sum. This gives

$$\begin{aligned} \text{r.h.s. (4.5.14)} &= \sup_{\|g\|_{L^2(m)}=1} \sum_{z\in\mathbb{Z}^d} g(z)^2 \sup_{\|f_z\|_{L^2(\nu_\rho)}=1} \int_\Omega \nu_\rho(d\eta) \\ &\qquad \times \left(W(\eta,z)\, f_z(\eta)^2 - \frac{1}{2} \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)[f_z(\eta^{x,y}) - f_z(\eta)]^2\right). \end{aligned} \tag{4.5.15}$$

By (4.5.11) and the shift-invariance of ν_ρ, we may replace z by 0 under the second supremum in (4.5.15), in which case the latter no longer depends on z, and we get

$$\begin{aligned} \text{r.h.s. (4.5.15)} &= \sup_{\|f\|_{L^2(\nu_\rho)}=1} \int_\Omega \nu_\rho(d\eta) \left[W(\eta,0)\, f(\eta)^2 - \frac{1}{2} \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)[f(\eta^{x,y}) - f(\eta)]^2\right] \\ &= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho}\left(\exp\left[\int_0^t ds\; W\left(\xi_{\frac{s}{\kappa}}, 0\right)\right]\right), \end{aligned} \tag{4.5.16}$$

where the second equality comes from the analogue of Proposition 2.2.2 with self-adjoint operator $\frac{1}{\kappa}L + W(\cdot,0)$ (instead of G_V^κ), cf. Remark 2.2.3. ∎

Lemma 4.5.3. *For any $\rho \in (0,1)$,*

$$\max_{\beta\in[0,1]} \big[\gamma\beta - \Psi_d(\beta)\big] \sim \gamma\rho \quad as \quad \gamma \downarrow 0. \tag{4.5.17}$$

Proof. First, using that $\Psi_d(\rho) = 0$, we obtain the lower bound

$$\max_{\beta\in[0,1]} \big[\gamma\beta - \Psi_d(\beta)\big] \geq \gamma\rho - \Psi_d(\rho) = \gamma\rho. \tag{4.5.18}$$

Next, for any $\delta > 0$, we have

$$\begin{aligned} \max_{\beta\in[0,1]} \big[\gamma\beta - \Psi_d(\beta)\big] &= \max_{\substack{\beta\in[0,1]\\ |\beta-\rho|\geq\delta}} \big[\gamma\beta - \Psi_d(\beta)\big] \vee \max_{\substack{\beta\in[0,1]\\ |\beta-\rho|<\delta}} \big[\gamma\beta - \Psi_d(\beta)\big] \\ &\leq \left(\gamma - \min_{\substack{\beta\in[0,1]\\ |\beta-\rho|\geq\delta}} \Psi_d(\beta)\right) \vee \big(\gamma(\rho+\delta)\big) \\ &\leq \gamma(\rho+\delta) \qquad \text{for } 0 < \gamma \leq \gamma_0(\delta), \end{aligned} \tag{4.5.19}$$

where in the second inequality we use that Ψ_d has a unique zero at ρ. Letting $\gamma \downarrow 0$ followed by $\delta \downarrow 0$, we get the desired upper bound. ∎

Lemma 4.5.4. *There exists $C > 0$ such that, for all $t \geq 0$ and $x, y \in \mathbb{Z}^d$,*

$$p_t(x,y) \leq \frac{C}{(1+t)^{\frac{d}{2}}}. \tag{4.5.20}$$

Proof. This is a standard fact. Indeed, we can decompose the transition kernel of simple random walk with step rate 1 as

$$p_{dt}(x,y) = \prod_{j=1}^{d} p_t^{(1)}(x^j, y^j), \qquad x = (x^1,\dots,x^d),\ y = (y^1,\dots,y^d), \tag{4.5.21}$$

where $p_t^{(1)}(x,y)$ is the transition kernel of 1-dimensional simple random walk with step rate 1. In Fourier representation,

$$p_t^{(1)}(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, e^{ik\cdot(y-x)}\, e^{-t\hat{\varphi}(k)}, \;\; \hat{\varphi}(k) = 1 - \cos k. \tag{4.5.22}$$

The bound in (4.5.20) follows from (4.5.21) and

$$p_t^{(1)}(x,y) \leq p_t^{(1)}(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, e^{-t\hat{\varphi}(k)} \leq \frac{C}{(1+t)^{\frac{1}{2}}}, \quad t \geq 0,\ x,y \in \mathbb{Z}^d. \tag{4.5.23}$$

∎

4.6 Proof of Proposition 4.4.2

The proof of Proposition 4.4.2 is given in Section 4.6.1 subject to four lemmas. The latter will be proved in Sections 4.6.2–4.6.5, respectively. All results are valid for $d \geq 3$.

4.6.1 Proof of Proposition 4.4.2

Lemma 4.6.1. *Uniformly in $\eta \in \Omega$ and $x \in \mathbb{Z}^d$, as $\kappa \to \infty$,*

$$\left[\left(e^{-\frac{r}{\kappa}\psi}\mathcal{A}\, e^{\frac{r}{\kappa}\psi}\right) - \mathcal{A}\left(\frac{r}{\kappa}\psi\right)\right](\eta, x) = \frac{r^2}{2\kappa^2} \sum_{e:\ \|e\|=1} \Big(\psi(\eta, x+e) - \psi(\eta, x)\Big)^2 + O\bigg(\Big(\frac{1}{\kappa}\Big)^3\bigg). \tag{4.6.1}$$

Lemma 4.6.2. *For any $\kappa, T > 0$, $\alpha \in \mathbb{R}$ and $r > 0$,*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho, 0}\Bigg(\exp\left[\frac{\alpha r}{2\kappa^2}\int_0^t ds \sum_{e:\ \|e\|=1} \Big(\psi\Big(\xi_{\frac{s}{\kappa}}, X_s + e\Big) - \psi\Big(\xi_{\frac{s}{\kappa}}, X_s\Big)\Big)^2\right]\Bigg) \\
&\le \limsup_{t\to\infty} \frac{1}{2t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{z\in\mathbb{Z}^d} K^{\kappa,T}_{\text{diag}}(z)\Big(\xi_{\frac{s}{\kappa}}(z) - \rho\Big)^2\right]\Bigg) \\
&\quad + \limsup_{t\to\infty} \frac{1}{2t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{\substack{z_1,z_2\in\mathbb{Z}^d \\ z_1\neq z_2}} K^{\kappa,T}_{\text{off}}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1) - \rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2) - \rho\Big)\right]\Bigg),
\end{aligned} \tag{4.6.2}$$

where

$$\begin{aligned}
K^{\kappa,T}_{\text{diag}}(z) &= \sum_{e:\ \|e\|=1} \Big(\chi(z+e) - \chi(z)\Big)^2, \\
K^{\kappa,T}_{\text{off}}(z_1,z_2) &= \sum_{e:\|e\|=1} \Big(\chi(z_1+e) - \chi(z_1)\Big)\Big(\chi(z_2+e) - \chi(z_2)\Big), \quad z_1 \neq z_2,
\end{aligned} \tag{4.6.3}$$

with

$$\chi(z) = \int_0^T du\, p_{2du1[\kappa]}(0,z). \tag{4.6.4}$$

Lemma 4.6.3. *For any $\alpha \in \mathbb{R}$ and $r > 0$,*

$$\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{z\in\mathbb{Z}^d} K^{\kappa,T}_{\text{diag}}(z)\Big(\xi_{\frac{s}{\kappa}}(z) - \rho\Big)^2\right]\Bigg) \le \alpha r\, \rho(1-\rho)\frac{1}{d}G_d. \tag{4.6.5}$$

Lemma 4.6.4. *For any $\alpha \in \mathbb{R}$ and $r > 0$,*

$$\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{\substack{z_1,z_2\in\mathbb{Z}^d \\ z_1\neq z_2}} K^{\kappa,T}_{\text{off}}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1) - \rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2) - \rho\Big)\right]\Bigg) \le 0. \tag{4.6.6}$$

Combining Lemmas 4.6.1–4.6.4, we obtain the claim in Proposition 4.4.2.

4.6.2 Proof of Lemma 4.6.1

Lemma 4.6.1 is immediate from (4.1.4) and the following two lemmas.

Lemma 4.6.5. *Uniformly in $\eta \in \Omega$ and $x \in \mathbb{Z}^d$, as $\kappa \to \infty$,*

$$\frac{1}{\kappa}\left[\left(e^{-\frac{r}{\kappa}\psi} L\, e^{\frac{r}{\kappa}\psi}\right) - L\left(\frac{r}{\kappa}\psi\right)\right](\eta,x) = O\left(\frac{1}{\kappa^3}\right). \tag{4.6.7}$$

Lemma 4.6.6. *Uniformly in $\eta \in \Omega$ and $x \in \mathbb{Z}^d$, as $\kappa \to \infty$,*

$$\left[\left(e^{-\frac{r}{\kappa}\psi} \Delta\, e^{\frac{r}{\kappa}\psi}\right) - \Delta\left(\frac{r}{\kappa}\psi\right)\right](\eta,x) = \frac{r^2}{2\kappa^2}\sum_{e\colon\, \|e\|=1}\Big(\psi(\eta,x+e)-\psi(\eta,x)\Big)^2 + O\left(\frac{1}{\kappa^3}\right). \tag{4.6.8}$$

Proof of Lemma 4.6.5. By (1.2.3–1.2.4), we have

$$\begin{aligned}&\left[\left(e^{-\frac{r}{\kappa}\psi} L\, e^{\frac{r}{\kappa}\psi}\right) - L\left(\frac{r}{\kappa}\psi\right)\right](\eta,x)\\ &\quad= \frac{1}{2d}\sum_{\{a,b\}}\left(e^{\frac{r}{\kappa}[\psi(\eta^{a,b},x)-\psi(\eta,x)]} - 1 - \frac{r}{\kappa}[\psi(\eta^{a,b},x)-\psi(\eta,x)]\right).\end{aligned} \tag{4.6.9}$$

Taylor expansion of the r.h.s. of (4.6.9) gives that uniformly in $\eta \in \Omega$ and $x \in \mathbb{Z}^d$,

$$\frac{1}{\kappa}\left[\left(e^{-\frac{r}{\kappa}\psi} L\, e^{\frac{r}{\kappa}\psi}\right) - L\left(\frac{r}{\kappa}\psi\right)\right](\eta,x) = \frac{r^2}{4d\kappa^3}\sum_{\{a,b\}}\Big(\psi(\eta^{a,b},x)-\psi(\eta,x)\Big)^2 e^{o(1)} = O\left(\frac{1}{\kappa^3}\right), \tag{4.6.10}$$

where we use (4.5.4–4.5.5). ∎

Proof of Lemma 4.6.6. By (1.1.2), we have

$$\begin{aligned}&\left[\left(e^{-\frac{r}{\kappa}\psi} \Delta\, e^{\frac{r}{\kappa}\psi}\right) - \Delta\left(\frac{r}{\kappa}\psi\right)\right](\eta,x)\\ &\quad= \sum_{e\colon\, \|e\|=1}\left(e^{\frac{r}{\kappa}[\psi(\eta,x+e)-\psi(\eta,x)]} - 1 - \frac{r}{\kappa}[\psi(\eta,x+e)-\psi(\eta,x)]\right).\end{aligned} \tag{4.6.11}$$

Taylor expansion of the r.h.s. of (4.6.11) gives that uniformly in $\eta \in \Omega$ and $x \in \mathbb{Z}^d$,

$$\left[\left(e^{-\frac{r}{\kappa}\psi} \Delta\, e^{\frac{r}{\kappa}\psi}\right) - \Delta\left(\frac{r}{\kappa}\psi\right)\right](\eta,x) = \frac{r^2}{2\kappa^2}\sum_{e\colon\, \|e\|=1}\Big(\psi(\eta,x+e)-\psi(\eta,x)\Big)^2 + R_{\kappa,T}(\eta,x) \tag{4.6.12}$$

with

$$|R_{\kappa,T}(\eta,x)| \le \frac{r^3}{6\kappa^3}\sum_{e\colon\, \|e\|=1}\Big|\psi(\eta,x+e)-\psi(\eta,x)\Big|^3 e^{o(1)} \le \frac{8dr^3}{3\kappa^3}T^3 e^{o(1)}, \tag{4.6.13}$$

where we use (4.5.3). Combining (4.6.12–4.6.13), we arrive at (4.6.8). ∎

4.6.3 Proof of Lemmas 4.6.2

Proof. By (4.4.3), we have for all $\eta \in \Omega$ and $x \in \mathbb{Z}^d$,

$$\begin{aligned}
&\sum_{e\colon \|e\|=1} \Big(\psi(\eta, x+e) - \psi(\eta, x)\Big)^2 \\
&\quad = \sum_{e\colon \|e\|=1} \sum_{z_1,z_2\in\mathbb{Z}^d} \int_0^T du \int_0^T dv \,\Big(p_{2du1[\kappa]}(z_1, x+e) - p_{2du1[\kappa]}(z_1, x)\Big) \\
&\qquad\qquad \times \Big(p_{2dv1[\kappa]}(z_2, x+e) - p_{2dv1[\kappa]}(z_2, x)\Big)\big[\eta(z_1)-\rho\big]\big[\eta(z_2)-\rho\big] \\
&\quad = \sum_{z_1,z_2\in\mathbb{Z}^d} K^{\kappa,T}(z_1,z_2)\,\big[\eta(z_1+x)-\rho\big]\big[\eta(z_2+x)-\rho\big],
\end{aligned} \tag{4.6.14}$$

where $K^{\kappa,T}\colon \mathbb{Z}^d\times\mathbb{Z}^d \mapsto \mathbb{R}$ is given by

$$K^{\kappa,T}(z_1,z_2) = \sum_{e:\|e\|=1} \Big(\chi(z_1+e)-\chi(z_1)\Big)\Big(\chi(z_2+e)-\chi(z_2)\Big). \tag{4.6.15}$$

Therefore, for all $\kappa, T > 0$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha r}{2\kappa^2}\int_0^t ds \sum_{e:\|e\|=1} \Big(\psi\big(\xi_{\frac{s}{\kappa}}, X_s+e\big) - \psi\big(\xi_{\frac{s}{\kappa}}, X_s\big)\Big)^2\right]\right) \\
&\quad = \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha r}{2\kappa^2}\int_0^t ds \sum_{z_1,z_2\in\mathbb{Z}^d} K^{\kappa,T}(z_1,z_2) \right.\right. \\
&\qquad\qquad \left.\left. \times \Big(\xi_{\frac{s}{\kappa}}(z_1+X_s)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2+X_s)-\rho\Big)\right]\right) \\
&\quad \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{2\kappa^2}\int_0^t ds \sum_{z_1,z_2\in\mathbb{Z}^d} K^{\kappa,T}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\right),
\end{aligned} \tag{4.6.16}$$

where in the last line we use Lemma 4.5.2 with

$$W(\eta,x) = \frac{\alpha r}{2\kappa^2} \sum_{z_1,z_2\in\mathbb{Z}^d} K^{\kappa,T}(z_1,z_2)\,\big[\eta(z_1+x)-\rho\big]\big[\eta(z_2+x)-\rho\big], \tag{4.6.17}$$

which satisfies $W(\eta,x) = W(\tau_x\eta, 0)$ as required in (4.5.11). Splitting the sum in the r.h.s. of (4.6.16) into its diagonal and off-diagonal part and using the Cauchy-Schwarz inequality, we arrive at (4.6.2). ∎

4.6.4 Proof of Lemma 4.6.3

The proof of Lemma 4.6.3 is based on the following two lemmas. Recall (2.1.1).

Lemma 4.6.7. *For any $T > 0$ there exists $C_T > 0$, satisfying $\lim_{T\to\infty} C_T = 0$, such that*

$$\lim_{\kappa\to\infty} \|K^{\kappa,T}_{\text{diag}}\|_1 = \frac{1}{d} G_d + C_T. \tag{4.6.18}$$

Lemma 4.6.8. *For any $T > 0$, $\alpha \in \mathbb{R}$ and $r > 0$,*

$$\limsup_{t,\kappa\to\infty} \frac{\kappa^2}{t} \log \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa}(1-2\rho)\|K^{\kappa,T}_{\text{diag}}\|_1 T_{t/\kappa}\right]\right) \le \alpha r\rho(1-2\rho) \lim_{\kappa\to\infty} \|K^{\kappa,T}_{\text{diag}}\|_1. \tag{4.6.19}$$

Before giving the proofs of Lemmas 4.6.7–4.6.8, we first prove Lemma 4.6.3.

Proof of Lemma 4.6.3. By Jensen's inequality, we have

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{z\in\mathbb{Z}^d} K^{\kappa,T}_{\text{diag}}(z)\Big(\xi_{\frac{s}{\kappa}}(z)-\rho\Big)^2\right]\right)\\
&\quad\le \sum_{z\in\mathbb{Z}^d} \frac{K^{\kappa,T}_{\text{diag}}(z)}{\|K^{\kappa,T}_{\text{diag}}\|_1}\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa^2}\|K^{\kappa,T}_{\text{diag}}\|_1\int_0^t ds\,\Big(\xi_{\frac{s}{\kappa}}(z)-\rho\Big)^2\right]\right)\\
&\quad= \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa^2}\|K^{\kappa,T}_{\text{diag}}\|_1\int_0^t ds\,\Big(\xi_{\frac{s}{\kappa}}(0)-\rho\Big)^2\right]\right)\\
&\quad= \exp\left[\frac{\alpha r}{\kappa^2}\rho^2\|K^{\kappa,T}_{\text{diag}}\|_1 t\right]\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa}(1-2\rho)\|K^{\kappa,T}_{\text{diag}}\|_1\int_0^{t/\kappa} ds\,\xi_s(0)\right]\right),
\end{aligned}\tag{4.6.20}$$

where the first equality uses the shift-invariance of ν_ρ. Therefore

$$\begin{aligned}
&\lim_{t,\kappa,T\to\infty} \frac{\kappa^2}{t}\log\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{z\in\mathbb{Z}^d} K^{\kappa,T}_{\text{diag}}(z)\Big(\xi_{\frac{s}{\kappa}}(z)-\rho\Big)^2\right]\right)\\
&\le \lim_{\kappa,T\to\infty} \alpha r\rho^2\|K^{\kappa,T}_{\text{diag}}\|_1 + \lim_{t,\kappa,T\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha r}{\kappa}(1-2\rho)\|K^{\kappa,T}_{\text{diag}}\|_1 T_{\frac{t}{\kappa}}\right]\right).
\end{aligned}\tag{4.6.21}$$

Now use Lemmas 4.6.7–4.6.8 to obtain (4.6.5). ∎

Proof of Lemma 4.6.7. By (4.6.3), we have

$$\begin{aligned}
\|K^{\kappa,T}_{\text{diag}}\|_1 &= 2\sum_{\{x,y\}}\int_0^T du\int_u^T dv\,\Big(p_{2du1[\kappa]}(0,y)-p_{2du1[\kappa]}(0,x)\Big)\Big(p_{2dv1[\kappa]}(0,y)-p_{2dv1[\kappa]}(0,x)\Big)\\
&= -4\int_0^T du\int_u^T dv\sum_{x\in\mathbb{Z}^d} p_{2du1[\kappa]}(0,x)\Big[\Delta_1 p_{2dv1[\kappa]}(0,x)\Big]\\
&= -\frac{4}{1[\kappa]}\int_0^T du\int_u^T dv\sum_{x\in\mathbb{Z}^d} p_{2du1[\kappa]}(0,x)\left[\frac{\partial}{\partial v}p_{2dv1[\kappa]}(0,x)\right],
\end{aligned}\tag{4.6.22}$$

where we recall the remark below (4.5.8). After performing the integration w.r.t. the variable v, we get

$$\begin{aligned}
\|K^{\kappa,T}_{\text{diag}}\|_1 &= \frac{4}{1[\kappa]}\left(\int_0^T du\sum_{x\in\mathbb{Z}^d} p^2_{2du1[\kappa]}(0,x) - \int_0^T du\sum_{x\in\mathbb{Z}^d} p_{2du1[\kappa]}(0,x)p_{2dT1[\kappa]}(0,x)\right)\\
&= \frac{4}{1[\kappa]}\left(\int_0^T du\; p_{4du1[\kappa]}(0,0) - \int_0^T du\; p_{2d(u+T)1[\kappa]}(0,0)\right).
\end{aligned}\tag{4.6.23}$$

Hence

$$\lim_{\kappa\to\infty} \|K^{\kappa,T}_{\rm diag}\|_1 = \frac{1}{d}\Bigg(\int_0^{2dT} du\; p_u(0,0) - \int_{2dT}^{4dT} du\; p_u(0,0)\Bigg), \tag{4.6.24}$$

which gives (4.6.18). ∎

Proof of Lemma 4.6.8. To derive (4.6.19), we use the large deviation principle for $(T_t)_{t\geq 0}$ stated in Section 2.1. By Varadhan's Lemma we have, for all $\kappa, T > 0$,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\alpha r}{\kappa}(1-2\rho)\|K^{\kappa,T}_{\rm diag}\|_1 T_{t/\kappa}\right]\Bigg) \\ &\qquad = \frac{1}{\kappa}\max_{\beta\in[0,1]}\Big[\frac{\alpha r}{\kappa}(1-2\rho)\|K^{\kappa,T}_{\rm diag}\|_1\beta - \Psi_d(\beta)\Big]. \end{aligned} \tag{4.6.25}$$

By Lemma 4.6.7, $(1/\kappa)\|K^{\kappa,T}_{\rm diag}\|_1 \downarrow 0$ as $\kappa\to\infty$ for any $T > 0$. Hence Lemma 4.5.3 can be applied to get (4.6.19). ∎

4.6.5 Proof of Lemma 4.6.4

The proof of Lemma 4.6.4 is based on the following two lemmas. Recall (4.6.3). For $z_1, z_2 \in \mathbb{Z}^d$ with $z_1 \neq z_2$ and $\gamma \in \mathbb{R}$, let

$$\begin{aligned} h_{\gamma,\kappa}(z_1,z_2) &= \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\gamma}{\kappa^2}\int_0^t ds\;\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\Bigg) \\ &= \limsup_{t\to\infty}\frac{1}{\kappa t}\log\mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\gamma}{\kappa}\int_0^t ds\;\big(\xi_s(z_1)-\rho\big)\big(\xi_s(z_2)-\rho\big)\right]\Bigg). \end{aligned} \tag{4.6.26}$$

Lemma 4.6.9. *For all $\kappa, T > 0$,*

$$\|K^{\kappa,T}_{\rm off}\|_1 \leq 8dT^2. \tag{4.6.27}$$

Lemma 4.6.10. *For any $z_1, z_2 \in \mathbb{Z}^d$ with $z_1 \neq z_2$ and any $\gamma \in \mathbb{R}$,*

$$\limsup_{\kappa\to\infty}\kappa^2 h_{\gamma,\kappa}(z_1,z_2) \leq 0. \tag{4.6.28}$$

Before giving the proof of Lemmas 4.6.9–4.6.10, we first prove Lemma 4.6.4.

Proof of Lemma 4.6.4. Let $K^{\kappa,T;+}_{\rm off}$ and $K^{\kappa,T;-}_{\rm off}$ denote, respectively, the positive and negative part of $K^{\kappa,T}_{\rm off}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\log\mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\alpha r}{\kappa^2}\int_0^t ds \sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}} K^{\kappa,T}_{\rm off}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\Bigg) \\ &\leq \frac{1}{2}\log\mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{2\alpha r}{\kappa^2}\sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}} K^{\kappa,T;+}_{\rm off}(z_1,z_2)\int_0^t ds\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\Bigg) \\ &\quad + \frac{1}{2}\log\mathbb{E}_{\nu_\rho}\Bigg(\exp\left[-\frac{2\alpha r}{\kappa^2}\sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}} K^{\kappa,T;-}_{\rm off}(z_1,z_2)\int_0^t ds\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\right]\Bigg). \end{aligned} \tag{4.6.29}$$

We estimate the first term in the r.h.s. of (4.6.29). For $R > 0$, let

$$B_R = \{(z_1, z_2) \in \mathbb{Z}^d \times \mathbb{Z}^d : \|z_1\| + \|z_2\| \leq R\}. \tag{4.6.30}$$

Then

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Bigg(\exp\Bigg[\frac{2\alpha r}{\kappa^2}\int_0^t ds \sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}} K_{\rm off}^{\kappa,T;+}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\Bigg]\Bigg)\\
&\quad\leq \exp\Bigg[\frac{2|\alpha| r t}{\kappa^2}\sum_{\substack{(z_1,z_2)\in B_R^c\\ z_1\neq z_2}} K_{\rm off}^{\kappa,T;+}(z_1,z_2)\Bigg]\\
&\qquad\times \mathbb{E}_{\nu_\rho}\Bigg(\exp\Bigg[\frac{2\alpha r}{\kappa^2}\int_0^t ds \sum_{\substack{(z_1,z_2)\in B_R\\ z_1\neq z_2}} K_{\rm off}^{\kappa,T;+}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\Bigg]\Bigg).
\end{aligned} \tag{4.6.31}$$

Applying Jensen's inequality, we get

$$\begin{aligned}
&\frac{\kappa^2}{t}\log \mathbb{E}_{\nu_\rho}\Bigg(\exp\Bigg[\frac{2\alpha r}{\kappa^2}\int_0^t ds \sum_{\substack{z_1,z_2\in\mathbb{Z}^d\\ z_1\neq z_2}} K_{\rm off}^{\kappa,T;+}(z_1,z_2)\Big(\xi_{\frac{s}{\kappa}}(z_1)-\rho\Big)\Big(\xi_{\frac{s}{\kappa}}(z_2)-\rho\Big)\Bigg]\Bigg)\\
&\leq 2|\alpha| r \sum_{\substack{(z_1,z_2)\in B_R^c\\ z_1\neq z_2}} K_{\rm off}^{\kappa,T;+}(z_1,z_2) + \frac{\kappa}{t/\kappa}\log \sum_{\substack{(z_1,z_2)\in B_R\\ z_1\neq z_2}} \frac{K_{\rm off}^{\kappa,T;+}(z_1,z_2)}{\|K_{{\rm off};R}^{\kappa,T;+}\|_1}\\
&\qquad\qquad \times \mathbb{E}_{\nu_\rho}\Bigg(\exp\Bigg[\frac{2\alpha r}{\kappa}\|K_{{\rm off};R}^{\kappa,T;+}\|_1 \int_0^{t/\kappa} ds\,\Big(\xi_s(z_1)-\rho\Big)\Big(\xi_s(z_2)-\rho\Big)\Bigg]\Bigg),
\end{aligned} \tag{4.6.32}$$

where

$$\|K_{{\rm off};R}^{\kappa,T;+}\|_1 = \sum_{\substack{(z_1,z_2)\in B_R\\ z_1\neq z_2}} K_{\rm off}^{\kappa,T;+}(z_1,z_2). \tag{4.6.33}$$

By Lemma 4.6.10 (with $\gamma = 2\alpha r\|K_{{\rm off};R}^{\kappa,T;+}\|_1$), the second term in the r.h.s. of (4.6.32) is asymptotically bounded by above by zero (as $t \to \infty$) for any $\kappa, T > 0$, $\alpha \in \mathbb{R}$ and $r > 0$, and any R finite. The first term in the r.h.s. of (4.6.32) does not depend on t and, by Lemma 4.6.9, tends to zero as $R \to \infty$. This shows that the first term in the r.h.s of (4.6.29) yields a zero contribution. The same is true for the second term by the same argument. This completes the proof of (4.6.6). ∎

Proof of Lemma 4.6.9. The claim follows from (4.6.3–4.6.4). ∎

Proof of Lemma 4.6.10. The proof of Lemma 4.6.10 is long, since it is based on three further lemmas. Let $z_1, z_2 \in \mathbb{Z}^d$ with $z_1 \neq z_2$. Without loss of generality, we may assume that

$$z_1 \in H^- \quad\text{and}\quad z_2 \in H^+ \tag{4.6.34}$$

with

$$H^- = \{z \in \mathbb{Z}^d\colon\ z^1 \leq 0\} \quad\text{and}\quad H^+ = \{z \in \mathbb{Z}^d\colon\ z^1 > 0\}. \tag{4.6.35}$$

Let

$$h^-_{\gamma,\kappa}(z_1) = \limsup_{t\to\infty} \frac{1}{3\kappa t} \log \mathbb{E}^{\mathrm{IRW}}_{\nu_\rho}\Bigg(\exp\left[-\frac{3\gamma}{\kappa}\rho \int_0^t ds\ \tilde{\xi}^-_s(z_1)\right]\Bigg),$$

$$h^+_{\gamma,\kappa}(z_2) = \limsup_{t\to\infty} \frac{1}{3\kappa t} \log \mathbb{E}^{\mathrm{IRW}}_{\nu_\rho}\Bigg(\exp\left[-\frac{3\gamma}{\kappa}\rho \int_0^t ds\ \tilde{\xi}^+_s(z_2)\right]\Bigg), \tag{4.6.36}$$

$$h^\pm_{\gamma,\kappa}(z_1,z_2) = \limsup_{t\to\infty} \frac{1}{3\kappa t} \log \mathbb{E}^{\mathrm{IRW}}_{\nu_\rho}\Bigg(\exp\left[\frac{3\gamma}{\kappa} \int_0^t ds\ \tilde{\xi}^-_s(z_1)\tilde{\xi}^+_s(z_2)\right]\Bigg),$$

where $(\tilde{\xi}^-_t)_{t\geq 0}$ and $(\tilde{\xi}^+_t)_{t\geq 0}$ are independent IRW's on H^- and H^+, respectively, with transition kernels $p^-(\cdot,\cdot)$ and $p^+(\cdot,\cdot)$ corresponding to simple random walks stepping at rate 1 such that steps outside H^- and H^+, respectively, are suppressed.

Lemma 4.6.11. *For all $\kappa>0$, $z_1\in H^-$, $z_2\in H^+$ and $\gamma\in\mathbb{R}$,*

$$h_{\gamma,\kappa}(z_1,z_2) \leq \frac{\gamma}{\kappa^2}\rho^2 + h^-_{\gamma,\kappa}(z_1) + h^+_{\gamma,\kappa}(z_2) + h^\pm_{\gamma,\kappa}(z_1,z_2). \tag{4.6.37}$$

Lemma 4.6.12. *For all $\gamma\in\mathbb{R}$,*

$$\limsup_{\kappa\to\infty} \kappa^2 \sup_{z_1\in H^-} h^-_{\gamma,\kappa}(z_1) \leq -\gamma\rho^2 \quad \textit{and} \quad \limsup_{\kappa\to\infty} \kappa^2 \sup_{z_2\in H^+} h^+_{\gamma,\kappa}(z_2) \leq -\gamma\rho^2. \tag{4.6.38}$$

Lemma 4.6.13. *For all $\gamma\in\mathbb{R}$,*

$$\limsup_{\kappa\to\infty} \kappa^2 \sup_{\substack{z_1\in H^-\\ z_2\in H^+}} h^\pm_{\gamma,\kappa}(z_1,z_2) \leq \gamma\rho^2. \tag{4.6.39}$$

Combining (4.6.37–4.6.39), we get (4.6.28). ∎

Proof of Lemma 4.6.11. Similarly as in the proof of Lemma 4.5.2, by cutting the bonds connecting H^- and H^+ in the analogue of the variational formula of Proposition 2.2.2 (cf. Remark 2.2.3), we get

$$\begin{aligned} h_{\gamma,\kappa}(z_1,z_2) &\leq \limsup_{t\to\infty} \frac{1}{\kappa t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\gamma}{\kappa}\int_0^t ds\ \big(\xi^-_s(z_1)-\rho\big)\big(\xi^+_s(z_2)-\rho\big)\right]\Bigg) \\ &= \limsup_{t\to\infty} \frac{1}{\kappa t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{\gamma}{\kappa}\int_0^t ds\ \Big(\rho^2 - \rho\,\xi^-_s(z_1) + \rho\,\xi^+_s(z_2) + \xi^-_s(z_1)\xi^+_s(z_2)\Big)\right]\Bigg), \end{aligned} \tag{4.6.40}$$

where $(\xi^-_t)_{t\geq 0}$ and $(\xi^+_t)_{t\geq 0}$ are independent exclusion processes in H^- and H^+, respectively, obtained from $(\xi_t)_{t\geq 0}$ by suppressing jumps between H^- and H^+. Applying Hölder's inequality in the r.h.s. of (4.6.40) to separate terms, we obtain

$$\begin{aligned} h_{\gamma,\kappa}(z_1,z_2) \leq \frac{\gamma}{\kappa^2}\rho^2 &+ \limsup_{t\to\infty} \frac{1}{3\kappa t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[-\frac{3\gamma}{\kappa}\rho\int_0^t ds\ \xi^-_s(z_1)\right]\Bigg) \\ &+ \limsup_{t\to\infty} \frac{1}{3\kappa t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[-\frac{3\gamma}{\kappa}\rho\int_0^t ds\ \xi^+_s(z_2)\right]\Bigg) \\ &+ \limsup_{t\to\infty} \frac{1}{3\kappa t} \log \mathbb{E}_{\nu_\rho}\Bigg(\exp\left[\frac{3\gamma}{\kappa}\int_0^t ds\ \xi^-_s(z_1)\xi^+_s(z_2)\right]\Bigg). \end{aligned} \tag{4.6.41}$$

In order to get (4.6.37), we apply Proposition 1.2.1 to the last three terms in the r.h.s. of (4.6.41). For the first two terms, pick, respectively $K(z,s) = -(3\gamma/\kappa)\rho\, 1_{z_1}$ and $K(z,s) = -(3\gamma/\kappa)\rho\, 1_{z_2}(z)$. For the last term, we have to apply Proposition 1.2.1 twice, once for the exclusion process $(\xi_t^+)_{t\geq 0}$ on H^+ with $K(z,s) = -(3\gamma/\kappa)\, \xi_s^-(z_1)\, 1_{z_2}(z)$ and once for the exclusion process $(\xi_t^-)_{t\geq 0}$ on H^- with $K(z,s) = -(3\gamma/\kappa)\, \tilde{\xi}_s^-(z_2)\, 1_{z_1}(z)$. Here, we in fact apply a *modification* of Proposition 1.2.1 by considering $(\xi_t^-)_{t\geq 0}$ and $(\xi_t^+)_{t\geq 0}$ on $\mathbb{Z}^d$ with particles not moving on H^+ and H^-, respectively. See the proof of Proposition 1.2.1 in Appendix A to verify that this modification holds true. ∎

Proof of Lemma 4.6.12. We prove the second line of (4.6.38). The first line follows by symmetry. Let

$$H_\eta^+ = \{x \in H^+ \colon \eta(x) = 1\}, \qquad \eta \in \Omega. \tag{4.6.42}$$

Fix $z \in H^+$. Then

$$\mathbb{E}_{\nu_\rho}^{\mathrm{IRW}}\left(\exp\left[-\frac{3\gamma}{\kappa}\rho \int_0^t ds\ \tilde{\xi}_s^+(z)\right]\right) = \int_\Omega \nu_\rho(d\eta) \prod_{x\in H_\eta^+} E_x^{\mathrm{RW},+}\left(\exp\left[-\frac{3\gamma}{\kappa}\rho \int_0^t ds\ 1_z(Y_s^+)\right]\right), \tag{4.6.43}$$

where $E_x^{\mathrm{RW},+}$ is expectation w.r.t. simple random walk $Y^+ = (Y_t^+)_{t\geq 0}$ on H^+ with transition kernel $p^+(\cdot,\cdot)$ and step rate 1 starting from $Y_0^+ = x \in H^+$. Using that ν_ρ is the Bernoulli product measure with density ρ, we get

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}^{\mathrm{IRW}}\left(\exp\left[-\frac{3\gamma}{\kappa}\rho \int_0^t ds\ \tilde{\xi}_s^+(z)\right]\right) \\ &= \int_\Omega \nu_\rho(d\eta) \prod_{x\in H^+} E_x^{\mathrm{RW},+}\left(\exp\left[-\eta(x)\frac{3\gamma}{\kappa}\rho \int_0^t ds\ 1_z(Y_s^+)\right]\right) \\ &= \prod_{x\in H^+}\Big(1-\rho+\rho v(x,t)\Big) \leq \exp\Big[\rho \sum_{x\in H^+}\big(v(x,t)-1\big)\Big] \end{aligned} \tag{4.6.44}$$

with

$$v(x,t) = E_x^{\mathrm{RW},+}\left(\exp\left[-\frac{3\gamma}{\kappa}\rho \int_0^t ds\ 1_z(Y_s^+)\right]\right). \tag{4.6.45}$$

By the Feynman-Kac formula, $v\colon H^+ \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t} v(x,t) = \frac{1}{2d}\Delta^+ v(x,t) - \left\{\frac{3\gamma}{\kappa}\rho\, 1_z(x)\right\} v(x,t), \qquad v(\cdot,0) \equiv 1, \tag{4.6.46}$$

where

$$\Delta^+ v(x,t) = \sum_{\substack{y\in H^+\\ \|y-x\|=1}} [v(y,t) - v(x,t)], \qquad x \in H^+. \tag{4.6.47}$$

Put

$$w(x,t) = v(x,t) - 1. \tag{4.6.48}$$

Then $w\colon H^+ \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial w}{\partial t}(x,t) = \frac{1}{2d}\Delta^+ w(x,t) - \left\{\frac{3\gamma}{\kappa}\rho\, 1_z(x)\right\}[w(x,t)+1], \qquad w(\cdot,0) \equiv 0. \tag{4.6.49}$$

Since $\sum_{x\in H^+} \Delta^+ f(x) = 0$ for all $f\colon H^+ \to \mathbb{R}$, (4.6.49) gives

$$\frac{\partial}{\partial t} \sum_{x\in H^+} w(x,t) = -\frac{3\gamma}{\kappa}\,\rho[w(z,t)+1]. \tag{4.6.50}$$

After integrating (4.6.50) w.r.t. t, we obtain

$$\sum_{x\in H^+} w(x,t) = -\frac{3\gamma}{\kappa}\rho\, t - \frac{3\gamma}{\kappa}\rho \int_0^t ds\ w(z,s). \tag{4.6.51}$$

Combining (4.6.36), (4.6.44), (4.6.48) and (4.6.51), we arrive at

$$h^+_{\gamma,\kappa}(z) \le -\frac{\gamma}{\kappa^2}\rho^2\Big(1 + \lim_{t\to\infty} \frac{1}{t}\int_0^t ds\ w(z,s)\Big). \tag{4.6.52}$$

The limit in the r.h.s. exists since, by (4.6.45) and (4.6.48), $w(z,t)$ is monotone in t.

We will complete the proof by showing that the second term in the r.h.s. of (4.6.52) tends to zero as $\kappa\to\infty$. This will rely on the following lemma, the proof of which is deferred to the end of this section.

Lemma 4.6.14. *Let $G^+(x,y)$ be the Green kernel on H^+ associated with $p^+_t(x,y)$. Then $\|G^+\|_\infty \le 2G_d < \infty$.*

Return to (4.6.45). If $\gamma > 0$, then by Jensen's inequality we have

$$1 \ge v(x,t) \ge \exp\left[-\frac{3\gamma}{\kappa}\rho\int_0^t ds\ p^+_s(x,z)\right] \ge \exp\left[-\frac{3\gamma}{\kappa}\rho\|G^+\|_\infty\right], \tag{4.6.53}$$

where $\|G^+\|_\infty < \infty$ by Lemma 4.6.14. To deal with the case $\gamma \le 0$, let $\mathcal{G}^+$ denote the Green operator acting on functions $V\colon H^+ \to [0,\infty)$ as

$$(\mathcal{G}^+V)(x) = \sum_{y\in H^+} G^+(x,y)V(y), \qquad x\in H^+. \tag{4.6.54}$$

We have

$$\left\|\mathcal{G}^+\Big(\frac{3\gamma}{\kappa}\rho\,1_z\Big)\right\|_\infty \le \frac{3|\gamma|}{\kappa}\rho\,\|G^+\|_\infty. \tag{4.6.55}$$

The r.h.s. tends to zero as $\kappa\to\infty$. Hence Lemma 8.2.1 in Gärtner and den Hollander (10) can be applied to (4.6.45) for κ large enough, to yield

$$1 \le v(x,t) \le \frac{1}{1-\frac{3|\gamma|}{\kappa}\rho\,\|G^+\|_\infty} \downarrow 1 \quad \text{as} \quad \kappa\to\infty. \tag{4.6.56}$$

Therefore, combining (4.6.53) and (4.6.56), we see that for all $\gamma\in\mathbb{R}$ and $\delta\in(0,1)$ there exists $\kappa_0 = \kappa_0(\gamma,\delta)$ such that

$$\|v-1\|_\infty \le \delta \qquad \forall\,\kappa > \kappa_0. \tag{4.6.57}$$

By (4.6.48–4.6.49), we have

$$w(z,t) = -\frac{3\gamma}{\kappa}\rho\int_0^t ds\ E^{\mathrm{RW},+}_z\Big(1_z(Y^+_s)\,v(Y^+_s,t-s)\Big). \tag{4.6.58}$$

Via (4.6.57) it therefore follows that

$$-\frac{3\gamma}{\kappa}\rho(1\pm\delta)G^+(z,z)\le\lim_{t\to\infty}\frac{1}{t}\int_0^t ds\; w(z,s)\le-\frac{3\gamma}{\kappa}\rho(1\mp\delta)G^+(z,z)\qquad\forall\,\kappa>\kappa_0,\quad(4.6.59)$$

where the choice of + or − in front of δ depends on the sign of γ. The latter shows that the second term in the r.h.s. of (4.6.52) is $O(1/\kappa)$. This proves (4.6.38). ∎

Proof of Lemma 4.6.13. The proof is similar to that of Lemma 4.6.12. Let

$$\begin{aligned}H^+_\eta&=\{x\in H^+\colon\ \eta(x)=1\},\quad \eta\in\Omega,\\ H^-_\eta&=\{x\in H^-\colon\ \eta(x)=1\},\quad \eta\in\Omega.\end{aligned}\qquad(4.6.60)$$

Fix $z_1\in H^-$ and $z_2\in H^+$. Then

$$\begin{aligned}&\mathbb{E}^{\mathrm{IRW}}_{\nu_\rho}\left(\exp\left[\frac{3\gamma}{\kappa}\int_0^t ds\;\tilde{\xi}^-_s(z_1)\tilde{\xi}^+_s(z_2)\right]\right)\\ &\quad=\int_\Omega\nu_\rho(d\eta)\prod_{x\in H^-_\eta}\prod_{y\in H^+_\eta}E^{\mathrm{RW},-}_x\,E^{\mathrm{RW},+}_y\left(\exp\left[\frac{3\gamma}{\kappa}\int_0^t ds\;1_{(z_1,z_2)}(Y^-_s,Y^+_s)\right]\right),\end{aligned}\qquad(4.6.61)$$

where Y^- on H^- and Y^+ on H^+ are simple random walks with step rate 1 and transition kernel $p^-(\cdot,\cdot)$ and $p^+(\cdot,\cdot)$ starting from $Y^-_0=x\in H^-$ and $Y^+_0=y\in H^+$, respectively. Using that ν_ρ is the Bernoulli product measure with density ρ, we get

$$\begin{aligned}&\mathbb{E}^{\mathrm{IRW}}_{\nu_\rho}\left(\exp\left[\frac{3\gamma}{\kappa}\int_0^t ds\;\tilde{\xi}^-_s(z_1)\tilde{\xi}^+_s(z_2)\right]\right)\\ &\quad=\int_\Omega\nu_\rho(d\eta)\prod_{x\in H^-}\prod_{y\in H^+}E^{\mathrm{RW},-}_x\,E^{\mathrm{RW},+}_y\left(\exp\left[\eta(x)\eta(y)\frac{3\gamma}{\kappa}\int_0^t ds\;1_{(z_1,z_2)}(Y^-_s,Y^+_s)\right]\right)\\ &\quad=\prod_{x\in H^-}\prod_{y\in H^+}\Big(1-\rho^2+\rho^2 v(z_1,z_2;t)\Big)\le\exp\left[\rho^2\sum_{x\in H^-}\sum_{y\in H^+}\big(v(z_1,z_2;t)-1\big)\right]\end{aligned}\qquad(4.6.62)$$

with

$$v(z_1,z_2;t)=E^{\mathrm{RW},-}_x\,E^{\mathrm{RW},+}_y\left(\exp\left[\frac{3\gamma}{\kappa}\int_0^t ds\;\int_0^t ds\;1_{(z_1,z_2)}(Y^-_s,Y^+_s)\right]\right).\qquad(4.6.63)$$

By the Feynman-Kac formula, $v\colon\ (H^-\times H^+)\times[0,\infty)\to\mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t}v(x,y;t)=\frac{1}{2d}(\Delta^-+\Delta^+)v(x,y;t)+\left\{\frac{3\gamma}{\kappa}1_{z_1,z_2}(x,y)\right\}v(x,y;t),\qquad v(\cdot,\cdot;0)\equiv1,\quad(4.6.64)$$

where

$$\begin{aligned}\Delta^-v(x;t)&=\sum_{\substack{y\in H^-\\ \|y-x\|=1}}[v(y,t)-v(x,t)],\qquad x\in H^-,\\ \Delta^+v(x;t)&=\sum_{\substack{y\in H^+\\ \|y-x\|=1}}[v(y,t)-v(x,t)],\qquad x\in H^+.\end{aligned}\qquad(4.6.65)$$

Put

$$w(x,y;t) = v(x,y;t) - 1. \tag{4.6.66}$$

Then, $w\colon\ (H^- \times H^+) \times [0,\infty) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial w}{\partial t}(x,y;t) = \frac{1}{2d}(\Delta^- + \Delta^+)w(x,y;t) + \left\{\frac{3\gamma}{\kappa}\, 1_{(z_1,z_2)}(x,y)\right\}\,[w(x,y;t)+1], \qquad w(\cdot,\cdot;0) \equiv 0. \tag{4.6.67}$$

By (4.6.65) and (4.6.67),

$$\frac{\partial}{\partial t}\sum_{x\in H^-}\sum_{y\in H^+} w(x,y;t) = \frac{3\gamma}{\kappa}[w(z_1,z_2;t)+1]. \tag{4.6.68}$$

After integrating (4.6.68) w.r.t. t, we obtain

$$\sum_{x\in H^-}\sum_{y\in H^+} w(x,y;t) = \frac{3\gamma}{\kappa}\,t + \frac{3\gamma}{\kappa}\int_0^t ds\ w(z_1,z_2;s). \tag{4.6.69}$$

Combining (4.6.36), (4.6.62), (4.6.66) and (4.6.69), we arrive at

$$h^{\pm}_{\gamma,\kappa}(z_1,z_2) \le \frac{\gamma}{\kappa^2}\rho^2\left(1 + \lim_{t\to\infty}\frac{1}{t}\int_0^t ds\ w(z_1,z_2;s)\right). \tag{4.6.70}$$

The limit in the r.h.s. exists, since $w(z_1,z_2;s)$ is monotone in s.
We will complete the proof by showing that the second term in the r.h.s. of (4.6.70) tends to zero as $\kappa\to\infty$. Return to (4.6.63). If $\gamma \le 0$, then by Jensen's inequality we have

$$1 \ge v(x,y;t) \ge \exp\left[-\frac{3|\gamma|}{\kappa}\int_0^\infty ds\ p_s^-(x,z_1)\,p_s^+(y,z_2)\right] \ge \exp\left[-\frac{3|\gamma|}{\kappa}\Big(\|G^-\|_\infty \wedge \|G^+\|_\infty\Big)\right], \tag{4.6.71}$$

where $\|G^-\|_\infty, \|G^+\|_\infty < \infty$ by Lemma 4.6.14. To deal with the case $\gamma > 0$, let $\mathcal{G}^\pm$ denote the Green operator acting on functions $V\colon\ H^-\times H^+ \to [0,\infty)$ as

$$(\mathcal{G}^\pm V)(x,y) = \sum_{\substack{a\in H^-\\ b\in H^+}} G^\pm(x,y;a,b)V(a,b), \qquad x\in H^-,\ y\in H^+, \tag{4.6.72}$$

where

$$G^\pm(x,y;a,b) = \int_0^\infty ds\ p_s^-(x,a)\,p_s^+(y,b). \tag{4.6.73}$$

We have

$$\left\|\mathcal{G}^\pm\Big(\frac{3\gamma}{\kappa}\,1_{(z_1,z_2)}\Big)\right\|_\infty \le \frac{3\gamma}{\kappa}\,\|G^\pm\|_\infty \le \frac{3\gamma}{\kappa}\Big(\|G^-\|_\infty \wedge \|G^+\|_\infty\Big). \tag{4.6.74}$$

The r.h.s. tends to zero as $\kappa\to\infty$. Hence Lemma 8.2.1 in Gärtner and den Hollander (10) can be applied to (4.6.63) for κ large enough, to yield

$$1 \le v(x,t) \le \frac{1}{1 - \frac{3\gamma}{\kappa}\Big(\|G^-\|_\infty \wedge \|G^+\|_\infty\Big)} \downarrow 1 \quad\text{as}\quad \kappa\to\infty. \tag{4.6.75}$$

Therefore, combining (4.6.71) and (4.6.75), we see that for all $\gamma \in \mathbb{R}$ and $\delta > 0$ there exists $\kappa_0 = \kappa_0(\gamma, \delta)$ such that

$$\|v - 1\|_\infty \le \delta \qquad \forall \kappa > \kappa_0. \tag{4.6.76}$$

By (4.6.66–4.6.67), we have

$$w(z_1, z_2; t) = \frac{3\gamma}{\kappa} \int_0^t ds\; E_{z_1}^{\mathrm{RW},-} E_{z_2}^{\mathrm{RW},+} \Big(1_{(z_1,z_2)}(Y_s^-, Y_s^+)\, v(Y_s^-, Y_s^+; t-s) \Big). \tag{4.6.77}$$

Via (4.6.76) it therefore follows that for all $\kappa > \kappa_0$,

$$\frac{3\gamma}{\kappa}(1 \pm \delta) G^{\pm}(z_1, z_1; z_2, z_2) \le \lim_{t\to\infty} \frac{1}{t} \int_0^t ds\; w(z_1, z_2; s) \le \frac{3\gamma}{\kappa}(1 \mp \delta) G^{\pm}(z_1, z_1; z_2, z_2). \tag{4.6.78}$$

Combining (4.6.70) and (4.6.78), we arrive at (4.6.39). ∎

Proof of Lemma 4.6.14. We have $G^+(x,y) = \sum_{n=0}^\infty p_n^+(x,y)$, $x, y \in H^+$, with $p_n^+(x,y)$ the n-step transition probability of simple random walk on H^+ whose steps outside H^+ are suppressed (i.e., the walk pauses when it attempts to leave H^+). Let $p_n(x,y)$ be the n-step transition probability of simple random walk on $\mathbb{Z}^d$. Then

$$p_n^+(x,y) \le 2 p_n(x,y), \qquad x, y \in H^+,\ n \in \mathbb{N}_0. \tag{4.6.79}$$

Indeed, if we reflect simple random walk in the $(d-1)$-dimensional hyperplane between H^+ and its complement, then we obtain precisely the random walk that pauses when it attempts to leave H^+. Hence, we have $p_n^+(x,y) = p_n(x,y) + p_n(x, y*)$, $x, y \in H^+$, $n \in \mathbb{N}_0$, with $y*$ the reflection image of y. Since $p_n(x, y*) \le p_n(x,y)$, $x, y \in H^+$, the claim in (4.6.79) follows. Sum on n, to get $G^+(x,y) \le 2G(x,y)$, $x, y \in H^+$. Now use that $G(x,y) \le G(0,0) = G_d$, $x, y \in \mathbb{Z}^d$. ∎

4.7 Proof of Proposition 4.4.3

The proof of Proposition 4.4.3 is given in Section 4.7.1 subject to three lemmas. The latter are proved in Sections 4.7.2–4.7.4, respectively. The first two lemmas are valid for $d \ge 3$, the third for $d \ge 4$.

4.7.1 Proof of Proposition 4.4.3

Lemma 4.7.1. *For all $t \ge 0$, $\kappa, T > 0$ and $\alpha \in \mathbb{R}$,*

$$\mathbb{E}_{\nu_\rho, 0}\left(\exp\left[\frac{\alpha}{\kappa} \int_0^t ds\; (\mathcal{P}_T \phi)(Z_s) \right] \right) \le E_0\left(\exp\left[\frac{\alpha}{\kappa} \rho \int_0^t ds \sum_{x\in\mathbb{Z}^d} p_{2dT1[\kappa]}(X_{t-s}, x)\, w^{(t)}(x,s) \right] \right), \tag{4.7.1}$$

where $w^{(t)}\colon \mathbb{Z}^d \times [0,t) \to \mathbb{R}$ is the solution of the Cauchy problem

$$\frac{\partial w^{(t)}}{\partial s}(x,s) = \frac{1}{2d\kappa} \Delta w^{(t)}(x,s) + \frac{\alpha}{\kappa} p_{2dT1[\kappa]}(X_{t-s}, x)[w^{(t)}(x,s) + 1], \qquad w^{(t)}(\cdot, 0) \equiv 0. \tag{4.7.2}$$

Lemma 4.7.2. *For all $t \geq 0$, $\kappa > 0$, T large enough and $\alpha \in \mathbb{R}$,*

$$\begin{aligned} &E_0\biggl(\exp\biggl[\frac{\alpha}{\kappa}\rho\int_0^t ds\sum_{x\in\mathbb{Z}^d} p_{2dT1[\kappa]}(X_{t-s},x)\,w^{(t)}(x,s)\biggr]\biggr) \\ &\qquad \leq E_0\biggl(\exp\biggl[\frac{2\alpha^2}{\kappa^2}\rho\int_0^t ds\int_s^t du\, p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u,X_s)\biggr]\biggr). \end{aligned} \tag{4.7.3}$$

Lemma 4.7.3. *If $d \geq 4$, then for any $\alpha \in \mathbb{R}$,*

$$\lim_{T,\kappa,t\to\infty}\frac{\kappa^2}{t}\log E_0\biggl(\exp\biggl[\frac{2\alpha^2}{\kappa^2}\rho\int_0^t ds\int_s^t du\,p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u,X_s)\biggr]\biggr) = 0. \tag{4.7.4}$$

Lemmas 4.7.1–4.7.3 clearly imply (4.4.16).

4.7.2 Proof of Lemma 4.7.1

For all $t \geq 0$, $\kappa, T > 0$ and $\alpha \in \mathbb{R}$, let $v^{(t)}\colon \mathbb{Z}^d \times [0,t) \to \mathbb{R}$ be such that

$$v^{(t)}(x,s) = w^{(t)}(x,s) + 1, \tag{4.7.5}$$

where $w^{(t)}$ is defined by (4.7.2). Then $v^{(t)}$ is the solution of the Cauchy problem

$$\frac{\partial v^{(t)}}{\partial s}(x,s) = \frac{1}{2d\kappa}\Delta v^{(t)}(x,s) + \frac{\alpha}{\kappa}p_{2dT1[\kappa]}(X_{t-s},x)\,v^{(t)}(x,s), \qquad v^{(t)}(\cdot,0)\equiv 1, \tag{4.7.6}$$

and has the representation

$$v^{(t)}(x,s) = E_x^{\mathrm{RW}}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\int_0^s du\,p_{2dT1[\kappa]}\Bigl(X_{t-s+u},Y_{\frac{u}{\kappa}}\Bigr)\biggr]\biggr). \tag{4.7.7}$$

Proof. By (4.1.3) and (4.4.5), we have

$$\mathbb{E}_{\nu_\rho,0}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\int_0^t ds\,(\mathcal{P}_T\phi)(Z_s)\biggr]\biggr) = \mathbb{E}_{\nu_\rho,0}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\sum_{z\in\mathbb{Z}^d}\int_0^t ds\,p_{2dT1[\kappa]}(X_s,z)\Bigl(\xi_{\frac{s}{\kappa}}(z)-\rho\Bigr)\biggr]\biggr). \tag{4.7.8}$$

Therefore, by Proposition 1.2.1 (with $K(z,s) = \alpha\,p_{2dT1[\kappa]}(X_{\kappa s},z)$), we get

$$\begin{aligned} &\mathbb{E}_{\nu_\rho,0}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\int_0^t ds\,(\mathcal{P}_T\phi)(Z_s)\biggr]\biggr) \\ &\quad\leq E_0\,\mathbb{E}_{\nu_\rho}^{\mathrm{IRW}}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\sum_{z\in\mathbb{Z}^d}p_{2dT1[\kappa]}(X_s,z)\int_0^t ds\,\Bigl(\tilde{\xi}_{\frac{s}{\kappa}}(z)-\rho\Bigr)\biggr]\biggr) \\ &\quad\leq \exp\Bigl[-\frac{\alpha}{\kappa}\rho t\Bigr]\,E_0\int_\Omega \nu_\rho(d\eta)\prod_{x\in A_\eta}E_x^{\mathrm{RW}}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\sum_{z\in\mathbb{Z}^d}p_{2dT1[\kappa]}(X_s,z)\int_0^t ds\,\delta_z\Bigl(Y_{\frac{s}{\kappa}}\Bigr)\biggr]\biggr) \\ &\quad= \exp\Bigl[-\frac{\alpha}{\kappa}\rho t\Bigr]\,E_0\int_\Omega \nu_\rho(d\eta)\prod_{x\in A_\eta}E_x^{\mathrm{RW}}\biggl(\exp\biggl[\frac{\alpha}{\kappa}\int_0^t ds\,p_{2dT1[\kappa]}\Bigl(X_s,Y_{\frac{s}{\kappa}}\Bigr)\biggr]\biggr), \end{aligned} \tag{4.7.9}$$

where $A_\eta = \{x \in \mathbb{Z}^d : \eta(x) = 1\}$ and $\mathbb{E}_x^{\rm RW}$ is expectation w.r.t. to simple random walk $Y = (Y_t)_{t\geq 0}$ on $\mathbb{Z}^d$ with step rate 1 starting from $Y_0 = x$. Using that ν_ρ is the Bernoulli product measure with density ρ, we get

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho,0}\bigg(\exp\left[\frac{\alpha}{\kappa}\int_0^t ds\ (\mathcal{P}_T\phi)(Z_s)\right]\bigg)\\
&\quad\leq \exp\left[-\frac{\alpha}{\kappa}\rho t\right]\int_\Omega \nu_\rho(d\eta)\,E_0\bigg(\prod_{x\in\mathbb{Z}^d} E_x^{\rm RW}\bigg(\exp\left[\eta(x)\frac{\alpha}{\kappa}\int_0^t ds\, p_{2dT1[\kappa]}\Big(X_s, Y_{\frac{s}{\kappa}}\Big)\right]\bigg)\bigg)\\
&\quad= \exp\left[-\frac{\alpha}{\kappa}\rho t\right]\int_\Omega \nu_\rho(d\eta)\,E_0\bigg(\prod_{x\in\mathbb{Z}^d}\Big[1+\eta(x)\,w^{(t)}(x,t)\Big]\bigg)\\
&\quad= \exp\left[-\frac{\alpha}{\kappa}\rho t\right]E_0\bigg(\prod_{x\in\mathbb{Z}^d}\Big[1+\rho\, w^{(t)}(x,t)\Big]\bigg) \leq \exp\left[-\frac{\alpha}{\kappa}\rho t\right]E_0\bigg(\exp\Big[\rho\sum_{x\in\mathbb{Z}^d} w^{(t)}(x,t)\Big]\bigg),
\end{aligned}\tag{4.7.10}$$

where $w^{(t)}\colon\ \mathbb{Z}^d\times[0,t)\to\mathbb{R}$ solves (4.7.2). From (4.7.2) we deduce that

$$\frac{\partial}{\partial s}\sum_{x\in\mathbb{Z}^d} w^{(t)}(x,s) = \frac{\alpha}{\kappa}\sum_{x\in\mathbb{Z}^d} p_{2dT1[\kappa]}(X_{t-s},x)\big[1+w^{(t)}(x,s)\big]. \tag{4.7.11}$$

Integrating (4.7.11) w.r.t. s and inserting the result into (4.7.10), we get (4.7.1). ∎

4.7.3 Proof of Lemma 4.7.2

Next, we consider $v^{(t)}$ and $w^{(t)}$ as defined in (4.7.5–4.7.7), but with $|\alpha|$ instead of α.

Proof. We begin by showing that, for T large enough and all x, s, t and $X_{(.)}$, we have $v^{(t)}(x,s)\leq 2$.

Do a Taylor expansion, to obtain ($s_0 = 0$)

$$v^{(t)}(x,s) = \sum_{n=0}^\infty \left(\frac{|\alpha|}{\kappa}\right)^n \left(\prod_{l=1}^n \int_{s_{l-1}}^s ds_l\right) E_x^{\rm RW}\left(\prod_{m=1}^n p_{2dT1[\kappa]}\Big(X_{t-s+s_m}, Y_{\frac{s_m}{\kappa}}\Big)\right). \tag{4.7.12}$$

In Fourier representation the transition kernel of simple random walk with step rate 1 reads

$$p_s(x,y) = \oint dk\ e^{-i\,k\cdot(y-x)}\,e^{-s\widehat{\varphi}(k)}, \tag{4.7.13}$$

where $\oint dk = (2\pi)^{-d}\int_{[-\pi,\pi)^d} dk$ and

$$\widehat{\varphi}(k) = \frac{1}{2d}\sum_{\substack{x\in\mathbb{Z}^d\\ \|x\|=1}}\Big(1-e^{i\,k\cdot x}\Big),\quad k\in[-\pi,\pi)^d. \tag{4.7.14}$$

Combining (4.7.12–4.7.13), we get

$$
\begin{aligned}
v^{(t)}(x,s) &= \sum_{n=0}^{\infty}\left(\frac{|\alpha|}{\kappa}\right)^n \left(\prod_{l=1}^{n}\int_{s_{l-1}}^{s} ds_l\right)\left(\prod_{m=1}^{n}\oint dk_m\right)\\
&\qquad \times \mathbb{E}_x^{\mathrm{RW}}\left(\exp\left[i\sum_{p=1}^{n}\Big(Y_{\frac{s_p}{\kappa}} - X_{t-s+s_p}\Big)\cdot k_p\right]\exp\left[-\Big(2dT1[\kappa]\Big)\sum_{q=1}^{n}\widehat{\varphi}(k_q)\right]\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{|\alpha|}{\kappa}\right)^n \left(\prod_{l=1}^{n}\int_{s_{l-1}}^{s} ds_l\right)\left(\prod_{m=1}^{n}\oint dk_m\right)\exp\left[-i\sum_{p=1}^{n}\Big(X_{t-s+s_p}-x\Big)\cdot k_p\right]\\
&\qquad \times \exp\left[-\Big(2dT1[\kappa]\Big)\sum_{q=1}^{n}\widehat{\varphi}(k_q)\right]E_0^{\mathrm{RW}}\left(\exp\left[i\sum_{r=1}^{n}Y_{\frac{s_r}{\kappa}}\cdot k_r\right]\right),
\end{aligned}
\tag{4.7.15}
$$

where in the last line we did a spatial shift of Y by x. Because Y has independent increments, we have

$$
\begin{aligned}
E_0^{\mathrm{RW}}\left(\exp\left[i\sum_{r=1}^{n}Y_{\frac{s_r}{\kappa}}\cdot k_r\right]\right) &= E_0^{\mathrm{RW}}\left(\exp\left[i\sum_{r=1}^{n}(k_r+\cdots+k_n)\cdot\Big(Y_{\frac{s_r}{\kappa}}-Y_{\frac{s_{r-1}}{\kappa}}\Big)\right]\right)\\
&= \prod_{r=1}^{n}E_0^{\mathrm{RW}}\left(\exp\left[i(k_r+\cdots+k_n)\cdot Y_{\frac{s_r-s_{r-1}}{\kappa}}\right]\right)\\
&= \prod_{r=1}^{n}\sum_{z\in\mathbb{Z}^d} p_{\frac{s_r-s_{r-1}}{\kappa}}(0,z)\,\exp\big[i(k_r+\cdots+k_n)\cdot z\big]\\
&= \prod_{r=1}^{n}\exp\left[-\frac{s_r-s_{r-1}}{\kappa}\,\widehat{\varphi}(k_r+\cdots+k_n)\right],
\end{aligned}
\tag{4.7.16}
$$

where the last line uses (4.7.13). Since the r.h.s. is non-negative, taking the modulus of the r.h.s. of (4.7.15), we obtain

$$
\begin{aligned}
v^{(t)}(x,s) &\le \sum_{n=0}^{\infty}\left(\frac{|\alpha|}{\kappa}\right)^n \left(\prod_{l=1}^{n}\int_{s_{l-1}}^{s} ds_l\right)\left(\prod_{m=1}^{n}\oint dk_m\right)\\
&\qquad \times \exp\left[-\Big(2dT1[\kappa]\Big)\sum_{q=1}^{n}\widehat{\varphi}(k_q)\right]E_0^{\mathrm{RW}}\left(\exp\left[i\sum_{r=1}^{n}Y_{\frac{s_r}{\kappa}}\cdot k_r\right]\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{|\alpha|}{\kappa}\right)^n \left(\prod_{l=1}^{n}\int_{s_{l-1}}^{s} ds_l\right)E_0^{\mathrm{RW}}\left(\prod_{m=1}^{n}p_{2dT1[\kappa]}\Big(0,Y_{\frac{s_m}{\kappa}}\Big)\right),
\end{aligned}
\tag{4.7.17}
$$

where the last line uses (4.7.13). Thus

$$
\begin{aligned}
v^{(t)}(x,s) &\le E_0^{\mathrm{RW}}\left(\exp\left[\frac{|\alpha|}{\kappa}\int_0^s du\,p_{2dT1[\kappa]}\Big(0,Y_{\frac{u}{\kappa}}\Big)\right]\right)\\
&\le E_0^{\mathrm{RW}}\left(\exp\left[|\alpha|\int_0^{\infty} du\,p_{2dT1[\kappa]}(0,Y_u)\right]\right).
\end{aligned}
\tag{4.7.18}
$$

Next, let $\mathcal{G}$ denote the Green operator acting on functions $V\colon\,\mathbb{Z}^d\to[0,\infty)$ as

$$(\mathcal{G}V)(x)=\sum_{y\in\mathbb{Z}^d}G(x,y)V(y),\qquad x\in\mathbb{Z}^d. \tag{4.7.19}$$

With p_t denoting the function $p_t(0,\cdot)$, we have

$$\Big\|\mathcal{G}\Big(|\alpha|\,p_{2dT1[\kappa]}\Big)\Big\|_\infty=|\alpha|\sup_{x\in\mathbb{Z}^d}\int_0^\infty ds\,\sum_{y\in\mathbb{Z}^d}p_s(x,y)\,p_{2dT1[\kappa]}(0,y)\le|\alpha|G_{2dT1[\kappa]} \tag{4.7.20}$$

with

$$G_t=\int_t^\infty ds\;p_s(0,0) \tag{4.7.21}$$

the truncated Green function at the origin. The r.h.s. of (4.7.20) tends to zero as $T\to\infty$. Hence Lemma 8.2.1 in Gärtner and den Hollander (10) can be applied to the r.h.s. of (4.7.18) for T large enough, to yield

$$v^{(t)}(x,s)\le\frac{1}{1-\Big\|\mathcal{G}\Big(|\alpha|\,p_{2dT1[\kappa]}\Big)\Big\|_\infty}\downarrow 1\quad\text{as}\quad T\to\infty,\quad\text{uniformly in }\kappa>0. \tag{4.7.22}$$

Thus, for T large enough and all x, s, t, κ and $X_{(.)}$, we have $v^{(t)}(x,s)\le 2$, as claimed earlier. For such T, recalling (4.7.5), we conclude from (4.7.2) that $w^{(t)}\le\bar{w}^{(t)}$, where $\bar{w}^{(t)}$ solves

$$\frac{\partial\bar{w}^{(t)}}{\partial s}(x,s)=\frac{1}{2d\kappa}\Delta\bar{w}^{(t)}(x,s)+\frac{2|\alpha|}{\kappa}\,p_{2dT1[\kappa]}(X_{t-s},x),\qquad\bar{w}^{(t)}(\cdot,0)\equiv 0, \tag{4.7.23}$$

The latter has the representation

$$\bar{w}^{(t)}(x,s)=\frac{2|\alpha|}{\kappa}\int_0^s du\,\sum_{z\in\mathbb{Z}^d}p_{\frac{s-u}{\kappa}}(x,z)\,p_{2dT1[\kappa]}(X_{t-u},z)=\frac{2|\alpha|}{\kappa}\int_0^s du\,p_{\frac{s-u}{\kappa}+2dT1[\kappa]}(x,X_{t-u}). \tag{4.7.24}$$

Hence,

$$\begin{aligned}&E_0\bigg(\exp\bigg[\frac{\alpha}{\kappa}\rho\int_0^t ds\sum_{x\in\mathbb{Z}^d}p_{2dT1[\kappa]}(X_{t-s},x)\,w^{(t)}(x,s)\bigg]\bigg)\\&\qquad\le E_0\bigg(\exp\bigg[\frac{|\alpha|}{\kappa}\rho\int_0^t ds\sum_{x\in\mathbb{Z}^d}p_{2dT1[\kappa]}(X_{t-s},x)\,\bar{w}^{(t)}(x,s)\bigg]\bigg)\\&\qquad=E_0\bigg(\exp\bigg[\frac{2\alpha^2}{\kappa^2}\rho\int_0^t ds\int_0^s du\,p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_{t-s},X_{t-u})\bigg]\bigg),\end{aligned} \tag{4.7.25}$$

which proves the claim in (4.7.3). ∎

4.7.4 Proof of Lemma 4.7.3

The proof of Lemma 4.7.3 is based on the following lemma. For $t \geq 0$, $\alpha \in \mathbb{R}$ and $a, \kappa, T > 0$, let

$$\Lambda_\alpha(t; a, \kappa, T) = \frac{1}{2t} \log E_0\bigg(\exp\left[\frac{4\alpha^2}{\kappa^2} \rho \int_0^t ds \int_s^{s+a\kappa^3} du\ p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \right] \bigg) \tag{4.7.26}$$

and

$$\lambda_\alpha(a, \kappa, T) = \limsup_{t\to\infty} \Lambda_\alpha(t; a, \kappa, T). \tag{4.7.27}$$

Lemma 4.7.4. *If $d \geq 4$, then for any $\alpha \in \mathbb{R}$ and $a, T > 0$,*

$$\limsup_{\kappa\to\infty} \kappa^2 \lambda_\alpha(a, \kappa, T) \leq 2\alpha^2 \rho\, G_{4dT}, \tag{4.7.28}$$

where G_t is the truncated Green function at the origin defined by (4.7.21). Before giving the proof of Lemma 4.7.4, we first prove Lemma 4.7.3.

Proof of Lemma 4.7.3. Return to (4.7.4). By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\frac{\kappa^2}{t} \log E_0\bigg(\exp\left[\frac{2\alpha^2}{\kappa^2} \rho \int_0^t ds \int_s^t du\, p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \right] \bigg) \\ &\quad\leq \frac{\kappa^2}{2t} \log E_0\bigg(\exp\left[\frac{4\alpha^2}{\kappa^2} \rho \int_0^t ds \int_s^{s+a\kappa^3} du\ p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \right] \bigg) \\ &\qquad + \frac{\kappa^2}{2t} \log E_0\bigg(\exp\left[\frac{4\alpha^2}{\kappa^2} \rho \int_0^t ds \int_{s+a\kappa^3}^{\infty} du\ p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \right] \bigg). \end{aligned} \tag{4.7.29}$$

Moreover, by Lemma 4.5.4 and the fact that $d \geq 3$, we have

$$\begin{aligned} \frac{1}{\kappa^2} \int_0^t ds \int_{s+a\kappa^3}^{\infty} du\ p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) &\leq \frac{1}{\kappa^2} \int_0^t ds \int_{s+a\kappa^3}^{\infty} du\ p_{\frac{u-s}{\kappa}}(0,0) \\ &\leq \frac{C}{\kappa}\, t \int_{a\kappa^2}^{\infty} du\ \frac{1}{(1+u)^{\frac{d}{2}}} \leq \frac{\tilde{C}}{a^{\frac{1}{2}}\kappa^2}\, t \end{aligned} \tag{4.7.30}$$

with $C, \tilde{C} > 0$. Combining (4.7.29–4.7.30) and Lemma 4.7.4, and letting $a \to \infty$, we get (4.7.4). ∎

The proof of Lemma 4.7.4 is based on one further lemma. For $\gamma \geq 0$ and $a, \kappa, T > 0$, let

$$\Lambda_\gamma(a, T) = \limsup_{\kappa\to\infty} \frac{1}{a\kappa} \log E_0\bigg(\frac{\gamma}{\kappa^2} \int_0^{a\kappa^3} ds \int_s^{\infty} du\ p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_s, X_u) \bigg). \tag{4.7.31}$$

Lemma 4.7.5. *If $d \geq 4$, then for any $\gamma \geq 0$ and $a, T > 0$,*

$$\Lambda_\gamma(a, T) \leq \gamma\, G_{4dT}. \tag{4.7.32}$$

Before giving the proof of Lemma 4.7.5, we first prove Lemma 4.7.4.

Proof of Lemma 4.7.4. Split the integral in the exponent in the r.h.s. of (4.7.26) as follows:

$$\begin{aligned}\int_0^t ds \int_s^{s+a\kappa^3} du\; p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \\ \leq \left(\sum_{\substack{k=1\\ \text{even}}}^{\lceil t/a\kappa^3 \rceil} + \sum_{\substack{k=1\\ \text{odd}}}^{\lceil t/a\kappa^3 \rceil} \right) \int_{(k-1)a\kappa^3}^{ka\kappa^3} ds \int_s^{s+a\kappa^3} du\; p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s).\end{aligned} \tag{4.7.33}$$

Note that in each of the two sums, the summands are i.i.d. Hence, substituting (4.7.33) into (4.7.26) and applying the Cauchy-Schwarz inequality, we get

$$\Lambda_\alpha(t; a, \kappa, T) \leq \frac{\lceil t/a\kappa^3 \rceil}{4t} \log E_0\left(\exp\left[\frac{8\alpha^2}{\kappa^2}\rho \int_0^{a\kappa^3} ds \int_s^{s+a\kappa^3} du\; p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \right] \right). \tag{4.7.34}$$

Letting $t \to \infty$ and recalling (4.7.27), we arrive at

$$\lambda_\alpha(a, \kappa, T) \leq \frac{1}{4a\kappa^3} \log E_0\left(\exp\left[\frac{8\alpha^2}{\kappa^2}\rho \int_0^{a\kappa^3} ds \int_s^{s+a\kappa^3} du\; p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_u, X_s) \right] \right). \tag{4.7.35}$$

Combining this with Lemma 4.7.5 (with $\gamma = 8\alpha^2\rho$), we obtain (4.7.28). ∎

The proof of Lemma 4.7.5 is based on two further lemmas.

Lemma 4.7.6. *For any $\beta > 0$ and $M \in \mathbb{N}$,*

$$\begin{aligned}E_0\left(\exp\left[\beta \sum_{k=1}^{M} \int_0^\infty ds\; p_{\frac{s}{\kappa}+4dT1[\kappa]}\big(U_{k-1}(0), U_{k-1}(s)\big) \right] \right) \\ \leq \prod_{k=1}^{M} \max_{y_1,\cdots,y_{k-1}\in\mathbb{Z}^d} E_0\left(\exp\left[\beta \sum_{l=0}^{k-1} \int_0^\infty ds\; p_{\frac{a\kappa^2}{M}l+\frac{s}{\kappa}+4dT1[\kappa]}(0, X_s + y_l) \right] \right),\end{aligned} \tag{4.7.36}$$

where $U_k(t) = X(\frac{k}{M}a\kappa^3 + s)$, $k \in \mathbb{N}_0$ and $y_0 = 0$.

Lemma 4.7.7. *For any $\beta > 0$, $M \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $y_0, \cdots, y_k \in \mathbb{Z}^d$,*

$$E_0\left(\exp\left[\beta \sum_{l=0}^{k-1} \int_0^\infty ds\; p_{\frac{a\kappa^2}{M}l+\frac{s}{\kappa}+4dT1[\kappa]}(0, X_s + y_l) \right] \right) \leq \exp\left[\frac{\beta \sum_{l=0}^{k} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]}}{1 - \beta \sum_{l=0}^{k} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]}} \right], \tag{4.7.37}$$

(recall (4.7.21)*), provided that*

$$\beta \sum_{l=0}^{k} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]} < 1. \tag{4.7.38}$$

The proofs of Lemmas 4.7.6–4.7.7 are similar to those of Lemmas 6.3.1–6.3.2 in Gärtner and den Hollander (10). We refrain from spelling out the details. We conclude by proving Lemma 4.7.5.

Proof of Lemma 4.7.5. As in the proof of Lemma 6.2.1 in Gärtner and den Hollander (10), using Lemmas 4.7.6–4.7.7 we obtain

$$\begin{aligned} &\frac{1}{a\kappa}\log E_0\Bigg(\exp\left[\frac{\gamma}{\kappa^2}\int_0^{a\kappa^3} ds\int_s^\infty du\; p_{\frac{u-s}{\kappa}+4dT1[\kappa]}(X_s,X_u)\right]\Bigg)\\ &\qquad\le \frac{\gamma\sum_{l=0}^{M-1} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]}}{1-\gamma\frac{a\kappa}{M}\sum_{l=0}^{M-1} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]}}, \end{aligned} \tag{4.7.39}$$

provided that

$$\gamma\frac{a\kappa}{M}\sum_{l=0}^{M-1} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]} < 1. \tag{4.7.40}$$

But (recall (4.7.21))

$$\sum_{l=0}^{M-1} G_{\frac{a\kappa^2}{M}l+4dT1[\kappa]} \le G_{4dT1[\kappa]} + \sum_{l=1}^{M-1} G_{\frac{a\kappa^2}{M}l}. \tag{4.7.41}$$

From Lemma 4.5.4 we get $G_t \le C/t^{\frac{d}{2}-1}$. Therefore

$$\frac{\kappa}{M}\sum_{l=1}^{M-1} G_{\frac{a\kappa^2}{M}l} \le \begin{cases} \frac{C_3}{a^{\frac{1}{2}}} & \text{if } d=3,\\ \frac{C_4}{a}\frac{1}{\kappa}\log M & \text{if } d=4,\\ \frac{C_d}{a^{\frac{d}{2}-1}}\frac{M^{\frac{d}{2}-2}}{\kappa^{d-3}} & \text{if } d\ge 5, \end{cases} \tag{4.7.42}$$

for some $C_d>0$, $d\ge 3$. Hence, picking $1 \ll M \le C\kappa^2$, (4.7.40) holds for κ large enough when $d\ge 4$, and so the claim (4.7.32) follows from (4.7.39) and (4.7.41–4.7.42). ∎

4.8 Extension to arbitrary p

In Sections 4.2–4.7 we proved Theorem 1.3.4 for $p=1$. We briefly indicate how the proof can be extended to arbitrary p.

As in (4.1.6), after time rescaling we have, for any $p\in\mathbb{N}$,

$$\lambda_p^*(\kappa) = \lim_{t\to\infty}\Lambda_p^*(\kappa;t) \quad\text{with}\quad \Lambda_p^*(\kappa;t) = \frac{1}{t}\log\mathbb{E}_{\nu_\rho,0,\cdots,0}\Bigg(\exp\left[\frac{1}{\kappa}\int_0^t ds\sum_{k=1}^p \xi_{\frac{s}{\kappa}}\big(X_k(s)\big)\right]\Bigg). \tag{4.8.1}$$

We are interested in the quantity

$$\lambda_p^*(\kappa) - \frac{\rho}{\kappa} = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\nu_\rho,0,\cdots,0}\Bigg(\exp\left[\frac{1}{\kappa}\int_0^t ds\sum_{k=1}^p\Big(\xi_{\frac{s}{\kappa}}\big(X_k(s)\big)-\rho\Big)\right]\Bigg). \tag{4.8.2}$$

As in (4.4.1), for $T>0$ let $\psi_p\colon\ \Omega\times(\mathbb{Z}^d)^p$ be defined by

$$\psi(\eta,x_1,\cdots,x_p) = \int_0^T ds\left(\mathcal{P}_s^{(p)}\phi_p\right)(\eta,x_1,\ldots,x_p) \quad\text{with}\quad \phi_p(\eta,x_1,\cdots,x_p) = \sum_{k=1}^p\big[\eta(x_k)-\rho\big], \tag{4.8.3}$$

where $(\mathcal{P}_s^{(p)})_{s\geq 0}$ is the semigroup with generator (compare with (4.1.4))

$$\mathcal{A}^{(p)} = \frac{1}{\kappa} L + \sum_{k=1}^{p} \Delta_k. \tag{4.8.4}$$

Using (1.2.6), we obtain the representation (compare with (4.4.3))

$$\psi_p(\eta, x_1, \cdots, x_p) = \int_0^T ds \sum_{z\in\mathbb{Z}^d} \sum_{k=1}^{p} p_{2ds1[\kappa]}(z, x_k)\big[\eta(z) - \rho\big] = \sum_{k=1}^{p} \psi(\eta, x_k). \tag{4.8.5}$$

Let (compare with (4.1.3))

$$Z_s^{(p)} = \big(\xi_{\frac{s}{\kappa}}, X_1(s), \cdots, X_p(s)\big). \tag{4.8.6}$$

First, we have the analogue of Proposition 4.4.1:

Proposition 4.8.1. *For any $p \in \mathbb{N}$, $\kappa, T > 0$,*

$$\begin{aligned}\lambda_p^*(\kappa) - \frac{\rho}{\kappa} \begin{array}{c}\leq\\ \geq\end{array} \frac{1}{2q} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\bigg(\exp\bigg[\frac{2q}{r}\int_0^t ds\, \Big[\Big(e^{-\frac{r}{\kappa}\psi_p}\mathcal{A}e^{\frac{r}{\kappa}\psi_p}\Big) - \mathcal{A}\Big(\frac{r}{\kappa}\psi_p\Big)\Big]\,(Z_s^{(p)})\bigg]\bigg)\\ + \frac{1}{4q} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\bigg(\exp\bigg[\frac{4q}{\kappa}\int_0^t ds\, \Big(\mathcal{P}_T^{(p)}\phi_p\Big)\,(Z_s^{(p)})\bigg]\bigg),\end{aligned} \tag{4.8.7}$$

where $1/r + 1/q = 1$, for any $r, q > 1$ in the first inequality and any $q < 0 < r < 1$ in the second inequality.

Next, using (4.8.5), the bound

$$\Big(\psi_p(\eta^{a,b}, x_1, \cdots, x_p) - \psi_p(\eta, x_1, \cdots, x_p)\Big)^2 \leq p \sum_{k=1}^{p} \Big(\psi(\eta^{a,b}, x_k) - \psi(\eta, x_k)\Big)^2, \tag{4.8.8}$$

and the estimate in (4.5.4), we also have the analogue of Lemma 4.6.1:

Lemma 4.8.2. *Uniformly in $\eta \in \Omega$ and $x_1, \cdots, x_p \in \mathbb{Z}^d$,*

$$\begin{aligned}&\Big[\Big(e^{-\frac{r}{\kappa}\psi_p}\mathcal{A}\,e^{\frac{r}{\kappa}\psi_p}\Big) - \mathcal{A}\,\Big(\frac{r}{\kappa}\psi_p\Big)\Big]\,(\eta, x_1, \cdots, x_p)\\ &\quad= \frac{r^2}{2\kappa^2} \sum_{k=1}^{p} \sum_{e:\ \|e\|=1} \Big(\psi(\eta, x_k + e) - \psi(\eta, x_k)\Big)^2 + O\bigg(\Big(\frac{1}{\kappa}\Big)^3\bigg).\end{aligned} \tag{4.8.9}$$

Using Hölder's inequality to separate terms, we may therefore reduce to the case $p = 1$ and deal with the first term in the r.h.s. of (4.8.7) to get the analogue of Proposition 4.4.2.
For the second term in (4.8.7), using (1.2.6) we have

$$(\mathcal{P}_T^{(p)}\phi_p)(\eta, x_1, \cdots, x_p) = \sum_{k=1}^{p} \sum_{z\in\mathbb{Z}^d} p_{2dT1[\kappa]}(z, x_k)\big[\eta(z) - \rho\big] = \sum_{k=1}^{p} \big(\mathcal{P}_T\phi\big)(\eta, x_k). \tag{4.8.10}$$

Using Hölder's inequality to separate terms, we may therefore again reduce to the case $p = 1$ and deal with the second term in the r.h.s. of (4.8.7) to get the analogue of Proposition 4.4.3.

A Appendix

In this appendix we give the proof of Proposition 1.2.1.

Proof. Fix $t \geq 0$, $\eta \in \Omega$ and $K\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ such that $S = \sum_{z\in\mathbb{Z}^d} \int_0^t ds\, |K(z,s)| < \infty$. First consider the case $K \geq 0$. Since the ξ-process and the $\tilde{\xi}$-process are both monotone in their initial configuration (as is evident from the graphical representation described in Section 1.2), it suffices to show that

$$\mathbb{E}_\eta\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\xi_s(z)\right]\right) \leq \mathbb{E}_\eta^{\mathrm{IRW}}\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\tilde{\xi}_s(z)\right]\right), \tag{A.0.11}$$

for all $\eta \in \Omega$ such that $|\{x \in \mathbb{Z}^d : \eta(x) = 1\}| < \infty$. This goes as follows.
Since $\xi_s(z) \in \{0,1\}$, we may write for any $r \in \mathbb{R} \setminus \{0\}$,

$$\mathbb{E}_\eta\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\xi_s(z)\right]\right) = \mathbb{E}_\eta\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\frac{e^{r\,\xi_s(z)}-1}{e^r-1}\right]\right). \tag{A.0.12}$$

By Taylor expansion, we get

$$\begin{aligned}
&\mathbb{E}_\eta\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\frac{e^{r\,\xi_s(z)}-1}{e^r-1}\right]\right)\\
&= \exp\left[\frac{-t}{e^r-1}\,S\right]\mathbb{E}_\eta\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\frac{e^{r\,\xi_s(z)}}{e^r-1}\right]\right)\\
&= \exp\left[\frac{-t}{e^r-1}\,S\right]\\
&\quad\times\sum_{n=0}^\infty\left(\frac{1}{e^r-1}\right)^n\frac{1}{n!}\left(\prod_{j=1}^n\int_0^t ds_j\sum_{z_j\in\mathbb{Z}^d}\right)\left(\prod_{j=1}^n K(z_j,s_j)\right)\mathbb{E}_\eta\left(\exp\left[r\sum_{j=1}^n\xi_{s_j}(z_j)\right]\right).
\end{aligned}\tag{A.0.13}$$

According to Lemma 4.1 in Landim (18), we have for any $r \in \mathbb{R}$,

$$\mathbb{E}_\eta\left(\exp\left[r\sum_{j=1}^n\xi_{s_j}(z_j)\right]\right) \leq \mathbb{E}_\eta^{\mathrm{IRW}}\left(\exp\left[r\sum_{j=1}^n\tilde{\xi}_{s_j}(z_j)\right]\right). \tag{A.0.14}$$

Picking $r \geq 0$, combining (A.0.12–A.0.14), and using the analogue of (A.0.13) for $(\tilde{\xi}_t)_{t\geq 0}$, we obtain

$$\mathbb{E}_\eta\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\xi_s(z)\right]\right) \leq \mathbb{E}_\eta^{\mathrm{IRW}}\left(\exp\left[\sum_{z\in\mathbb{Z}^d}\int_0^t ds\; K(z,s)\,\frac{e^{r\,\tilde{\xi}_s(z)}-1}{e^r-1}\right]\right). \tag{A.0.15}$$

Now let $r \downarrow 0$ and use the dominated convergence theorem to arrive at (A.0.11).
For the case $K \leq 0$ we can use the same argument with

$$-\,\xi_s = \frac{e^{-r\xi_s}-1}{1-e^{-r}}. \tag{A.0.16}$$

∎

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