Electronic Journal of Probability

Vol. 6 (2001) Paper no. 6, pages 1–27.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol6/paper6.abs.html

STRICT INEQUALITY FOR PHASE TRANSITION BETWEEN FERROMAGNETIC AND FRUSTRATED SYSTEMS[1]

Emilio De Santis
Department of Mathematics, University of Roma "La Sapienza"
Piazzale Aldo Moro 2, 00185 Roma, Italy
desantis@mat.uniroma1.it

Abstract We consider deterministic and disordered frustrated systems in which we can show some strict inequalities with respect to related ferromagnetic systems. A case particularly interesting is the Edwards-Anderson spin-glass model in which it is possible to determine a region of uniqueness of the Gibbs measure, which is strictly larger than the region of uniqueness for the related ferromagnetic system. We analyze also deterministic systems with $|J_b| \in [J_A, J_B]$, where $0 < J_A \leq J_B < \infty$, for which we prove strict inequality for the critical points of the related FK model. The results are obtained for the Ising models but some extensions to Potts models are possible.

Keywords Phase transition, Ising model, disordered systems, stochastic order.

AMS subject classification 82B26, 82B31, 82B43, 82B44, 82C20.

Submitted to EJP on December 8, 2000. Final version accepted on February 7, 2001.

[1]THE WORK WAS PARTIALLY SUPPORTED BY THE ITALIAN CNR GRANT.

1 Introduction

The problem of proving inequalities in probabilistic models is very common; especially in statistical mechanics, there are models in which many qualitative properties, such as the existence of a phase transition, are only shown by using inequalities. In particular, during the last ten years there was a large effort to prove strict inequalities between critical points (see [AG91, BGK93]). In these works, inequalities are proved between the critical points in percolation and in the Ising model and there is an extension to the Potts model and to many-body interactions in [Gr94].

There are some recent papers that prove this kind of results for disordered systems. These are models in which the interactions are themselves random variables, so that the Gibbs measure becomes a function of these random variables. Important results for disordered models are in [Ca98, Gr99]. In the second work there is a general strategy to study this kind of problems in a wide context.

In this paper, we will prove a strict inequality between phase transitions in the ferromagnetic Ising and in the Edwards-Anderson models (see [Is25, EA75]). We have to stress that it is not rigorously known whether there is a phase transition in the Edwards-Anderson model at some positive temperature, but in any case our result makes sense. In a future paper we will also define a frustrated Potts model and a frustrated many-body model for which it is possible to extended our results for the two-body interactions.

Our results, for the strict inequality between the percolation critical points of the random cluster models, are related in spirit and partly in methodology to the recent work of Grimmett [Gr99] (see also [BGK93]) and to work of Campanino [Ca98]. Then we use a work of Newman [Ne94] (see also [Ne97]) to show, as a consequence of the random cluster percolation result, a strict inequality for the phase transition of the related Gibbs measures. The main differences between our work and [Ca98, Gr99] are:

a) we prove a strict inequality for the phase transition, i.e. for uniqueness of the Gibbs distribution, and not only for symmetry breaking;

b) we show also a strict inequality for the phase transitions of disordered ferromagnetic Ising models (such a result, but with a different methodology, is also proved by Gandolfi [Ga98]);

c) our method can be extended to Potts models and to frustrated many-body models; in a future paper we will provide these extensions (this is not explicit in [Ca98] and we do not know if it is possible to find a related Gibbs measure for the random cluster measure in [Ca98] besides the Ising model). Now we will deal only with the Ising models for the sake of clarity and not to add unnecessary difficulties.

We present here the main result on disordered systems which was suggested as an open problem by Newman (see C. Newman, *Topics in Disordered Systems.* Lecture in Mathematics, 1997 [Ne97]).

Theorem 1 *Let J be an interaction configuration with $|J_e| = 1$. The J_e are i.i.d. random variables with the probability $Q(J_e = -1) = \tilde{p}$ and $0 < \tilde{p} < 1$. Then the Gibbs measure $\pi_{J\beta}$ is unique in a region strictly larger than the uniqueness region of the ferromagnetic Gibbs measure $\pi^F_{|J|\beta}$ Q-a.e.:*

$$\beta_c(J) > \beta_c(|J|) \qquad Q - a.e. \tag{1}$$

We remark that the standard Edwards-Anderson model has $p = 1/2$, so we obtain that the region of uniqueness for the E-A model is larger than for the ferromagnetic Ising model. We will prove this theorem at the end of section 4.

2 General definitions and main results

The Graph. We consider the infinite graph $\mathbf{E}^d = (\mathbf{Z}^d, E(\mathbf{Z}^d))$ embedded in $\mathbf{R}^d$; the graph has an edge for every pair of vertices having Euclidean distance equal to 1. We will often abbreviate $E(\mathbf{Z}^d)$ with E. We will use the following definitions: the *distance between two vertices* x, $y \in \mathbf{Z}^d$ is $d(x, y) = \max_{i=1,\dots,d} |x_i - y_i|$ and the *distance between two sets* A, $B \subset \mathbf{Z}^d$ is $d(A, B) = \min_{x \in A, y \in B} d(x, y)$. The same definition of distance applies also to edges thought as a set of two vertices. We say that the vertices i and k belong to an edge e if $e = \{i, k\}$. For a given set of edges $F \subset E$ we will call

$$V(F) = \{x : \exists y \in \mathbf{Z}^d \text{ such that } \{x, y\} \in F\} \tag{2}$$

the vertex set of F.

Given a set of vertices $A \subset \mathbf{Z}^d$, we indicate with $E(A)$ the edges which have both vertices in A and with ∂A the edges which have one vertex in A and one vertex in A^c, where A^c is the complement of A.

A *connected set of vertices* is a subset $A \subset \mathbf{Z}^d$ with the property that for all the pairs of vertices $v_i, v_f \in A$ there exists a finite sequence of vertices $v_1, v_2, \dots v_n \in A$ such that $D(v_i, v_1) = D(v_n, v_f) = 1$ and $D(v_k, v_{k+1}) = 1$ for all $k = 1, \dots, n-1$, where $D(,)$ is the Euclidean distance.

Given a finite set of edges γ we say that it is a *minimal cut set* if there exists a finite connected set of vertices A such that $\gamma = \partial A$; in $\mathbf{E}^d$, given a minimal cut set γ, there is only one connected set of vertices A verifying the previous condition, so we will call this finite connected set of vertices A the inner part of γ -writing $Int(\gamma) = A$- and A^c the outer part of γ -writing $Out(\gamma) = A^c$.

Blocks and cover of $\mathbf{Z}^d$. We need to introduce the following subsets of $E(\mathbf{Z}^d)$: a *block* $B_n(v)$ is a set of edges

$$B_n(v) = \{e \in E(\mathbf{Z}^d) : d(e, v) \leq n\} \tag{3}$$

with *center* $v \in \mathbf{Z}^d$ and *size* $n \in \mathbf{N}$. We will call $\partial^E B_n(v)$, defined as follows

$$\partial^E B_n(v) = \{\{i, k\} \in E(\mathbf{Z}^d) : \{i, k\} \notin B_n(v) \text{ and } i, k \in V(B_n(v))\} \tag{4}$$

the *outer boundary of* $B_n(v)$ and

$$\partial^I B_n(v) = \{(i,k) \in E(\mathbf{Z}^d) : d(v,i) + d(v,k) = 2n+1\} \tag{5}$$

the *inner boundary of* $B_n(v)$. Notice that $\partial^E B_n(v) \cap B_n(v) = \emptyset$ and $\partial^I B_n(v) \subset B_n(v)$. The collection of blocks $\{B_n(k)\}_{k\in C_n}$ where C_n is the sub-lattice of $\mathbf{Z}^d$ defined as

$$C_n = \{k = (k_1, k_2, \dots, k_d) \in \mathbf{Z}^d : k_i = (2n+1)h_i \text{ with } h_i \in \mathbf{Z} \text{ for } i = 1, \dots, d\}$$

is a cover of $E(\mathbf{Z}^d)$, that is $E(\mathbf{Z}^d) = \cup_{v\in C_n} B_n(v)$. Two blocks of size n belonging to the cover are said to be *adjacent* if the Euclidean distance between their centers is equal to $2n+1$. It is easy to see that the intersection of adjacent blocks is not empty and is equal to the intersection of their inner boundaries. We remark also that $\partial^I B_n(v)$ has the important property of being a minimal cut set.

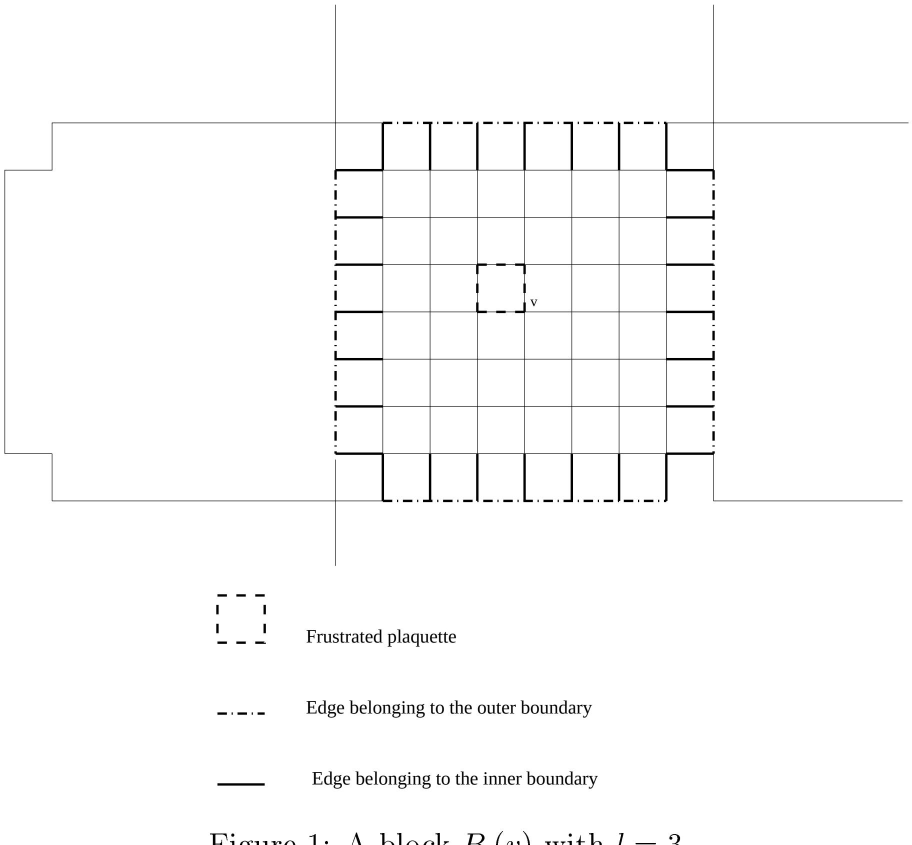

Figure 1: A block $B_l(v)$ with $l = 3$.

Spin State Space. The *spin state space* is $\Omega = \{-1,1\}^{\mathbf{Z}^d}$, a *spin configuration* is indicated with $\sigma \in \Omega$ and a *spin* on a vertex $i \in \mathbf{Z}^d$ is indicated with $\sigma_i \in \{-1,1\}$ or with $\sigma(i)$. With σ_A or $\sigma(A)$ we will indicate the restriction of σ to the vertices A.

The *edge state space* is $H_E = \{0,1\}^E$ and an *edge configuration* is indicated with $\eta \in H_E$ with $\eta_e \in \{0,1\}$ for $e \in E$; we call an edge e *open* if $\eta_e = 1$ and *closed* otherwise. We will consider, for the spaces Ω and H_E, the σ-algebras $\mathcal{F}$ and $\mathcal{F}_E$ generated by all finite cylinders.

We will also consider a finite set of vertices $\Lambda \subset \mathbf{Z}^d$ but we will always think of it as a subset of $\mathbf{Z}^d$ so that the definition of inner and outer part makes sense; we define $\Omega_\Lambda = \{-1,1\}^\Lambda$ and $H_{E,\Lambda} = \{0,1\}^{E(\Lambda)\cup\partial\Lambda}$ (in these cases of finite spaces the σ-algebras are the space of all subsets of Ω_Λ and $H_{E,\Lambda}$). We define $[\sigma\tau]_\Lambda$ as a spin configuration in Ω having $[\sigma\tau]_\Lambda(i) = \sigma(i)$ if $i \in \Lambda$ and $[\sigma\tau]_\Lambda(i) = \tau(i)$ otherwise.

A *cluster* C is the maximal connected set of vertices having the property that for all the pairs of vertices $v_i, v_f \in C$ there exists a finite sequence of vertices $v_1, \ldots v_n \in C$ such that $D(v_i, v_1) = D(v_n, v_f) = 1$ and $D(v_k, v_{k+1}) = 1$ for all $k = 1, \ldots, n-1$ and all the pairs of vertices $\{v_i, v_1\}$, $\{v_1, v_2\}, \ldots,$ $\{v_n, v_f\}$ are open edges.

Let $J_e = J_{\{i,k\}} \in \mathbf{R}$ be the *interaction* between the spins i and k. For all $e \in E$ define $p_e = 1 - \exp(-\beta|J_e|)$ where the parameter $\beta \in (0,\infty)$ is the *inverse temperature*. We denote the configuration of all $\{J_e\}_{e\in E}$ with J.

The *potential* $\phi = (\phi_{e_1}, \phi_{e_2}, \ldots)$, where $\{e_i\}_i$ is the set of all the edges $E(\mathbf{Z}^d)$, is a function; for every edge e we put $\phi_e : \{-1,1\}\times\{-1,1\} \to \{-1,1\}$ defined as: $\phi_e(\sigma) = J_e\sigma_i\sigma_k$ where $e = \{i,k\} \in E$.

We say that a potential $\tilde{\phi}$ is a *gauge transformation* of the potential ϕ if for each vertex i there exists a 1:1 mapping $f_i : \{-1,1\} \to \{-1,1\}$ such that $\tilde{\sigma}_i = f_i(\sigma_i)$ and $\tilde{\phi}(\tilde{\sigma}) = \phi(\sigma)$ for all $\sigma \in \Omega$.

Random Cluster Measure or FK-*measure.* We will follow the exposition in [Ne97]. Let $\Lambda \subset \mathbf{Z}^d$ be a finite set of vertices and τ a configuration in Ω. Define the *random cluster model* (or the FK-*measure*) on the finite space $H_{E,\Lambda}$ with τ boundary conditions as:

$$\mu^\tau_{\Lambda,J\beta}(\eta) = \frac{I^\tau_{\eta\sim J}\ 2^{k(\eta)} \prod_{b\in E} p_e^{\eta_e}(1-p_e)^{(1-\eta_e)}}{Z^\tau_{\Lambda,J\beta}}, \qquad \forall \eta \in H_{E,\Lambda} \tag{6}$$

where $k(\eta)$ is the number of clusters which do not have open edges in $\partial\Lambda$ (the clusters not touching the boundary), and $Z^\tau_{\Lambda,J\beta}$ is the normalizing factor (or *partition function*) and $I^\tau_{\eta\sim J}$ is the indicator function

$$I^\tau_{\eta\sim J} = \begin{cases} 1 & \text{if } \exists\ \sigma : \forall e \in E(\Lambda)\cup\partial\Lambda\ \phi_e([\sigma\tau]_\Lambda)\eta_e \geq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

We remark that the random cluster measure can depend on the boundary conditions because the forbidden configurations can change.

We will say that there is a constant boundary condition if $\tau \equiv i$ and $i = -1, 1$. We say that the system is *unfrustrated* if for $\eta \equiv 1$ (we mean that for each $e \in E$: $\eta_e = 1$) we have $I^{\tau\equiv 1}_{\eta\sim J} = 1$. The systems with $J \equiv 1$ and the systems which are gauge transformations of those are unfrustrated. A *plaquette* Pl is a square of four edges and it is called *frustrated* if $I^{\tau\equiv 1}_{\eta\sim J} = 0$ for all configurations η with $\eta(Pl) = 1$. In what follows we will show an application of the FK measure to a family of Ising models.

The Hamiltonian and the Gibbs Measure. We define the Hamiltonian on the finite set of vertices Λ with boundary condition τ. Let $\sigma, \tau \in \Omega_\Lambda$; the Hamiltonian is:

$$H_{\Lambda,J}(\sigma|\tau) = -\sum_{e\in E(\Lambda)} \phi_e(\sigma) - \sum_{e\in\partial\Lambda} \phi_e([\sigma\tau]_\Lambda). \tag{8}$$

The related *Gibbs measure* $\pi^\tau_{\Lambda,J\beta}$ is

$$\pi^\tau_{\Lambda,J\beta}(\sigma) = \frac{1}{Y^\tau_{\Lambda,J\beta}} \exp(-\beta H_{\Lambda,J}(\sigma|\tau)), \tag{9}$$

where $Y^\tau_{\Lambda,J\beta}$ is the partition function for this Gibbs measure. If $J_e > 0$ for all $e \in E$ we say that the system is ferromagnetic and we add an index F to the Gibbs measures, writing π^F; the ferromagnetic FK measures is μ^F. We will use also the *free boundary conditions* for the Gibbs and the FK measures; these measures with free boundary condition are defined following the same arguments in (6)-(9) just imposing that all the interactions on edges in $\partial\Lambda$ are identically equal to zero. In this case we simply do not label the boundary condition.

From [Ne97] we know that there is a joint measure of the Gibbs measure and of the FK measure verifying interesting properties (see Proposition 3.3 p. 33 [Ne97]). So we could obtain a spin configuration $\sigma \in \Omega_\Lambda$ with Gibbs distribution π just taking η with FK distribution μ and *coloring* the single clusters independently of each other; for each cluster let us choose a vertex k and with uniform probability a spin value $\sigma_k \in \{-1, 1\}$ and then color the other vertices in the cluster C according to the rule that $\phi_e(\sigma) > 0$ for all $e \in E(C)$.

Let us introduce, as in [Ne97], a coupling of $\mu^\tau_{\Lambda,J\beta}$ and $\pi^\tau_{\Lambda,J\beta}$ on the finite space $H_{E,\Lambda} \times \Omega_\Lambda$

$$\nu^\tau_{\Lambda,J\beta}(\eta,\sigma) = \frac{L^\tau_{\eta\sim J\sigma} \prod_{e\in E} p_e^{\eta_e}(1-p_e)^{(1-\eta_e)}}{W^\tau_{\Lambda,J\beta}} \tag{10}$$

with

$$L^\tau_{\eta\sim J\sigma} = \begin{cases} 1 \text{ if } \forall e \in E(\Lambda) \cup \partial\Lambda, \forall v \in \Lambda^c \; \sigma_v = \tau_v \;\; \eta_e\phi_e(\sigma) \geq 0 \\ 0 \text{ otherwise} \end{cases}. \tag{11}$$

In [Ne97] it is proved that $\nu_{\Lambda,J\beta}$ is really a joint measure of $\mu_{\Lambda,J\beta}$ and $\pi_{\Lambda,J\beta}$.

Stochastic Order. Let us define a partial order on the configuration space $H_{E,\Lambda}$: $\eta_1 \preceq^* \eta_2$ if $\eta_{1,e} \leq \eta_{2,e}$ for all $e \in E(\Lambda)$, where Λ can be all $\mathbf{Z}^d$.

We will introduce a dynamics on the edge configurations. We denote by $H^T_{E,\Lambda} = \{0,1\}^{E(\Lambda)\times\mathbf{N}}$ the space of the trajectories; the single trajectory is denoted by $\eta^T \in H^T_{E,\Lambda}$, and $\eta_e(t)$ is the value of the configuration at time t and corresponding to the edge e. We have an analogous partial order for the trajectories; we write $\eta^T_1 \preceq^* \eta^T_2$ if for all $e \in E(\Lambda)$ and for all $t \in \mathbf{N}$ we have $\eta_{1,e}(t) \leq \eta_{2,e}(t)$.

An event $C \in \mathcal{F}_E(\Lambda)$ is *increasing* - write $C \uparrow$ - if $\eta \in C$ and $\eta \preceq^* \eta'$ imply that $\eta' \in C$; an analogous definition applies to the trajectory space $H^T_{E,\Lambda}$.

Associated with the partial order we define the *stochastic order* between two measures μ_1 and μ_2: $\mu_1 \preceq \mu_2$ if for all increasing events $A \in \mathcal{F}_E$ we have $\mu_1(A) \leq \mu_2(A)$; in the following we will refer to this definition as Definition A.

Let us define the event $M \in \mathcal{F}_E \times \mathcal{F}_E$ as

$$M = \{(\eta_1, \eta_2) \in (H_{E,\Lambda} \times H_{E,\Lambda}) : \eta_1 \preceq^* \eta_2\} \tag{12}$$

then by means of Strassen's theorem (see [St65], [Li85], [Lin92]) we have the stochastic order $\mu_1 \preceq \mu_2$ if and only if there exists a probability measure $(H_{E,\Lambda} \times H_{E,\Lambda}, \mathcal{F}_{E,\Lambda} \times \mathcal{F}_{E,\Lambda}, \bar{\mu})$ which has μ_1, μ_2 as marginals ($\mu_1(\cdot) = \bar{\mu}(\cdot, H_{E,\Lambda})$, $\mu_2(\cdot) = \bar{\mu}(H_{E,\Lambda}, \cdot)$) and $\bar{\mu}(M) = 1$. We call $\bar{\mu}$ the *joint representation of the stochastic order* $\mu_1 \preceq \mu_2$ or *the coupling measure.* We will refer to this equivalent characterization of the stochastic order as Definition B.

Related to the same partial order we have this definition: for transition kernels K and K' of two Markov chains in $H_{E,\Lambda}$, we say that K is dominated by K' if

$$K(\eta, \cdot) \preceq K'(\eta', \cdot) \text{ for all } \eta', \eta \in H_{E,\Lambda} \text{ such that } \eta \preceq^* \eta' \tag{13}$$

If (13) is true for $K = K'$, we say that K is attractive.

Notice that the stochastic order is a partial order but the relation of domination is not. We will use both $K(\eta, \eta')$ and $K_{\eta,\eta'}$ for the kernel.

We quote standard results in coupling theory (see [Lin92]): if $\mu_1 \preceq \mu_2$ and there are two sequences of kernels $\{K_1^{(i)}\}$ and $\{K_2^{(i)}\}$ with $K_1^{(i)}$ dominated by $K_2^{(i)}$ for every $i = 1, 2, \ldots$ we have:

A1) for all $n \in \mathbf{N}$, $\mu_1 K_1^{(1)} \ldots K_1^{(n)} \preceq \mu_2 K_2^{(1)} \ldots K_2^{(n)}$;

A2) the induced processes μ_1^T and μ_2^T on the trajectory space $H^T_{E,\Lambda}$ are stochastically ordered: $\mu_1^T \preceq \mu_2^T$;

Using Strassen's theorem it is easy to see that there exists a joint representation of the stochastic order $\mu_1^T \preceq \mu_2^T$. We stress that the relations A1) and A2) are also verified using a single kernel if it is attractive. We will call a Markov chain attractive if its kernel is attractive.

Given two measures ν_1 and ν_2 on the dyadic space $\{0, 1\}$ we will indicate with *standard coupling* this joint measure P:

$$\begin{gathered} P(1,1) = \inf\{\nu_1(1), \nu_2(1)\}; P(0,0) = \inf\{\nu_1(0), \nu_2(0)\}; \\ P(1,0) = \sup\{\nu_1(1) - \nu_2(1), 0\}; P(0,1) = \sup\{\nu_2(1) - \nu_1(1), 0\}. \end{gathered} \tag{14}$$

In the same way we will use the expression *standard coupling* also for the coupling of the trajectories of two Markov chains with only two states. In this case we will think the coupling, as usual, to be driven by the extraction of a uniform random variable in $[0,1]$ that updates at the same time the coupled Markov chains (see [Li85]). We try to explain the construction of the coupling; by hypothesis we have two independent Markov chains $\{X_n\}_{n\in N}$ and $\{Y_n\}_{n\in N}$ with values in $\{0,1\}$ and transition probabilities

$$P(X_{n+1}=0|X_n)=p_1(X_n,0) \text{ and}$$
$$P(Y_{n+1}=0|Y_n)=p_2(Y_n,0).$$

Obviously, the transition probability in 1 are also known. We consider i.i.d. r.v.'s $\{\theta_n\}_{n\in N}$ uniform in $[0,1]$ that are also independent of $\{X_n\}_{n\in N}$ and $\{Y_n\}_{n\in N}$. We couple the Markov chains using the variables θ_n; if $\theta_n < p_1(X_n,0)$ then $X_{n+1}=0$ otherwise $X_{n+1}=1$ and also if $\theta_n < p_2(Y_n,0)$ then $Y_{n+1}=0$ otherwise $Y_{n+1}=1$.

We quote without a proof some properties of the FK measures (see for more details [Ne97]). If for each $e\in E(\Lambda)$, $p'_e = 1-e^{-\beta'|J'_e|} \le p_e = 1-e^{-\beta|J_e|}$, then the ferromagnetic FK measures are stochastically ordered:

$$\mu^{\tau\equiv i,F}_{\Lambda,J'\beta'} \preceq \mu^{\tau\equiv i,F}_{\Lambda,J\beta} \qquad \text{for } i=-1,1. \tag{15}$$

The same stochastic order relation is true in the case of free boundary conditions:

$$\mu^{F}_{\Lambda,J'\beta'} \preceq \mu^{F}_{\Lambda,J\beta}. \tag{16}$$

Moreover a FK measure with the same absolute value of the interactions $|J'_e| = J_e$ for all $e\in E$ and the same β and with any boundary conditions τ' verifies the relation:

$$\mu^{\tau'}_{\Lambda,J'\beta} \preceq \mu^{\tau\equiv i,F}_{\Lambda,J\beta}; \tag{17}$$

these relations will play an important role in the paper. Also, the ferromagnetic FK measures with free boundary condition or with $\tau\equiv i$ satisfy the FKG inequalities (see [FKG71, Gr95, Ne97]).

For the ferromagnetic Ising model we recall Proposition 3.8 p. 35 [Ne97] (see also the original proof [LM72]) that prove the equivalence of the transition phase in the Ising model with percolation in the related FK model.

Proposition 2.1 *For the ferromagnetic Ising model there is a unique infinite Gibbs measure at inverse temperature β if and only if*

$$E(\sigma_x) \equiv \lim_{\Lambda\to\mathbf{Z}^d} \mu^{\tau\equiv i,F}_{\Lambda,J\beta}(x \longleftrightarrow \Lambda^c) = 0. \tag{18}$$

We will not prove Proposition 2.1. A proof can be found in [Ne97] pp. 35-37. In any case we will use some related ideas in the proof of Theorem 3 that is presented in the appendix.

Ferromagnetic and frustrated measures. Let us define two Gibbs measures and their related FK measures. We define these measures on a finite set of vertices $\Lambda\subset\mathbf{Z}^d$ -*finite volume*- fixing the configurations J and the boundary conditions τ. We take a ferromagnetic and a frustrated system, and we will use the label $i = 1,2$ to indicate respectively the ferromagnetic and the frustrated system. Take $J_{1,e} = |J_{2,e}| \in [J_A, J_B]$ for all $e \in E(\Lambda) \cup \partial\Lambda$, where $0 < J_A \leq J_B < \infty$. Let $J_{1,e} > 0$ for all edges e -so that system 1 is ferromagnetic- and J_2 is such that in each block $B_n(v) \subset E(\Lambda)$ there is a frustrated plaquette. We remind that we have covered the edges $E(\mathbf{Z}^d)$ with blocks of a fixed size n. We are ready to write the finite volume Gibbs measures $\pi^{\tau\equiv i}_{1,\Lambda,J_1\beta}$, $\pi^{\tau'}_{2,\Lambda,J_2\beta}$ and the related FK measures $\mu^{\tau\equiv i}_{1,\Lambda,J_1\beta}$, $\mu^{\tau'}_{2,\Lambda,J_2\beta}$. We denote with $\pi_{1,J_1\beta}$ any infinite volume Gibbs measure obtained as a weak limit along any subsequence $\{\Lambda_L\}_L$ and any boundary condition τ, i.e. $\pi^{\tau}_{1,\Lambda_L,J\beta} \to_{L\to\infty} \pi_{1,J\beta}$; we follow the same notation for the other measures taking a weak limit of finite volume measures. The above construction for the FK and Gibbs measures shall be called hypothesis (H1). Sometimes we use the label τ of the boundary condition also for the infinite volume measure if it is relevant. The *critical point* is

$$\beta_c(J) = \sup\{\beta > 0 : \text{ the Gibbs measure } \pi_{J\beta} \text{ is unique}\} \tag{19}$$

i.e. the weak limit of the measures is independent of the boundary conditions and of the sequence $\{\Lambda_L\}_L$; the *percolation critical point* of the FK measure is

$$\beta_c^{FK}(J) = \sup\{\beta > 0 : \exists\ \mu_{J\beta} \text{ and } \exists\ x \in\ \mathbf{Z}^d \text{ with } \mu_{J\beta}(|C_x| = \infty) = 0\}. \tag{20}$$

The measure $\mu_{J\beta}$ need not be translation invariant. In any case, using a lemma of [DG99] it is known that if the probability to percolate is larger than zero at a vertex, then it is larger than zero for all the vertices if all the interactions are different from zero; so, in this case, we could indicate only the cluster at the origin in (20).

For ferromagnetic systems, using Proposition 2.1, one has that $\beta_c^{FK}(J) = \beta_c(J)$ (see [Ne97]). When the Gibbs measure is not unique one says that there is a *phase transition*; for the Ising models one can also define the region of parameter β in which there is *broken symmetry*, that is $E(\sigma_x) > 0$ for some boundary conditions. It is known that broken symmetry implies a phase transition but in general for the Ising models it is not known if the converse is true; only for ferromagnetic systems, by Proposition 2.1, it is known that *phase transition* and *broken symmetry* are equivalent. For notational convenience we will write $\beta_c^{FK}(1,|J|)$, $\beta_c^{FK}(2,J)$ in place of $\beta_c^{FK}(J_1)$ and $\beta_c^{FK}(J_2)$ if the two systems satisfy the hypothesis (H1), and analogously for the critical point β_c. We have this first strict inequality for the FK measures (see for the same result with a different proof [Ca98]).

Theorem 2 *Let us consider the graph $\mathbf{E}^d$ with $d \geq 2$, let the FK measures $\mu^{\tau\equiv i}_{1,|J|\beta}$ and $\mu^{\tau'}_{2,J\beta}$ verify the hypothesis (H1), and let $[\beta_A, \beta_B]$ be an interval with $0 < \beta_A < \beta_B < \infty$ then there exists $\alpha \in (0,1)$ such that for each $\beta \in [\beta_A, \beta_B]$*

$$\mu^{\tau'}_{2,J\beta} \preceq \mu^{\tau\equiv i}_{1,\alpha|J|\beta}, \tag{21}$$

and $\beta_c^{FK}(2,J) > \beta_c^{FK}(1,|J|)$.

We will give the proof of the stochastic order in Theorem 2 in the next section. It is known that for $d \geq 2$, $0 < \beta_c^{FK}(J) < \infty$, for the ferromagnetic systems with $J_e > \epsilon > 0$ for all the edges (see [Ne97]). So the second part of the Theorem is nontrivial.

Now we show that $\beta_c^{FK}(2,J) > \beta_c^{FK}(1,|J|)$ is a consequence of (21). By the stochastic order, formula (21), we have the inequality $\beta_c^{FK}(2,J) > \beta_c^{FK}(1,|J|)$ because percolation is an increasing event and the following relation is obviously true:

$$\alpha\beta_c^{FK}(1,|J|) = \beta_c^{FK}(1,\alpha|J|). \tag{22}$$

So $\alpha\beta_c^{FK}(1,|J|) = \beta_c^{FK}(1,\alpha|J|) \leq \beta^{FK}(2,J)$. We know also that $\beta^{FK}(2,J) < \infty$, this follows from a modification of [DG99] in which only FK measures without boundary conditions are analyzed. The idea is that all the boundary conditions can be thought of as a particular choice of J interactions and the configuration η on some edges connecting the vertices of the boundary. So we can do the same kind of proof as with free boundary conditions [DG99]. We have also Theorem 3, which is similar to Proposition 2.1 in the results and it is also related to [Ne94] in the idea of the proof and in the kind of the result.

Theorem 3 *Let $\mu^{\tau\equiv i,F}_{\Lambda,J_1\beta_1}$ a ferromagnetic FK measure with boundary condition $\tau \equiv i$ with $i=-1,1$. If for all vertices $x \in \mathbf{Z}^d$*

$$\lim_{\Lambda\to\mathbf{Z}^d} \mu^{\tau\equiv i,F}_{\Lambda,J_1\beta_1}(x \longleftrightarrow \Lambda^c) = 0, \tag{23}$$

and

$$\forall\Lambda, \forall\tau \qquad \mu^{\tau}_{\Lambda,J\beta} \preceq \mu^{\tau\equiv i,F}_{\Lambda,J_1\beta_1} \tag{24}$$

then for all finite sets of vertices $A \subset \mathbf{Z}^d$:

$$\forall\tau,\tau' \qquad \lim_{\Lambda\to\mathbf{Z}^d} |\pi^{\tau}_{\Lambda,J\beta}(\sigma_A) - \pi^{\tau'}_{\Lambda,J\beta}(\sigma_A)| = 0 \tag{25}$$

therefore the related infinite volume Gibbs measure $\pi_{J\beta}$ is unique.

We will give the proof of Theorem 3 in the appendix. The next result is an immediate consequence of Theorem 2 and Theorem 3, but we want to point it out because it is relevant for Statistical Mechanics (for the non strict inequality see [Ne94])).

Corollary 2.1 *Let us consider the graph $\mathbf{E}^d$ with $d \geq 2$. Let $\pi_{1,J_1\beta}$ and $\pi_{2,J_2\beta}$ be two Gibbs measures satisfying $(H1)$. Then $\beta_c(2,J) > \beta_c(1,|J|)$.*

3 Non disordered systems

In the previous section we have defined a ferromagnetic and a frustrated measure on the space $H_{E,\Lambda}$ for a particular choice of interactions $J = \{J_e\}_{e\in E}$. Now we want to construct non homogeneous Markov chains having the finite volume FK measures $\mu^{\tau'}_{2,\Lambda,J\beta}$ and $\mu^{\tau\equiv i}_{1,\Lambda,J\beta}$ as stationary measures. The Markov chains that we will construct will be irreducible and so the stationary measure will be unique. We will study the coupling between these

Markov chains to deduce stochastic order between their stationary measures. In what follows Λ will indicate a set of vertices of shape $\mathbf{Z}^d \cap [-L, L]^d$ for any $L > 2$.

We consider Markov chains that update the configurations on the edges or on the frustrated plaquettes. Before giving the kernels of the Markov chains we will fix the order in which the configurations are updated. Let's give a lexicographic order to the blocks $B_n(v) \subset E(\Lambda)$ that are used to cover $E(\mathbf{Z}^d)$; we write $\{B_1, B_2, \ldots, B_N\}$. In every block B_n we mark a frustrated plaquette that we will call pl_n. We update the configuration in the given lexicographic order, and for each block we use this procedure

(a) update the configuration on the edges $e \in \partial^E B_n$ in a lexicographic order;

(b) update the configuration on the edges $e \in B_n$ in a lexicographic order;

(c) update the configuration on the marked frustrated plaquette pl_n;

(d) update the configuration on every plaquette belonging to B_n and then mark it. The first marked plaquette is pl_n updated in point (c). Update the edges on the unmarked plaquettes that are adjacent to any marked plaquette and that are in the block which we are considering. Let's update the edges of this plaquette in the order e_1, e_2, e_3, e_1, e_2 (see figure 2) leaving unchanged the edges belonging to marked edges;

(e) update the configuration on the edges $e \in \partial^I B_n$ in this order: e_1, e_2, e_1, where e_1 and e_2 are two edges touching the same side of $\partial^E B$ and then let update in a lexicographic order all the other edges touching the same side. Then repeat this procedure for the four sides of $\partial^I B$ (see fig. 2).

(f) update the configuration on the edges $e \in \partial^E B_n$ in a lexicographic order;

(h) let's take the next block and repeat all the procedure.

Now we will write the kernels of three Markov chains which are $P^{(1)}_{\Lambda,\beta}$, $P^{(2)}_{\Lambda,\beta}$ and $\widetilde{P}_{\Lambda,\beta}$; the first and second Markov chains have respectively $\mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}$ and $\mu^{\tau'}_{2,\Lambda,J\beta}$ as stationary measures. We will drop the labels Λ and β in the kernels for simplicity.

In general we will indicate with P the Markov kernel that is the result of the composition

$$P_{D_1} P_{D_2} \ldots P_{D_n},$$

where P_{D_i} updates the configuration on the set of edges D_i. On each set of edges D -it will be an edge or a plaquette- we will use as kernel the conditional probability (*Gibbs sampler*); for every η_{ini}, $\eta_{fin} \in \{0,1\}^\Lambda$ and every region $D \subset \Lambda$ we use this kernel to update the configuration:

$$P^{(1)}_D(\eta_{ini}, \eta_{fin}) = \mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}(\eta(D) = \eta_{fin}(D) | \eta_{ini}(E(\Lambda) \setminus D)); \tag{26}$$

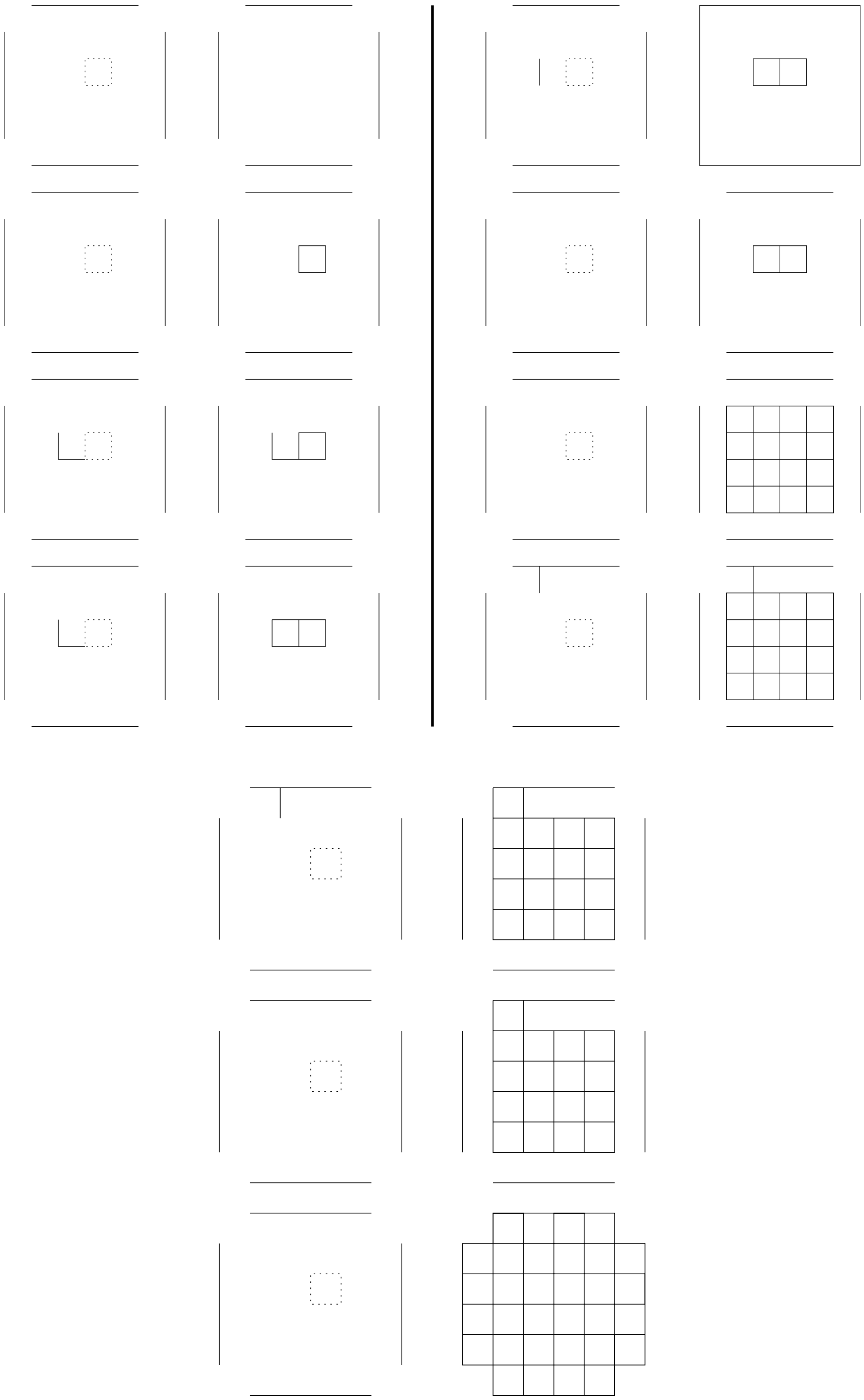

Figure 2: the coupling described in points (d) and (e) that will be used in Lemma 3.5

so the kernel (or *transition probability*) is independent of the initial value on D but it depends on $\eta_{ini}(E(\Lambda)\setminus D)$; we also point out that $\eta_{ini}(E(\Lambda)\setminus D)) = \eta_{fin}(E(\Lambda)\setminus D))$. Analogously we proceed for the Markov chain $P^{(2)}$ substituting $\mu^{\tau'}_{2,\Lambda,J\beta}$ to the $\mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}$ in (26). Finally we define the kernel $\widetilde{P}$ that has

$$\widetilde{P}_{pl}(\eta_{ini},\eta_{fin}) = \begin{cases} \mu^{\tau'}_{2,\Lambda,J\beta}(\eta(pl)) = \eta_{fin}(pl)|\eta_{ini}(E(\Lambda)\setminus pl)) \text{ if } \eta(B\setminus pl) = 0 \\ \mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}(\eta(pl)) = \eta_{fin}(pl)|\eta_{ini}(E(\Lambda)\setminus pl)) \text{ otherwise} \end{cases} \tag{27}$$

for the marked frustrated plaquette pl and $\widetilde{P}_e = P^{(1)}_e$ for all the edges that are not in a marked frustrated plaquette. The technique, that we will explain later, is applicable also to obtain strict inequality for two ferromagnetic systems (see [BGK93, Gr95, Gr99]) but in that case we do not need to introduce the measure $\widetilde{P}$. For the proofs we could also choose a continuous time Markov process as in [BGK93] but it seems to us that with Markov chains there are more explicit bounds for the differences of the percolation critical points $\beta^{FK}_c(2,J) - \beta^{FK}_c(1,|J|)$. Recall also that the measures μ_1 and μ_2 are reversible with respect to the defined Markov chains $P^{(1)}$ and $P^{(2)}$, so they are stationary measures for them. The stationary measure is unique for each chain because the chain is irreducible.

In the next lemma we prove, adding an hypothesis, the Strassen's theorem in a stronger form which is useful to prove strict inequalities.

Lemma 3.1 *Let μ_1 and μ_2 be measures on the finite space $X = \{0,1\}^B$ with $|B| < \infty$. The following assertions are equivalent:*

(a) There exists $\varepsilon > 0$ such that for each increasing event $E \subset X$ with $E \neq \emptyset, X$ $\mu_1(E) + \varepsilon \leq \mu_2(E)$.

(b) There exists a coupling P_{12} on the space $X \times X$ with $P_{12}(\cdot, X) = \mu_1(\cdot)$ e $P_{12}(X,\cdot) = \mu_2(\cdot)$ and P_{12} is a measure supported on the set M with $P_{12}(\eta_1 \equiv 0, \eta_2 \equiv 1) \geq \varepsilon$.

Proof. (b) $\Rightarrow$(a) is a consequence of this direct computation; let E be as in the hypothesis of the theorem

$$\mu_1(E) = P_{12}(E,X) = P_{12}(E,E) \tag{28}$$

$$\leq P_{12}(\eta_1 \equiv 0, \eta_2 \equiv 1) - \varepsilon + P_{12}(E,E) \tag{29}$$

$$\leq P_{12}(X,E) - \varepsilon = \mu_2(E) - \varepsilon. \tag{30}$$

the equalities in (28)-(30) and the inequality (29) are true because of the hypothesis, and the inequality in (30) because

$$\{\eta_1 \equiv 0, \eta_2 \equiv 1\} \cap \{E \times E\} = \emptyset \qquad \text{and} \qquad \{\eta_1 \equiv 0, \eta_2 \equiv 1\} \cup \{E \times E\} \subset \{X \times E\}.$$

(a) $\Rightarrow$(b). There exists P_{12} which respect the Strassen's theorem with $\mu_1(E) = P_{12}(E,X)$ and $P_{12}(X,E) = \mu_2(E)$ because $\mu_1 \preceq \mu_2$; so we have only to prove that there is such a coupling measure verifying $P_{12}(\eta_1 \equiv 0, \eta_2 \equiv 1) \geq \varepsilon$. Let's define for each event $A \in X$ the function

$$\nu_\varepsilon(A) = \mu_2(A) - \varepsilon I(\{\eta \equiv 1\} \subset A) + \varepsilon I(\{\eta \equiv 0\} \subset A) \tag{31}$$

where I is the indicator function and ε is given by point (a) of the theorem. It is easy to see that ν_ε is a probability measure coinciding with μ_2 on the configurations that are not constant ($\eta \not\equiv 0, 1$). Moreover,

$$\nu_\varepsilon(\eta \equiv 1) = \mu_2(\eta \equiv 1) - \varepsilon \geq 0 \quad \text{and}$$

$$\nu_\varepsilon(\eta \equiv 0) = \mu_2(\eta \equiv 0) + \varepsilon.$$

Then ν_ε is a probability measure and by construction:

$$\mu_1 \preceq \nu_\varepsilon \preceq \mu_2; \tag{32}$$

in fact for every increasing event $E \neq \emptyset, X$ by hypothesis $\mu_1(E) \leq \mu_2(E) - \varepsilon = \nu_\varepsilon(E)$ whereas for the events $\emptyset, X$ we have the equality.

We can construct a coupling measure P_0 between the measures μ_1 and ν_ε, because they are stochastically ordered. Now we can construct the coupling P_{12} using this relation:

$$P_{12}(\eta_1, \eta_2) = \begin{cases} P_0(\eta_1, \eta_2) \text{ if } (\eta_1, \eta_2) \neq (0,1), (0,0) \\ P_0(0,0) - \varepsilon \text{ if } \eta_1 \equiv 0 \text{ and } \eta_2 \equiv 0 \\ P_0(0,1) + \varepsilon \text{ if } \eta_1 \equiv 0 \text{ and } \eta_2 \equiv 1 \end{cases}.$$

P_{12} is a joint probability measure of μ_1 and μ_2 verifying all the conditions in point (b); in fact all the components of P_{12} under the diagonal are equal to zero, $P_{12}(\eta_1, X) = P_0(\eta_1, X) = \mu_1(\eta_1)$ and

$$P_{12}(X, \eta_2) = P_0(X, \eta_2) + \varepsilon I(\eta_2 \equiv 1) - \varepsilon I(\eta_2 \equiv 0) = \mu_2(\eta_2).$$

This ends the proof. $\square$

It is known that $P^{(1)}$ is attractive (see [Gr95]), we want now to prove that also $\widetilde{P}$ has this relevant property.

Lemma 3.2 *The kernel* $\widetilde{P}$ *is attractive.*

Proof. We shall show that for all configurations $\eta \preceq \xi$, $\widetilde{P}(\eta, \cdot) \preceq \widetilde{P}(\xi, \cdot)$. It is enough to consider only configurations which are different on a single edge e, i.e. $\eta_e = 0$ and $\xi_e = 1$, showing on these configurations the monotone property. At time n if an edge e is updated then $\widetilde{P}$ is equal to the ferromagnetic kernel ($\widetilde{P}_e = P^{(1)}_e$ for all the edges e). But $P^{(1)}_e$ is attractive (see [Gr95, Ne97]) so also $\widetilde{P}_e$ is attractive. If at time n the frustrated plaquette pl_i is updated there are two case:

1) $\eta(B_i) \neq 0$; again $\widetilde{P}_{pl}$ is equal to the ferromagnetic kernel, so it is attractive.

2) $\eta(B_i) \equiv 0$; if the edge e in which $\eta_e \neq \xi_e$ is not in B_i then it does not influence $\widetilde{P}_{pl}$

because the plaquette is inside a minimal cut set (in two dimensions a dual circuit) γ with all the edges closed and the kernels $\widetilde{P}(\eta,\cdot)$ and $\widetilde{P}(\xi,\cdot)$ are equal; this is equivalent to free boundary conditions on the plaquette. If $e \in B_i$:

$$\mu_2(\cdot|\eta(E \setminus pl) = 0) \preceq \mu_1(\cdot|\eta(E \setminus pl) = 0) \preceq \mu_1(\cdot|\xi(E \setminus pl));$$

the first stochastic order follows from $\mu_2^F \preceq \mu_1^F$ with free boundary condition (16), the second stochastic order follows from the attractiveness of the ferromagnetic kernel $P_D^{(1)} = \mu_1(\cdot|\eta(E \setminus D))$. $\square$

Now we can proof the domination between the kernels $P^{(2)}$, $\widetilde{P}$ and $P^{(1)}$. In the next lemma we denote with δ_η the probability measure supported on the configuration η.

Lemma 3.3 *For all the initial conditions $\eta \in \{0,1\}^{E(\Lambda)}$ and for all $t \in \mathbf{N}$*

$$\delta_\eta(P^{(2)})^t \preceq \delta_\eta(\widetilde{P})^t \preceq \delta_\eta(P^{(1)})^t.$$

Proof. $\delta_\eta(P^{(2)})^t \preceq \delta_\eta(\widetilde{P})^t$. Using the condition A1) we have only to prove the result on the update of a single set D_i. So we should prove that for each D_i, η and η' we have:

$$P_{D_i}^{(2)}(\eta,\cdot) \preceq \widetilde{P}_{D_i}(\eta',\cdot)$$

with $\eta \preceq^* \eta'$. If $\eta = \eta'$, we have $P_{D_i}^{(2)}(\eta,\cdot) \preceq \widetilde{P}_{D_i}(\eta,\cdot)$ as a consequence of the stochastic order (17); in fact the conditional probabilities are FK measure on the particular set of edges D_i (plaquette or the single edge). The kernel $\widetilde{P}_{D_i}$ is attractive, so we have $\widetilde{P}_{D_i}(\eta,\cdot) \preceq \widetilde{P}_{D_i}(\eta',\cdot)$, which shows the right stochastic order. An analogous argument works also for the second domination. $\square$

We remaind that in the rest of the section the configuration J verifies $|J_e| \in [J_A, J_B]$ where $0 < J_A < J_B < \infty$.

Lemma 3.4 *For every inverse temperature $\beta \in (0,\infty)$ there exists $\varepsilon_{1,J}(\beta) > 0$ such that for each frustrated plaquette pl and for each increasing event $E \in H_{pl} = \{0,1\}^{pl}$ and $E \neq \emptyset, H_{pl}$ we have:*

$$\mu_{1,\Lambda,|J|\beta}^{\tau\equiv i}(E|\eta(B \setminus pl) \equiv 0) \geq \mu_{2,\Lambda,J\beta}^{\tau'}(E|\eta(B \setminus pl) \equiv 0) + \varepsilon_{1,J}(\beta) \tag{33}$$

Proof. The conditional probabilities in (33) are independent of the boundary conditions and the size of the box Λ because the plaquette is isolated being inside a minimal cut set (in two dimension a dual circuit) γ in which $\eta(\gamma) = 0$. There is only a forbidden configuration on H_{pl}, that is the configuration $\eta(pl) \equiv 1$. Let's write $\mu_1 = \mu_{1,\Lambda,|J|\beta}^{\tau\equiv i}$ and $\mu_2 = \mu_{2,\Lambda,J\beta}^{\tau'}$. Let $\mu_1(E|\eta(B\backslash pl) \equiv 0) = \frac{A(E)}{Z}$ and being $\mu_1(\eta(pl) \equiv 0|\eta(B\backslash pl) \equiv 0) = \delta > 0$ we have $Z > A(E)$ for all E that respect the hypothesis; in fact $\{\eta(pl) = 0\}$ is not in any increasing event $E \neq \emptyset, H_{pl}$. Then $\mu_2(E|\eta(B\backslash pl) \equiv 0) = \frac{A(E)-\varepsilon}{Z-\varepsilon}$ and $\varepsilon = \prod_{b\in pl} p_b > 0$ that is the weight of the configuration $\eta(pl) \equiv 1$ for the measure μ_1. Instead, for the probability measure μ_2 the configuration $\eta(pl) \equiv 1$ has a null weight because such a configuration is forbidden. For all E respecting the hypothesis

$$\mu_1(E|\eta(B \setminus pl) \equiv 0) > \mu_2(E|\eta(B \setminus pl) \equiv 0). \tag{34}$$

So there exists a constant $\varepsilon_{1,J}(\beta) > 0$ verifying (33) because the number of events E verifying the inequality (34) is finite. □

In the next lemma η is a configuration with $\eta(pl) = 0$ and η' is a configuration with $\eta'(pl) = 1$. Let's define

$$P^{(1)}_{B\setminus pl} := P^{(1)}_{e(1)} P^{(1)}_{e(2)} \dots P^{(1)}_{e(n)} \tag{35}$$

where all the $\{e(i)\}_{i=1,\dots,n}$ are different edges inside $B \setminus pl$.

Lemma 3.5 *For all $\beta \in (0,\infty)$ there exists $\varepsilon_{2,J}(\beta) > 0$ such that for each increasing event E in $H_B = \{0,1\}^B$ with $E \neq \emptyset, H_B$ and for each pair of configurations $\eta, \eta' \in \{0,1\}^{\Lambda\setminus(\partial^E B \cup B)}$ (external to $B \cup \partial^E B$) with $\eta \preceq \eta'$:*

$$\mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}(E|\eta, \eta(pl) = 0) + \varepsilon_{2,J}(\beta) < \mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}(E|\eta', \eta'(pl) = 1). \tag{36}$$

Proof. We will use in the proof a dynamic argument. We will find a lower bound for

$$\mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}(E|\eta', \eta'(pl) = 1) - \mu^{\tau\equiv i}_{1,\Lambda,|J|\beta}(E|\eta, \eta(pl) = 0) \tag{37}$$

using the same kernel $P^{(1)}_e$ to update the configuration for two systems with a different initial configuration on the plaquette pl and eventually out of the considered block B. Being $P^{(1)}_{B\setminus pl}$ irreducible, aperiodic and with finite number of states we know that, for every configuration on pl and on B^c, its stationary measure is unique and in the limit $t \to \infty$ the measure at time t of the Markov chain converges to this stationary measure. We use the standard coupling for the kernel $P^{(1)}_e$; this kernel is attractive so at every time t we will have $\eta(t) \preceq^* \eta'(t)$.

There is a positive probability that $\eta' \equiv 0$ on all the edges $e \in B \setminus pl$ and so, using the coupling, also $\eta \equiv 0$ on $B \setminus pl$; there is a positive probability that also $\eta = \eta' \equiv 1$ on the set $\partial^E B$, in fact all the configurations on a finite set of edges has a positive probability to be realized (*finite energy*). We start at such a configuration, i.e. $\eta'(B \setminus pl) = \eta(B \setminus pl) = 0$ and $\eta'(\partial^E B) = \eta(\partial^E B) = 1$. Let's update the configuration on plaquettes having at least one edge belonging to a plaquette already updated (see point (d) for the description of the dynamics). If in all the previous steps the updated plaquettes in the two systems verify $\eta = 0$ and $\eta' = 1$, there is a different probability to have $\eta_e = 1$ and $\eta'_e = 1$ with e belonging to the plaquette that we are updating (see [BGK93]). This is so because updating the first two edges there is a positive probability to have $\eta'_{e(1)} = \eta'_{e(2)} = 1$ (see figure 2). So on the third edge, conditioning on the event $\eta'_{e(1)} = \eta'_{e(2)} = 1$ and on all the updated plaquettes in the block $\eta = 0$ and $\eta' = 1$, we obtain that the probability to have $\eta(e(3)) = 0$ and $\eta'(e(3)) = 1$ is

$$p_{e(3)} - \frac{p_{e(3)}}{p_{e(3)} + 2(1 - p_{e(3)})} > 0. \tag{38}$$

Formula (38) is a consequence of the fact that the vertices in i, k belonging to $e(3)$ are connected in the configuration η' independently of $\eta'(e_3)$, whereas they are not in the configuration η (see [BGK93] and [Gr95] p. 1471).

Now updating the configurations on the edges $e(1)$ and $e(2)$ we will find the same bound for the probability of the event $\eta_{e(1)} < \eta'_{e(1)}$.

So we can bound the the probability of the event $\eta_{(e(i))} < \eta'_{(e(i))}$ for $i = 1, 2, 3$ with

$$p_{e(1)} p_{e(2)} \prod_{i=1,2,3} \left[\frac{p_{e(i)}(1 - p_{e(i)})}{2 - p_{e(i)}} \right]. \tag{39}$$

We can use this procedure for all the plaquettes in B. A slightly different strategy is needed for the edges in $\partial^I B$ that belong to plaquettes that are not inside B. Let's update the first edge $e(1) \in \partial^I B$. There is a positive probability for the event $\eta_{e(1)} = 1$. Now for all the successive edges we have the bound (38) for the probability to have $\eta_e = 0$ and $\eta'_e = 0$ for all $e \in \partial^I B$. Now we update the edges in $\partial^E B$ and there is a positive probability to obtain all the possible configurations $\eta(\partial^E B)$.

This bound is uniform in the time t so, for the unique stationary measures, we have that there is a probability larger than zero to have $\eta(B) \equiv 0$, $\eta'(B) \equiv 1$ and in each case the configurations are ordered $\eta \preceq^* \eta'$; in fact for every updated region we can obtain only ordered configurations (doing a comparison on all the terms), so the coupling has all the measure supported on the space M. Now using the Lemma 3.1 we have (36). It is clear that all the bounds are also uniform in Λ and in the configurations $\eta,\ \ \eta' \in \{0,1\}^{\Lambda \setminus (\partial^E B \cup B)}$ outside $\Lambda \setminus (\partial^E B \cup B)$. $\square$

Let's remark that the lower bound (39) does not hold if $\beta = 0$ or $\beta = \infty$ (means zero temperature) because the inequality (38) becomes equal to zero (remind that $p_e = 1 - e^{-|J_e|\beta}$). Also, (36) falls for $\beta = 0, \infty$, so the condition $\beta \in (0, \infty)$ is necessary for Theorem 3.5.

The proof of Theorem 2 will be similar to [BGK93, Gr99]; as a matter of fact we want to prove the strict inequality using only the attractive Markov chains $\widetilde{P}$ and $P^{(1)}$. Now we are ready to prove the main result of this section.

Proof of Theorem 2. Let's put back the labels β, J and Λ on the measures μ_1 and μ_2, and therefore on the kernels $\widetilde{P}$ and $P^{(1)}$. The kernel $P^{(1)}_{\Lambda, J\beta}$ is

$$P^{(1)}_{\Lambda,|J|\beta} = P^{(1)}_{pl_1,|J|\beta} P^{(1)}_{B_1 \setminus pl_1,|J|\beta} \dots P^{(1)}_{pl_N,|J|\beta} P^{(1)}_{B_N \setminus pl_N,|J|\beta} \tag{40}$$

where $P^{(1)}_{B_i,|J|\beta}$ is defined in (35); analogously for $\widetilde{P}_{\Lambda,J\beta}$. Since the Markov chains are irreducible, aperiodic and with a finite number of states

$$\mu_{1,\Lambda,|J|\beta} = \lim_{k\to\infty} \delta_\eta (P^{(1)}_{\Lambda,|J|\beta})^k; \quad \mu_{2,\Lambda,J\beta} = \lim_{k\to\infty} \delta_\eta (P^{(2)}_{\Lambda,J\beta})^k$$

$$\widetilde{\mu}_{\Lambda,J\beta} = \lim_{k\to\infty} \delta_\eta (\widetilde{P}_{\Lambda,J\beta})^k; \tag{41}$$

for all η and for all the parameters $\beta > 0$ and $J > 0$. It is easy to see that the FK measures are continuous in the parameters $\beta > 0$ and $J > 0$, in fact p_e is continuous in β and J and there are only sums, products and a division, but with the denominator larger than zero for all $\beta > 0$.

By Lemma 3.4, Lemma 3.5 and Lemma 3.1 for E increasing and different from $\emptyset$, $H_{E,B}$

$$[\delta_\eta (\widetilde{P}_{pl,J\beta} \widetilde{P}_{B\setminus pl,J\beta})^k](E) + \varepsilon_{3,J(B)}(\beta) \leq [\delta_\eta (P^{(1)}_{pl,|J|\beta} P^{(1)}_{B\setminus pl,|J|\beta})^k](E) \tag{42}$$

for all $n > 0$, with $\varepsilon_{3,J(B)}(\beta) > 0$; $\varepsilon_{3,J(B)}(\beta)$ depends only on β and the interactions on the block B. As a matter of fact Lemma 3.1 and Lemma 3.4 guarantee that there is a coupling which gives positive probability to the event that the plaquette has all the edge closed in the frustrated system and all the edges open in the ferromagnetic system; Lemma 3.5 guarantees the same result for $B \setminus pl$. The kernels are continuous being equal to the FK measure (with some boundary conditions). They are also uniformly continuous on the compact parameter's region $[\beta_A, \beta_B] \times [J_A, J_B]^B$. So for all the interactions in $[J_A, J_B]$ and for all $\beta \in [\beta_A, \beta_B]$ there exists $\alpha \in (0,1)$ such that

$$[\delta_\eta (\widetilde{P}_{pl,J\beta} \widetilde{P}_{B\setminus pl,J\beta})^k](E) < [\delta_\eta (P^{(1)}_{pl,\alpha|J|\beta} P^{(1)}_{B\setminus pl,\alpha|J|\beta})^k](E) \tag{43}$$

Let's remark that not all the boxes $[-L, L]^d \cap \mathbf{Z}^d$ can be divided into blocks of fixed size n, but in any case we can take boxes of different sizes k with $n \leq k \leq 2n$ and, using all these types of blocks, we can cover every box $[-L, L]^d \cap \mathbf{Z}^d$ if L is larger than n. On each block B, we will find a different value of $\varepsilon_{3,J(B)}(\beta)$, but in any case the different sizes are a finite number (between n and $2n$), so the minimum exists and is larger than zero. We redefine $\alpha := \min_B \alpha(B)$.

On each block we can replace the kernel $\widetilde{P}_{B,J\beta}$ with $P^{(1)}_{B,\alpha|J|\beta}$ getting a domination for any single change. Therefore the domination remains also replacing $\widetilde{P}_{B,J\beta}$ with $P^{(1)}_{B,|J|\beta}$ in all the boxes, since the kernels are all attractive. So, using property A1), we obtain the stochastic order $\widetilde{\mu}_{\Lambda,J\beta} \preceq \mu_{1,\Lambda,\alpha|J|\beta}$ for the stationary measures. We deduce

$$\mu^{\tau'}_{2,\Lambda,J\beta_1} \preceq \widetilde{\mu}^{\tau\equiv i}_{1,\Lambda,|J|\beta} \preceq \mu^{\tau\equiv i}_{1,\Lambda,\alpha|J|\beta} \tag{44}$$

for all boundary conditions τ', where the first stochastic order is a consequence of Lemma 3.3. Remember that α in (44) is defined uniformly in Λ, so we can take the weak limit on every subsequence of boxes Λ_k, on which the limit exists, to have

$$\mu^{\tau'}_{2,J\beta_1} \preceq \mu^{\tau\equiv i}_{1,\alpha|J|\beta}. \tag{45}$$

This ends the proof. $\square$

Notice that it is possible to use the stochastic order (45) for every increasing event, not only for percolation. For the increasing events in the tail σ-algebra one can define a critical point analogous to (20) and one has a strict inequality in the critical point for the frustrated and the ferromagnetic measure if the ferromagnetic critical point is different from zero and infinite. For the Edwards-Anderson model, if one wants only to show a strict inequality between the regions in which there is broken symmetry, one could use the result in [Ca98].

4 Strict inequality for disordered systems

In this section we will find a strict inequality in the phase transition for disordered systems. In all the section we will consider the configuration of the interactions $J = \{J_e\}_{e\in E}$ as random variables. We will denote with $(\Theta, \mathcal{A}, Q)$ the abstract probability measure on which the random variables $\{J_e\}_{e\in E}$ are defined. The random variables $\{J_e\}_{e\in E}$ are *i.i.d.* and the support of the distribution is $\{-J_1, -J_2, J_1, J_2\}$ with $0 < J_2 < J_1 < \infty$; let us define $p = 2Q(J_e = -J_2) = 2Q(J_e = J_2)$.

The second claim of Proposition 3.9 in [Ne97] is given for the Ising model but the same proof works for the frustrated Ising model. So we will write the proposition without proof.

Proposition 4.1 *Given the random variable J, the percolation critical points $\beta_c^{FK}(J)$ and $\beta_c^{FK}(|J|)$ are Q-a.e. constant.*

The second point of the next Theorem could be obtained using the paper [OPR83] in which a strict inequality for the magnetization of some ferromagnetic systems is proved. We need a similar estimate for the FK measure. We will give the proof of the theorem because it is different from [OPR83] and it is directly applicable to the FK measures that we will use at the end of this section. We will write $\mathbf{J_1}$ for the configuration with all the interactions equal to J_1. Analogously for $\mathbf{J_2}$.

Theorem 4 *Let J be the absolute value of the random variable defined before, then*

$$\beta_c^{FK}(\mathbf{J}_2) > \beta_c^{FK}(J) > \beta_c^{FK}(\mathbf{J}_1) \qquad Q-a.e. \tag{46}$$

$$and \ \ \beta_c(\mathbf{J}_2) > \beta_c(J) > \beta_c(\mathbf{J}_1) \qquad Q-a.e. \tag{47}$$

Proof. We will abbreviate the notation writing $\eta_{\setminus b} = \eta(E\setminus b)$, $J_{\setminus b} = J(E\setminus b)$ and omitting the labels β and Λ in the measures. We will prove (46) for the FK measures with free boundary conditions; then we will finish the proof using the fact that the percolation is an increasing event, the equality between $\mu_{J\beta}^F$ and $\mu_{J\beta}^{\tau\equiv i,F}$ for almost all the β, and the fact that $\mu_{J\beta}^{\tau\equiv i,F}$ respects the relation (15).

Let's define the integrated measure μ of the ferromagnetic measures

$$\mu(\eta) = \sum_{\mathbf{J}} Q(\mathbf{J})\mu_{\mathbf{J}}(\eta), \tag{48}$$

where $\mathbf{J}$ is the configuration of the interactions and the single values are equal to J_1 or J_2. We have omitted the labels β and Λ in the integrated measure. The sum in (48) is finite because we are considering, as in the previous section, a finite box Λ. We will see that we only need to prove the stochastic order between the integrated measure μ and the ferromagnetic measures (see for the same technique [GKN92, DG99]). The conditional probability measure of μ is

$$\mu(\eta_e = 1|\eta_{\setminus e}) = \frac{\sum_{J_{\setminus e}} pQ(J_{\setminus e})\mu_{J_2,J_{\setminus e}}(\eta_e = 1, \eta_{\setminus e}) + (1-p)Q(J_{\setminus e})\mu_{J_1,J_{\setminus e}}(\eta_e = 1, \eta_{\setminus e})}{\sum_{J_{\setminus e}} pQ(J_{\setminus e})\mu_{J_2,J_{\setminus e}}(\eta_{\setminus e}) + (1-p)Q(J_{\setminus e})\mu_{J_1,J_{\setminus e}}(\eta_{\setminus e})}, \tag{49}$$

where $J_1, J_{\setminus e}$ means that $J_e = J_1$ and on the edges $E \setminus e$ the interactions are equal to a fixed $J_{\setminus e}$. We have written $\mu_{J_2}(\eta_e = 1|\eta_{\setminus e})$ in place of $\mu_{J_2,J_{\setminus e}}(\eta_e = 1|\eta_{\setminus e})$ because this conditional probability is independent of $J_{\setminus e}$ (see [Ne97]).

$$\mu(\eta_e = 1|\eta_{\setminus e}) = \frac{\sum_{J_{\setminus e}} pQ(J_{\setminus e})\mu_{J_2}(\eta_e = 1|\eta_{\setminus e})\mu_{J_2,J_{\setminus e}}(\eta_{\setminus e}) + (1-p)Q(J_{\setminus e})\mu_{J_1}(\eta_e = 1|\eta_{\setminus e})\mu_{J_1,J_{\setminus e}}(\eta_{\setminus e})}{\sum_{J_{\setminus e}} pQ(J_{\setminus e})\mu_{J_2,J_{\setminus e}}(\eta_{\setminus e}) + (1-p)Q(J_{\setminus e})\mu_{J_1,J_{\setminus e}}(\eta_{\setminus e})} \tag{50}$$

So we have a convex linear combination of the probability measures $\mu_{J_2}(\eta_e = 1|\eta_{\setminus e})$ and $\mu_{J_1}(\eta_e = 1|\eta_{\setminus e})$. Let's define

$$p_a(\eta_{\setminus e}) = \frac{\sum_{J_{\setminus e}} pQ(J_{\setminus e})\mu_{J_2,J_{\setminus e}}(\eta_{\setminus e})}{\sum_{J_{\setminus e}} pQ(J_{\setminus e})\mu_{J_2,J_{\setminus e}}(\eta_{\setminus e}) + (1-p)Q(J_{\setminus e})\mu_{J_1,J_{\setminus e}}(\eta_{\setminus e})}. \tag{51}$$

Then

$$\mu(\eta_e = 1|\eta_{\setminus e}) = p_a(\eta_{\setminus e})\mu_{J_2}(\eta_e = 1|\eta_{\setminus e}) + (1 - p_a(\eta_{\setminus e}))\mu_{J_1}(\eta_e = 1|\eta_{\setminus e}). \tag{52}$$

We want to show that $0 < p_a < 1$. We will first prove that $p_a > 0$. Let

$$F = \max_{\eta} \frac{\mu_{J_1,J_{\setminus e}}(\eta_{\setminus e})}{\mu_{J_2,J_{\setminus e}}(\eta_{\setminus e})}. \tag{53}$$

It is known that

$$\frac{\sum_{i=1}^{N} \alpha_i A_i}{\sum_{i=1}^{N} \alpha_i B_i} \leq max_i \left\{ \frac{A_i}{B_i} \right\} \quad \text{with} \quad A_i, B_i, \alpha_i \geq 0. \tag{54}$$

We apply the inequality (54) to (51) to obtain, with some algebraic calculations,

$$p_a(\eta_{\setminus e}) \geq \frac{1}{1 + \frac{1-p}{p}F}. \tag{55}$$

Let's apply (54) to (53) using also the definition of the ferromagnetic FK measure

$$F \leq \max\left\{1, \frac{1+e^{-\beta J_1}}{1+e^{-\beta J_2}}\right\} \frac{Z(J_2, J_{\setminus e})}{Z(J_1, J_{\setminus e})} = \frac{Z(J_2, J_{\setminus e})}{Z(J_1, J_{\setminus e})} \tag{56}$$

where the first part in the max function corresponds to a configuration $\eta_{\setminus e}$ in which the connection of $\{x, y\} = e$ is independent of the value of η_e; the number $\frac{1+e^{-\beta J_1}}{1+e^{-\beta J_2}}$, instead, depends on the configurations $\eta_{\setminus e}$ connecting the vertices x and y -with $\{x, y\} = e$- if and only if $\eta_e = 1$. Applying again the inequality (54) we have

$$\frac{Z(J_2, J_{\setminus e})}{Z(J_1, J_{\setminus e})} \leq \max\left\{1, \frac{1+e^{-\beta J_2}}{1+e^{-\beta J_1}}\right\} = \frac{1+e^{-\beta J_2}}{1+e^{-\beta J_1}} \leq 2 \tag{57}$$

for all $\beta > 0$ and $0 < J_2 < J_1$. Then

$$p_a(\eta_{\setminus e}) \geq \frac{p}{p + 2(1-p)}. \tag{58}$$

In the same way we can show that $1 - p_a(\eta_{\setminus e}) > 0$.

As in the last section, we will define two processes P_1 and $\overline{P}$, where the elements of the kernels are conditional probabilities. Remember that $[\eta'\eta]_e$ is the configuration which coincides with η' on the edge e and is equal to η on all the other edges. Let's define

$$P_{1,e,\mathbf{J_1}\beta}(\eta, [\eta'\eta]_e) = \mu_{\mathbf{J}_1\beta}(\eta'_e|\eta_{\setminus e})$$

for the kernel (transition matrix) that updates the configuration on the edge e; analogously for the integrated measure

$$\overline{P}_{e,\beta}(\eta, [\eta'\eta]_e) = \mu_\beta(\eta'_e|\eta_{\setminus e}).$$

If $\eta' \equiv 1$ for each $e \in E$

$$P_{1,e,\mathbf{J_1}\beta}(\eta, [\eta'\eta]_e) - \overline{P}_{e,\beta}(\eta, [\eta'\eta]_e) = \tag{59}$$

$$\mu_{\mathbf{J_1}\beta}(\eta_e = 1|\eta_{\setminus e}) - \mu_\beta(\eta_e = 1|\eta_{\setminus e}) \geq \tag{60}$$

$$\inf_{\eta_{\setminus e}}[p(\eta_{\setminus e})] \inf_{\eta_{\setminus e}}[\mu_{\mathbf{J}_1\beta}(\eta_e = 1|\eta_{\setminus e}) - \mu_{\mathbf{J}_2\beta}(\eta_e = 1|\eta_{\setminus e})] \geq \tag{61}$$

$$\frac{p}{p+2(1-p)} \inf_{\eta_{\setminus e}}\left[e^{-\beta J_2} - e^{-\beta J_1}, \frac{1-e^{-\beta J_1}}{1+e^{-\beta J_1}} - \frac{1-e^{-\beta J_2}}{1+e^{-\beta J_2}}\right] > 0. \tag{62}$$

(62) is continuous with respect to the variables β, J_1, J_2, and uniformly continuous if the variables are defined in a compact space. So there exists $\alpha \in (0, 1)$ such that, changing $\mu_{\mathbf{J_1}\beta}$ with $\mu_{\alpha\mathbf{J_1}\beta}$ in (60), the inequality (62) is verified for all $\beta \in [\beta_A, \beta_B]$ (with $0 < \beta_A < \beta_B < \infty$).

We can choose β_A and β_B such that $\beta_c^{FK}(\mathbf{J_1}), \beta_c^{FK}(\mathbf{J_2}) \in [\beta_A, \beta_B]$ with $0 < \beta_A < \beta_B < \infty$. In fact $\beta_c^{FK}(\mathbf{J_1})$ and $\beta_c^{FK}(\mathbf{J_2})$ are finite positive numbers. So due to the stochastic order of the FK measures (15), also all the ferromagnetic measures with $J_e \in [J_1, J_2]$ have $\beta_c^{FK}(J) \in [\beta_A, \beta_B]$. The Markov chain $P_{1,\alpha J_1 \beta}$ is attractive and dominates $\overline{P}_{e,\beta}$ on every edge because of (59)-(62); here the edges play the role of the blocks in the previous section. Following the exposition of the previous section, we have that the stationary measures of the processes $P_{1,\alpha J_1 \beta}$ and $\overline{P}_{e,\beta}$ are stochastically ordered and, rewriting the parameters Λ and β, one has $\mu_{\Lambda,\beta} \preceq \mu_{\Lambda,\alpha \mathbf{J}_1 \beta}$ $\forall \beta \in [\beta_A, \beta_B]$. We remark that the inequalities (59)-(62) are independent of Λ, so we can take the limit of $\Lambda \to \mathbf{Z}^d$ preserving the stochastic order between the measures. So, for all the weak limits of the integrated measure μ_β, one has

$$\beta_c^{FK}(\mu) \geq \beta_c^{FK}(\alpha \mathbf{J}_1) = \beta_c^{FK}(\mathbf{J}_1)/\alpha > \beta_c^{FK}(\mathbf{J}_1), \tag{63}$$

where $\beta_c^{FK}(\mu) := \sup\{\beta > 0 : \mu_\beta(\{O \leftrightarrow \infty\}) = 0\}$.

We will denote with I the event that there exists an infinite cluster (there is percolation). We will prove that $\beta_c^{FK}(J) = \beta_c^{FK}(\mu)$ Q almost everywhere. Since $\beta_c^{FK}(J)$ is Q-a.e. constant, by Proposition 4.1, we have

$$\beta_c^{FK}(\mu) = \sup\{\beta > 0 : \mu_\beta(\{O \leftrightarrow \infty\}) = 0\} = \sup\left\{\beta > 0 : \int_\Theta Q(d\theta)\mu_{J(\theta)\beta}(I) = 0\right\}. \tag{64}$$

We can integrate (64) only on the subset of Θ where $\beta_c^{FK}(J(\theta))$ is a constant, neglecting a set of measure zero. Then it is easy to see that $\beta_c^{FK}(J) = \beta_c^{FK}(\mu)$ Q almost everywhere. So we have proved that $\beta_c^{FK}(J) > \beta_c^{FK}(\mathbf{J}_1)$. Using the fact that for almost every β the FK measures with free boundary conditions and with $\tau \equiv i$ boundary conditions are equal and that I is an increasing event, we deduce the second inequality in (46) (see [ACCN88] and also [Gr95]). The other inequality in (46) follows using the same arguments; the inequalities in (47) are a consequence of the equality between β_c and β_c^{FK} for the ferromagnetic Ising model or using a construction due to Newman presented in the Appendix. □

Notice that for Theorem 4 we do not need the explicit definition of the percolation but only the fact that it is an increasing event in the tail σ-algebra $\mathcal{A}_\infty$. So, for other incresing events in $\mathcal{A}_\infty$, we could define a critical point and use the same idea to proof a strict inequality for the critical points of the ferromagnetic and frustrated system.

In the next theorem we denote with $\pi_{J\beta}^F$ the Gibbs measure of a ferromagnetic Ising model and with $\pi_{J\beta}$ the Gibbs measure of a frustrated Ising model with equal absolute value of the interactions J. For the interaction of a ferromagnetic system we define the partial order $J^{(1)} \preceq^* J^{(2)}$ if on all the edges $J_e^{(1)} \leq J_e^{(2)}$. We consider the i.i.d. random variables $\{J_e\}_{e \in E}$ on the space $\{-1, 1\}$ with distribution $Q(J_e = -1) = \tilde{p}$ with $0 < \tilde{p} < 1$.

Proof of Theorem 1. Let's consider a specific realization of the random variable J and the related Gibbs measure $\pi_{J\beta}$. We will construct a disordered ferromagnetic Ising model related to this configuration of the interactions J. Let's cover the graph $\mathbf{E}^d$ with blocks of a fixed size, we can choose $n = 6$. In each block there is a positive probability to find a frustrated plaquette because $0 < Q(J_e = -1) < 1$. In all the blocks that do not have frustrated plaquettes we set the interaction equal to the constant 1; on all the remaining edges we set $J_e = J_2 < 1$. We will indicate this configuration with $\tilde{J}$. If J_2 is chosen close to 1, we know from the previous section that the kernel $\widetilde{P}_{B,J}$ is dominated by $P^{(1)}_{B,J_2}$ and the kernel $\widetilde{P}_{B,J_1}$ dominates the kernel $P^{(2)}_{B,J}$. So, replacing on all the blocks the kernel $P^{(2)}_{B,J}$ with $P^{(1)}_{B,J\equiv 1}$ if there aren't frustrated plaquettes and with $P^{(1)}_{B,J\equiv J_2}$ if there is at least one frustrated plaquette, we have a kernel that dominates the previous one. We can calculate, using the reversibility, the stationary measure of this new kernel which turn out to be a ferromagnetic FK measure. So we find that this last FK measure and the original systems are stochastically ordered by taking the limit for $N \to \infty$ in A1). Again, the limit exist and is unique because the Markov chain is irreducible and aperiodic. So we have

$$\mu_{J\beta} \preceq \mu^F_{\tilde{J}\beta}. \tag{65}$$

We want to compare the ferromagnetic measure $\mu^F_{J\beta}$ with $\mu^F_{\tilde{J}\beta}$; we know by Theorem 4 that if $\tilde{J}$ are dyadic i.i.d. random variables then we have the strict inequality for the critical point.

From the construction one can see that on each block, independently from the others, there is a positive probability to find a frustrated plaquette. Therefore, in the coupling on each block B there is a positive probability to have $J_b = J_2 < 1$. The considered procedure to construct the configuration $\tilde{J}$ on the ferromagnetic system induces a measure P_1 on the interactions. It is simple to see that the measure P_1 is stochastically ordered with a dyadic product measure B_{p_1} (independent on all the edges) where $B_{p_1}(J_b = 1) = p_1$ and $B_{p_1}(J_b = J_2) = 1 - p_1$. Obviously the stochastic order is with respect to the partial order on the configurations of interactions. To have $P_1 \preceq B_{p_1}$ it is enough that

$$|p_1|^{|B|} \geq P\left(\{\text{ There is a frustrated plaquette in } B \setminus \partial^I B\}\right),$$

where $|B|$ is the number of edges in a block. So there is a ferromagnetic FK measure, called $\mu^F_{(J_1,J_2)\beta}$, with distribution of the interactions B_{p_1} satisfying the following stochastic order

$$\mu_{J\beta} \preceq \mu^F_{\tilde{J}\beta} \preceq \mu^F_{(J_1,J_2)\beta}. \tag{66}$$

Using Theorem 4 one obtains that for almost all the realizations of J

$$\beta_c^{FK}(J) \geq \beta_c^{FK}((J_1, J_2)) > \beta_c^{FK}(J \equiv 1). \tag{67}$$

Finally from the formula (66) using Theorem 3 one has

$$\beta_c(J) \geq \beta_c((J_1, J_2)) > \beta_c(J \equiv 1). \tag{68}$$

This ends the proof. □

The strategy of the proofs is quite general and can be applied to other models. For example with this method it is possible to study strict inequalities in the phase transition of disordered systems having also a random magnetic field (see [BK88] for the definition of the model).

A Appendix

We begin with some definitions that will be used in the proof of Theorem 3. Let's define the event

$$F(\Lambda_1) = \{\eta : \forall e \in E(\Lambda_1)\eta_e = 1\}; \tag{69}$$

let's $\Lambda_1 \subset \Lambda_2 \subset \mathbf{Z}^d$; let define the following event in which we write $m.c.s.$ in place of minimal cut set

$$F(\Lambda_1, \Lambda_2) = \{\eta : \exists\, \gamma \;\; m.c.s. \text{ with } \Lambda_1 \subset Int(\gamma) \subset \Lambda_2 \text{ and } \eta(\gamma) \equiv 0\}. \tag{70}$$

The event $F(\Lambda_1)$ is an increasing event, while $F(\Lambda_1, \Lambda_2)$ is decreasing. Define the event

$$F_\gamma = \{\eta : \gamma \text{ is the most external } m.c.s \text{ with } \eta_\gamma = 0 \text{ and } \Lambda_1 \subset Int(\gamma) \subset \Lambda_2\}; \tag{71}$$

it is easy to see that $F_\gamma \cap F_{\gamma'} = \emptyset$ for all $\gamma \neq \gamma'$ and $F(\Lambda_1, \Lambda_2) = \cup_\gamma F_\gamma$ with $\Lambda_1 \subset Int(\gamma) \subset \Lambda_2$; so we will divide the space $H_{E,\Lambda}$ with the events F_γ and $H_{E,\Lambda} \setminus F(\Lambda_1, \Lambda_2)$. It is known that, in the uniqueness region, as in (18) of Proposition 2.1 for all $\Lambda_1 \subset \mathbf{Z}^d$

$$\lim_{\Lambda \to \mathbf{Z}^d} \mu^{\tau \equiv i, F}_{\Lambda, J\beta}(F(\Lambda_1, \Lambda)) = 1. \tag{72}$$

Using the definition of FK measure with free boundary conditions we have

$$\mu^{\tau}_{\Lambda,\mathbf{J}\beta}(\eta_{E(A)}|\eta_\gamma = 0, \eta_{Est(\gamma)}) = \mu^{\tau'}_{\Lambda,\mathbf{J}\beta}(\eta_{E(A)}|\eta_\gamma = 0, \eta'_{Est(\gamma)}) \tag{73}$$

$$\nu^{\tau}_{\Lambda,\mathbf{J}\beta}(\eta_{E(A)}, \sigma_A|\eta_\gamma = 0, \eta_{Est(\gamma)}) = \nu^{\tau'}_{\Lambda,\mathbf{J}\beta}(\eta_{E(A)}, \sigma_A|\eta_\gamma = 0, \eta'_{Est(\gamma)}) \tag{74}$$

where γ is a minimal cut set, $A \subset Int(\gamma) \subset \Lambda$ so the conditional FK measure is equal to the FK measure with free boundary conditions $Int(\gamma)$

$$\mu_{Int(\gamma),\mathbf{J}\beta}(\eta_{E(A)}) = \mu^{\tau}_{\Lambda,\mathbf{J}\beta}(\eta_{E(A)}|\eta_\gamma = 0, \eta_{Est(\gamma)}). \tag{75}$$

In what fallows we will write H in place of $H_{E,\Lambda}$.

Proof of Theorem 3. We know, using the stochastic order $\forall\Lambda, \forall\tau' \mu^{\tau'}_{\Lambda,J\beta} \preceq \mu^{\tau\equiv i,F}_{\Lambda,J_1\beta_1}$, that there is a coupling $P^{+,\tau'}_\Lambda$ verifying Strassen's theorem. So the measure is supported on the set M defined in (12). The joint measure $P^{+,\tau'}_\Lambda$ has the following marginal measures $\mu^{\tau\equiv i,F}_{\Lambda,J_1\beta_1}(\cdot) = P^{+,\tau'}_\Lambda(\cdot, H)$ and $\mu^{\tau'}_{\Lambda,J\beta}(\cdot) = P^{+,\tau'}_\Lambda(H, \cdot)$.
To prove the uniqueness of the Gibbs measure, it is enough to show that

$$|\pi^{\tau}_{\Lambda,J\beta}(\sigma_A) - \pi^{\tau'}_{\Lambda,J\beta}(\sigma_A)| \tag{76}$$

tends to zero in the limit $\Lambda \to \mathbf{Z}^d$ for all boundary conditions τ, τ' and for all finite sets of vertices A. Let's use the joint measure $\nu^{\tau}_{\Lambda,\mathbf{J}\beta}(\eta, \sigma)$ on the space $H \times \Omega_\Lambda$ (see definition in (10)), so that (76) is equal to

$$\left|\sum_\eta \left(\nu^{\tau}_{\Lambda,J\beta}(\eta, \sigma_A) - \nu^{\tau'}_{\Lambda,J\beta}(\eta, \sigma_A)\right)\right|. \tag{77}$$

We will write $\eta_E = \eta_{E(\Lambda_1)}$ and $\eta_{\setminus E} = \eta_{\setminus E(\Lambda_1)}$; then (77) becomes:

$$= \left| \sum_{\eta_E, \eta_{\setminus E}} \left(\nu^{\tau}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) \nu^{\tau}_{\Lambda,J\beta}(\eta_{\setminus E}) - \nu^{\tau'}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) \nu^{\tau'}_{\Lambda,J\beta}(\eta_{\setminus E}) \right) \right| \tag{78}$$

$$= \left| \sum_{\eta_E, \eta_{\setminus E}} \left(\nu^{\tau}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) \mu^{\tau}_{\Lambda,J\beta}(\eta_{\setminus E}) - \nu^{\tau'}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) \mu^{\tau'}_{\Lambda,J\beta}(\eta_{\setminus E}) \right) \right| \tag{79}$$

$$= \left| \sum_{\eta_E, \eta_{\setminus E}, \eta'} \left(\nu^{\tau}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) P^{+,\tau}_{\Lambda}(\eta', \eta_{\setminus E}) - \nu^{\tau'}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) P^{+,\tau'}_{\Lambda}(\eta', \eta_{\setminus E}) \right) \right|. \tag{80}$$

Consider a minimal cut set γ having $\Lambda_2 \supset Int(\gamma) \supset \Lambda_1 \supset A$; then (80) is less than or equal to the expression

$$\le \left| \sum_{\eta_E, \eta_{\setminus E}, \gamma} \left(\nu^{\tau}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) P^{+,\tau}_{\Lambda}(F_\gamma, \eta_{\setminus E}) - \nu^{\tau'}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_{\setminus E}) P^{+,\tau'}_{\Lambda}(F_\gamma, \eta_{\setminus E}) \right) \right|$$
$$+ P^{+,\tau'}_{\Lambda}((H \setminus F(\Lambda_1, \Lambda)),\ H). \tag{81}$$

Since $P^{+,\tau'}_{\Lambda}(\eta' \succeq \eta) = 1$ if $\eta'_\gamma = 0$, then also $\eta_\gamma = 0$. So we get

$$\le | \textstyle\sum_{\eta_E, \eta_{\setminus E}, \gamma} \nu^{\tau}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_\gamma = 0, \eta_{\setminus E}) P^{+,\tau}_{\Lambda}(F_\gamma, \eta_{\setminus E}) -$$
$$\nu^{\tau'}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_\gamma = 0, \eta_{\setminus E}) P^{+,\tau'}_{\Lambda}(F_\gamma, \eta_{\setminus E})|$$
$$+ P^{+,\tau'}_{\Lambda}((H \setminus F(\Lambda_1, \Lambda)),\ H); \tag{82}$$

and using (74), the formula (82) is equal to

$$\left| \sum_{\eta_E, \eta_{\setminus E}, \gamma} \left[\nu^{\tau}_{\Lambda,J\beta}(\eta_E, \sigma_A|\eta_\gamma = 0) \left(P^{+,\tau}_{\Lambda}(F_\gamma, \eta_{\setminus E}) - P^{+,\tau'}_{\Lambda}(F_\gamma, \eta_{\setminus E}) \right) \right] \right| + \mu^{\tau \equiv i}_{\Lambda, J_1\beta_1}(H \setminus F(\Lambda_1, \Lambda)). \tag{83}$$

Performing all the sums on $\eta_{\setminus E}$ one obtains

$$\sum_{\eta_{\setminus E}} P^{+,\tau}_{\Lambda}(F_\gamma, \eta_{\setminus E}) = P^{+,\tau}_{\Lambda}(F_\gamma, H) = \mu^{\tau \equiv i, F}_{\Lambda, J_1\beta_1}(F_\gamma)$$

and in the same way $P^{+,\tau'}_{\Lambda}(F_\gamma, \eta_{\setminus E})$; then the sum is equal to zero and (83) is equal to

$$\mu^{\tau \equiv i, F}_{\Lambda, J_1\beta_1}(H \setminus F(\Lambda_1, \Lambda)).$$

But for $\Lambda \to \mathbf{Z}^d$ the expression $\mu^{\tau \equiv i, F}_{\Lambda, J_1\beta_1}(H \setminus F(\Lambda_1, \Lambda))$ goes to zero. Then on all finite cylinders, Gibbs measures with different boundary conditions are equal; so they are the same measures. This ends the proof. $\square$

Acknowledgments. *I wish to thank Alberto Gandolfi for discussing with me some ideas and mainly for suggesting me to use the integrated measure in Theorem 4. I am especially indebted to Charles Newman for many helpful and insightful comments. The research of the author was supported by a grant from the Italian CNR (Centro Nazionale di Ricerca).*

References

[ACCN88] M. Aizenman, J. Chayes, L. Chayes, C.M. Newman, *Discontinuity of the magnetization in one-dimensional* $1/|x-y|^2$ *Ising and Potts models.* J. Statis. Phys. **50**, 1988, pp. 1-40.

[AG91] M. Aizenman, G. Grimmett, *Strict monotonicity for critical points in percolation and ferromagnetic models.* J. Statist. Phys., **168**, 1991, pp. 39-55.

[BGK93] C. Bezuidenhout, G. Grimmett, H. Kesten, *Strict monotonicity for critical values of Potts models and random cluster processes.* Comm. Math. Phys., **158**, 1993, pp. 1-16.

[BK88] J. Bricmont, A. Kupiainen, *Phase transition in 3d random field Ising model.* Comm. Math. Phys., **116**, 1988, pp. 539-572.

[Ca98] M. Campanino, *Strict inequality for critical percolation values in frustrated random cluster models.* Markov Processes and Related Fields, **4**, 1998, pp. 395-410.

[De98] E. De Santis, *Strict inequalities in phase transition between ferromagnetic and frustrated systems.* Ph.D. thesis, Rome, 1998.

[DG99] E. De Santis, A. Gandolfi, *Bond percolation in frustrated systems.* Ann. Probab., **27**, 1999, pp. 1781-1808

[Di97] R. Diestel, *Graph Theory.* Springer-Verlag, 1997.

[EA75] S. Edwards, P. Anderson, *Theory of spin glasses.* J. Phys. F **5**, 1975, pp. 965-974.

[ES88] G. Edwards, A. Sokal, *Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm.* Phys. Rev. D, **38**, 1988, pp. 2009-2012.

[FKG71] C. Fortuin, P. Kasteleyn, J. Ginibre, *Correlation inequality on some partially ordered sets.* Comm. Math. Phy., **22**, 1971, pp. 89-103.

[FK72] C. Fortuin, P. Kasteleyn, *On the random cluster model. I. Introduction and relation to other models.* Physica, **57**, 1972, pp. 536-564.

[Fo72a] C. Fortuin, *On the random cluster model. II. The percolation model.* Physica, **58**, 1972, pp. 393-418.

[Fo72b] C. Fortuin, *On the random cluster model. III. The simple random-cluster process.* Physica, **59**, 1972, pp. 545-570.

[Ga98] A. Gandolfi, *Inequalities for critical points in disordered ferromagnets.* preprint 1998.

[GKN92] A. Gandolfi, M. Keane, C. Newman, *Uniqueness of the infinite component in a random graph with applications to percolation and spin glass.* Probab. Theory Related Fields, **92**, 1992, pp. 511-527.

[Gr94] G. Grimmett, *Potts models and random-cluster processes with many-body interactions.* J. Stat. Phys., **75**, 1994, pp. 67-121

[Gr95] G. Grimmett, *The stochastic random-cluster process and the uniqueness of random-cluster measures.* Ann. Prob. **23**, 4, 1995, pp. 1461-1510.

[Gr99] G. Grimmett, *Inequalities and entanglements for percolation and random-cluster models.* Perplexing Problems In Probability, Progr. Probab., **44**, 1999, pp. 91–105,

[Is25] E. Ising, *Beitrag zur Theorie des Ferromagnetismus.* Z. Phys., **31**, 1925, pp. 253-258.

[KF69] P. Kasteleyn, C. Fortuin, *Phase transitions in lattice systems with random local properties.* J. Phys. Soc. Japan **26** (Suppl.) 1969, pp. 11-14.

[LM72] J. Lebowitz and A. Martin-Lof, *On the uniqueness of the equilibrium state for Ising systems*, Comm. Math. Phy., **25**, 1972, pp. 276-282.

[Li85] T. Liggett, *Interacting Particle Systems.* Springer-Verlag, Berlin, 1985.

[Lin92] T. Lindvall, *Lecture on the coupling Method.* Series in Probability and Mathematical Statistics, 1992.

[Ne94] C.M. Newman, *Disordered Ising system and random cluster representation.* In Probability and Phase Transition, G. Grimmett (ed.), Kluwer, Dordrecht, 1994, pp. 247-260,

[Ne97] C. Newman, *Topics in Disordered Systems.* Lectures in Mathematics, Birkhäuser, ETH Zürich, 1997.

[OPR83] E. Olivieri, J.F. Perez, S. Goulart Rosa Jr., *Some rigorous results on the phase diagram of the dilute Ising model.* Phys. lett., **94A**, 1983, pp. 309-311.

[St65] V. Strassen, *The existence of probability measures with given marginals.* Ann. Math. Statist. **36**, 1965, pp. 423-439.