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Stochastic FitzHugh-Nagumo equations on networks with impulsive noise

Stefano Bonaccorsi
Dipartimento di Matematica
Università di Trento, via Sommarive 14
38100 Povo (TN), Italy.
bonaccor@science.unitn.it

Carlo Marinelli*
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6, D-53115 Bonn, Germany
marinelli@wiener.iam.uni-bonn.de

Giacomo Ziglio
Dipartimento di Matematica
Università di Trento, via Sommarive 14
38100 Povo (TN), Italy.
ziglio@science.unitn.it

Abstract

We consider a system of nonlinear partial differential equations with stochastic dynamical boundary conditions that arises in models of neurophysiology for the diffusion of electrical potentials through a finite network of neurons. Motivated by the discussion in the biological literature, we impose a general diffusion equation on each edge through a generalized version of the FitzHugh-Nagumo model, while the noise acting on the boundary is described by a generalized stochastic Kirchhoff law on the nodes. In the abstract framework of matrix operators theory, we rewrite this stochastic boundary value problem as a stochastic evolution equation in infinite dimensions with a power-type nonlinearity, driven by an additive Lévy noise. We prove global well-posedness in the mild sense for such stochastic partial differential equation by monotonicity methods.

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1 Introduction

In this paper we study a system of nonlinear diffusion equations on a finite network in the presence of an impulsive noise acting on the nodes of the system. We allow a rather general nonlinear drift term of polynomial type, including functions of FitzHugh-Nagumo type (i.e. $f(u) = -u(u-1)(u-a)$) arising in various models of neurophysiology (see e.g. the monograph [19] for more details).

Electric signaling by neurons has been studied since the 50s, starting with the now classical Hodgkin-Huxley model [16] for the diffusion of the transmembrane electrical potential in a neuronal cell. This model consists of a system of four equations describing the diffusion of the electrical potential and the behaviour of various ion channels. Successive simplifications of the model, trying to capture the key phenomena of the Hodgkin-Huxley model, lead to the reduced FitzHugh-Nagumo equation, which is a scalar equation with three stable states (see e.g. [27]).

Among other papers dealing with the case of a whole neuronal network (usually modeled as a graph with m edges and n nodes), which is intended to be a simplified model for a large region of the brain, let us mention a series of recent papers by Mugnolo et al. [21; 24], where the well-posedness of the isolated system is studied.

Note that, for a diffusion on a network, other conditions must be imposed in order to define the behaviour at the nodes. We impose a continuity condition, that is, given any node in the network, the electrical potentials of all its incident edges are equal. Each node represents an active soma, and in this part of the cell the potential evolves following a generalized Kirchhoff condition that we model with dynamical boundary conditions for the internal dynamics.

Since the classical work of Walsh [28], stochastic partial differential equations have been an important modeling tool in neurophysiology, where a random forcing is introduced to model external perturbations acting on the system. In our neuronal network, we model the electrical activity of background neurons with a stochastic input of impulsive type, to take into account the stream of excitatory and inhibitory action potentials coming from the neighbors of the network. The need to use models based on impulsive noise was already pointed out in several papers by Kallianpur and coauthors – see e.g. [17; 18]. On the other hand, from a mathematical point of view, the addition of a Brownian noise term does not affect the difficulty of the problem. In fact, in section 2 below, a Wiener noise could be added taking $q \neq 0$, introducing an extra term that does not modify the estimates obtained in section 3, which are the basis for the principal results of this paper. Let us also recall that the existence and uniqueness of solutions to reaction-diffusion equations with additive Brownian noise is well known – see e.g. [8; 10; 12].

Following the approach of [5], we use the abstract setting of stochastic PDEs by semigroup techniques (see e.g. [10; 11]) to prove existence and uniqueness of solutions to the system of stochastic equations on a network. In particular, the specific stochastic dynamics is rewritten in terms of a stochastic evolution equation driven by an additive Lévy noise on a certain class of Hilbert spaces.

The rest of the paper is organized as follows: in section 2 we introduce the problem and we motivate our assumptions in connection with the applications to neuronal networks. Then we provide a suitable abstract setting and we prove, following [24], that the linear operator appearing as leading drift term in the stochastic PDE generates an analytic semigroup of contractions. Section 3 contains our main results. First we prove existence and uniqueness of mild solution for the problem under Lipschitz conditions on the nonlinear drift term. This result (essentially already known) is used to obtain existence and uniqueness in the mild sense for the SPDE with a locally Lipschitz drift of FitzHugh-Nagumo type by monotonicity techniques.

2 Setting of the problem

Let us begin introducing some notation used throughout the paper. We shall denote by $\rightharpoonup$ and $\stackrel{*}{\rightharpoonup}$, respectively, weak and weak* convergence of functions. All stochastic elements are defined on a (fixed) filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual hypotheses. Given a Banach space E, we shall denote by $\mathbb{L}^p(E)$ the space of E-valued random variables with finite p-th moment.

The network is identified with the underlying graph G, described by a set of n vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$ and m oriented edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ which we assume to be normalized, i.e., $\mathsf{e}_j = [0, 1]$. The graph is described by the *incidence matrix* $\Phi = \Phi^+ - \Phi^-$, where $\Phi^+ = (\phi^+_{ij})_{n\times m}$ and $\Phi^- = (\phi^-_{ij})_{n\times m}$ are given by

$$\phi^-_{ij} = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(1) \\ 0, & \text{otherwise} \end{cases} \qquad \phi^+_{ij} = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(0) \\ 0, & \text{otherwise.} \end{cases}$$

The degree of a vertex is the number of edges entering or leaving the node. We denote

$$\Gamma(\mathsf{v}_i) = \{j \in \{1, \dots, m\} \ : \ \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i\}$$

hence the degree of the vertex v_i is the cardinality $|\Gamma(\mathsf{v}_i)|$.

The electrical potential in the network shall be denoted by $\bar{u}(t, x)$ where $\bar{u} \in (L^2(0, 1))^m$ is the vector $(u_1(t, x), \dots, u_m(t, x))$ and $u_j(t, \cdot)$ is the electrical potential on the edge e_j. We impose a general diffusion equation on every edge

$$\frac{\partial}{\partial t} u_j(t, x) = \frac{\partial}{\partial x}\left(c_j(x)\frac{\partial}{\partial x} u_j(t, x)\right) + f_j(u_j(t, x)), \tag{1}$$

for all $(t, x) \in \mathbb{R}_+ \times (0, 1)$ and all $j = 1, ..., m$. The generality of the above diffusion is motivated by the discussion in the biological literature, see for example [19], who remark, in discussing some concrete biological models, that the basic cable properties is not constant throughout the dendritic tree. The above equation shall be endowed with suitable boundary and initial conditions. Initial conditions are given for simplicity at time $t = 0$ of the form

$$u_j(0, x) = u_{j0}(x) \in C([0, 1]), \qquad j = 1, ..., m. \tag{2}$$

Since we are dealing with a diffusion in a network, we require first a continuity assumption on every node

$$p_i(t) := u_j(t, \mathsf{v}_i) = u_k(t, \mathsf{v}_i), \qquad t > 0,\ j, k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n \tag{3}$$

and a stochastic generalized Kirchhoff law in the nodes

$$\frac{\partial}{\partial t} p_i(t) = -b_i p_i(t) + \sum_{j\in\Gamma(\mathsf{v}_i)} \phi_{ij}\mu_j c_j(\mathsf{v}_i)\frac{\partial}{\partial x} u_j(t, \mathsf{v}_i) + \sigma_i \frac{\partial}{\partial t} L(t, \mathsf{v}_i), \tag{4}$$

for all $t > 0$ and $i = 1, \ldots, n$. Observe that the plus sign in front of the Kirchhoff term in the above condition is consistent with a model of purely excitatory node conditions, i.e. a model of a neuronal tissue where all synapses depolarize the postsynaptic cell. Postsynaptic potentials can have graded amplitudes modeled by the constants $\mu_j > 0$ for all $j = 1, ..., m$.

Finally, $L(t, \mathsf{v}_i),\ i = 1, ..., n$, represent the stochastic perturbation acting on each node, due to the external surrounding, and $\frac{\partial}{\partial t} L(t, \mathsf{v}_i)$ is the formal time derivative of the process L, which takes a meaning only in integral sense. Biological motivations lead us to model this term by a Lévy process. In fact, the evolution of the electrical potential on the molecular membrane can be perturbed by different types of random terms, each modeling the influence, at different time scale, of the surrounding medium. On a fast time scale, vesicles of neurotransmitters released by external neurons cause electrical impulses which arrive randomly at the soma causing a sudden change in the membrane voltage potential of an amount, either positive or negative, depending on the composition of the vesicle and possibly even on the state of the neuron. We model this behaviour perturbing the equation by an additive n-dimensional impulsive noise of the form

$$L(t) = \int_{\mathbb{R}^n} x \tilde{N}(t, \mathrm{d}x). \tag{5}$$

See Hypothesis 2.2 below for a complete description of the process and [18] for a related model.

Although many of the above reasonings remain true also when considering the diffusion process on the fibers, we shall not pursue such generality and assume that the random perturbation acts only on the boundary of the system, i.e. on the nodes of the network.

Let us state the main assumptions on the data of the problem.

Hypothesis 2.1.

1. *In (1), we assume that $c_j(\cdot)$ belongs to $C^1([0,1])$, for $j = 1, \ldots, m$ and $c_j(x) > 0$ for every $x \in [0,1]$.*
2. *There exists constants $\eta \in \mathbb{R}$, $c > 0$ and $s \geq 1$ such that, for $j = 1, \ldots, m$, the functions $f_j(u)$ satisfy $f_j(u) + \eta u$ is continuous and decreasing, and $|f_j(u)| \leq c(1 + |u|^s)$.*
3. *In (4), we assume that $b_i \geq 0$ for every $i = 1, \ldots, n$ and at least one of the coefficients b_i is strictly positive.*
4. *$\{\mu_j\}_{j=1,\ldots,m}$ and $\{\sigma_i\}_{i=1,\ldots,n}$ are real positive numbers.*

Given a Hilbert space $\mathcal{H}$, let us define the space $L^2_{\mathcal{F}}(\Omega \times [0,T]; \mathcal{H})$ of adapted processes $Y : [0,T] \to \mathcal{H}$ endowed with the natural norm

$$|Y|_2 = \left(\mathbb{E} \int_0^T |Y(t)|^2_{\mathcal{H}} \mathrm{d}t \right)^{1/2}.$$

We shall consider a Lévy process $\{L(t),\ t \geq 0\}$ with values in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, i.e., a stochastically continuous, adapted process starting almost surely from 0, with stationary independent increments and càdlàg trajectories. By the classical Lévy-Itô decomposition theorem, one has

$$L(t) = mt + qW_t + \int_{|x| \leq 1} x[N(t, \mathrm{d}x) - t\nu(\mathrm{d}x)] + \int_{|x|>1} x N(t, \mathrm{d}x), \qquad t \geq 0 \tag{6}$$

where $m \in \mathbb{R}^n$, $q \in M_{n\times n}(\mathbb{R})$ is a symmetric, positive defined matrix, $\{W_t,\ t \geq 0\}$ is an n-dimensional centered Brownian motion, $N(t, \mathrm{d}x)$ is a Poisson measure and the Lévy measure $\nu(\mathrm{d}x)$ is σ-finite on $\mathbb{R}^n \setminus \{0\}$ and such that $\int \min(1, x^2)\nu(\mathrm{d}x) < \infty$. We denote by $\tilde{N}(\mathrm{d}t, \mathrm{d}x) := N(\mathrm{d}t, \mathrm{d}x) - \mathrm{d}t\nu(\mathrm{d}x)$ the compensated Poisson measure.

Hypothesis 2.2. *We suppose that the measure ν has finite second order moment, i.e.*

$$\int_{\mathbb{R}^n} |x|^2 \nu(\mathrm{d}x) < \infty. \tag{7}$$

Condition (7) implies that the generalized compound Poisson process $\int_{|x|>1} x\, N(t, \mathrm{d}x)$ has finite moments of first and second order. Then, with no loss of generality, we assume that

$$\int_{|x|>1} x\nu(\mathrm{d}x) = 0. \tag{8}$$

We also assume throughout that the Lévy process is a pure jump process, i.e. $m \equiv 0$ and $q \equiv 0$, which leads to the representation (5) in view of assumptions (7) and (8).

2.1 Well-posedness of the linear deterministic problem

We consider the product space $\mathbb{H} = (L^2(0,1))^m$. A vector $\bar{u} \in \mathbb{H}$ is a collection of functions $\{u_j(x),\ x \in [0,1],\ j = 1, \ldots, m\}$ which represents the electrical potential inside the network.

Remark 2.3. For any real number $s \geq 0$ we define the Sobolev spaces

$$\mathbb{H}^s = (H^s(0,1))^m,$$

where $H^s(0,1)$ is the fractional Sobolev space defined for instance in [22]. In particular we have that $\mathbb{H}^1 \subset (C[0,1])^m$. Hence we are allowed to define the boundary evaluation operator $\Pi : \mathbb{H}^1 \to \mathbb{R}^n$ defined by

$$\Pi\bar{u} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \text{where } p_i = \bar{u}(\mathsf{v}_i) = u_k(\mathsf{v}_i) \quad \text{for } k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n.$$

On the space $\mathbb{H}$ we introduce the linear operator $(A, D(A))$ defined by

$$D(A) = \{\bar{u} \in \mathbb{H}^2 \mid \exists\, p \in \mathbb{R}^n \text{ such that } \Pi\bar{u} = p\}$$
$$A\bar{u} = \left(\quad \frac{\partial}{\partial x}\left(c_j(x)\frac{\partial}{\partial x}u_j(t,x)\right)\right)_{j=1,\ldots,m}$$

As discussed in [24], the diffusion operator A on a network, endowed with active nodes, fits the abstract mathematical theory of parabolic equations with dynamic boundary conditions, and in particular it can be discussed in an efficient way by means of sesquilinear forms.

Notice that no other condition except continuity on the nodes is imposed on the elements of $D(A)$. This is often stated by saying that the domain is *maximal*.

The so called feedback operator, denoted by C, is a linear operator from $D(A)$ to $\mathbb{R}^n$ defined as

$$C\bar{u} = \left(\sum_{j\in\Gamma(\mathsf{v}_i)} \phi_{ij}\mu_j c_j(\mathsf{v}_i) \frac{\partial}{\partial x} u_j(t,\mathsf{v}_i) \right)_{i=1,\dots,n}.$$

On the vector space $\mathbb{R}^n$ we also define the diagonal matrix

$$B = \begin{pmatrix} -b_1 & & \\ & \ddots & \\ & & -b_n \end{pmatrix}.$$

With the above notation, problem (1)–(4) can be written as an abstract Cauchy problem on the product space $\mathcal{H} = \mathbb{H} \times \mathbb{R}^n$ endowed with the natural inner product

$$\langle X, Y\rangle_{\mathcal{H}} = \langle \bar{u}, \bar{v}\rangle_{\mathbb{H}} + \langle p, q\rangle_{\mathbb{R}^n}, \qquad \text{where } X, Y \in \mathcal{H} \text{ and } X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix},\ Y = \begin{pmatrix} \bar{v} \\ q \end{pmatrix}.$$

We introduce the matrix operator $\mathcal{A}$ on the space $\mathcal{H}$, given in the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \{X = (\bar{u}, p) \in \mathcal{H} \,:\, \bar{u} \in D(A), u_j(\mathsf{v}_i) = p_i \text{ for every } j \in \Gamma(\mathsf{v}_i)\}.$$

Then the linear deterministic part of problem (1)–(4) becomes

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t) = \mathcal{A}X(t) \\ \quad X(0) = x_0 \end{cases} \tag{9}$$

where $x_0 = (u_j(0,x))_{j=1,\dots,m} \in C([0,1])^m$ is the vector of initial conditions. This problem is well posed, as the following result shows.

Proposition 2.4. *Under Hypotheses 2.1.1 and 2.1.2 the operator $(\mathcal{A}, D(\mathcal{A}))$ is self-adjoint, dissipative and has compact resolvent. In particular, it generates a C_0 analytic semigroup of contractions.*

Proof. For the sake of completeness, we provide a sketch of the proof following [24]. The idea is simply to associate the operator $(\mathcal{A}, D(\mathcal{A}))$ with a suitable form $\mathfrak{a}(X,Y)$ having dense domain $\mathcal{V} \subset \mathcal{H}$.

The space $\mathcal{V}$ is defined as

$$\mathcal{V} = \left\{ X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \mid \bar{u} \in (H^1(0,1))^m,\ u_k(\mathsf{v}_i) = p_i \text{ for } i = 1,\dots,n,\ k \in \Gamma(\mathsf{v}_i) \right\}$$

and the form $\mathfrak{a}$ is defined as

$$\mathfrak{a}(X,Y) = \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) u_j'(x) v_j'(x)\,\mathrm{d}x + \sum_{l=1}^{n} b_l p_l q_l, \qquad X = \begin{pmatrix} \bar u \\ p \end{pmatrix},\ Y = \begin{pmatrix} \bar v \\ q \end{pmatrix}.$$

The form $\mathfrak{a}$ is clearly positive and symmetric; furthermore it is closed and continuous. Then a little computation shows that the operator associated with $\mathfrak{a}$ is $(\mathcal{A}, D(\mathcal{A}))$ defined above. Classical results in Dirichlet forms theory, see for instance [25], lead to the desired result. □

The assumption that $b_l > 0$ for some l is a dissipativity condition on $\mathcal{A}$. In particular it implies the following result (for a proof see [24]).

Proposition 2.5. *Under Hypotheses 2.1.1 and 2.1.3, the operator $\mathcal{A}$ is invertible and the semigroup $\{\mathcal{T}(t),\ t \geq 0\}$ generated by $\mathcal{A}$ is exponentially bounded, with growth bound given by the strictly negative spectral bound of the operator $\mathcal{A}$.*

3 The stochastic Cauchy problem

We can now solve the system of stochastic differential equations (1)– (4). The functions $f_j(u)$ which appear in (1) are assumed to have a polynomial growth. We remark that the classical FitzHugh-Nagumo problem requires

$$f_j(u) = u(u-1)(a_j - u) \qquad j = 1, ..., m$$

for some $a_j \in (0,1)$, and satisfies Hypothesis 2.1.2 with

$$\eta \leq -\max_j \frac{(a_j^3+1)}{3(a_j+1)}, \qquad s = 3.$$

We set

$$F(\bar u) = \big(f_j(u_j)\big)_{j=1,\dots,m} \quad \text{and} \quad \mathcal{F}(X) = \begin{pmatrix} -F(\bar u) \\ 0 \end{pmatrix} \quad \text{for } X = \begin{pmatrix} \bar u \\ p \end{pmatrix}, \tag{10}$$

and we write our problem in abstract form

$$\begin{cases} \mathrm{d}X(t) = [\mathcal{A}X(t) - \mathcal{F}(X(t))]\,\mathrm{d}t + \Sigma\,\mathrm{d}\mathcal{L}(t) \\ X(0) = x_0, \end{cases} \tag{11}$$

where Σ is the matrix defined by

$$\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(\sigma_1, \dots, \sigma_n) \end{pmatrix},$$

and $\mathcal{L}(t)$ is the natural embedding in $\mathcal{H}$ of the n-dimensional Lévy process $L(t)$, i.e.

$$\mathcal{L}(t) = \begin{pmatrix} 0 \\ L(t) \end{pmatrix}.$$

Remark 3.1. Note that in general $\mathcal{F}$ is only defined on its domain $D(\mathcal{F})$, which is strictly smaller than $\mathcal{H}$.

Let us recall the definition of mild solution for the stochastic Cauchy problem (11).

Definition 3.2. *An $\mathcal{H}$-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* mild solution *of (11) if*

$$\int_0^T |\mathcal{F}(X(s))|\,\mathrm{d}s < +\infty \tag{12}$$

and

$$X(t) = \mathcal{T}(t)x_0 - \int_0^t \mathcal{T}(t-s)\mathcal{F}(X(s))\,\mathrm{d}s + \int_0^t \mathcal{T}(t-s)\Sigma\,\mathrm{d}\mathcal{L}(s) \tag{13}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, where $\mathcal{T}(t)$ is the semigroup generated by $\mathcal{A}$.

Condition (12) implies that the first integral in (13) is well defined. The second integral, which we shall refer to as stochastic convolution, is well defined as will be shown in the following subsection.

3.1 The stochastic convolution process

In our case the stochastic convolution can be written as

$$Z(t) = \int_0^t \int_{\mathbb{R}^n} \mathcal{T}(t-s)\begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}s, \mathrm{d}x).$$

The definition of stochastic integral with respect to a compensated Poisson measure has been discussed by many authors, see for instance [1; 2; 3; 9; 14; 15]. Here we limit ourselves to briefly recalling some conditions for the existence of such integrals. In particular, in this paper we only integrate deterministic functions, such as $\mathcal{T}(\cdot)\Sigma$, taking values in (a subspace of) $L(\mathcal{H})$, the space of linear operators from $\mathcal{H}$ to $\mathcal{H}$. In order to define the stochastic integral of this class of processes with respect to the Lévy martingale-valued measure

$$M(t,B) = \int_B x\,\tilde{N}(t, \mathrm{d}x), \tag{14}$$

one requires that the mapping $\mathcal{T}(\cdot)\Sigma : [0,T] \times \mathbb{R}^n \ni (t,x) \mapsto \mathcal{T}(t)(0, \sigma x)$ belongs to the space $L^2((0,T) \times B; \langle M(\mathrm{d}t, \mathrm{d}x)\rangle)$ for every $B \in \mathcal{B}(\mathbb{R}^n)$, i.e. that

$$\int_0^T \int_B \left| \mathcal{T}(s)\begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{H}}^2 \nu(\mathrm{d}x)\,\mathrm{d}s < \infty. \tag{15}$$

Thanks to (7), one has

$$\begin{aligned} &\int_0^T \int_B \left| \mathcal{T}(s)\begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{H}}^2 \nu(\mathrm{d}x)\,\mathrm{d}s \\ &\qquad \le |\sigma|^2 \left(\int_0^T |\mathcal{T}(s)|_{L(\mathcal{H})}^2\,\mathrm{d}s \right) \left(\int_B |x|^2\,\nu(\mathrm{d}x) \right) < \infty, \end{aligned}$$

thus the stochastic convolution $Z(t)$ is well defined for all $t \in [0, T]$.

We shall now prove a regularity property (in space) of the stochastic convolution. Below we will also see that the stochastic convolution has càdlàg paths.

Let us define the product spaces $\mathcal{E} := (C[0,1])^m \times \mathbb{R}^n$ and $C_{\mathcal{F}}([0,T]; L^2(\Omega; \mathcal{E}))$, the space of $\mathcal{E}$-valued, adapted mean square continuous processes Y on the time interval $[0,T]$ such that

$$|Y|^2_{C_{\mathcal{F}}} := \sup_{t \in [0,T]} \mathbb{E}|Y(t)|^2_{\mathcal{E}} < \infty.$$

Lemma 3.3. *For all $t \in [0,T]$, the stochastic convolution $\{Z(t),\ t \in [0,T]\}$ belongs to the space $C_{\mathcal{F}}([0,T]; L^2(\Omega; \mathcal{E}))$. In particular, $Z(t)$ is predictable.*

Proof. Let us recall that the (unbounded) matrix operator $\mathcal{A}$ on $\mathcal{H}$ is defined by

$$\mathcal{A} = \begin{pmatrix} \partial_x^2 & 0 \\ -\partial_\nu & B \end{pmatrix}$$

with domain $D(\mathcal{A}) = \{X = (\bar{u}, p) \in \mathcal{H} : \bar{u} \in D(A), u_l(\mathsf{v}_i) = p_i \text{ for every } l \in \Gamma(\mathsf{v}_i)\}$, and, by proposition 2.4, it generates a C_0-analytic semigroup of contractions on $\mathcal{H}$.

Let us introduce the interpolation spaces $\mathcal{H}_\theta = (\mathcal{H}, D(\mathcal{A}))_{\theta,2}$ for $\theta \in (0,1)$. By classical interpolation theory (see e.g. [23]) it results that, for $\theta < 1/4$, $\mathcal{H}_\theta = \mathbb{H}^{2\theta} \times \mathbb{R}^n$ while for $\theta > 1/4$ the definition of $\mathcal{H}_\theta$ involves boundary conditions, that is

$$\mathcal{H}_\theta = \left\{ \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \in H^{2\theta} \times \mathbb{R}^n : \Pi \bar{u} = p \right\}.$$

Therefore, one has $(0, \sigma x) \in \mathcal{H}_\theta$ for $\theta < 1/4$. Furthermore, for $\theta > 1/2$, one also has $\mathcal{H}_\theta \subset \mathbb{H}^1 \times \mathbb{R}^n \subset (C[0,1])^m \times \mathbb{R}^n$ by Sobolev embedding theorem. Moreover, for all $x \in \mathcal{H}_\theta$ and $\theta + \gamma \in (0,1)$, it holds

$$|\mathcal{T}(t)x|_{\theta+\gamma} \leq t^{-\gamma} |x|_\theta e^{\omega_{\mathcal{A}} t},$$

where $\omega_{\mathcal{A}}$ is the spectral bound of the operator $\mathcal{A}$.

Let θ, γ be real numbers such that $\theta \in (0, 1/4)$, $\gamma \in (0, 1/2)$ and $\theta + \gamma \in (1/2, 1)$. Then for all $t \in [0,T]$

$$|Z(t)|_{\theta+\gamma} \leq \int_0^t \int_{\mathbb{R}^n} \left| \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma} \tilde{N}(\mathrm{d}x, \mathrm{d}s) \qquad \mathbb{P}\text{-a.s.}$$

The right hand side of the above inequality is well defined if and only if

$$\mathbb{E} \left| \int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma} \tilde{N}(\mathrm{d}x, \mathrm{d}s) \right|^2 = \int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|^2_{\theta+\gamma} \nu(\mathrm{d}x)\mathrm{d}s < \infty,$$

where the identity follows by the classical isometry for Poisson integrals. On the other hand, one has

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|^2_{\theta+\gamma} \nu(\mathrm{d}x)\mathrm{d}s &\leq \int_0^T \int_{\mathbb{R}^n} s^{-2\gamma} \left| \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|^2_{\theta} e^{2\omega_{\mathcal{A}} s} \nu(\mathrm{d}x)\mathrm{d}s \\ &\leq |\sigma|^2 \int_0^T s^{-2\gamma} e^{2\omega_{\mathcal{A}} s} \mathrm{d}s \int_{\mathbb{R}^n} |x|^2 \nu(\mathrm{d}x) < \infty \end{aligned}$$

using $\gamma \in (0,1/2)$ and assumption (7). So $Z(t) \in \mathcal{H}_{\theta+\gamma}$ for $\theta+\gamma > 1/2$ and then $Z(t) \in (C[0,1])^m \times \mathbb{R}^n = \mathcal{E}$. It remains to prove that $Z(t)$ is mean square continuous as $\mathcal{E}$-valued process. For $0 \le s < t \le T$ we can write

$$\begin{aligned}
\mathbb{E}|Z(t)-Z(s)|_{\mathcal{E}}^2 &= \mathbb{E}\left|\int_0^t \mathcal{T}(t-r)\Sigma\,\mathrm{d}\mathcal{L}(r) - \int_0^s \mathcal{T}(s-r)\Sigma \mathrm{d}\mathcal{L}(r)\right|_{\mathcal{E}}^2 \\
&\le 2\mathbb{E}\left|\int_0^s \int_{\mathbb{R}^n} [\mathcal{T}(t-r)-\mathcal{T}(s-r)] \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}x,\mathrm{d}r)\right|_{\mathcal{E}}^2 \\
&\quad +2\mathbb{E}\left|\int_s^t \int_{\mathbb{R}^n} \mathcal{T}(t-r) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}x,\mathrm{d}r)\right|_{\mathcal{E}}^2 \\
&= 2\int_0^s \int_{\mathbb{R}^n} \left|[\mathcal{T}(t-r)-\mathcal{T}(s-r)] \begin{pmatrix} 0 \\ \sigma x \end{pmatrix}\right|_{\mathcal{E}}^2 \nu(\mathrm{d}x)\mathrm{d}r \\
&\quad +2\int_s^t \int_{\mathbb{R}^n} \left|\mathcal{T}(t-r) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix}\right|_{\mathcal{E}}^2 \nu(\mathrm{d}x)\mathrm{d}r \longrightarrow 0
\end{aligned}$$

by the strong continuity of the semigroup $\mathcal{T}(t)$. Since the stochastic convolution $Z(t)$ is adapted and mean square continuous, it is predictable. ☐

3.2 Existence and uniqueness in the Lipschitz case

We consider as a preliminary step the case of Lipschitz continuous nonlinear term and we prove existence and uniqueness of solutions in the space $C_{\mathcal{F}}$ of adapted mean square continuous processes taking values in $\mathcal{H}$. We would like to mention that this result is included only for the sake of completeness and for the simplicity of its proof (which is essentially based only on the isometry defining the stochastic integral). In fact, a much more general existence and uniqueness result was proved by Kotelenez in [20].

Theorem 3.4. *Assume that Hypothesis 2.2 holds, and let x_0 be an $\mathcal{F}_0$-measurable $\mathcal{H}$-valued random variable such that $\mathbb{E}|x_0|^2 < \infty$. Let $G : \mathcal{H} \to \mathcal{H}$ be a function satisfying Lipschitz and linear growth conditions:*

$$|G(x)| \le c_0(1+|x|), \qquad |G(x)-G(y)| \le c_0|x-y|, \qquad x,y \in \mathcal{H}. \tag{16}$$

for some constant $c_0 > 0$. Then there exists a unique mild solution $X \in C^0([0,T];L^2(\Omega,\mathcal{H}))$ to equation (11) with $-\mathcal{F}$ replaced by G. Moreover, the solution map $x_0 \mapsto X(t)$ is Lipschitz continuous.

Proof. We follow the semigroup approach of [11, Theorem 7.4] where the case of Wiener noise is treated. We emphasize only the main differences in the proof.

The uniqueness of solutions reduces to a simple application of Gronwall's inequality. To prove existence we use the classical Banach's fixed point theorem in the space $C_{\mathcal{F}}([0,T];L^2(\Omega;\mathcal{H}))$. Let $\mathcal{K}$ be the mapping

$$\mathcal{K}(Y)(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)G(Y(s))\,\mathrm{d}s + Z(t)$$

where $Y \in C_{\mathcal{F}}([0,T];L^2(\Omega;\mathcal{H}))$ and $Z(t)$ is the stochastic convolution. $Z(\cdot)$ and $\mathcal{T}(\cdot)x_0$ belong to $C_{\mathcal{F}}([0,T];L^2(\Omega;\mathcal{H}))$ respectively in view of Lemma 3.3 and the assumption on x_0. Moreover, setting

$$\mathcal{K}_1(Y)(t) = \int_0^t \mathcal{T}(t-s)G(Y(s))\,\mathrm{d}s,$$

it is sufficient to note that

$$|\mathcal{K}_1(Y)|^2_{C_{\mathcal{F}}} \leq (Tc_0)^2(1+|Y|^2_{C_{\mathcal{F}}})$$

by the linear growth of G and the contractivity of $\mathcal{T}(t)$. Then we obtain that $\mathcal{K}$ maps the space $C_{\mathcal{F}}([0,T];L^2(\Omega;\mathcal{H}))$ to itself. Furthermore, using the Lipschitz continuity of G, it follows that for arbitrary processes Y_1 and Y_2 in $C_{\mathcal{F}}([0,T];L^2(\Omega;\mathcal{H}))$ we have

$$|\mathcal{K}(Y_1)-\mathcal{K}(Y_2)|^2_{C_{\mathcal{F}}} = |\mathcal{K}_1(Y_1)-\mathcal{K}_1(Y_2)|^2_{C_{\mathcal{F}}} \leq (c_0T)^2|Y_1-Y_2|^2_{C_{\mathcal{F}}}.$$

If we choose an interval $[0,\tilde{T}]$ such that $\tilde{T} < c_0^{-1}$, it follows that the mapping $\mathcal{K}$ has a unique fixed point $X \in C_{\mathcal{F}}([0,\tilde{T}];L^2(\Omega;\mathcal{H}))$. The extension to an arbitrary interval $[0,T]$ follows by patching together the solutions in successive time intervals of length $\tilde{T}$.

The Lipschitz continuity of the solution map $x_0 \mapsto X$ is again a consequence of Banach's fixed point theorem, and the proof is exactly as in the case of Wiener noise.

It remains to prove the mean square continuity of X. Observe that $\mathcal{T}(\cdot)x_0$ is a deterministic continuous function and it follows, again from Lemma 3.3, that the stochastic convolution $Z(t)$ is mean square continuous. Hence it is sufficient to note that the same holds for the term $\int_0^t \mathcal{T}(t-s)G(X(s))\,\mathrm{d}s$, that is $\mathbb{P}$-a.s. a continuous Bochner integral and then continuous as the composition of continuous functions on $[0,T]$. □

Remark 3.5. By standard stopping time arguments one can actually show existence and uniqueness of a mild solution assuming only that x_0 is $\mathcal{F}_0$-measurable.

In order to prove that the solution constructed above has càdlàg paths, unfortunately one cannot adapt the factorization technique developed for Wiener integrals (see e.g. [11]). However, the càdlàg property of the solution was proved by Kotelenez [20], under the assumption that $\mathcal{A}$ is dissipative. Therefore, thanks to proposition 2.4, the solution constructed above has càdlàg paths. One could also obtain this property proving the following a priori estimate, which might be interesting in its own right.

Theorem 3.6. *Under the assumptions of theorem 3.4 the unique mild solution of problem* (11) *verifies*

$$\mathbb{E}\sup_{t\in[0,T]}|X(t)|^2_{\mathcal{H}} < \infty.$$

Proof. Let us consider the Itô formula for the function $|\cdot|^2_{\mathcal{H}}$, applied to the process X. Although our computations are only formal, they can be justified using a classical approximation argument. We obtain

$$\mathrm{d}|X(t)|^2_{\mathcal{H}} = 2\langle X(t-),\mathrm{d}X(t)\rangle_{\mathcal{H}} + \mathrm{d}[X](t).$$

By the dissipativity of the operator $\mathcal{A}$ and the Lipschitz continuity of G, we obtain

$$\begin{aligned}\langle X(t-),\mathrm{d}X(t)\rangle_{\mathcal{H}} &= \langle \mathcal{A}X(t),X(t)\rangle_{\mathcal{H}}\mathrm{d}t + \langle G(X(t)),X(t)\rangle_{\mathcal{H}}\mathrm{d}t + \langle X(t-),\Sigma\mathrm{d}\mathcal{L}(t)\rangle_{\mathcal{H}}\\ &\leq c_0|X(t)|^2_{\mathcal{H}} + \langle X(t-),\Sigma\mathrm{d}\mathcal{L}(t)\rangle_{\mathcal{H}}.\end{aligned}$$

Therefore

$$|X(t)|^2_{\mathcal{H}} \leq |x_0|^2_{\mathcal{H}} + 2c_0 \int_0^t |X(s)|^2_{\mathcal{H}} \mathrm{d}s + 2 \int_0^t \langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}} + \int_0^t |\Sigma|^2 \mathrm{d}[\mathcal{L}](s)$$

and

$$\begin{aligned}\mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} \leq & \mathbb{E}|x_0|^2_{\mathcal{H}} + 2c_0 T \mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} \\ & + 2\mathbb{E} \sup_{t\leq T} \Big| \int_0^t \langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}} \Big| + T \int_{\mathbb{R}^n} |\Sigma|^2 |x|^2 \, \nu(dx),\end{aligned} \tag{17}$$

where we have used the relation

$$\mathbb{E} \sup_{t\leq T} [X](t) \leq \mathbb{E} \int_0^T |\Sigma|^2 \, d[\mathcal{L}](t) = \mathbb{E} \int_0^T |\Sigma|^2 \, d\langle \mathcal{L}\rangle(t) = T \int_{\mathbb{R}^n} \left| \Sigma \begin{pmatrix} 0 \\ x \end{pmatrix} \right|^2 \nu(dx).$$

By the Burkholder-Davis-Gundy inequality applied to $M_t = \int_0^t \langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}$, there exists a constant c_1 such that

$$\begin{aligned}\mathbb{E} \sup_{t\leq T} \Big| \int_0^t \langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}} \Big| &\leq c_1 \mathbb{E} \left(\left[\int_0^\cdot \langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}} \right] (T) \right)^{1/2} \\ &\leq c_1 \mathbb{E} \left(\sup_{t\leq T} |X(t)|^2_{\mathcal{H}} \int_0^T |\Sigma|^2 \mathrm{d}[\mathcal{L}](s) \right)^{1/2} \\ &\leq c_1 \left(\varepsilon \mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} + \frac{1}{4\varepsilon} \mathbb{E} \int_0^T |\Sigma|^2 \mathrm{d}[\mathcal{L}](s) \right) \\ &= c_1 \varepsilon \mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} + \frac{c_1 T}{4\varepsilon} \int_{\mathbb{R}^n} |\Sigma|^2 |x|^2 \nu(\mathrm{d}x),\end{aligned} \tag{18}$$

where we have used the elementary inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$. Then by (17) and (18) we have

$$\begin{aligned}\mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} \leq & \mathbb{E}|x_0|^2_{\mathcal{H}} + 2c_0 T \mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} + 2c_1 \varepsilon \mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} \\ & + \left(\frac{c_1}{2\varepsilon} + 1 \right) T \int_{\mathbb{R}^n} |\Sigma|^2 |x|^2 \, \nu(dx),\end{aligned}$$

hence

$$\mathbb{E} \sup_{t\leq T} |X(t)|^2_{\mathcal{H}} \leq N \left[\mathbb{E}|x_0|^2_{\mathcal{H}} + T \left(1 + \frac{c_1}{2\varepsilon} \right) \right] < +\infty,$$

where

$$N = N(c_0, c_1, T, \varepsilon) = \frac{1}{1 - 2c_0 T - 2c_1 \varepsilon}.$$

Choosing $\varepsilon > 0$ and $T > 0$ such that $N < 1$, one obtains the claim for a small time interval. The extension to arbitrary time interval follows by classical extension arguments. $\square$

3.3 FitzHugh-Nagumo type nonlinearity

Let us now consider the general case of a nonlinear quasi-dissipative drift term $\mathcal{F}$.

Theorem 3.7. *Let $\mathcal{F} : D(\mathcal{F}) \subset \mathcal{H} \to \mathcal{H}$ be defined as in (10). Then the equation*

$$\begin{cases} \mathrm{d}X(t) = [\mathcal{A}X(t) - \mathcal{F}(X(t))]\,\mathrm{d}t + \Sigma\,\mathrm{d}\mathcal{L}(t), \qquad t \in [0,T], \\ X(0) = x_0 \end{cases} \tag{19}$$

admits a unique mild solution, denoted by $X(t, x_0)$, which satisfies the estimate

$$\mathbb{E}|X(t,x) - X(t,y)|^2 \le e^{2\eta t}\mathbb{E}|x-y|^2.$$

for all x, $y \in \mathcal{H}$.

Proof. As observed in section 3 above, there exists $\eta > 0$ such that $F + \eta I$ is accretive. By a standard argument one can reduce to the case of $\eta = 0$ (see e.g. [4]), which we shall assume from now on, without loss of generality. Let us set, for $\lambda > 0$, $F_\lambda(u) = F((1+\lambda F)^{-1}(u))$ (Yosida regularization). $\mathcal{F}_\lambda$ is then defined in the obvious way.

Let $\mathcal{G}y = -\mathcal{A}y + \mathcal{F}(y)$. Then $\mathcal{G}$ is maximal monotone on $\mathcal{H}$. In fact, since $\mathcal{A}$ is self-adjoint, setting

$$\varphi(u) = \begin{cases} |\mathcal{A}^{1/2}u|^2, & u \in D(\mathcal{A}^{1/2}) \\ +\infty, & \text{otherwise,} \end{cases}$$

one has $\mathcal{A} = \partial\varphi$. Let us also set $F = \partial g$, where $g : \mathbb{R}^m \to \mathbb{R}$ is a convex function, the construction of which is straightforward. Well-known results on convex integrals (see e.g. [4, sec. 2.2]) imply that F on H is equivalently defined as $F = \partial I_g$, where

$$I_g(u) = \begin{cases} \displaystyle\int_{[0,1]^m} g(u(x))\,\mathrm{d}x, & \text{if } \ g(u) \in L^1([0,1]^m), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us recall that

$$\mathcal{F} = \begin{pmatrix} -F \\ 0 \end{pmatrix}.$$

Since $D(\mathcal{F}) \cap D(\mathcal{A})$ is not empty, $\mathcal{G}$ is maximal monotone if $\varphi((I+\lambda\mathcal{F})^{-1}(u)) \le \varphi(u)$ (see e.g. [6, Thm. 9]), which is verified by a direct (but tedious) calculation using the explicit form of $\mathcal{A}$, since $(I+\lambda f_j)^{-1}$ is a contraction on $\mathbb{R}$ for each $j = 1, \dots, m$.

Let us consider the regularized equation

$$dX_\lambda(t) + \mathcal{G}_\lambda X_\lambda(t)\,dt = \Sigma\,d\mathcal{L}(t).$$

Appealing to Itô's formula for the square of the norm one obtains

$$|X_\lambda(t)|^2 + 2\int_0^t \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s)\rangle\,ds = |x_0|^2 + 2\int_0^t \langle X_\lambda(s-), \Sigma\,d\mathcal{L}(s)\rangle + [X_\lambda](t)$$

for all $t \in [0,T]$. Taking expectation on both sides yields

$$\mathbb{E}|X_\lambda(t)|^2 + 2\mathbb{E}\int_0^t \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s)\rangle\, ds = |x_0|^2 + t\int_{\mathbb{R}^n} |\Sigma|^2\, |z|^2\, \nu(dz), \tag{20}$$

where we have used the identity

$$\mathbb{E}[X_\lambda](t) = \mathbb{E}\Big[\int_0^\cdot \Sigma\, d\mathcal{L}(s)\Big](t) = t\int_{\mathbb{R}^n} |\Sigma|^2\, |z|^2\, \nu(dz).$$

Since by (20) we have that $\{X_\lambda\}$ is a bounded subset of $L^\infty([0,T],\mathbb{L}^2(\mathcal{H}))$, and $\mathbb{L}^2(\mathcal{H})$ is separable, Banach-Alaoglu's theorem implies that

$$X_\lambda \overset{*}{\rightharpoonup} X \qquad \text{in } L^\infty([0,T],\mathbb{L}^2(\mathcal{H})),$$

on a subsequence still denoted by λ. Thanks to the assumptions on f_j, one can easily prove that $\langle F(u),u\rangle \geq c|u|^{p+1}$ for some $c>0$ and $p \geq 1$, hence (20) also gives

$$\mathbb{E}\int_0^T |X_\lambda(s)|_{p+1}^{p+1}\, ds < C,$$

which implies that

$$X_\lambda \rightharpoonup X \qquad \text{in } L^{p+1}(\Omega\times[0,T]\times D, \mathbb{P}\times dt\times d\xi), \tag{21}$$

where $D = [0,1]^m \times \mathbb{R}^n$. Furthermore, (20) and (21) also imply

$$\mathcal{G}_\lambda X_\lambda \rightharpoonup \eta \qquad \text{in } L^{\frac{p+1}{p}}(\Omega\times[0,T]\times D, \mathbb{P}\times dt\times d\xi).$$

The above convergences immediately imply that X and η are predictable, then in order to complete the proof of existence, we have to show that $\eta(\omega,t,\xi) = \mathcal{G}(X(\omega,t,\xi))$, $\mathbb{P}\times dt\times d\xi$-a.e.. For this it is enough to show that

$$\limsup_{\lambda\to 0} \mathbb{E}\int_0^T \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s)\rangle\, ds \leq \mathbb{E}\int_0^T \langle \eta(s), X(s)\rangle\, ds.$$

Using again Itô's formula we get

$$\mathbb{E}|X(T)|^2 + 2\mathbb{E}\int_0^T \langle \eta(s), X(s)\rangle\, ds = |x_0|^2 + T\int_{\mathbb{R}^n} |\Sigma|^2\, |z|^2\, \nu(dz). \tag{22}$$

However, (21) implies that

$$\liminf_{\lambda\to 0} \mathbb{E}|X_\lambda(T)|^2 \geq \mathbb{E}|X(T)|^2$$

(see e.g. [7, Prop. 3.5]), from which the claim follows comparing (20) and (22).

The Lipschitz dependence on the initial datum as well as (as a consequence) uniqueness of the solution is proved by observing that $X(t,x) - X(t,y)$ satisfies $\mathbb{P}$-a.s. the deterministic equation

$$\frac{d}{dt}(X(t,x)-X(t,y)) = \mathcal{A}(X(t,x)-X(t,y)) - \mathcal{F}(X(t,x)) + \mathcal{F}(X(t,y)),$$

hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|X(t,x)-X(t,y)|^2 &= \big\langle \mathcal{A}(X(t,x)-X(t,y)), X(t,x)-X(t,y)\big\rangle \\ &\quad -\big\langle \mathcal{F}(X(t,x)-\mathcal{F}(X(t,y)), X(t,x)-X(t,y)\big\rangle \\ &\leq \eta|X(t,x)-X(t,y)|^2,\end{aligned}$$

where $X(\cdot,x)$ stands for the mild solution with initial datum x. By Gronwall's lemma we have

$$\mathbb{E}|X(t,x)-X(t,y)|^2 \leq e^{2\eta t}\mathbb{E}|x-y|^2,$$

which concludes the proof of the theorem. □

Remark 3.8. An alternative method to solve stochastic evolution equations with a dissipative nonlinear drift term is developed in [11; 12], for the case of Wiener noise, and in the recent book [26] for the case of Lévy noise. This approach consists essentially in the reduction of the stochastic PDE to a deterministic PDE with random coefficients, by "subtracting the stochastic convolution". To carry out this plan one has to find a reflexive Banach space $\mathcal{V}$, continuously embedded in $\mathcal{H}$, which is large enough to contain the paths of the stochastic convolution, and at the same time not too large so that it is contained in the domain of the nonlinearity $\mathcal{F}$. In particular, in the case of equation (19), theorem 10.14 in [26] yields existence and uniqueness of a mild solution provided, among other conditions, that

$$\int_0^T |Z(t)|^{18}\,dt < \infty \qquad \mathbb{P}\text{-a.s.}$$

The result could also be obtained applying theorem 10.15 of op. cit., provided one can prove that $\mathcal{L}$ has càdlàg trajectories in the domain of a fractional power of a certain operator defined in terms of $\mathcal{A}$. In some specific cases, such condition is implied by suitable integrability conditions of the Lévy measure. Unfortunately it seems to us rather difficult to verify such conditions, a task that we have not been able to accomplish. On the other hand, our approach, while perhaps less general, yields the well-posedness result under seemingly natural assumptions.

Remark 3.9. By arguments similar to those used in the proof of theorem 3.6 one can also obtain that

$$\mathbb{E}\sup_{t\leq T}|X_\lambda(t)|^2 < C,$$

i.e. that $\{X_\lambda\}$ is bounded in $\mathbb{L}^2(L^\infty([0,T];\mathcal{H}))$. By means of Banach-Alaoglu's theorem, one can only conclude that $X_\lambda \xrightarrow{*} X$ in $\mathbb{L}^2(L^1([0,T];\mathcal{H}))'$, which is larger than $\mathbb{L}^2(L^\infty([0,T];\mathcal{H}))$. In fact, from [13, Thm. 8.20.3], being $L^1([0,T];\mathcal{H})$ a separable Banach space, one can only prove that if F is a continuous linear form on $\mathbb{L}^2(L^1([0,T];\mathcal{H}))$, then there exists a function f mapping Ω into $L^\infty([0,T];\mathcal{H})$ that is weakly measurable and such that

$$F(g) = \mathbb{E}\langle f, g\rangle$$

for each $g \in \mathbb{L}^2(L^1([0,T];\mathcal{H}))$.

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