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STATISTICALLY STATIONARY SOLUTIONS TO THE 3-D NAVIER-STOKES EQUATION DO NOT SHOW SINGULARITIES

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Abstract If μ is a probability measure on the set of suitable weak solutions of the 3-D Navier-Stokes equation, invariant for the time-shift, with finite mean dissipation rate, then at every time t the set of singular points is empty μ-a.s. The existence of a measure μ with the previous properties is also also proved; it may describe a turbulent asymptotic regime.

Keywords Navier-Stokes equations, suitable weak solutions, stationary solutions.

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1 Introduction

Consider the three dimensional Navier-Stokes equation in a domain D, describing the dynamic of a viscous incompressible Newtonian fluid:

$$\begin{gathered}\frac{\partial u}{\partial t}+(u\cdot\nabla)\,u+\nabla p=\nu\triangle u+f\\ \operatorname{div}u=0.\end{gathered}\tag{1}$$

The vector field $u(t,x)$ is the velocity field, $p(t,x)$ is the pressure, $f(t,x)$ the body force, $\nu>0$ the kynematic viscosity. We assume that D is a bounded regular open domain, with homogeneous Dirichlet boundary conditions; however it seems that the results proved below can be adapted to the case of the full space $\mathbf{R}^3$, or the torus $\mathbf{T}^3$ with periodic boundary conditions (see Remarks 6 and 10).

One of the main open questions concerning equation (1) is the possible emergence of singularities. This problem is fundamental from the viewpoint of the mathematical analysis (the well-posedness of equation (1), with given initial and boundary conditions, is an open problem), but it is also related to the understanding of the evolution of 3-D structures like thin vortex filaments, the intensity of vortex stretching, the rate of transfer of energy from larger to smaller scales, so it is relevant for the physical understanding of fluids.

In the sequel, a point $(t,x)\in(0,\infty)\times D$ will be called *regular* if it has a neighborhood where u is essentially bounded. This mild regularity condition implies stronger local regularity of u and its derivatives as soon as the data have a certain regularity (see [16], and comments in [2]). The points which are not regular will be called *singular*, and $S\subset(0,\infty)\times D$ will denote the set of all singular points. For every t, we denote by $S_t\subset D$ the set of singular points at time t:

$$S_t=\{\,x\in D\,|\,(t,x)\in S\,\}.$$

Since we deal with families of solutions, we also write $S(u)$ and $S_t(u)$ to denote the sets S and S_t corresponding to a solution u. We shall recall below the definition of suitable weak solution; such solutions exist globally in time, but their uniqueness is not known.

It is not known whether S is empty or not. When $\nu=0$ some numerical investigations (see for instance [1], [3], [8], [9]) support the belief that S is not empty, but for $\nu>0$ the answer is even less clear. Scheffer (see [14] and references therein) and Caffarelli, Kohn, and Nirenberg [2] have proved that, for certain weak solutions, S is small in the sense of the Hausdorff dimension. The result of [2], the best known at present, states that for the suitable weak solutions $\mathcal{H}^1(S)=0$, $\mathcal{H}^d(\cdot)$ denoting the d-dimensional Hausdorff measure. See for instance also [7], [11] for other proofs or comments.

Having in mind the research of Lanford [10], Sigmund-Schultze [17] and other authors who prove existence of solutions to very difficult dynamical problems for a.e. initial condition with respect to some time-invariant or space-homogeneous measure, we study the problem of singularities not for an individual solution, but for a stochastic process solution to (1), stationary in time. The case of space-homogeneous fields seems to be tractable as well, but we do not solve it here. We recall also the intensive activity of C. Foias, M. I. Vishik, A. V. Fursikov and others in similar directions.

Let us state our main result. A stochastic process u, solution of equation (1), can be identified with a probability measure on the set W of all (deterministic) solution of (1). We prefer to express the result in terms of measures instead of processes. With the definitions and notations given in the next section, we prove the following result.

Theorem 1 *On the set W of all suitable weak solutions u of equation (1), let μ be a probability measure, invariant for the time-shift, with finite mean dissipation rate. Then, at every given time $t \geq 0$, we have*

$$S_t(u) = \emptyset \qquad \text{for } \mu - a.e.\ u \in W.$$

In Theorem 5 below we prove that there exists a measure μ with such properties.

Roughly speaking, in terms of processes, we prove that a time-stationary suitable weak solution u of equation (1), with finite mean dissipation rate, has the property that, at every given time $t \geq 0$, the set S_t is empty with probability one. In a sense, at every time $t \geq 0$, we cannot *see* the singularities: only a negligible set of paths may have singularities at time t.

On the contrary, we do not prove that μ-almost all the individual trajectories do not have singularities in time-space ($S = \emptyset$). The counterexample at the end of the paper shows that our approach is not strong enough to attack this more difficult question.

Our result partially confirms (and it has been strongly motivated by) some physical intuition, see for instance Chorin [4], p. 93, about the fact that very strong vorticity intensification (and possible blow-up) typical of 3-D fluids is mitigated in the stationary regime.

Finally we remark that $S = \emptyset$ is known for constant in time solutions (a particular case of stationary solutions). In contrast, the stationary solutions considered here may describe a fluid in the turbulent regime. In a work in progress we also generalize the result of [2] and the previous theorem to stochastic Navier-Stokes equations.

1.1 Notations

We shall use the function spaces

$$H = \{\, \phi : D \to \mathbf{R}^3 \mid \phi \in [L^2(D)]^3,\ \operatorname{div} \phi = 0,\ \phi \cdot n|_{\partial D} = 0 \,\}$$

where n is the outer normal to ∂D (cf. [19] for more details, and in particular for the interpretation of the condition $\phi \cdot n|_{\partial D} = 0$), and

$$V = \{\, \phi \in [H^1(D)]^3 \mid \operatorname{div} \phi = 0,\ \phi|_{\partial D} = 0 \,\}$$

($H^\alpha(D)$ denotes the classical Sobolev space). We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the norm and inner product in H. Identifying H with its dual space H', and identifying H' with a subspace of V' (the dual space of V), we have $V \subset H \subset V'$ and we can denote the dual pairing between V and V' by $\langle \cdot, \cdot \rangle$ when no confusion may arise. Moreover, we set $D(A) = [H^2(D)]^3 \cap V$, we denote by $D(A^{-1})$ the dual space of $D(A)$, and we perform identifications as above to get the dense continuous inclusions

$$D(A) \subset V \subset H \subset V' \subset D(A^{-1}).$$

With $D(A)$ defined above, we define the linear operator $A : D(A) \subset H \to H$ as $Au = -P\Delta u$, where P is the orthogonal projection in $[L^2(D)]^3$ over H. The operator A is positive selfadjoint with compact resolvent (see [21], Ch. III, Section 2.1); we denote by $0 < \lambda_1 \le \lambda_2 \le \dots$ the eigenvalues of A, and by $e_1, e_2, \dots$ a corresponding complete orthonormal system of eigenvectors.

The fractional powers A^α of A, $\alpha \ge 0$, are simply defined by

$$A^\alpha x = \sum_{i=1}^{\infty} \lambda_i^\alpha \langle x, e_i \rangle e_i$$

with domain

$$D(A^\alpha) = \{ x \in H \mid \|x\|_{D(A^\alpha)} < \infty \}$$

where

$$\|x\|^2_{D(A^\alpha)} = \sum_{i=1}^{\infty} \lambda_i^{2\alpha} \langle x, e_i \rangle^2 = |A^\alpha x|^2.$$

The space $D(A^\alpha)$ is an Hilbert space with the inner product

$$\langle x, y \rangle_{D(A^\alpha)} = \langle A^\alpha x, A^\alpha y \rangle,$$

with $x, y \in D(A^\alpha)$.

Since V coincides with $D(A^{1/2})$ (see [20] Section 2.2, or [21], Ch. III, Section 2.1), we can endow V with the norm $\|u\| = |A^{1/2} u|$.

We remark that

$$\|u\|^2 \ge \lambda_1 |u|^2.$$

2 Definition and existence of a stationary measure on weak solutions

Let us first recall the definition of suitable weak solution given in [2] (the regularity of p can be improved, see [18], [11], but we do not need it here). A minor difference with respect to [2] is that we do not call solution the pair (u, p), but only the velocity u (the pressure appears as an auxiliary variable). The definitions are equivalent for all purposes, but it is slightly easier to put a topology only on the set W of fields u.

Throughout the paper, we assume that f is given and time-independent, with the regularity

$$f \in \left[L^2(D) \right]^3, \qquad \operatorname{div} f = 0^1.$$

We have assumed that f is independent of time since we are interested in stationary solutions, so we need an autonomous dynamic; the extension to the case when f is a stationary stochastic process (possibly a generalized one, as a white noise) is meaningful, and will be treated in a work in preparation.

[1] The regularity given by [2] is $q > \frac{5}{2}$, but it is easy to see that, going along their proofs, the same results are valid for $f \in L^{q_1}(0, T; L^{q_2}(D))$, with q_1, $q_2 > \frac{3}{2}$ and $\frac{2}{q_1} + \frac{3}{q_2} < 2$. In this situation we have $q_1 = \infty$ and so the previous condition means $q_2 > \frac{3}{2}$; moreover we need $f \in L^2(D)$ in order to get existence for the solutions (see also [13]).

Definition 2 *A suitable weak solution to equation (1) in* $(0,\infty)\times D$ *is a vector field*

$$u\in L^{\infty}\left(0,T;H\right)\cap L^{2}\left(0,T;V\right)\ \text{ for all } T>0,$$

weakly continuous in H*, such that there exists*

$$p\in L^{\frac{5}{4}}_{loc}((0,\infty)\times D)$$

such that equation (1) holds true in the sense of distributions on $(0,\infty)\times D$*, the classical energy inequality*

$$\int_D |u(t)|^2+2\nu\int_s^t\int_D|\nabla u|^2\leq\int_D|u(s)|^2+2\int_s^t\int_D f\cdot u$$

holds true for almost all $s\geq 0$ *and all* $t>s$*, and also the following Local Energy Inequality holds true*

$$\begin{aligned}\int_D |u(t)|^2\varphi+2\nu\int_0^t\int_D\varphi|\nabla u|^2 \quad\leq\quad & \int_0^t\int_D|u|^2\left(\frac{\partial\varphi}{\partial t}+\nu\triangle\varphi\right)\\ & +\int_0^t\int_D(|u|^2+2p)(u\cdot\nabla)\varphi+2\int_0^t\int_D(f\cdot u)\varphi\end{aligned}$$

for every smooth function $\varphi:\boldsymbol{R}\times D\to\boldsymbol{R}$*,* $\varphi\geq 0$*, with compact support in* $(0,\infty)\times D$*, and for all* $t>0$*.*

We have included in the definition the classical energy inequality and the weak continuity of u (as a function of t) with values in H, properties not stated in the definition of [2], since they hold true for the solutions constructed in [2] and are classical in the theory of 3-dimensional Navier-Stokes equations.

Denote by W the set of all suitable weak solutions of the Navier-Stokes equation (1) in $(0,\infty)\times D$. It has been proved in [2] that $W\neq\emptyset$. Let us define the following metric on W (many of the ideas below concerning the framework of the path space are taken from Sell [15]; see also [5]):

$$d(u^{(1)},u^{(2)})=\sum_{n=1}^{\infty}2^{-n}\left(1\wedge\int_0^n\int_D|u^{(1)}-u^{(2)}|^2dx\,dt\right),\qquad u^{(1)},u^{(2)}\in W.$$

The metric space (W,d) is presumably not complete; a little modification of the definition of suitable weak solution would give us a complete metric space: one has to exclude $t=0$ as in [15]. Here we do not need completeness.

Let $C_{\mathrm{b}}(W)$ be the space of all bounded continuous functions $\phi:W\to\mathbf{R}$, with the uniform topology. Let $\mathcal{B}$ denote the Borel σ-algebra of (W,d), and let $M_1(W)$ be the set of all probability measures on $(W,\mathcal{B})$. We denote by $\mu(\phi)$ the integral $\int_W\phi(u)\,\mu(du)$, for all $\mu\in M_1(W)$ and $\phi\in C_{\mathrm{b}}(W)$.

Let $\tau_t:W\to W$, $t\geq 0$, be the time-shift, defined as $(\tau_t u)(s,x)=u(t+s,x)$. It is easy to verify the following properties:

i) $\tau_t u\in W$ for all $u\in W$ and $t\geq 0$,

ii) the mapping $(t,u)\mapsto\tau_t u$ is continuous from $[0,\infty)\times W$ to W.

See also [15]. We denote by τ_t also the induced mapping on $C_{\rm b}(W)$ defined as

$$(\tau_t\phi)(u) = \phi(\tau_t u) \qquad \phi \in C_{\rm b}(W).$$

We write $\tau_t\mu$ for the image measure of μ under τ_t (often denoted by $\mu(\tau_t^{-1})$). We have

$$(\tau_t\mu)(\phi) = \mu(\tau_t\phi)$$

for all $\mu \in M_1(W)$ and $\phi \in C_{\rm b}(W)$.

Definition 3 *A probability measure $\mu \in M_1(W)$ will be called time-stationary if $\tau_t\mu = \mu$ for all $t \geq 0$. We say that μ has finite mean dissipation rate if*

$$\int_W \left(\int_0^T \int_D |\nabla u|^2 dx\, dt \right) \mu(du) < \infty \quad \textit{for all } T > 0.$$

The measurability of $\int_0^T \int_D |\nabla u|^2 dx\, dt$ as a function from $(W, \mathcal{B})$ to $\mathbf{R}$ is obvious by the lower semicontinuity of the integral.

We prepare the existence of time-stationary measures with the following compactness criterium.

Lemma 4 *Given a function $k : [0,\infty) \to [0,\infty)$, the set*

$$K = \left\{ u \in W \mid \|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)} \leq k(T),\ \forall T > 0 \right\}$$

is relatively compact in (W, d).

Proof. Given a sequence $\{u_n\} \subset K$, we have to prove that there exists a subsequence $\{u_\nu\}$ which converges to some $u \in W$ in the metric d, i.e. in the strong topology of $L^2(0,T;H)$ for all $T > 0$.

Let $\{p_n\} \subset L^{5/4}_{\rm loc}((0,\infty) \times D)$ be a sequence corresponding to $\{u_n\}$ as in the definition of suitable weak solution. Since u_n satisfies equation (1) in the sense of distributions, one has $\left\| \frac{du_n}{dt} \right\|_{L^2(0,T;D(A^{-1}))} \leq C(n,T)$ where $C(n,T)$ depends only on the norms of u_n in $L^\infty(0,T;H)$ and $L^2(0,T;V)$. The proof of this result is classical; see for instance [11] Lemma 2.3, that we use again in the sequel of this proof. From the bound in the definition of K it follows that there exists a constant $C(T)$ independent of n such that

$$\left\| \frac{du_n}{dt} \right\|_{L^2(0,T;D(A^{-1}))} \leq C(T).$$

The boundedness of $\{u_n\}$ in $L^2(0,T;V)$ and $L^2(0,T;D(A^{-1}))$ implies the existence of a strongly convergent subsequence in $L^2(0,T;H)$, by a classical compactness result (see [19]). We apply this argument iteratively on $[0,n]$: first we extract a subsequence $\{u_{n^{(1)}}\}$ strongly convergent in $L^2(0,1;H)$ to some $u^{(1)}$, then from $\{u_{n^{(1)}}\}$ we can extract a subsequence $\{u_{n^{(2)}}\}$ strongly convergent to some $u^{(2)}$ in $L^2(0,2;H)$, and so on. Since a fortiori $\{u_{n^{(2)}}\}$ also converges strongly to $u^{(2)}$ in $L^2(0,1;H)$, $u^{(1)}$ and $u^{(2)}$ coincide over $[0,1]$; a similar identification holds true for the other limit functions. The procedure can be formalized by induction. Then it is sufficient to take the diagonal sequence to have a subsequence $\{u_\nu\}$ of the original sequence $\{u_n\}$, and a function u, such that $\{u_\nu\}$ converges strongly to u in $L^2(0,T;H)$ for every $T>0$. If we pass to a further subsequence, we also have that $\{u_\nu\}$ converges strongly to u in H for a.e. t, weakly to u in $L^2(0,T;V)$ and in $W^{1,2}(0,T;D(A^{-1}))$, and weak star in $L^\infty(0,T;H)$, for every $T>0$, so in particular u has these regularity properties. By a classical result, from the properties $L^\infty(0,T;H)$ and $C([0,T];D(A^{-1}))$ (that follows from $W^{1,2}(0,T;D(A^{-1}))$), there exists a representative in the equivalence class of u that is weakly continuous in H; we consider it in the sequel.

From [11] Lemma 2.3, for every closed set $B\subset D$ there exists a constant $C(B,n,T)$, depending only on the norms of u_n in $L^\infty(0,T;H)$ and $L^2(0,T;V)$, such that

$$\|p_n\|_{L^{5/4}((0,T)\times B)} \le C(B,n,T).$$

From the bound in the definition of K it follows that there exists a constant $C(B,T)$ independent of n such that

$$\|p_n\|_{L^{\frac{5}{4}}((0,T)\times B)} \le C(B,T).$$

Therefore, up to a further subsequence, we can assume that $\{p_\nu\}$ converges weakly to some p in $L^{5/4}_{\mathrm{loc}}((0,\infty)\times D)$.

The proof is complete if we show that u is a suitable weak solution of equation (1) (with the pressure p). We have already proved all the regularity properties of u and p, so we have only to prove, by a passage to the limit, that u satisfies equation (1) in the sense of distributions, that the classical energy inequality holds true, and that the local energy inequality holds true. The passage to the limit in the local energy inequality is less classical and more difficult, and essentially contains all the main ingredients or ideas to prove the others (that can be found for instance in [19]), so we restrict ourselves to this point. We follow the proof of a similar fact given in the appendix of [2].

Fix some $T>0$. Since p_n are bounded in $L^{5/4}_{\mathrm{loc}}$, then, up to a subsequence, p_n converges to p in $L^{5/4}(0,T;L^{5/3}_{\mathrm{loc}}(D))$ (the argument to show this can be found in [2], p. 781-782). Moreover, since u_n converges to u in $L^2((0,T)\times D)$ and the u_n are bounded in $L^{10/3}((0,T)\times D)$ by Sobolev inequalities and the bounds in the definition of K, we have

$$u_n \to u \qquad \text{strongly in } L^s((0,T)\times D), \qquad s\in[2,\frac{10}{3}). \tag{2}$$

Using again Sobolev inequalities and the bounds in the definition of K, we know that the u_n are bounded in $L^{20/3}(0,T;L^{5/2}(D))$. Since we know that u_n converges to u in $L^{5/2}((0,T)\times D)$, we can conclude that

$$u_n \to u \qquad \text{strongly in } L^5(0,T;L^{5/2}(D)). \tag{3}$$

Now we have all the convergence properties we need to prove that for each positive function $\varphi\in C_c^\infty((0,\infty)\times D)$,

$$\begin{aligned} 2\nu\int_0^t\int_D \varphi|\nabla u|^2 \;\le\; & \int_0^t\int_D |u|^2\left(\frac{\partial\varphi}{\partial t}+\nu\triangle\varphi\right)+\int_0^t\int_D(|u|^2+2p)(u\cdot\nabla)\varphi \\ & +2\int_0^t\int_D (f\cdot u)\varphi. \end{aligned}$$

In fact the integral $\int\int \varphi|\nabla u|^2$ is lower semicontinuous and so

$$\int_0^t \int_D \varphi \left|\nabla u\right|^2 \leq \liminf_{n\to\infty} \int_0^t \int_D \varphi \left|\nabla u_n\right|^2,$$

while the other terms converge thanks to (2), (3) and the fact that p_n converges to p in $L^{5/4}(0,T;L^{5/3}_{\text{loc}}(D))$. At last the complete local energy inequality can be obtained with an argument similar to the one used in [2], p. 783. ■

Theorem 5 *There exists a time stationary probability measure $\mu \in M_1(W)$, with finite mean dissipation rate, for the Navier-Stokes equation (1). For any such measure μ, there exists a constant $C_\mu > 0$ such that for all $t \geq s \geq 0$ we have*

$$\int_W \left(\int_s^t \int_D |\nabla u|^2 dx\, dt\right) \mu(du) = C_\mu (t-s). \tag{4}$$

Proof. *Step 1.* We apply the classical Krylov-Bogoliubov method to the semigroup τ_t in W. Let $v_0 \in M_1(W)$ be the δ-Dirac measure concentrated at some given element $u^* \in W$ (W is non empty): $v_0 = \delta_{u^*}$. Let

$$v_t = \tau_t v_0 = \delta_{\tau_t u^*}$$

$$\mu_t = \frac{1}{t}\int_0^t v_s\, ds, \quad t \geq 0.$$

It is not difficult to see (using property (ii) of τ_t) that v_t and μ_t are well defined elements of $M_1(W)$. We shall show in *Step 2* that there exists a compact set $K \subset W$ such that $\tau_t u^* \in K$ for all $t \geq 0$, so $v_t(K) = 1$ and thus

$$\mu_t\,(K) = 1$$

for all $t \geq 0$. Then $(\mu_t)_{t\geq 0}$ is tight. We can apply Prohorov theorem (notice that completeness of (W, d) is not required in the part of Prohorov theorem we use here); hence there is a sequence μ_{t_n} weakly convergent to some $\mu \in M_1(W)$:

$$\mu_{t_n}(\phi) \to \mu(\phi) \quad \forall \phi \in C_{\mathrm{b}}(W).$$

Now for $t \geq 0$ we have ($\phi \in C_{\mathrm{b}}(W)$)

$$\begin{aligned}
(\tau_t \mu)(\phi) &= \mu(\tau_t \phi) \\
&= \lim_{n \to \infty} \mu_{t_n}(\tau_t \phi) \qquad (\text{since } \tau_t \phi \in C_{\mathrm{b}}(W)) \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} (\tau_s v_0)\,(\tau_t \phi)\, ds \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} v_0(\tau_s \tau_t \phi)\, ds \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} v_0(\tau_{s+t} \phi)\, ds \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_t^{t+t_n} v_0(\tau_r \phi)\, dr \\
&= \lim_{n \to \infty} \mu_{t_n}(\phi) + \lim_{n \to \infty} \frac{1}{t_n} \left(\int_{t_n}^{t+t_n} v_0(\tau_r \phi)\, dr - \int_0^t v_0(\tau_r \phi)\, dr \right) \\
&= \mu(\phi) \qquad (v_0(\tau_r \phi) \text{ is bounded in } r).
\end{aligned}$$

Therefore μ is time-stationary.

Step 2. We denote u^* by u for simplicity. As announced in *Step 1*, we have to prove that $\tau_t u \in K$ for all $t \geq 0$, for a suitable compact set K. In view of Lemma 4 and the definition of $\tau_t u$, it is sufficient to prove that for all $T > 0$ there exists a constant $C(T) > 0$ such that

$$\sup_{s \in [t, t+T]} \int_D |u(s,x)|^2 dx + \int_t^{t+T} \int_D |\nabla u|^2 dx\, dr \leq C(T), \tag{5}$$

uniformly in $t \geq 0$.

From the classical energy inequality, for almost all $s \geq 0$ and all $t > s$ we have

$$\int_D |u(t,x)|^2 dx + \nu \int_s^t \int_D |\nabla u|^2 dx\, dr \leq \int_D |u(s,x)|^2 dx + C_\nu \int_s^t \|f\|_{V'}^2 dr \tag{6}$$

for some constant C_ν. Poincaré inequality implies the existence of a constant $\lambda_1 > 0$ such that

$$\int_D |u(t,x)|^2 dx + \nu \lambda_1 \int_s^t \int_D |u|^2 dx\, dr \leq \int_D |u(s,x)|^2 dx + C_\nu \|f\|_{V'}^2 (t-s).$$

Denote $\int_D |u(t)|^2 dx$ by $v(t)$, $\nu\lambda_1$ by λ, $C_\nu \|f\|_{V'}^2$ by C. We have, for all $s \geq 0$ and all $t > s$,

$$v(t) \leq v(s) - \int_s^t \lambda v(r) dr + C(t-s). \tag{7}$$

Let $t \notin \mathcal{N}$, where $\mathcal{N}$ is the set of Lebesgue measure zero for which (7) does not hold, and let $u(s) = -v(t-s)$ for $s \in [0,t]$. It is easy to see that for each $s \in [0,t]$ such that $t - s \notin \mathcal{N}$, we have

$$u(s) \leq u(0) + \int_0^s \lambda u(r) dr + Cs$$

and by Gronwall lemma

$$u(s) \le u(0)\mathrm{e}^{\lambda s} + \frac{C}{\lambda}(\mathrm{e}^{\lambda s} - 1).$$

Come back to function v to obtain

$$v(t) \le v(t-s)\mathrm{e}^{-\lambda s} + \frac{C}{\lambda}(1 - \mathrm{e}^{-\lambda s})$$

and then we can conclude that for all $t \notin \mathcal{N}$

$$v(t) \le \sup_{0 \le s \le 1} v(s) + \frac{C}{\lambda}.$$

In our application, this means that

$$\int_D |u(t,x)|^2 dx \le \frac{C_\nu \|f\|_{V'}^2}{\nu \lambda_1} + \sup_{0 \le s \le 1} \int_D |u(s,x)|^2 dx \tag{8}$$

holds true for all $t \ge 0$ (by the weak continuity of u in H). Given $T > 0$, for all $t > 0$ we have from (6) and (8)

$$\nu \int_t^{t+T} \int_D |\nabla u|^2 dx\, dr \le \frac{C_\nu \|f\|_{V'}^2}{\nu \lambda_1} + \sup_{0 \le s \le 1} \int_D |u(s,x)|^2 dx + C_\nu T \|f\|_{V'}^2$$

so there exists a constant $C_1(T)$ such that

$$\int_t^{t+T} \int_D |\nabla u|^2 dx\, dr \le C_1(T)$$

uniformly in $t \ge 0$. A similar estimate

$$\sup_{s \in [t,t+T]} \int_D |u(s,x)|^2 dx \le C_2(T)$$

is a straightforward consequence of (8), so (5) is proved. This proves the claim left open in *Step 1* and completes the proof of the existence of a time-stationary measure μ.

Step 3. Let us prove an important identity. Recall some facts and notations from Section 1.1. For all $h \in V$ we have

$$\int_D |\nabla h|^2 dx = \lim_{N \to \infty} \sum_{n=1}^{n} \lambda_n \langle h, e_n \rangle^2.$$

From the monotone convergence theorem it follows that for every function $h \in L^2(0,T;V)$ we have

$$\int_0^t \int_D |\nabla h|^2 dx\, dt = \lim_{N \to \infty} \sum_{n=1}^{n} \lambda_n \int_0^t \langle h(t), e_n \rangle^2 dt.$$

The shift invariance on μ implies that $\int_W \phi(\tau_t u)\mu(du) = \int_W \phi(u)\mu(du)$ for all $t \ge 0$ and all continuous bounded functions ϕ on W. Therefore, for every real $R > 0$ and natural $N > 0$, we have

$$\int_W \left(R \wedge \sum_{n=1}^{n} \int_s^t \lambda_n \langle u(r), e_n \rangle^2 dr \right) \mu(du) = \int_W \left(R \wedge \sum_{n=1}^{n} \int_0^{t-s} \lambda_n \langle u(r), e_n \rangle^2 dr \right) \mu(du)$$

for all $t > s \geq 0$. The same identity holds true, by monotone convergence, without cut-off:

$$\int_W \left(\int_s^t \int_D |\nabla u|^2 dr \right) \mu(du) = \int_W \left(\int_0^{t-s} \int_D |\nabla u|^2 dr \right) \mu(du). \tag{9}$$

Both members can be infinite, at this stage of the proof.

Step 4. We have to prove that μ has finite mean dissipation rate. Given $T > 0$, from (6) there exists a constant $C_{T,f} > 0$ such that

$$\int_0^t \int_s^t \int_D |\nabla u(r,x)|^2 dx\, dr\, ds \leq \int_0^t \int_D \|u(s,x)|^2 dx\, ds + C_{T,f}$$

whence

$$\int_0^t \int_0^r \int_D |\nabla u(r,x)|^2 dx\, ds\, dr \leq \int_0^t \int_D |u(s,x)|^2 dx\, ds + C_{T,f}$$

yielding

$$\int_0^t r \int_D |\nabla u(r,x)|^2 dx\, dr \leq \int_0^t \int_D |u(s,x)|^2 dx\, ds + C_{T,f}$$

Therefore, for all $\delta > 0$,

$$\int_\delta^t \int_D |\nabla u(r,x)|^2 dx\, dr \leq \frac{1}{\delta} \left(\int_0^t \int_D |u(s,x)|^2 dx\, ds + C_{T,f} \right). \tag{10}$$

The claim of finite mean dissipation rate will follow as soon as we prove that

$$\int_W \left(\int_0^t \int_D |u(s,x)|^2 dx\, ds \right) \mu(du) < \infty \tag{11}$$

for all $T > 0$. Indeed, the left-hand-side of (10) will be μ-integrable, so by the identity (9) we get

$$\int_W \left(\int_0^{T-\delta} \int_D |\nabla u(r,x)|^2 dx\, dr \right) \mu(du) < \infty$$

which is the required property. We have

$$\int_W \left(\int_0^t \int_D |u(s,x)|^2 dx\, ds \right) \nu_t(du) = \int_t^{t+T} \int_D |u(s,x)|^2 dx\, ds \leq C_t$$

by the estimates of *Step 2.* By convex combination, the same estimate is true for μ_t in place of ν_t. Thus for every $R > 0$ we have

$$\int_W \left(\int_0^t \int_D (|u(s,x)|^2 \wedge R)\, dx\, ds \right) \mu_t(du) \leq C_t.$$

Taking $t = t_n$, the sequence found in *Step 1*, we get in the limit as $n \to \infty$

$$\int_W \left(\int_0^t \int_D (|u(s,x)|^2 \wedge R)\, dx\, ds \right) \mu(du) \leq C_t$$

for every $R > 0$, which implies (11) by monotone convergence.

Step 5. Let us prove (4). Let $\vartheta(t)$ be the real valued function

$$\vartheta(t) = \int_W \left(\int_0^t \int_D |\nabla u|^2 dr \right) \mu(du).$$

By identity (9) we get $\vartheta(t+s) = \vartheta(t) + \vartheta(s)$, for all $t, s \geq 0$. Moreover, $\vartheta(t)$ is non decreasing. These two properties imply that it is linear. The proof of the theorem is complete. ∎

Remark 6 In the same way an existence result can be obtained for stationary solutions in the periodic case. An existence theorem for deterministic suitable weak solutions with periodic boundary conditions can be found in P. L. Lions (Theorem 3.2, [12]).

As it may concern the case of the whole space, a different approach has to be used, since it is not possible to use Poincarè inequality to obtain the uniform estimates used to control the mean dissipation rate. A way to show the claim is to proceed by approximating a stationary solution in the whole space by means of (statistically) stationary solutions with periodic boundary conditions and then taking the limit as the characteristic length goes to infinity (a more detailed proof of these facts can be found in [13]).

In order to be convinced that the mean dissipation rate remain finite also in the case of the whole space, we use the classical energy inequality. In fact, once we have proved that the mean dissipation rate is finite, we can see that it does not depend on the geometry of the domain we consider, but only on the amplitude of the external force and on the viscosity. Consider for example a stationary solution u_{st}^L of law μ^L with periodic boundary conditions. If we take the expectation in the classical energy inequality, using stationarity, we get

$$\nu \mathbf{E} \int_0^T \int_{Q_L} |\nabla u|^2 \leq T \|f\|_{V_L'}^2,$$

where $Q_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^3$ is the periodic domain, and so

$$C_{\mu^L} \leq \frac{1}{\nu} \|f\|_{V_L'}^2.$$

3 Proof of Theorem 1

Let us recall the fundamental criterium of regularity proved by Caffarelli, Kohn and Nirenberg [2]: if u is a suitable weak solution, there exists $\varepsilon > 0$ such that any point $(t, x) \in (0, \infty) \times D$ satisfying

$$\limsup_{r \to 0} \frac{1}{r} \int_{t-r^2}^{t+r^2} \int_{B_r(x)} |\nabla u|^2 \leq \varepsilon \tag{12}$$

is a regular point for u. Here $B_r(x)$ is the ball of radius r centered at x.

Lemma 7 *Let μ be a time-stationary measure for the Navier-Stokes equation (1), with finite mean dissipation rate. Let $r_n = 2^{-n}$. For every $t \geq 0$, we have*

$$\lim_{n \to \infty} \frac{1}{r_n} \int_{t-r_n^2}^{t+r_n^2} \int_D |\nabla u|^2 = 0 \textit{ for } \mu\textit{-a.e. } u \in W.$$

Proof. Let us introduce the random variable

$$X_n(u) = \frac{1}{r_n} \int_{t-r_n^2}^{t+r_n^2} \int_D |\nabla u|^2,$$

defined on W. We have

$$\int_W X_n \, d\mu = C_\mu r_n$$

where C_μ is the constant given by Theorem 5. Given $\delta > 0$, we thus have

$$\mu(X_n > \delta) \leq \frac{C_\mu}{\delta} r_n$$

so that

$$\sum_{n=1}^{\infty} \mu(X_n > \delta) < \infty.$$

By Borel-Cantelli Lemma, there exists a set $W_t \subset W$ of μ-measure one, such that for all $u \in W_t$ there exists $n_0(u)$ such that for all $n > n_0(u)$ we have $u \in (X_n < \delta)$, i.e.

$$X_n(u) < \delta.$$

Taking a sequence $\delta_k \to 0$, the previous statement (with $n_0(u)$ depending on k) holds true for all k and all u in a set of μ-measure one, proving the claim. ∎

Corollary 8 *For every $t \geq 0$, we have*

$$\lim_{r \to 0} \frac{1}{r} \int_{t-r^2}^{t+r^2} \int_D |\nabla u|^2 = 0 \textit{ for } \mu\textit{-a.e. } u \in W.$$

The proof of the first claim follows from the inequality

$$\frac{1}{r} \int_{t-r^2}^{t+r^2} \int_D |\nabla u|^2 \leq 2 \frac{1}{r_n} \int_{t-r_n^2}^{t+r_n^2} \int_D |\nabla u|^2$$

for $r \in (r_{n+1}, r_n)$. The claim of Theorem 1 is now a simple consequence of the criterium of Caffarelli, Kohn and Nirenberg [2] recalled above and the previous corollary.

Remark 9 With a similar proof as in Corollary 8, it is possible to give a stronger estimate of the quantity in (12), namely

$$\lim_{r \to 0} \frac{1}{r^{2-\varepsilon}} \int_{t-r^2}^{t+r^2} \int_D |\nabla u|^2 = 0 \text{ for } \mu\text{-a.e. } u \in W$$

for every $\varepsilon > 0$.It is not known if this estimate can improve the result in Theorem 1 (see also the counterexample in the next section).

Remark 10 In view of Remark 6, we can see that Theorem 1 holds also for statistically stationary solutions with periodic boundary conditions and in the whole space.

3.1 A counterexample

We do not know whether the following result is true: for μ-a.e. $u \in W$ the set $S(u)$ is empty. In this section we only show that it is difficult to obtain such a result with the present approach. More precisely, we show that the following abstract condition, corresponding to the basic properties we have in our approach, are not sufficient to prove the better statement on $S(u)$. Having in mind the process $Y_t(u) = \int_D |\nabla u(t)|^2 dx$, assume that a real valued stochastic process $(Y_t)_t$ on $(W, \mathcal{B}, \mu)$ fulfills the conditions:

1) Y_t is stationary,

2) $Y_t \geq 0$,

3) $t \to Y_t$ is integrable μ-a.s.

4) $\int_W Y_t d\mu < \infty$.

In fact these properties are a little stronger than those satisfied in the Navier-Stokes problem. We show that these conditions *do not* imply that for μ-a.e. $u \in W$

$$\lim_{r \to 0} \frac{1}{r} \int_{t-r^2}^{t+r^2} Y_s = 0 \text{ for all } t, \tag{13}$$

i.e. the condition of Corollary 8 that implies Theorem 1. Consider the torus $\mathbf{T}^1$ and let $\phi : \mathbf{T}^1 \to \mathbf{R}^+$ be a function which is regular except at 0, where it has a singularity such that

$$\int_{\mathbf{T}^1} \phi < \infty \text{ but } \int_{-r^2}^{r^2} \phi = \frac{1}{\log r^{-1}}.$$

We *do not* have

$$\lim_{r \to 0} \frac{1}{r} \int_{-r^2}^{r^2} \phi = 0$$

(and even with arbitrary powers of r). On $W = \mathbf{T}^1$ consider the Borel sets with the Haar measure μ and the process

$$Y_t(u) = \phi(t + u).$$

For all $u \in \mathbf{T}^1$ it has a singularity at $t = -u$ and condition (13) is violated. On the other side it satisfies the assumptions (1)-(4) above. Finally, notice that it does not contradict the theorems proved above: given t, for μ-a.e. u we have

$$\lim_{r \to 0} \frac{1}{r} \int_{t-r^2}^{t+r^2} Y_s(u) = 0.$$

Indeed, it holds true for all $u \neq -t$:

$$\frac{1}{r} \int_{t-r^2}^{t+r^2} Y_s(u)\, ds = \frac{1}{r} \int_{t-r^2}^{t+r^2} \phi(s+u)\, ds = \frac{1}{r} \int_{t+u-r^2}^{t+u+r^2} \phi(s')\, ds'$$

and at $t + u \neq 0$ the function ϕ is regular. This expression diverges only at $t + u = 0$.

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