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Approximation at first and second order of m-order integrals of the fractional Brownian motion and of certain semimartingales

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Abstract: Let X be the fractional Brownian motion of any Hurst index $H \in (0,1)$ (resp. a semimartingale) and set $\alpha = H$ (resp. $\alpha = \frac{1}{2}$). If Y is a continuous process and if m is a positive integer, we study the existence of the limit, as $\varepsilon \to 0$, of the approximations

$$I_\varepsilon(Y,X) := \left\{ \int_0^t Y_s \left(\frac{X_{s+\varepsilon} - X_s}{\varepsilon^\alpha} \right)^m ds,\, t \geq 0 \right\}$$

of m-order integral of Y with respect to X. For these two choices of X, we prove that the limits are almost sure, uniformly on each compact interval, and are in terms of the m-th moment of the Gaussian standard random variable. In particular, if m is an odd integer, the limit equals to zero. In this case, the convergence in distribution, as $\varepsilon \to 0$, of $\varepsilon^{-\frac{1}{2}} I_\varepsilon(1,X)$ is studied. We prove that the limit is a Brownian motion when X is the fractional Brownian motion of index $H \in (0, \frac{1}{2}]$, and it is in term of a two dimensional standard Brownian motion when X is a semimartingale.

Key words and phrases: m-order integrals; (fractional) Brownian motion; limit theorems; stochastic integrals.

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1 Introduction

In this paper we investigate the accurate convergence of some approximations of m-order integrals which appear when one performs stochastic calculus with respect to processes which are not semimartingales, for instance the fractional Brownian motion. We explain below our main motivation of this study.

1.1 Preliminaries

Recall that the fractional Brownian motion with Hurst index $0 < H < 1$ is a continuous centered Gaussian process $B^H = \{B_t^H : t \geq 0\}$ with covariance function given by $\mathrm{Cov}(B_s^H, B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})$. It is well known that B^H is a semi-martingale if and only if $H = \frac{1}{2}$ (see [16], pp. 97-98). Moreover, if $H > \frac{1}{2}$, B^H is a zero quadratic variation process. Hence, for $H \geq \frac{1}{2}$, a Stratonovich type formula involving symmetric stochastic integrals holds (see, for instance [17] or [5]):

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\circ B_s^H. \tag{1.1}$$

On the other hand, if $0 < H < \frac{1}{2}$, some serious difficulties appear. It is a quite technical computation to prove that (1.1) is still valid when $H > \frac{1}{6}$ (see [8] or [3]). In fact, for $H > \frac{1}{6}$, in [8] was proved, on the one hand,

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\circ B_s^H - \frac{1}{12}\int_0^t f^{(3)}(B_s^H) d^{\circ(3)} B_s^H + \frac{1}{120}\int_0^t f^{(5)}(B_s^H) d^{\circ(5)} B_s^H$$

and on the other hand,

$$\int_0^t f^{(3)}(B_s^H) d^{\circ(3)} B_s^H = \int_0^t f^{(5)}(B_s^H) d^{\circ(5)} B_s^H = 0,$$

where, for X, Y continuous processes and $m \geq 1$, the m-order symmetric integral is given by

$$\int_0^t Y_s d^{\circ(m)} X_s := \lim_{\varepsilon \to 0} \mathrm{prob} \frac{1}{2}\int_0^t (Y_s + Y_{s+\varepsilon}) \frac{(X_{s+\varepsilon} - X_s)^m}{\varepsilon} ds, \tag{1.2}$$

and the m-forward integral is given by

$$\int_0^t Y_s d^{-(m)} X_s := \lim_{\varepsilon \to 0} \mathrm{prob} \int_0^t Y_s \frac{(X_{s+\varepsilon} - X_s)^m}{\varepsilon} ds, \tag{1.3}$$

(if $m = 1$, we write $\int_0^t Y_s d^\circ X_s$ (resp. $\int_0^t Y_s d^- X_s$) instead of $\int_0^t Y_s d^{\circ(1)} X_s$ (resp. $\int_0^t Y_s d^{-(1)} X_s$)).

In [8] one studies, for the fractional Brownian motion, the existence of the m-order symmetric integrals. According to evenness of m, it is proved that

- if $mH = 2nH > 1$ then integral $\int_0^t f(B_s^H) d^{\circ(2n)} B_s^H$ exists and vanishes;
- if $mH = (2n-1)H > \frac{1}{2}$ then integral $\int_0^t f(B_s^H) d^{\circ(2n-1)} B_s^H$ exists and vanishes.

Moreover it is also emphasized that integrals $\int_0^t f(B_s^H) d^{\circ(2n)} B_s^H$ (resp. $\int_0^t f(B_s^H) d^{\circ(2n-1)} B_s^H$) do not exist in general if $2nH \leq 1$ (resp. $(2n-1)H \leq \frac{1}{2}$). This last statement is in fact a direct consequence of results of the present paper (as it is mentioned in the proof of Theorem 4.1 in [8]). An important consequence is that $H = \frac{1}{6}$ is a barrier of validity for the formula (1.1) (see [1], [3], [4], [7] and [8]).

1.2 First order approximation: almost sure convergence

In the definitions (1.2) and (1.3), limits are in probability. One can ask a natural question: is it possible to have almost sure convergence? For instance, in [10], the almost sure convergence of a generalized quadratic variation of a Gaussian process is proved using a discrete observation of one sample path; in particular, their result applies to the fractional Brownian motion. Here, we prove (see Theorems 2.1 and 2.2 below for precise statements) that, as $\varepsilon \to 0$,

$$\int_0^t Y_s\, f(\frac{X_{s+\varepsilon} - X_s}{\varepsilon^\alpha})ds \tag{1.4}$$

converge almost surely, uniformly on each compact interval, to an explicit limit when f belongs to a sufficiently large class of functions (including the case of polynomial functions, for instance $f(x) = x^m$), Y is any continuous process, $X = B^H$ is the fractional Brownian motion with $H \in (0,1)$ (resp. $X = Z$ a semimartingale) and $\alpha = H$ (resp. $\alpha = \frac{1}{2}$). Let us remark that the case when X is a semimartingale is a non Gaussian situation unlike was the case in [10] or other papers (at our knowledge).
If $m = 2n$ is an even integer the previous result suffices to study the existence of $2n$-order integrals for the fractional Brownian motion with all $0 < H < 1$. Indeed, by choosing $f(x) = x^{2n}$, we can write the following equivalent, as $\varepsilon \downarrow 0$:

$$\frac{1}{\varepsilon}\int_0^t Y_s(B^H_{s+\varepsilon} - B^H_s)^{2n} \sim \varepsilon^{2nH-1}\frac{(2n)!}{2^n n!}\int_0^t Y_s ds, \text{ if } \int_0^t Y_s ds \neq 0.$$

On the other hand, if $m = 2n-1$ is an odd integer, we need to refine our analysis (especialy for Hurst index $0 < H \leq \frac{1}{2}$) because, in this case, we do not have an almost sure non-zero equivalent.

1.3 Second order approximation: convergence in distribution

Set $Y \equiv 1$ and $f(x) = x^m$ in (1.4), with $m \geq 3$ an odd integer. For the two same choices of X (that is $X = B^H$ with $\alpha = H$ or X a semimartingale with $\alpha = \frac{1}{2}$), we have, for all $T > 0$,

$$\text{a.s., } \forall t \in [0,T],\ \lim_{\varepsilon\to 0}\int_0^t \left(\frac{X_{s+\varepsilon} - X_s}{\varepsilon^\alpha}\right)^m ds = 0.$$

After correct renormalization, is it possible to obtain the convergence in distribution of our approximation? We prove (see Theorems 2.4 and 2.5 below for precise statements) that the family of processes

$$\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^t \left(\frac{X_{s+\varepsilon} - X_s}{\varepsilon^\alpha}\right)^m ds : t \geq 0\right\} \tag{1.5}$$

converges in distribution, as $\varepsilon \to 0$, to an explicit limit:

- If $X = B^H$ is the fractional Brownian motion with $H \in (0, \frac{1}{2}]$, we obtain a Brownian motion and our approach is different to those given by [6] or [19];
- If X is a semimartingale, we express the limiting process in terms of a two-dimensional standard Brownian motion. This also give an example of a non-Gaussian situation when the convergence in distribution is studied.

We can see that

$$\left\{\int_0^t \left(\frac{B^H_{s+\varepsilon}-B^H_s}{\varepsilon^H}\right)^m ds,\ \varepsilon>0\right\} \text{ equals in law to } \left\{\int_0^{\frac{t}{\varepsilon}} \left(B^H_{s+1}-B^H_s\right)^m ds,\ \varepsilon>0\right\}$$

by using the self-similarity of the fractional Brownian motion (that is, for all $c>0$, B^H_{ct} equals in law - as a process - to $c^H B^H_t$).

In [6], the authors study the convergence in distribution (but only for finite dimensional marginals) of the discrete version of our problem, that is of the sum $\sum_{n=1}^{[t/\varepsilon]} f(B^H_{n+1}-B^H_n)$ with f a real function. On the other hand, in [19], the Hermite rank of f is used to discuss the existence of the limit in distribution of $\int_0^{t/\varepsilon} f(B^H_{s+1}-B^H_s)ds$ for $H>\frac{1}{2}$ (recall that, in the present paper, we assume that H is smaller than $\frac{1}{2}$).

2 Statement of results

2.1 Almost sure convergence

In the following, we shall denote by N a standard Gaussian random variable independent of all processes which will appear, by B^H the fractional Brownian motion with Hurst index H and by $B=B^{\frac{1}{2}}$ the Brownian motion.

Theorem 2.1 *Assume that $H\in(0,1)$. Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying for all $x,y\in\mathbb{R}$*

$$|f(x)-f(y)|\le L|x-y|^a(1+x^2+y^2)^b,\ (L>0, 0<a\le 1, b>0), \tag{2.1}$$

and $\{Y_t: t\ge 0\}$ be a continuous stochastic process. Then, as $\varepsilon\to 0$,

$$\int_0^t Y_s\, f(\frac{B^H_{s+\varepsilon}-B^H_s}{\varepsilon^H})ds \to \mathrm{E}\left[f(N)\right]\int_0^t Y_s ds, \tag{2.2}$$

almost surely, uniformly in t on each compact interval.

The following result contains a similar statement for continuous martingales:

Theorem 2.2 *Let $f:\mathbb{R}\to\mathbb{R}$ be a polynomial function. Assume that $\{Y_t: t\ge 0\}$ is a continuous process and that $\{J_t: t\ge 0\}$ is an adapted locally Hölder continuous paths process. Let $\{Z_t: t\ge 0\}$ be a continuous martingale given by $Z_t=Z_0+\int_0^t J_s dB_s$. Then, as $\varepsilon\to 0$,*

$$\int_0^t Y_s\, f(\frac{Z_{s+\varepsilon}-Z_s}{\sqrt{\varepsilon}})ds \to \int_0^t Y_s \mathrm{E}\left[f(N\, J_s)|\mathcal{F}_s\right] ds \tag{2.3}$$

almost surely, uniformly in t on each compact interval. Here, $\mathcal{F}_t=\sigma(J_s,\ s\le t)$.

Remarks: 1. For instance, if $f(x)=x^m$ then the right hand side of (2.3) equals to $\mathrm{E}\left[N^m\right]\int_0^t Y_s\, J_s^m\, ds$.
2. A similar result holds for continuous semimartingales of type $Z_t=Z_0+\int_0^t J_s dB_s+\int_0^t K_s ds$ because the finite variation part $\int_0^t K_s ds$ does not have any contribution to the limit. □

If we apply Theorem 2.1 with $f(x) = x$, we obtain, almost surely on each compact interval,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^H}\int_0^t Y_s(B^H_{s+\varepsilon} - B^H_s)ds = 0.$$

In the following result, we prove that, by replacing ε^{-H} by ε^{-1}, we obtain a non-zero limit for integrands of the form $Y_s = g(B^H_s)$:

Corollary 2.3

1) *Assume that* $g \in \mathrm{C}^2(\mathbb{R})$ *and that* H *belongs to* $[\frac{1}{2}, 1)$*. Then*

$$\int_0^t g(B^H_s)\frac{B^H_{s+\varepsilon} - B^H_s}{\varepsilon}ds$$

converges, as $\varepsilon \to 0$*, almost surely, on each compact interval. Consequently, the forward integral* $\int_0^t g(B^H_s)d^- B^H_s$ *can be defined path to path.*

2) *Assume that* $g \in \mathrm{C}^3(\mathbb{R})$ *and that* H *belongs to* $[\frac{1}{3}, 1)$*. Then*

$$\frac{1}{2}\int_0^t (g(B^H_{s+\varepsilon}) + g(B^H_s))\frac{B^H_{s+\varepsilon} - B^H_s}{\varepsilon}ds$$

converges, as $\varepsilon \to 0$*, almost surely, on each compact interval. Consequently, the symmetric integral* $\int_0^t g(B^H_s)d^\circ B^H_s$ *can be defined path to path.*

3) *Let* $\{Z_t : t \geq 0\}$ *be a continuous martingale as in Theorem 2.2. Assume that* $g \in \mathrm{C}^2(\mathbb{R})$*. Then*

$$\int_0^t g(Z_s)\frac{Z_{s+\varepsilon} - Z_s}{\varepsilon}ds$$

converges, as $\varepsilon \to 0$*, almost surely, on each compact interval, to the classical Itô integral* $\int_0^t g(Z_s)dZ_s$*.*

Remarks: 1. In [2], it is proved, for g regular enough, that

$$\int_0^t g(B^H_s)d^\circ B^H_s = \int_0^t g(B^H_s)\delta B^H_s + \mathrm{Tr}Dg(B^H)_t, \tag{2.4}$$

where $\int_0^t g(B^H_s)\delta B^H_s$ denotes the usual divergence integral with respect to B^H and $\mathrm{Tr}Dg(B^H)_t$ is defined as the limit in probability, as $\varepsilon \to 0$, of

$$\frac{1}{2\varepsilon}\int_0^t g'(B^H_s)\left[R(s, (s+\varepsilon)\wedge t) - R(s, (s-\varepsilon)\vee 0)\right]ds$$

with $R(s,t) = \mathrm{Cov}(B^H_s, B^H_t)$. For any $H \in (0,1)$, it is a simple computation to see that the previous limit exists almost surely, on each compact interval and equals to $H\int_0^t g'(B^H_s)s^{2H-1}ds$. Consequently, by using (2.4) and the part 2 of Corollary 2.3, we see that the divergence integral $\int_0^t g(B^H_s)\delta B^H_s$ can be defined path-wise.
2. In [20], it was introduced a path-wise stochastic integral with respect to B^H when the integrator has γ-Hölder continuous paths with $\gamma > 1 - H$. When the integrator is of the form $g(B^H_t)$, the condition on γ implies that $H > \frac{1}{2}$ (see p. 354). Hence, the first two parts of Corollary 2.3 could be viewed as improvements of the results in [20]. □

2.2 Convergence in distribution

Let m be an odd integer. It is well known that the monomial x^m may be expanded in terms of the Hermite polynomials:

$$x^m = \sum_{k=1}^{m} a_{k,m} H_k(x), \quad \text{with } H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}\left(e^{-\frac{x^2}{2}}\right), \; k = 0, 1, 2, \ldots \tag{2.5}$$

Note that the sum begin with $k = 1$ since m is odd (for instance $x = H_1(x)$, $x^3 = 3H_1(x) + H_3(x)$ and so on).

Theorem 2.4 *Let $m \geq 3$ be an odd integer and assume that H belongs to $(0, \frac{1}{2}]$. Then, as $\varepsilon \to 0$,*

$$\{\frac{1}{\sqrt{\varepsilon}} \int_0^t \left(\frac{B^H_{s+\varepsilon} - B^H_s}{\varepsilon^H}\right)^m ds : t \geq 0\} \xrightarrow{\mathcal{L}} \{\sqrt{c_{m,H}}\beta_t : t \geq 0\}. \tag{2.6}$$

Here $\{\beta_t : t \geq 0\}$ denotes a one-dimensional standard Brownian motion starting from 0 and $c_{m,H}$ is given by

$$c_{m,H} := 2 \sum_{k=1}^{m} \frac{a^2_{k,m}}{k!} \int_0^\infty \left[(x+1)^{2H} + |x-1|^{2H} - 2x^{2H}\right]^k dx,$$

where the coefficients $a_{k,m}$ are given by (2.5).

Remark: Let us note that if $0 < H < \frac{1}{2}$,

$$(x+1)^{2H} + |x-1|^{2H} - 2x^{2H} \sim H(2H-1)x^{-2(1-H)}, \quad \text{as } x \to \infty,$$

and if $H = \frac{1}{2}$,

$$(x+1)^{2H} + |x-1|^{2H} - 2x^{2H} = 0, \quad \text{as } x \geq 1.$$

Hence $c_{m,H} < \infty$ if and only if $0 < H \leq \frac{1}{2}$. □

Finally, let us state the result concerning martingales:

Theorem 2.5 *Let $m \geq 3$ be an odd integer and assume that σ is an element of $\mathrm{C}^2(\mathbb{R};\mathbb{R})$. Let $\{Z_t : t \geq 0\}$ be a continuous martingale given by $Z_t = Z_0 + \int_0^t \sigma(B_s)dB_s$. Then, as $\varepsilon \to 0$,*

$$\{\frac{1}{\sqrt{\varepsilon}} \int_0^t \left(\frac{Z_{s+\varepsilon} - Z_s}{\sqrt{\varepsilon}}\right)^m ds : t \geq 0\} \xrightarrow{\mathcal{L}} \{\int_0^t \sigma(\beta^{(1)}_s)^m d(\kappa_1 \beta^{(1)}_s + \kappa_2 \beta^{(2)}_s) : t \geq 0\}. \tag{2.7}$$

Here $\{(\beta^{(1)}_t, \beta^{(2)}_t) : t \geq 0\}$ denotes a two-dimensional standard Brownian motion starting from (0,0) and κ_i, $i = 1, 2$ are some constants.

3 Proofs

3.1 Proof of almost sure convergence

The idea to obtain almost sure convergence is firstly, to verify L^2-type convergence and secondly, to use a Borel-Cantelli type argument and the regularity of paths (see Lemma 3.1 below).

To begin with, let us recall a classical definition: the **local Hölder index** γ_0 of a continuous paths process $\{W_t : t \geq 0\}$ is the supremum of the exponents γ verifying, for any $T > 0$:

$$\mathrm{P}\left(\{\omega : \exists\, L(\omega) > 0,\ \forall s, t \in [0, T],\ |W_t(\omega) - W_s(\omega)| \leq L(\omega)|t - s|^{\gamma}\}\right) = 1. \tag{3.1}$$

We can state now the following almost sure convergence criterion which will be used in proving Theorems 2.1 and 2.2:

Lemma 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying (2.1), $\{W_t : t \geq 0\}$ be a locally Hölder continuous paths process with index γ_0 and $\{V_t : t \geq 0\}$ be a bounded variation continuous paths process. Set*

$$W_\varepsilon^{(f)}(t) := \int_0^t f\left(\frac{W_{s+\varepsilon} - W_s}{\varepsilon^{\gamma_0}}\right) ds,\ t \geq 0,\ \varepsilon > 0, \tag{3.2}$$

and assume that for each $t \geq 0$, as $\varepsilon \to 0$,

$$\left\| W_\varepsilon^{(f)}(t) - V_t \right\|_{\mathrm{L}^2}^2 = O(\varepsilon^\alpha), \quad \text{with } \alpha > 0. \tag{3.3}$$

Then, for any $t \geq 0$, $\lim_{\varepsilon \to 0} W_\varepsilon^{(f)}(t) = V_t$ almost surely, and if f is non-negative, for any continuous process $\{Y_t : t \geq 0\}$, as $\varepsilon \to 0$,

$$\int_0^t Y_s dW_\varepsilon^{(f)}(s) \to \int_0^t Y_s dV_s \tag{3.4}$$

almost surely, uniformly in t on every compact interval.

Proof. We split the proof in several steps.

Step 1. We set, for $n \in \mathbb{N}^*$, $\varepsilon_n := n^{-2/\alpha}$. For every $\delta > 0$

$$\mathrm{P}\left(|W_{\varepsilon_n}^{(f)}(t) - V_t| > \delta\right) \leq \frac{1}{\delta^2} \mathrm{E}\left[(W_{\varepsilon_n}^{(f)}(t) - V_t)^2\right] \leq \frac{\text{cst.}}{\delta^2} \varepsilon_n^\alpha.$$

Since $\sum \varepsilon_n^\alpha < +\infty$, we deduce, by applying Borel-Cantelli lemma that, for each $t \geq 0$, $\lim_{n \to \infty} W_{\varepsilon_n}^{(f)}(t) = V(t)$ almost surely.

Step 2. Fix $\varepsilon > 0$ and consider $n \in \mathbb{N}^*$ such that $\varepsilon_{n+1} < \varepsilon \leq \varepsilon_n$. Let us fix $\omega \in \Omega$. We shall denote, for each $t \geq 0$,

$$W_\varepsilon^{(f)}(t)(\omega) = W_{\varepsilon_n}^{(f)}(t)(\omega) + \xi_n(t)(\omega) + \zeta_n(t)(\omega),$$

with

$$\xi_n(t)(\omega) := \int_0^t \left[f\left(\frac{W_{s+\varepsilon}^{(f)}(\omega) - W_s^{(f)}(\omega)}{\varepsilon^{\gamma_0}}\right) - f\left(\frac{W_{s+\varepsilon_n}^{(f)}(\omega) - W_s^{(f)}(\omega)}{\varepsilon^{\gamma_0}}\right) \right] ds,$$

$$\zeta_n(t)(\omega) := \int_0^t \left[f(\frac{W^{(f)}_{s+\varepsilon_n}(\omega) - W^{(f)}_s(\omega)}{\varepsilon^{\gamma_0}}) - f(\frac{W^{(f)}_{s+\varepsilon_n}(\omega) - W^{(f)}_s(\omega)}{\varepsilon_n^{\gamma_0}}) \right] ds.$$

We prove that $\xi_n(t)(\omega), \zeta_n(t)(\omega)$ tend to zero, as $n \to \infty$, hence we shall deduce that, for each $t \geq 0$, $\lim_{\varepsilon\to 0} W^{(f)}_\varepsilon(t) = V(t)$ almost surely.

For notational convention, we will drop the argument ω and the superscript (f). We can write, for $\delta > 0$,

$$\begin{aligned} |\xi_n(t)| &\leq \frac{\text{cst.}}{\varepsilon^{a\gamma_0}} \int_0^t |W_{s+\varepsilon} - W_{s+\varepsilon_n}|^a \left[1 + \left(\frac{W_{s+\varepsilon} - W_s}{\varepsilon^{\gamma_0}} \right)^2 + \left(\frac{W_{s+\varepsilon_n} - W_s}{\varepsilon^{\gamma_0}} \right)^2 \right]^b ds \\ &\leq \frac{\text{cst.}}{\varepsilon_{n+1}^{(a+2b)\gamma_0}} |\varepsilon_n - \varepsilon_{n+1}|^{a(\gamma_0-\delta)} \varepsilon_n^{2b(\gamma_0-\delta)}\, t. \end{aligned}$$

Since $\varepsilon_n - \varepsilon_{n+1} = O(n^{-1-2/\alpha})$, we have $|\xi_n(t)| = O(n^{-a\gamma_0 + \delta(a+\frac{2a+4b}{\alpha})})$ as $n \to \infty$. Hence, $\lim_{n\to\infty} \xi_n(t) = 0$ by choosing δ small enough. Similarly,

$$\begin{aligned} |\zeta_n(t)| &\leq \text{cst.} \left(\frac{1}{\varepsilon^{\gamma_0}} - \frac{1}{\varepsilon_n^{\gamma_0}} \right)^a \int_0^t |W_{s+\varepsilon_n} - W_s|^a \left[1 + \left(\frac{W_{u+\varepsilon} - W_u}{\varepsilon^{\gamma_0}} \right)^2 + \left(\frac{W_{s+\varepsilon_n} - W_s}{\varepsilon^{\gamma_0}} \right)^2 \right]^b ds \\ &\leq \frac{\text{cst.}}{\varepsilon_{n+1}^{2b\gamma_0}} \left(\frac{1}{\varepsilon_{n+1}^{\gamma_0}} - \frac{1}{\varepsilon_n^{\gamma_0}} \right)^a \varepsilon_n^{(a+2b)(\gamma_0-\delta)}\, t. \end{aligned}$$

Since $\varepsilon_n^{-\gamma_0} - \varepsilon_{n+1}^{-\gamma_0} = O(n^{-1-2\gamma_0/\alpha})$, we have $|\zeta_n(t)| = O(n^{-a+\delta(a+\frac{2a+4b}{\alpha})})$ as $n \to \infty$. Again, $\lim_{n\to\infty} \zeta_n(t) = 0$ by choosing δ small enough.

Step 3. We will show that the exceptional set of the almost sure convergence $W^{(f)}_\varepsilon(t) \to V(t)$ can be choosed independent of t. Let Ω^* the set of probability 1, such that for every $\omega \in \Omega^*$, $\lim_{\varepsilon\to 0} W^{(f)}_\varepsilon(t)(\omega) = V_t(\omega)$, $\forall t \in \mathbb{Q}\cap\mathbb{R}^+$. Fix such a $\omega \in \Omega^*$, $t \in \mathbb{R}^+$ and assume that $\{s_n\}$ and $\{t_n\}$ are rational sequences such that $s_n \uparrow t$ and $t_n \downarrow t$. Clearly,

$$W^{(f)}_\varepsilon(s_n)(\omega) \leq W^{(f)}_\varepsilon(t)(\omega) \leq W^{(f)}_\varepsilon(t_n)(\omega).$$

First, letting ε goes to zero we get

$$V_{s_n}(\omega) \leq \liminf_{\varepsilon\to 0} W^{(f)}_\varepsilon(t)(\omega) \leq \limsup_{\varepsilon\to 0} W^{(f)}_\varepsilon(t)(\omega) \leq V_{t_n}(\omega),$$

and then, letting n goes to infinity we deduce that for each $\omega \in \Omega^*$ and each $t \in \mathbb{R}^+$, $\lim_{\varepsilon\to 0} W^{(f)}_\varepsilon(t)(\omega) = V_t(\omega)$.

Step 4. If f is non-negative we can apply Dini's theorem to obtain that $W^{(f)}_\varepsilon(t)$ converges almost surely toward V_t, uniformly on every compact interval.

Step 5. Further, the reasoning is pathwise, hence we fix $\omega \in \Omega$, we drop the argument ω and write small letters instead capital letters. Since $w^{(f)}_\varepsilon$ simply converges toward v, the distribution function of the measure $dw^{(f)}_\varepsilon$ converges toward the distribution function of the measure dv, hence $dw^{(f)}_\varepsilon$ weakly converges toward dv. Clearly, the measure dv does not charge points and the function $s \mapsto y_s \mathbf{1}_{[0,t]}(s)$ is dv-almost everywhere continuous. Consequentely,

$$\lim_{\varepsilon\to 0} \int_0^\infty y_s\, \mathbf{1}_{[0,t]}(s)\, dw^{(f)}_\varepsilon(s) = \int_0^t y_s\, dv_s.$$

The proof of the almost sure convergence criterion is done. ∎

Proof of Theorem 2.1.

First, let us note that if f satisfies (2.1) then the positive part f_+ and the negative part f_- also satisfy (2.1). Hence by linearity, we can assume that f is a non-negative function. We shall apply Lemma 3.1 to $W = B^H$, the fractional Brownian motion which is a locally Hölder continuous paths process with index H (as we can see by applying the classical Kolmogorov theorem, see [15], p. 25) and to the process $V_t = \mathrm{E}\,[f(N)]\; t$. We need to verify (3.3). First we note that

$$\mathrm{Var}(B^H_{u+\varepsilon} - B^H_u) = \varepsilon^{2H}$$

and, if $u + \varepsilon \leq u + \sqrt{\varepsilon} < v$,

$$\mathrm{Cov}(B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v) = (v-u-\varepsilon)^{2H} + (v-u+\varepsilon)^{2H} - 2(v-u)^{2H} \leq \frac{\mathrm{cst}.\varepsilon^2}{(v-u)^{2-2H}} \leq \mathrm{cst}.\varepsilon^{1+H},$$

as we can see by using Taylor expansion. Hence, by classical linear regression we obtain, for $u + \sqrt{\varepsilon} < v$,

$$\frac{B^H_{v+\varepsilon} - B^H_v}{\varepsilon^H} = O(\varepsilon^{1+H}) N_{u,\varepsilon} + \left(1 + O(\varepsilon^{2(1+H)})\right) M_{u,\varepsilon}, \tag{3.5}$$

uniformly with respect to u, where $N_{u,\varepsilon} = \frac{B^H_{u+\varepsilon} - B^H_u}{\varepsilon^H}$ and $M_{u,\varepsilon}$ are two independent standard Gaussian random variables. We write

$$\mathrm{E}\left\{\left[\int_0^t \left(f\Big(\frac{B^H_{s+\varepsilon} - B^H_s}{\varepsilon^H}\Big) - \mathrm{E}\,[f(N)]\right) ds\right]^2\right\} = T_1(\varepsilon) + T_2(\varepsilon) + T_3, \tag{3.6}$$

where

$$T_1(\varepsilon) := 2 \iint_{[0,t]^2} \mathbf{1}_{u<v<u+\sqrt{\varepsilon}} \mathrm{E}\left[f\Big(\frac{B^H_{u+\varepsilon} - B^H_u}{\varepsilon^H}\Big) f\Big(\frac{B^H_{v+\varepsilon} - B^H_v}{\varepsilon^H}\Big)\right] du dv,$$

$$T_2(\varepsilon) := 2 \iint_{[0,t]^2} \mathbf{1}_{u+\sqrt{\varepsilon}<v} \mathrm{E}\left[f\Big(\frac{B^H_{u+\varepsilon} - B^H_u}{\varepsilon^H}\Big) f\Big(\frac{B^H_{v+\varepsilon} - B^H_v}{\varepsilon^H}\Big)\right] du dv \text{ and } T_3 := -\mathrm{E}\,[f(N)]^2\, t^2.$$

Using (2.1) (which implies that $|f(x)| \leq \mathrm{cst}.(1+|x|^a(1+x^2)^b)$) and Cauchy-Schwarz inequality we can prove that $T_1(\varepsilon) = O(\sqrt{\varepsilon})$. Using again (2.1) and (3.5) we deduce that $T_2(\varepsilon) = \mathrm{E}\,[f(N)]^2\, t^2 + O(\varepsilon^{a(1+H)})$. Replacing in (3.6) we see that (3.3) is verified, hence Lemma 3.1 applies. The proof of (2.2) is done. ■

Proof of Theorem 2.2.

By linearity it suffices to prove the result for $f(x) = x^m$ and by classical localization argument (see for instance [8], p. 8), it suffices to prove the result for J bounded continuous process. Recall that N denotes a standard Gaussian random variable.

Thanks to Theorem 2.1, (2.3) is true for the Brownian motion $B = B^{\frac{1}{2}}$. More precisely, as $\varepsilon \to 0$,

$$\int_0^t J_s^m \left(\frac{B_{s+\varepsilon} - B_s}{\sqrt{\varepsilon}}\right)^m ds \to \mathrm{E}\,[N^m] \int_0^t J_s^m ds \tag{3.7}$$

almost surely uniformly on each compact interval.

At this point we need the following technical but simple:

Lemma 3.2 *Denote by $\mathcal{P}$ the set of finite sequences δ with values in $\{1,2\}$. For $\delta \in \mathcal{P}$ we denote by $k = k(\delta)$ the length of the support of the sequence δ and $n(\delta) := \sum_{j=1}^{k(\delta)} \delta(j)$. Let M be a martingale, $a < b$ be two real numbers and we shall denote*

$$I^{(M)}_{a,b,\delta} := \int_a^b dM^{(\delta(1))}_{t_1} \int_a^{t_1} dM^{(\delta(2))}_{t_2} \dots \int_a^{t_{k-1}} dM^{(\delta(k))}_{t_k}, \tag{3.8}$$

with the convention $dM^{(1)} = dM$ (Itô's differential) and $dM^{(2)} = d[M,M]$ (Riemann-Stieltjes differential). Then, for each $n \in \mathbb{N}^*$,

$$(M_b - M_a)^n = \sum_{\delta \in \mathcal{P}, n(\delta)=n} c_n(\delta) I^{(M)}_{a,b,\delta}, \tag{3.9}$$

where $c_n(\delta)$ is a contant depending only on δ and n.

Proof. We make an induction with respect to n. If $n = 1$, $M_b - M_a = \int_a^b dM_s$. Assume that (3.9) is true for n and we verify it for $n+1$:

$$(M_b - M_a)^{n+1} = (n+1) \int_a^b (M_s - M_a)^n dM_s + \frac{n(n+1)}{2} \int_a^b (M_s - M_a)^{n-1} d[M,M]_s$$

$$= (n+1) \int_a^b \left(\sum_{\delta \in \mathcal{P}, n(\delta)=n} c_n(\delta) I^{(M)}_{a,s,\delta} \right) dM_s + \frac{n(n+1)}{2} \int_a^b \left(\sum_{\delta \in \mathcal{P}, n(\delta)=n-1} c_n(\delta) I^{(M)}_{a,s,\delta} \right) d[M,M]_s$$

$$= \sum_{\delta \in \mathcal{P}, n(\delta)=n+1} c_{n+1}(\delta) I^{(M)}_{a,b,\delta}.$$

■

We can finish the proof of Theorem 2.2. By (3.9) we can write

$$J_s^m (B_{s+\varepsilon} - B_s)^m - (Z_{s+\varepsilon} - Z_s)^m = \sum_{\delta \in \mathcal{P}, n(\delta)=m} c(\delta) (J_s^m I^{(B)}_{s,s+\varepsilon,\delta} - I^{(Z)}_{s,s+\varepsilon,\delta}),$$

where

$$J_s^m I^{(B)}_{s,s+\varepsilon,\delta} - I^{(Z)}_{s,s+\varepsilon,\delta} = \sum_{j=1}^k \int_s^{s+\varepsilon} J_s^{\delta(1)} dB^{(\delta(1))}(t_1) \dots \int_s^{t_{j-2}} J_s^{\delta(j-1)} dB^{(\delta(j-1))}(t_{j-1})$$

$$\times \int_s^{t_{j-1}} (J_s^{\delta(j)} - J_{t_j}^{\delta(j)}) dB^{(\delta(j))}(t_j) \int_s^{t_j} J_{t_{j+1}}^{\delta(j+1)} dB^{(\delta(j+1))}(t_{j+1}) \dots \int_s^{t_{k-1}} J_{t_k}^{\delta(k)} dB^{(\delta(k))}(t_k).$$

By hypothesis, paths of J are locally Hölder continuous and by using the isometry property of Itô's integral we can deduce (3.3): as $\varepsilon \to 0$,

$$\mathrm{E}\left\{ \left[\int_0^t J_s^m \left(\frac{B_{s+\varepsilon} - B_s}{\sqrt{\varepsilon}} \right)^m ds - \int_0^t \left(\frac{Z_{s+\varepsilon} - Z_s}{\sqrt{\varepsilon}} \right)^m ds \right]^2 \right\} = O(\varepsilon^\alpha),\ \alpha > 0.$$

We need now the following simple modification of Lemma 3.1:

Lemma 3.3 *Let us made the same assumptions on the function f and on the process W as in Lemma 3.1. Moreover, we assume that $\{\tilde{W}_t : t \geq 0\}$ is another locally Hölder continuous paths process with same index γ_0 and assume that $\tilde{W}_\varepsilon^{(f)}(t)$ denotes the associated process to $\tilde{W}$ as in (3.2). If f is non-negative and if for each $t \geq 0$, as $\varepsilon \to 0$,*

$$\left\| W_\varepsilon^{(f)}(t) - \tilde{W}_\varepsilon^{(f)}(t) \right\|_{\mathrm{L}^2}^2 = O(\varepsilon^\alpha),\ \alpha > 0, \tag{3.10}$$

then, we have

$$\lim_{\varepsilon \to 0} \left(W_\varepsilon^{(f)}(t) - \tilde{W}_\varepsilon^{(f)}(t) \right) = 0 \tag{3.11}$$

almost surely, uniformly on every compact interval.

The proof is straightforward and we leave it to the reader. Using this result, we obtain that

$$\lim_{\varepsilon \to 0} \left[\int_0^t J_s^m \left(\frac{B_{s+\varepsilon} - B_s}{\sqrt{\varepsilon}} \right)^m ds - \int_0^t \left(\frac{Z_{s+\varepsilon} - Z_s}{\sqrt{\varepsilon}} \right)^m ds \right] = 0 \tag{3.12}$$

almost surely, uniformly on each compact interval. Combining (3.12) with (3.7), we get (2.3). ■

Proof of Corollary 2.3. To prove the first part, we set $Y_s = g''(B_s^H)$ and $f(x) = x^2$ in (2.2). Then, we obtain the existence of

$$\lim_{\varepsilon \to 0} \int_0^t g''(B_s^H) \frac{(B_{s+\varepsilon}^H - B_s^H)^2}{\varepsilon} ds, \text{ almost surely, uniformly on each compact interval.}$$

On the other hand, we can write, for $a, b \in \mathbb{R}$, $a < b$,

$$g(b) = g(a) + g'(a)(b-a) + \frac{g''(\theta)}{2}(b-a)^2,$$

with $\theta_{a,b} \in (a, b)$. Setting $a = B_s^H$ and $b = B_{s+\varepsilon}^H$, integrating in s on $[0, t]$ and dividing by ε we get:

$$\frac{1}{\varepsilon} \int_0^t (g(B_{s+\varepsilon}^H) - g(B_s^H)) ds = \frac{1}{\varepsilon} \int_0^t g'(B_s^H)(B_{s+\varepsilon}^H - B_s^H) ds + \frac{1}{2\varepsilon} \int_0^t g''(\theta_{B_s^H, B_{s+\varepsilon}^H})(B_{s+\varepsilon}^H - B_s^H)^2 ds.$$

By a simple change of variable we can transform the left-hand side as

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} g(B_s^H) ds - \frac{1}{\varepsilon} \int_0^\varepsilon g(B_s^H) ds,$$

which tends, as $\varepsilon \to 0$, almost surely uniformly on each compact interval toward $g(B_t^H) - g(0)$. The last term on the right-hand side of the previous equality converges, almost surely and uniformly on each compact interval and therefore the term which remains on the right-hand side is also forced to have a limit. The third part can be proved in a similar way.
Let us turn to the second part. By setting $Y_s = g^{(3)}(B_s^H)$ and $f(x) = x^3$ in (2.2), we obtain the existence of

$$\lim_{\varepsilon \to 0} \int_0^t g^{(3)}(B_s^H) \frac{(B_{s+\varepsilon}^H - B_s^H)^3}{\varepsilon} ds, \text{ almost surely, uniformly on each compact interval.}$$

On the other hand, by setting $Y_s \equiv 1$ and $f(x) = |x|^3$ in (2.2) we obtain that

$$\lim_{\varepsilon\to 0}\int_0^t \frac{|B^H_{s+\varepsilon} - B^H_s|^3}{\varepsilon}ds < +\infty, \text{ almost surely, } \forall t > 0.$$

Consequently, it suffices to use the following Taylor formula

$$g(b) = g(a) + \frac{g'(a) + g'(b)}{2}(b-a) - \frac{g^{(3)}(\theta)}{12}(b-a)^3$$

and the dominated convergence, in order to conclude as previously. ■

3.2 Proofs of the convergence in distribution

Proof of Theorem 2.4. First, let us explain the main ideas in the simpliest situation of the Brownian motion ($H = \frac{1}{2}$). In this case we are studying $M_T(t) = T^{-1/2}\int_0^{tT}(B_{s+1} - B_s)^m\,ds$ (see *Step 1* below) and we write it, thanks to succesive applications of Itô's formula and of the stochastic version of Fubini theorem, as $\int_0^{tT} R_T(s)dB_s$ plus a remainder which tends to zero in L^2, as $T \to \infty$. Then we can show that $\lim_{T\to\infty}\int_0^{tT} R_T(s)^2 ds = \mathrm{cst}.t$, hence, by Dubins-Schwarz theorem, we obtain that $M_T \to \sqrt{\mathrm{cst}.}\beta$, as $T \to \infty$, in the sense of finite dimensional time marginals. Finally we prove the tightness. Let us remark that similar technics have been used in [14], precisely in the proof of Proposition 3 (see also *Step 10* below).

For the fractional Brownian motion case ($0 < H < \frac{1}{2}$) technical difficulties appear because the kernel K in its moving average representation ($B^H_t = \int_0^t K(s,t)dB_s$) is singular at the points $s = 0$ and $s = t$. Again we split the proof in several steps.

Step 1. By the self-similarity of the fractional Brownian motion, that is $\{B^H_{ct} : t \geq 0\} \stackrel{\mathcal{L}}{=} \{c^H B^H_t : t \geq 0\}$, for all $c > 0$, we can see that

$$\{\frac{1}{\sqrt{\varepsilon}}\int_0^t\left(\frac{B^H_{s+\varepsilon} - B^H_s}{\varepsilon^H}\right)^m ds : t \geq 0\} \stackrel{\mathcal{L}}{=} \{\sqrt{\varepsilon}\int_0^{\frac{t}{\varepsilon}}\left(B^H_{s+1} - B^H_s\right)^m ds : t \geq 0\} = \{M_{\frac{1}{\varepsilon}}(t) : t \geq 0\},$$

where

$$M_T(t) := \frac{1}{\sqrt{T}}\int_0^{tT}\left(B^H_{s+1} - B^H_s\right)^m ds,\ t \geq 0. \tag{3.13}$$

Hence, to get (2.6) it suffices to prove that

$$\{M_T(t) : t \geq 0\} \xrightarrow{\mathcal{L}} \{\sqrt{c_{m,H}}\,\beta_t : t \geq 0\}, \text{ as } T \to \infty. \tag{3.14}$$

Moreover, this convergence is a consequence of the following two facts:

i) as $T \to \infty$, $\{M_T(t) : t \geq 0\} \to \{\sqrt{c_{m,H}}\,\beta_t : t \geq 0\}$ in law in sense of finite dimensional time marginals; (3.15)

ii) for $T \geq 1$, the family of distributions of processes M_T is tight. (3.16)

Step 2. Before proceeding with the proof of (3.15), let us show how the constant $c_{m,H}$ appears. We claim that, for each $t \geq 0$,

$$\lim_{T\to\infty} \mathrm{E}\left[M_T(t)^2\right] = c_{m,H}t. \tag{3.17}$$

Set $G_1 = B^H_{u+1} - B^H_u$, $G_2 = B^H_{v+1} - B^H_v$ and $\theta(u,v) = \mathrm{Cov}(G_1, G_2)$. We need to estimate the expectation of the product $G_1^m G_2^m$. Thanks to (2.5), we have

$$\mathrm{E}[G_1^m G_2^m] = \sum_{1\leq k,\ell\leq m} a_{k,m}a_{\ell,m}\mathrm{E}[H_k(G_1)H_\ell(G_2)] = \sum_{k=1}^m \frac{a_{k,m}^2}{k!}\theta(u,v)^k.$$

Replacing this in the expression of the second moment of $M_T(t)$, we obtain, noting also that $\theta(u,v) = \frac{1}{2}(|v-u+1|^{2H} + |v-u-1|^{2H} - 2|v-u|^{2H})$,

$$\begin{aligned}\mathrm{E}\left[M_T(t)^2\right] &= \frac{2}{T}\iint_{0\leq u<v\leq tT} dudv \sum_{k=1}^m \frac{a_{k,m}^2}{k!}\theta(u,v)^k \\ &= 2\int_0^t dy \sum_{k=1}^m \frac{a_{k,m}^2}{k!}\int_0^{T(t-y)} \left[(x+1)^{2H} + |x-1|^{2H} - 2x^{2H}\right]^k dx,\end{aligned}$$

by the change of variables $x = v-u, y = u/T$. Letting T goes to infinity we get on the right hand side of the previous equality $c_{m,H}t$.

Step 3. We proceed now to the first technical notation which will be useful in the next step. We write $M_T(t) = M_T(b) + M_T^{(b)}(t)$, where

$$M_T^{(b)}(t) := \frac{1}{\sqrt{T}}\int_{bT}^{tT}\left(B^H_{s+1} - B^H_s\right)^m ds,\ t\geq 0. \tag{3.18}$$

Choose an arbitrary, but fixed $\varrho > 0$. Let us note that, by (3.17), we can choose $b > 0$ small enough such that

$$\lim_{T\to\infty} \mathrm{E}\left[M_T(b)^2\right] \leq \varrho. \tag{3.19}$$

Step 4. Let us recall (see, for instance, [1], p. 122) that the fractional Brownian motion can be written as

$$B^H_t = A_t + \check{B}^H_t = \gamma_H\int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]dB_s + \gamma_H\int_0^t (t-s)^{H-\frac{1}{2}}dB_s,\ t\geq 0, \tag{3.20}$$

where, here and elsewhere we denote by γ_H the constant $\Gamma(H+\frac{1}{2})^{-1}$. We can write $M_T^{(b)}(t) = \check{M}_T^{(b)}(t) + D_T^{(b)}(t)$, where

$$\check{M}_T^{(b)}(t) := \frac{1}{\sqrt{T}}\int_{bT}^{tT}\left(\check{B}^H_{s+1} - \check{B}^H_s\right)^m ds,\ t\geq 0. \tag{3.21}$$

Since the process A has absolutely continuous trajectories and using the fact that A and $\check{B}$ are independent as stochastic integrals on disjoint intervals, it is not difficult to prove that, for each $t \geq 0$,

$$\lim_{T\to\infty} \mathrm{E}\left[D_T^{(b)}(t)^2\right] = \lim_{T\to\infty} \mathrm{E}\left[(M_T^{(b)}(t) - \check{M}_T^{(b)}(t))^2\right] = 0. \tag{3.22}$$

Step 5. By (3.21) and (3.20) we can write

$$\check{M}_T^{(b)}(t) = \frac{\gamma_H}{\sqrt{T}} \int_{bT}^{tT} \left(\int_0^{s+1} (s+1-u)^{H-\frac{1}{2}} dB_u - \int_0^s (s-u)^{H-\frac{1}{2}} dB_u \right)^m ds \tag{3.23}$$

$$= \frac{\gamma_H}{\sqrt{T}} \int_{bT}^{tT} \left(\int_0^s [(s+1-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}] dB_u + \int_s^{s+1} (s+1-u)^{H-\frac{1}{2}} dB_u \right)^m ds, \; t \geq 0.$$

We need to introduce a second technical notation. Let us denote:

$$\check{N}_T^{(b,c)}(t) := \frac{\gamma_H}{\sqrt{T}} \int_{bT}^{tT} \left(\int_{(s-c)\vee 0}^s [(s+1-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}] dB_u \right. \tag{3.24}$$

$$\left. + \int_s^{s+1} (s+1-u)^{H-\frac{1}{2}} dB_u \right)^m ds, \; t \geq 0,$$

where the positive constant c will be fixed and specified by the statement *a)* below. We shall prove successively the following statements:

a) for each $t \geq 0$, there exists $c > 0$ large enough, such that

$$\limsup_{T\to\infty} \mathrm{E}\left[(\check{M}_T^{(b)}(t) - \check{N}_T^{(b,c)}(t))^2 \right] \leq \mathrm{cst}.\varrho; \tag{3.25}$$

b) there exists two families of stochastic processes $\{R_T(t) : t \geq 0\}$ and $\{S_T(t) : t \geq 0\}$ such that, for each $t \geq 0$,

$$\check{N}_T^{(b,c)}(t) = \int_0^{tT} R_T(s) dB_s + S_T(t), \;\text{ with }\; \lim_{T\to\infty} \mathrm{E}[S_T(t)^2] = 0; \tag{3.26}$$

c) for each $t \geq 0$,

$$\lim_{T\to\infty} \mathrm{Var}\left(\int_0^{tT} R_T(s)^2 ds \right) = 0. \tag{3.27}$$

Step 6. Suppose for a moment that *a), b), c)* are proved and let us finish the proof of the convergence in law in sense of finite dimensional time marginals (3.15). First, we can write,

$$\int_0^{tT} R_T(s)^2 ds - c_{m,H}\, t = \{ \int_0^{tT} R_T(s)^2 ds - \mathrm{E}[\int_0^{tT} R_T(s)^2 ds] \} + \{ \mathrm{E}[\left(\int_0^{tT} R_T(s) dB_s \right)^2]$$

$$- \mathrm{E}[\check{M}_T^{(b)}(t)^2] \} + \{ \mathrm{E}[\check{M}_T^{(b)}(t)^2 - M_T^{(b)}(t)^2] \} + \{ \mathrm{E}[M_T^{(b)}(t)^2 - M_T(t)^2] \} + \{ \mathrm{E}\left[M_T(t)^2 \right] - c_{m,H}\, t \}.$$

By using (3.27) for the first term, (3.25)-(3.26) for the second term, (3.22) for the third term, (3.19) for the forth term and (3.17) for the fifth one, we obtain

$$\limsup_{T\to\infty} \mathrm{E}[(\int_0^{tT} R_T(s)^2 ds - c_{m,H}\, t)^2] \leq \mathrm{cst}.\varrho,$$

or equivalently, for each $t \geq 0$,

$$\limsup_{T\to\infty} \mathrm{E}[(a_{(T)}(t) - a(t))^2] \leq \mathrm{cst}.\varrho, \tag{3.28}$$

with notations $a_{(T)}(t) := \int_0^{tT} R_T(s)^2 ds$ and $a(t) := c_{m,H} t$. Second, we fix $d \in \mathbb{N}^*$ and $0 \le t_1 < t_2 < \ldots < t_d$ and we shall denote for any $\mathbf{u} \in \mathbb{R}^d$ and $f : \mathbb{R}_+ \to \mathbb{R}$, $\mathbf{u} \bullet f := \sum_{j=1}^d u_j f(t_j)$. We consider the characteristic functions:

$$|\mathrm{E}\left[\exp(i\mathbf{u} \bullet M_T)\right] - \mathrm{E}\left[\exp(i\mathbf{u} \bullet (\beta \circ a))\right]| \le \mathrm{E}\left[|\exp(i\mathbf{u} \bullet (M_T - M_T^{(b)})) - 1|\right]$$

$$+\mathrm{E}\left[|\exp(i\mathbf{u} \bullet (M_T^{(b)} - \check{M}_T^{(b)})) - 1|\right] + \mathrm{E}\left[|\exp(i\mathbf{u} \bullet (\check{M}_T^{(b)} - \check{N}_T^{(b,c)})) - 1|\right]$$

$$+\mathrm{E}\left[|\exp(i\mathbf{u} \bullet S_T) - 1|\right] + |\mathrm{E}[\exp(i\mathbf{u} \bullet \int_0^{\cdot T} R_T(s) dB_s)] - \mathrm{E}[\exp(i\mathbf{u} \bullet (\beta \circ a))]|.$$

By (3.19), (3.22), (3.25), (3.26) and using the classical inequality $|e^{ix} - 1| \le |x|$, we obtain, for T large enough

$$|\mathrm{E}\left[\exp(i\mathbf{u} \bullet M_T)\right] - \mathrm{E}\left[\exp(i\mathbf{u} \bullet (\beta \circ a))\right]| \le \mathrm{cst}.\varrho \tag{3.29}$$

$$+|\mathrm{E}[\exp(i\mathbf{u} \bullet \int_0^{\cdot T} R_T(s) dB_s)] - \mathrm{E}[\exp(i\mathbf{u} \bullet (\beta \circ a))]|.$$

By Dubins-Schwarz theorem, we can write, for each T, $\int_0^{\cdot T} R_T(s) dB_s = \beta_{(T)} \circ a_{(T)}$, with $\beta_{(T)}$ a one-dimensional standard Brownian motion starting from 0. Therefore, we have

$$|\mathrm{E}[\exp(i\mathbf{u} \bullet \int_0^{\cdot T} R_T(s) dB_s)] - \mathrm{E}[\exp(i\mathbf{u} \bullet (\beta \circ a))]| \tag{3.30}$$

$$\le 2\mathrm{P}(\|a_{(T)} - a\| > \delta) + \mathrm{E}\left[|\exp(i\mathbf{u} \bullet (\beta_{(T)} \circ a_{(T)})) - \exp(i\mathbf{u} \bullet \beta_{(T)} \circ a))| : \|a_{(T)} - a\| \le \delta\right]$$

$$\le \frac{2}{\delta^2} \mathrm{E}\left[\|a_{(T)} - a\|^2\right] + \|\mathbf{u}\| \, \mathrm{E}[\sup_{\|\mathbf{v}-\mathbf{w}\| \le \delta} \|(\beta_{v_1}, \ldots, \beta_{v_d}) - (\beta_{w_1}, \ldots, \beta_{w_d})\|].$$

Combining (3.29), (3.30) and letting $\varrho \to 0$, (3.15) follows.

Step 7. We verify (3.16), that is, the tightness of the family of distributions of processes M_T. It suffices to verify the classical Kolmogorov criterion (see [15], p. 489):

$$\sup_{T \ge 1} \mathrm{E}\left[(M_T(t) - M_T(s))^4\right] \le c_R |t - s|^2, \; \forall 0 \le s, t \le R. \tag{3.31}$$

Let $s, t \in [0, R]$. Then, by (3.13),

$$\mathrm{E}\left[(M_T(t) - M_T(s))^4\right] = \frac{1}{T^2} \iiiint_{[sT,tT]^4} \mathrm{E}\Big[(B^H_{u_1+1} - B^H_{u_1})^m (B^H_{u_2+1} - B^H_{u_2})^m (B^H_{u_3+1} - B^H_{u_3})^m$$

$$\times (B^H_{u_4+1} - B^H_{u_4})^m\Big] du_1 du_2 du_3 du_4 = \frac{1}{T^2} \iiiint_{[sT,tT]^4} \mathrm{E}\left[G_1^m G_2^m G_3^m G_4^m\right] du_1 du_2 du_3 du_4$$

where, as in *Step 2*, we denoted the standard Gaussian random variables $G_i = B^H_{u_i+1} - B^H_{u_i}$, $i = 1, 2, 3, 4$. Let us also denote $\theta_{ij} = \mathrm{Cov}(G_i, G_j)$, $i, j = 1, \ldots, 4$. We need to estimate the expectation of the product $G_1^m G_2^m G_3^m G_4^m$. By using (2.5), we get

$$\mathrm{E}\left[G_1^m G_2^m G_3^m G_4^m\right] = \sum_{k_1,k_2,k_3,k_4 \ge 1} a_{k_1,m} a_{k_2,m} a_{k_3,m} a_{k_4,m} \mathrm{E}\left[H_{k_1}(G_1) H_{k_2}(G_2) H_{k_3}(G_3) H_{k_4}(G_4)\right]$$

and we need to estimate $\mathrm{E}[H_{k_1}(G_1)H_{k_2}(G_2)H_{k_3}(G_3)H_{k_4}(G_4)]$. Using the result in [18], p. 210, we can write

$$\mathrm{E}\left[H_{k_1}(G_1)H_{k_2}(G_2)H_{k_3}(G_3)H_{k_4}(G_4)\right] \tag{3.32}$$
$$= \begin{cases} \frac{k_1!k_2!k_3!k_4!}{2^q q!}\sum_1 \theta_{i_1j_1}\ldots\theta_{i_qj_q}, & \text{if } k_1+k_2+k_3+k_4 = 2q \text{ and } 0\le k_1,k_2,k_3,k_4\le q \\ 0 & \text{otherwise,} \end{cases}$$

where $\sum_1$ is the sum over all indices $i_1, j_1, \ldots, i_q, j_q \in \{1,2,3,4\}$ such that $i_1 \neq j_1, \ldots, i_q \neq j_q$ and there are k_1 indices $1, \ldots, k_4$ indices 4. For instance $\mathrm{E}\left[H_1(G_1)H_1(G_2)H_1(G_3)H_1(G_4)\right] = \frac{1}{8}(\theta_{12}\theta_{34}+\theta_{13}\theta_{24}+\theta_{14}\theta_{23})$. Similarly, we can compute $\mathrm{E}\left[H_3(G_1)H_3(G_2)H_3(G_3)H_3(G_4)\right]$ in terms of θ_{ij} and so on. Since G_i have variance 1, we deduce, using the conditions on the indices appearing in (3.32), that

$$\mathrm{E}\left|H_{k_1}(G_1)H_{k_2}(G_2)H_{k_3}(G_3)H_{k_4}(G_4)\right| \le \text{cst.} \sum_{\{i,j\}\neq\{k,\ell\}} |\theta_{ij}|\,|\theta_{k\ell}|$$

Therefore, to get (3.31), we need to consider the following two type of terms: $\{i,j\}\cap\{k,\ell\} = \emptyset$, for instance $i=1, j=2, k=3, \ell=4$, or $\{i,j\}\cap\{k,\ell\}\neq\emptyset$, for instance $i=1, j=2, k=1, \ell=3$. Clearly, by simple change of variables,

$$\frac{1}{T^2}\iiiint_{[sT,tT]^4} |\theta_{12}|\,|\theta_{34}| du_1du_2du_3du_4 = \left(\frac{1}{T}\iint_{[sT,tT]^2} |\theta_{12}| du_1du_2\right)^2$$
$$= \left(\frac{1}{2T}\iint_{[sT,tT]^2} \left||u_2-u_1+1|^{2H} + |u_2-u_1-1|^{2H} - 2|u_2-u_1|^{2H}\right| du_1du_2\right)^2$$
$$= \left(\frac{1}{2T}\int_{sT}^{tT} du_1 \int_0^{(t-s)T} \left|(x+1)^{2H} + |x-1|^{2H} - 2x^{2H}\right| dx\right)^2 \le \lambda(t-s)^2,$$

where $\lambda := (\frac{1}{2}\int_0^\infty |(x+1)^{2H} + |x-1|^{2H} - 2x^{2H}| dx)^2$, and, similarly,

$$\frac{1}{T^2}\iiiint_{[sT,tT]^4} |\theta_{12}|\,|\theta_{13}| du_1du_2du_3du_4$$
$$= \frac{t-s}{4T}\iiint_{[sT,tT]^3} \left||u_2-u_1+1|^{2H} + |u_2-u_1-1|^{2H} - 2|u_2-u_1|^{2H}\right|$$
$$\times \left||u_3-u_1+1|^{2H} + |u_3-u_1-1|^{2H} - 2|u_3-u_1|^{2H}\right| du_1du_2du_3 \le \lambda(t-s)^2.$$

Hence (3.31) is verified so the family of distributions of processes M_T is tight.
The proof of Theorem 2.4 will be finished once we prove statements *a)-c)* in *Step 5.*

Step 8. We prove (3.25) and at the same time we precise the choice of the constant c. For notational convenience we will drop superscripts "(b)" or "(b,c)" during the proof. Using again (3.23) and (3.24) we can write

$$\check{M}_T(t) - \check{N}_T(t) = \sum_{k=1}^{m} \binom{m}{k} \check{P}_T^{(k)}(t), \tag{3.33}$$

with

$$\check{P}_T^{(k)}(t) = \frac{\gamma_H}{\sqrt{T}} \int_{bT}^{tT} ds \left[\int_0^{(s-c)\vee 0} \mathcal{K}(s,u) dB_u \right]^k \left[\int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u) dB_u \right]^{m-k}$$

and where we denoted

$$\mathcal{K}(s,u) := \begin{cases} (s+1-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}, & \text{if } 0 \le u \le s \\ (s+1-u)^{H-\frac{1}{2}}, & \text{if } s \le u \le s+1 \end{cases}. \tag{3.34}$$

We shall prove that the second moment of each term in (3.33) can be made small enough and then (3.25) will follows.

Step 9. We can write

$$\mathrm{E}\left[\check{P}_T^{(k)}(t)^2\right] = \frac{2\gamma_H^2}{T} \left[\iint_{bT<s<s'<s+c+1} \Delta(s,s') ds ds' + \iint_{bT<s, s+c+1<s'<tT} \Delta(s,s') ds ds' \right] \tag{3.35}$$

where

$$\Delta(s,s') = \mathrm{E}\left\{ \left[\int_0^{(s-c)\vee 0} \mathcal{K}(s,u) dB_u \right]^k \left[\int_0^{(s'-c)\vee 0} \mathcal{K}(s',u) dB_u \right]^k \right. \tag{3.36}$$

$$\left. \times \left[\int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u) dB_u \right]^{m-k} \left[\int_{(s'-c)\vee 0}^{s'+1} \mathcal{K}(s',u) dB_u \right]^{m-k} \right\}$$

and we shall study each term in (3.35). We need the following lemma:

Lemma 3.4 *There exists a positive constant κ such that, for all $s \ge 0$,*

$$\mathrm{E}\left[\left(\int_0^{s+1} \mathcal{K}(s,u) dB_u \right)^2 \right] = \int_0^{s+1} \mathcal{K}(s,u)^2 du \le \kappa. \tag{3.37}$$

Proof. By (3.34) and change of variables,

$$\int_0^{s+1} \mathcal{K}(s,u)^2 du = \int_0^s [(s+1-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}]^2 du + \int_s^{s+1} (s+1-u)^{2H-1} du =$$

$$\int_0^s [(v+1)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}}]^2 dv + \int_0^1 v^{2H-1} dv \le \int_0^\infty [(v+1)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}}]^2 dv + \int_0^1 v^{2H-1} dv = \kappa < \infty.$$

■

By using Cauchy-Schwarz inequality and (3.37), we can prove that there exists $c > 0$ large enough, such that

$$\frac{2\gamma_H^2}{T} \iint_{bT<s<s'<s+c+1} \Delta(s,s') ds ds' \le \mathrm{cst.} \left(\int_c^\infty [(v+1)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}}]^2 dv \right)^k$$

$$\le \mathrm{cst.} c^{1-k(2-2H)} \le \mathrm{cst.} \varrho,$$

since $1-k(2-2H)<0$, for all $k\geq 1$ (recall that $H<\frac{1}{2}$).

Consider now the second term in (3.35). Since m is an odd integer, the expectation equals zero for each even integer k, by independence of stochastic integrals on disjoint intervals. Hence we need to consider only odd integers k. We can write, for $bT<s, s+c+1<s'<tT$, using again the independence and (3.37),

$$\Delta(s,s') \leq \text{cst.E}\left\{X_1^k(X_2+Y_1)^k Y_2^{m-k}\right\}, \tag{3.38}$$

where

$$X_1 := \int_0^{(s-c)\vee 0} \mathcal{K}(s,u)dB_u,\ X_2 := \int_0^{(s-c)\vee 0} \mathcal{K}(s',u)dB_u$$

and

$$Y_1 := \int_{(s-c)\vee 0}^{s'-c} \mathcal{K}(s',u)dB_u,\ Y_2 := \int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u)dB_u.$$

We state the following simple result:

Lemma 3.5 *Let m,k be odd integers and assume that $(X_1,X_2),(Y_1,Y_2)$ are two independent centered Gaussian random vectors. Set $\theta=\mathrm{Cov}(X_1,X_2)$. Then*

$$\left|\mathrm{E}\left[X_1^k(X_2+Y_1)^k Y_2^{m-k}\right]\right| \leq \text{cst.}\left(|\theta|+\ldots+|\theta|^k\right).$$

Let us return to the study of the second term in (3.35). Using the upper result and (3.38), it can be bounded as follows:

$$\frac{2\gamma_H^2}{T}\iint_{bT<s,s+c+1<s'<tT}\Delta(s,s')dsds' \leq \frac{\text{cst.}}{T}\iint_{bT<s,s+c+1<s'<tT} dsds'\left(|\theta|+\ldots+|\theta|^k\right),$$

where, with the same notation as in previous lemma,

$$\begin{aligned}\theta &:= \int_0^{(s-c)\vee 0} du[(s+1-u)^{H-\frac{1}{2}}-(s-u)^{H-\frac{1}{2}}][(s'+1-u)^{H-\frac{1}{2}}-(s'-u)^{H-\frac{1}{2}}]\\ &= \int_{c\wedge s}^{s} dv[(v+1)^{H-\frac{1}{2}}-v^{H-\frac{1}{2}}][(v+1+s'-s)^{H-\frac{1}{2}}-(v+s'-s)^{H-\frac{1}{2}}].\end{aligned}$$

It suffices to show that there exists $c>0$ large enough such that

$$\frac{1}{T}\iint_{bT<s,s+c+1<s'<tT}|\theta|^j dsds' \leq \text{cst.}\varrho,\ j=1,\ldots,k. \tag{3.39}$$

If $j=1$, by successive change of variables $y=\frac{s}{T}$ and $x=s'-yT$, we get

$$\begin{aligned}\frac{1}{T}\int_{bT}^{tT}ds\int_{s+c+1}^{tT}|\theta|ds' &= \int_b^t dy\int_{c+1}^{(t-y)T}dx\\ &\times\left|\int_{c\vee yT}^{yT}dv[(v+1)^{H-\frac{1}{2}}-v^{H-\frac{1}{2}}][(x+v+1)^{H-\frac{1}{2}}-(x+v)^{H-\frac{1}{2}}]\right|\\ &\leq \text{cst.}\int_0^\infty dx\int_c^\infty dv[v^{H-\frac{1}{2}}-(v+1)^{H-\frac{1}{2}}][(x+v)^{H-\frac{1}{2}}-(x+v+1)^{H-\frac{1}{2}}]\to_{c\to\infty}0.\end{aligned}$$

If $j \geq 2$, we make a similar reasoning. Hence (3.39) is verified and (3.25) follows.

Step 10. We prove now the statement *b)* in *Step 5*, that is (3.26). Again, we will drop the superscripts "(b,c)". Using (3.34), (3.24) can be written as

$$\check{N}_{T}(t) = \frac{\gamma_H}{\sqrt{T}} \int_{bT}^{tT} ds \left(\int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u) dB_u \right)^m .$$

First we assume that $m = 3$. By using succesively the classical Itô's formula and the stochastic version of Fubini theorem, we can write

$$\left(\int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u) dB_u \right)^3 = 6 \int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u) dB_u \int_{(s-c)\vee 0}^{u} \mathcal{K}(s,v) dB_v \int_{(s-c)\vee 0}^{v} \mathcal{K}(s,w) dB_w$$

$$+3 \int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,u) dB_u \int_{(s-c)\vee 0}^{s+1} \mathcal{K}(s,v)^2 dv. \tag{3.40}$$

Remark: The previous equality (3.40) can be also written in terms of multiple stochastic integrals, as follows:

$$\left(\int_E \mathcal{K}(s,\cdot) dB \right)^3 = I_1(\mathcal{K}(s,\cdot))^3 = I_3(\mathcal{K}(s,\cdot)^{\otimes 3}) + 3\, I_1(\mathcal{K}(s,\cdot)^{\otimes 2} \times_{(1)} \mathcal{K}(s,\cdot))$$

$$= \int_{E^3} \mathcal{K}(s,\cdot)^{\otimes 3} dB + 3 \int_E \mathcal{K}(s,\cdot) dB \int_E \mathcal{K}(s,\cdot)^2 d\lambda, \text{ with } E = [(s-c)\vee 0, s+1].$$

Here λ denotes the Lebesgue measure and the multiple integrals I_p, the tensor product $f \otimes g$ and the contractions $f \times_{(p)} g$ are defined in [11] or [13]. □

Therefore, by applying the stochastic version of Fubini theorem in (3.40), we get $\check{N}_T(t) = \int_0^{tT} R_T(u) dB_u + S_T(t)$, where

$$R_T(u) := \frac{6\gamma_H}{\sqrt{T}} \int_{(u-1-c)\vee 0}^{u} dB_v \int_{(u-1-c)\vee 0}^{v} dB_w \int_{(u-1)\vee bT}^{(w+c)\wedge tT} ds \mathcal{K}(s,u)\mathcal{K}(s,v)\mathcal{K}(s,w) \tag{3.41}$$

$$+\frac{3\gamma_H}{\sqrt{T}} \left[\int_{(u-1-c)\vee 0}^{u} dv \int_{(u-1)\vee bT}^{(v+c)\wedge tT} ds \mathcal{K}(s,u)\mathcal{K}(s,v)^2 + \int_u^{u+c+1} dv \int_{(v-1)\vee bT}^{(u+c)\wedge tT} ds \mathcal{K}(s,v)^2 \mathcal{K}(s,u) \right].$$

It is not difficult to prove that $\mathrm{E}[S_T(t)^2] \leq (\text{cst.}/T) \to 0$, as $T \to \infty$, by using successively the stochastic version of Fubini theorem, (3.37) and Jensen inequality. Hence (3.26) is proved for $m = 3$. For m an odd integer strictly bigger than 3, (3.26) is obtained by using Lemma 3.2 and a similar reasoning as previously (using eventually the notation with multiple integrals, as in the previous remark).

Step 11. To end the proof of Theorem 2.4 we need to verify the statement *c)* in *Step 5*, that is (3.27). Assume that $m = 3$, the general case being similar (by using Lemma 3.2). By (3.41) we can write

$$R_T(u) = \frac{1}{\sqrt{T}} \left\{ \int_{(u-1-c)\vee 0}^{u} \mathfrak{B}(u,v) dB_v + \mathfrak{C}(u) \right\},$$

where we use the following notations:

$$\mathfrak{B}(u,v) = 6\gamma_{_H} \int_{(u-1-c)\vee 0}^{v} dB_w \int_{(u-1)\vee bT}^{(w+c)\wedge tT} ds \mathcal{K}(s,u)\mathcal{K}(s,v)\mathcal{K}(s,w) = \int_{(u-1-c)\vee 0}^{v} \mathfrak{D}(u,v,w) dB_w$$

and

$$\mathfrak{C}(u) = 3\gamma_{_H} \left[\int_{(u-1-c)\vee 0}^{u} dv \int_{(u-1)\vee bT}^{(v+c)\wedge tT} ds \mathcal{K}(s,u)\mathcal{K}(s,v)^2 \right.$$
$$\left. + \int_{u}^{u+c+1} dv \int_{(v-1)\vee bT}^{(u+c)\wedge tT} ds \mathcal{K}(s,v)^2 \mathcal{K}(s,u) \right].$$

Firstly, by Itô formula and secondly, by the stochastic version of Fubini theorem we can write

$$\int_0^{tT} R_{_T}(u)^2 du - \mathrm{E}\left[\int_0^{tT} R_{_T}(u)^2 du\right] = \frac{2}{T}\left[\int_0^{tT} du \mathfrak{C}(u) \int_{(u-1-c)\vee 0}^{u} \mathfrak{B}(u,v) dB_v\right.$$
$$\left. + \int_0^{tT} du \int_{(u-1-c)\vee 0}^{u} \mathfrak{B}(u,v) dB_v \int_{(u-1-c)\vee 0}^{v} \mathfrak{B}(u,z) dB_z \right] = \frac{2}{T}\int_0^{tT} (\mathfrak{E}(v) + \mathfrak{F}(v)) dB_v.$$

Indeed, for instance the first term can be written as follows:

$$\int_0^{tT} dB_v \int_v^{(v+c+1)\wedge tT} du \mathfrak{C}(u)\mathfrak{B}(u,v) = \int_0^{tT} dB_v \int_v^{(v+c+1)\wedge tT} du \mathfrak{C}(u)$$
$$\times \int_{(u-1-c)\vee 0}^{v} \mathfrak{D}(u,v,w) dB_w = \int_0^{tT} dB_v \int_{(v-1-c)\vee 0}^{v} dB_w \int_v^{(w+c+1)\wedge tT} du \mathfrak{C}(u)\mathfrak{D}(u,v,w).$$

Hence

$$\mathrm{Var}\left[\int_0^{tT} R_{_T}(u)^2 du\right] \le \frac{\mathrm{cst.}}{T^2}\int_0^{tT} \mathrm{E}\left[\mathfrak{E}(v)^2 + \mathfrak{F}(v)^2\right] dv.$$

By isometry formula we obtain

$$\frac{1}{T^2}\int_0^{tT} \mathrm{E}\left[\mathfrak{E}(v)^2\right] dv = \frac{1}{T^2}\int_0^{tT} dv \int_{(v-1-c)\vee 0}^{v} dw \left[\int_v^{(w+c+1)\wedge tT} du \mathfrak{C}(u)\mathfrak{D}(u,v,w)\right]^2.$$

Using again the stochastic version of Fubini theorem we can write

$$\frac{1}{3\gamma_{_H}}\mathfrak{C}(u) = \int_{(u-1)\vee bT}^{(u+c)\wedge tT} \mathcal{K}(s,u) ds \int_{(s-c)\vee 0}^{u} \mathcal{K}(s,v)^2 dv \le \kappa \int_{(u-1)\vee bT}^{(u+c)\wedge tT} \mathcal{K}(s,u) ds,$$

by (3.37). Hence, by using this last inequality, the stochastic version of Fubini theorem one more time, (3.37) and also Jensen inequality, we get $\frac{1}{T^2}\int_0^{tT} \mathrm{E}\left[\mathfrak{E}(v)^2\right] dv \le \mathrm{cst.}/T$. We can prove a similar bound for the term containing $\mathrm{E}[\mathfrak{F}(v)^2]$. The convergence in the statement *c)* is now established.

The proof of Theorem 2.4 is now complete for $0 < H < \frac{1}{2}$.

For $H = \frac{1}{2}$, the proof can be performed in a similar way with several simplifications of technical order (for instance, there are no more singularities at the extremities points 0 and t, so we do not need to introduce neither parameters b and c, nor $\breve{B}$, there are no more technical estimates). Clearly, one uses the same ideas given at the begining of the proof and details are left to the reader. ■

Proof of Theorem 2.5. Once again we split the proof in several steps. As usual, by localization, we assume that $\sigma, \sigma', \sigma''$ are bounded.

Step 1. Let us introduce the following processes:

$$\mathcal{X}_\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}} \int_0^t \sigma(B_s)^m \left(\frac{B_{s+\varepsilon} - B_s}{\sqrt{\varepsilon}} \right)^m ds,\, t \geq 0 \tag{3.42}$$

and

$$\mathcal{Y}_\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}} \left[\int_0^t \left(\frac{Z_{s+\varepsilon} - Z_s}{\sqrt{\varepsilon}} \right)^m ds - \int_0^t \sigma(B_s)^m \left(\frac{B_{s+\varepsilon} - B_s}{\sqrt{\varepsilon}} \right)^m ds \right],\, t \geq 0. \tag{3.43}$$

We shall prove that, as $\varepsilon \to 0$,

$$\{\mathcal{X}_\varepsilon(t) : t \geq 0\} \xrightarrow{\mathcal{L}} \left\{ \sqrt{c_{m,\frac{1}{2}}} \int_0^t \sigma(\beta_s^{(1)})^m d\beta_s^{(2)} : t \geq 0 \right\} \tag{3.44}$$

and, for each $T \in (0, \infty)$,

$$\lim_{\varepsilon \to 0} \mathrm{E}[\sup_{t \in [0,T]} |\mathcal{Y}_\varepsilon(t)|^2] = 0. \tag{3.45}$$

By (3.44) and (3.45) we deduce (2.7) using the following (classical) simple result concerning the convergence in distribution of a sum of two stochastic processes:

Lemma 3.6 *Consider $\{\mathcal{X}_\varepsilon(t) : t \geq 0\}$ and $\{\mathcal{Y}_\varepsilon(t) : t \geq 0\}$ two families of continuous real stochastic processes, starting from 0, such that, as $\varepsilon \to 0$, $\mathcal{X}_\varepsilon$ converges to $\mathcal{X}$ in law as processes and, for each $T \in (0,\infty)$, $\mathrm{E}[\sup_{t\in[0,T]} |\mathcal{Y}_\varepsilon(t)|^2] \to 0$. Then, as $\varepsilon \to 0$, $\mathcal{X}_\varepsilon + \mathcal{Y}_\varepsilon$ converges to $\mathcal{X}$ in law as processes.*

Proof. It is a classical argument to show that, as $\varepsilon \to 0$, $\mathcal{X}_\varepsilon + \mathcal{Y}_\varepsilon$ converges to $\mathcal{X}$ in law in sense of finite dimensional time marginals. Hence we need to verify the tightness for the family of processes $\{\mathcal{X}_\varepsilon + \mathcal{Y}_\varepsilon\}_{\varepsilon > 0}$ (see also [15], pp. 488). For a continuous function $g : [0,\infty) \to \mathbb{R}$ we denote by $\rho^T(g;\delta) := \sup\{|g(t) - g(s)| : 0 \leq s,t \leq T \text{ with } |s-t| \leq \delta\}$ its modulus of continuity. Since the process $\mathcal{X}_\varepsilon$ starts from 0, its convergence in law is equivalent to the following version of Prohorov's criterion:

$$\text{for each } \eta, T > 0,\ \lim_{\delta\to 0} \limsup_{\varepsilon \to 0} \mathrm{P}\left(\rho^T(\mathcal{X}_\varepsilon;\delta) > \eta\right) = 0. \tag{3.46}$$

Clearly, by Markov inequality,

$$\mathrm{P}\left(\rho^T(\mathcal{X}_\varepsilon + \mathcal{Y}_\varepsilon;\delta) > \eta\right) \leq \mathrm{P}(\rho^T(\mathcal{X}_\varepsilon;\delta) > \frac{\eta}{2}) + \frac{16}{\eta^2}\mathrm{E}[\sup_{t\in[0,T]} |\mathcal{Y}_\varepsilon(t)|^2].$$

The conclusion follows by using again (3.46) and the L^2-convergence of $\mathcal{Y}_\varepsilon$. ■

Step 2. We prove here (3.44). By Theorem 2.4, (2.7) is true for the martingale $Z = B$. This means that, for an odd integer $m \geq 3$, as $\varepsilon \to 0$,

$$\{\mathtt{N}_\varepsilon(t) : t \geq 0\} := \left\{ \frac{1}{\sqrt{\varepsilon}} \int_0^t \left(\frac{B_{s+\varepsilon} - B_s}{\sqrt{\varepsilon}} \right)^m ds : t \geq 0 \right\} \xrightarrow{\mathcal{L}} \{\sqrt{c_{m,\frac{1}{2}}}\, \beta_t : t \geq 0\}. \tag{3.47}$$

We will write that $\mathtt{N}_\varepsilon(t) = \mathtt{M}_\varepsilon(t) + \kappa_1 B_t + S_\varepsilon(t)$, $t \geq 0$, with κ_1 an explicit real constant, $\mathtt{M}_\varepsilon$ a martingale and, for each $t \geq 0$, $\lim_{\varepsilon\to 0} \mathrm{E}[S_\varepsilon(t)^2] = 0$. Moreover, we shall prove that:

(i) for each $t \geq 0$, $\lim_{\varepsilon \to 0} [\mathtt{M}_\varepsilon, \mathtt{M}_\varepsilon](t) = \kappa_2^2 t$ in L^2 so in probability (with κ_2 a positive constant);

(ii) for each $t \geq 0$, $\lim_{\varepsilon \to 0} [B, \mathtt{M}_\varepsilon](t) = 0$ in probability;

(iii) for each $t \geq 0$, $\lim_{\varepsilon \to 0} [B, \mathtt{M}_\varepsilon] \left([\mathtt{M}_\varepsilon, \mathtt{M}_\varepsilon]^{-1}(t) \right) = 0$ in probability.

Before proving *(i)-(iii)* let us finish the proof of (2.7). Let us denote β_ε the Dubins-Schwarz Brownian motion associated to $\mathtt{M}_\varepsilon$. *(i)-(iii)* and the asymptotic version of Knight's theorem (see [15], pp. 495-496) imply that, as $\varepsilon \to 0$,

$$\{(B_t, \beta_\varepsilon(t)) : t \geq 0\} \xrightarrow{\mathcal{L}} \{(\beta_t^{(1)}, \beta_t^{(2)}) : t \geq 0\}.$$

By *(i)* we deduce

$$\{(B_t, \mathtt{M}_\varepsilon(t)) : t \geq 0\} \xrightarrow{\mathcal{L}} \{(\beta_t^{(1)}, \kappa_2 \beta_t^{(2)}) : t \geq 0\},$$

where $\{(\beta_t^{(1)}, \beta_t^{(2)}) : t \geq 0\}$ denotes a two-dimensional Brownian motion starting from $(0,0)$. Since σ is continuous we deduce, as $\varepsilon \to 0$,

$$\{(\sigma(B_t)^m, \mathtt{M}_\varepsilon(t) + \kappa_1 B_t) : t \geq 0\} \xrightarrow{\mathcal{L}} \{(\sigma(\beta_t^{(1)})^m, \kappa_1 \beta_t^{(1)} + \kappa_2 \beta_t^{(2)}) : t \geq 0\},$$

and, thanks to Lemma 3.6, as $\varepsilon \to 0$,

$$\{(\sigma(B_t)^m, \mathtt{N}_\varepsilon(t)) : t \geq 0\} \xrightarrow{\mathcal{L}} \{(\sigma(\beta_t^{(1)})^m, \kappa_1 \beta_t^{(1)} + \kappa_2 \beta_t^{(2)}) : t \geq 0\}.$$

Moreover, $\forall t \geq 0$, $\forall J$ predictable process bounded by 1, $\forall \varepsilon > 0$,

$$\begin{aligned}
\mathrm{P}\left(\left|\int_0^t J_s d\mathtt{N}_\varepsilon(s)\right| > R \right) &\leq \frac{1}{R^2} \mathrm{E}\left\{ \left| \int_0^t J_s d\mathtt{N}_\varepsilon(s) \right|^2 \right\} = \frac{1}{R^2} \frac{1}{\varepsilon^{m+1}} \mathrm{E}\left\{ \left| \int_0^t J_s (B_{s+\varepsilon} - B_s)^m ds \right|^2 \right\} \\
&= \frac{2}{R^2} \frac{1}{\varepsilon^{m+1}} \iint_{0<s<s'<t} ds ds' \mathrm{E}\left[J_s J_{s'} (B_{s+\varepsilon} - B_s)^m (B_{s'+\varepsilon} - B_{s'})^m \right] \\
&\leq \frac{\text{cst.}}{R^2} \frac{1}{\varepsilon^{m+1}} \iint_{0<s<s'<s+\varepsilon<t} ds ds' \mathrm{E}\left[(B_{s+\varepsilon} - B_s)^{2m} (B_{s'+\varepsilon} - B_{s'})^{2m} \right]^{\frac{1}{2}} \\
&+ \frac{2}{R^2} \frac{1}{\varepsilon^{m+1}} \iint_{0<s<s+\varepsilon<s'<t} ds ds' \mathrm{E}\left[J_s J_{s'} (B_{s+\varepsilon} - B_s)^m \right] \mathrm{E}\left[(B_{s'+\varepsilon} - B_{s'})^m \right].
\end{aligned}$$

by Cauchy-Schwarz inequality for the first term and by independence for the second term,

$$\leq \frac{\text{cst.}}{R^2} \frac{1}{\varepsilon} \iint_{0<s<s'<s+\varepsilon<t} ds ds' + 0 \leq \frac{\text{cst.}}{R^2}.$$

Therefore, using the result concerning the convergence in distribution of stochastic integrals (see [12], p. 125), we obtain, as $\varepsilon \to 0$,

$$\left\{ \int_0^t \sigma(B_s)^m d\mathtt{N}_\varepsilon(s) : t \geq 0 \right\} \xrightarrow{\mathcal{L}} \left\{ \int_0^t \sigma(\beta_s^{(1)})^m (\kappa_1 d\beta_s^{(1)} + \kappa_2 d\beta_s^{(2)}) : t \geq 0 \right\},$$

which is (3.44).

Step 3. We shall prove the decomposition of $\mathtt{N}_\varepsilon$ and *(i)-(iii)*. As previously, we write down the proof for $m=3$. By using Itô's formula and the stochastic version of Fubini theorem, we can write

$$\mathtt{N}_\varepsilon(t) = \frac{3}{\varepsilon^2}\int_0^t ds \left[2\int_s^{s+\varepsilon} dB_u \int_s^u dB_v \int_s^v dB_w + \int_s^{s+\varepsilon} dB_u \int_s^u dv + \int_s^{s+\varepsilon} dv \int_s^v dB_v\right]$$

$$= \frac{3}{\varepsilon^2}\int_0^{t+\varepsilon} dB_u \left[2\int_{(u-\varepsilon)\vee 0}^u dB_v \int_{(u-\varepsilon)\vee 0}^v dB_w \int_{(u-\varepsilon)\vee 0}^{w\wedge t} ds + \int_{(u-\varepsilon)\vee 0}^u dv \int_{(u-\varepsilon)\vee 0}^{v\wedge t} ds \right.$$

$$\left. + \int_u^{u+\varepsilon} dv \int_{(v-\varepsilon)\vee 0}^{u\wedge t} ds\right] =: \mathtt{M}_\varepsilon(t) + \kappa_1 B_t + S_\varepsilon(t),$$

with

$$\mathtt{M}_\varepsilon(t) = \frac{6}{\varepsilon^2}\int_0^t dB_u \int_{(u-\varepsilon)\vee 0}^u dB_v \int_{(u-\varepsilon)\vee 0}^v dB_w \int_{(u-\varepsilon)\vee 0}^w ds =: \int_0^t R_\varepsilon(u) dB_u,$$

with $\mathrm{E}[\mathtt{M}_\varepsilon(t)^2] = 3t + O(\varepsilon)$, as $\varepsilon \to 0$. To verify the L^2-convergence of S_ε toward 0 in this case it is a simple computation (for general m we proceed as in *Step 10* of the proof of Theorem 2.4). To verify *(i)* we write

$$\int_0^t R_\varepsilon(u)^2 du - \kappa_2^2 t = \{\int_0^t R_\varepsilon(u)^2 du - \mathrm{E}[\int_0^t R_\varepsilon(u)^2 du]\} + \{\mathrm{E}[\mathtt{M}_\varepsilon(t)^2] - \kappa_2^2 t\}$$

Here we choose $\kappa_2 = \sqrt{3}$ such that the second term tends to zero as $\varepsilon \to 0$. On the other hand, the first term equals $\mathrm{Var}\left(\int_0^t R_\varepsilon(u)^2 du\right)$ and tends to zero as $\varepsilon \to 0$, as we can see by a similar reasoning in proving (3.27).
Clearly, by the stochastic version of Fubini theorem,

$$[B, \mathtt{M}_\varepsilon](t) = \frac{6}{\varepsilon^2}\int_0^t dB_v \int_{(v-\varepsilon)\vee 0}^v dB_w \int_w^t du \int_{(u-\varepsilon)\vee 0}^w ds.$$

Then, by isometry formula and Jensen inequality,

$$\mathrm{E}\left([B, \mathtt{M}_\varepsilon]^2(t)\right) \le \frac{\mathrm{cst.}}{\varepsilon^2}\int_0^t dv \int_{(v-\varepsilon)\vee 0}^v dw \int_w^t du \int_{(u-\varepsilon)\vee 0}^w ds \le \mathrm{cst.}\varepsilon$$

and the convergence in probability *(ii)* follows. Moreover, since $\lim_{\varepsilon\to 0}[\mathtt{M}_\varepsilon, \mathtt{M}_\varepsilon](t) = \mathrm{cst.}t$, we can obtain *(iii)*.

Step 4. We turn now to the proof of (3.45). Again by Lemma 3.2, we split

$$\mathcal{Y}_\varepsilon(t) := \frac{1}{\varepsilon^{\frac{m+1}{2}}}\int_0^t ds \sum_{\delta\in\mathcal{P}, n(\delta)=m} c(\delta)\left[I^{(Z)}_{s,s+\varepsilon,\delta} - \sigma(B_s)^m I^{(B)}_{s,s+\varepsilon,\delta}\right] =: \sum_{\delta\in\mathcal{P}, n(\delta)=m} \mathcal{Y}^{(\delta)}_\varepsilon(t)$$

Moreover, we can write $\mathcal{Y}^{(\delta)}_\varepsilon(t) = \sum_{j=1}^{k(\delta)} \mathcal{Y}^{(\delta,j)}_\varepsilon(t)$, where

$$\mathcal{Y}^{(\delta,j)}_\varepsilon(t) := \frac{1}{\varepsilon^{\frac{m+1}{2}}}\int_0^t ds \int_s^{s+\varepsilon} \sigma(B_s)^{\delta(1)} dB^{(\delta(1))}(t_1) \ldots \int_s^{t_{j-2}} \sigma(B_s)^{\delta(j-1)} dB^{(\delta(j-1))}(t_{j-1})$$

$$\times \int_s^{t_{j-1}} [\sigma(B_s)^{\delta(j)} - \sigma(B_{t_j})^{\delta(j)}] dB^{(\delta(j))}(t_j)$$

$$\times \int_s^{t_j} \sigma(B_{t_{j+1}})^{\delta(j+1)} dB^{(\delta(j+1))}(t_{j+1}) \dots \int_s^{t_{k-1}} \sigma(B_{t_k})^{\delta(k)} dB^{(\delta(k))}(t_k).$$

Fix $\delta \in \mathcal{P}$ such that $n(\delta) = m$ and set $i_0 = \inf\{k \geq 1 : \delta(k) = 1\}$. We need to prove that, for each $j \in \{1, \dots, k(\delta)\}$, for each $T \in (0, \infty)$,

$$\lim_{\varepsilon \to 0} \mathrm{E}[\sup_{t \in [0,T]} |\mathcal{Y}_\varepsilon^{(\delta,j)}(t)|^2] = 0. \tag{3.48}$$

In the following we will distinguish four types of terms.

Step 5. Assume that $i_0 = 1$. We illustrate the proof of (3.48) for $m = 3$, $\delta = (1, 2)$ and $j = 1$, the general case being similar. In this case

$$\mathcal{Y}_\varepsilon^{((1,2),1)}(t) = \frac{1}{\varepsilon^2} \int_0^t ds \int_s^{s+\varepsilon} [\sigma(B_u) - \sigma(B_s)] dB_u \int_s^u \sigma(B_v)^2 dv.$$

Thanks to the stochastic version of Fubini theorem and also using Burkholder-Davis-Gundy and Jensen inequalities,

$$\mathrm{E}[\sup_{t \in [0,T]} |\mathcal{Y}_\varepsilon^{((1,2),1)}(t)|^2] \leq \frac{\text{cst.}}{\varepsilon^4} \int_0^{T+1} du \mathrm{E} \left\{ [\int_{(u-\varepsilon)\vee 0}^u dv \sigma(B_v)^2 \int_{(u-\varepsilon)\vee 0}^{v \wedge t} ds (\sigma(B_u) - \sigma(B_s))]^2 \right\}$$

$$\leq \text{cst.}\mathrm{E}\left[\sup_{s,s' \in [0,T+1], |s-s'| < \varepsilon} |\sigma(B_s) - \sigma(B_{s'})|^2 \right],$$

since σ is bounded. Then (3.48) is a consequence of the following

Lemma 3.7 *Under the hypothesis of Theorem 2.5,*

$$\mathrm{E}\left[\sup_{s,s' \in [0,T+1], |s-s'| < \varepsilon} |\sigma(B_s) - \sigma(B_{s'})|^2 \right] \leq \text{cst.}\varepsilon.$$

Proof. We have $|\sigma(B_s) - \sigma(B_{s'})|^2 \leq \|\sigma'\|_\infty^2 |B_s - B_{s'}|^2 \leq \text{cst.}|B_s - B_{s'}|^2$. Moreover, by Doob's inequality $\mathrm{E}\left[\sup_{s,s'} |\sigma(B_s) - \sigma(B_{s'})|^2\right] \leq \text{cst.}\mathrm{E}\left[\sup_{s,s'} |B_s - B_{s'}|^2\right] \leq \text{cst.}\varepsilon$. ■

Step 6. Assume that $1 < i_0 < j$. We make the proof of (3.48) for $m = 5$, $\delta = (2, 1, 2)$ and $j = 3$. Clearly

$$\mathcal{Y}_\varepsilon^{((2,1,2),3)}(t) = \frac{1}{\varepsilon^3} \int_0^t ds \int_s^{s+\varepsilon} \sigma(B_u)^2 du \int_s^u \sigma(B_v) dB_v \int_s^v [\sigma(B_w)^2 - \sigma(B_s)^2] dw$$

$$= \mathcal{Y}_{\varepsilon,1}^{((2,1,2),3)}(t) + \mathcal{Y}_{\varepsilon,2}^{((2,1,2),3)}(t),$$

where

$$\mathcal{Y}_{\varepsilon,1}^{((2,1,2),3)}(t) := \frac{1}{\varepsilon^3} \int_0^t ds \int_s^{s+\varepsilon} \sigma(B_s)^2 du \int_s^u \sigma(B_v) dB_v \int_s^v [\sigma(B_w)^2 - \sigma(B_s)^2] dw,$$

and

$$\mathcal{Y}_{\varepsilon,2}^{((2,1,2),3)}(t) := \frac{1}{\varepsilon^3}\int_0^t ds \int_s^{s+\varepsilon} [\sigma(B_u)^2 - \sigma(B_s)^2]du \int_s^u \sigma(B_v)dB_v \int_s^v [\sigma(B_w)^2 - \sigma(B_s)^2]dw.$$

By the stochastic version of Fubini theorem we get

$$\mathcal{Y}_{\varepsilon,1}^{((2,1,2),3)}(t) = \frac{1}{\varepsilon^3}\int_0^{t+\varepsilon} \sigma(B_v)dB_v \int_{(v-\varepsilon)\vee 0}^v dw \int_v^{(w\wedge t)+\varepsilon} du$$

$$\times \int_{(u-\varepsilon)\vee 0}^{w\wedge t} ds\, \sigma(B_s)^2[\sigma(B_w)^2 - \sigma(B_s)^2]$$

and

$$\mathrm{E}[\sup_{t\in[0,T]} |\mathcal{Y}_{\varepsilon,1}^{((2,1,2),3)}(t)|^2] \leq \frac{\text{cst.}}{\varepsilon^6}\int_0^{T+1} dv \mathrm{E}\left\{\sigma(B_v)^2 \left[\int_{(v-\varepsilon)\vee 0}^v dw \int_v^{(w\wedge t)+\varepsilon} du\right.\right.$$

$$\left.\left.\times \int_{(u-\varepsilon)\vee 0}^{w\wedge t} ds\, \sigma(B_s)^2[\sigma(B_w)^2 - \sigma(B_s)^2]\right]^3\right\} \leq \text{cst.}\mathrm{E}\left[\sup_{s,s'\in[0,T+1],|s-s'|<\varepsilon} |\sigma(B_s) - \sigma(B_{s'})|^2\right],$$

which tends to zero, as $\varepsilon \to 0$, by Lemma 3.7. In a similar way we prove that as $\varepsilon \to 0$, $\mathrm{E}[\sup_{t\in[0,T]} |\mathcal{Y}_{\varepsilon,2}^{((2,1,2),3)}(t)|^2] \to 0$.

Step 7. Assume that $i_0 = j > 1$ and we prove of (3.48) for $m = 3$, $\delta = (2,1)$ and $j = 2$, the general case being similar. In this case

$$\begin{aligned}\mathcal{Y}_\varepsilon^{((2,1),2)}(t) &= \frac{1}{\varepsilon^2}\int_0^t ds \int_s^{s+\varepsilon} \sigma(B_u)^2 du \int_s^u [\sigma(B_v) - \sigma(B_s)]dB_v \\ &= \frac{1}{\varepsilon^2}\int_0^t ds \int_s^{s+\varepsilon} \sigma(B_s)^2 du \int_s^u [\sigma(B_v) - \sigma(B_s)]dB_v \\ &+\frac{1}{\varepsilon^2}\int_0^t ds \int_s^{s+\varepsilon} [\sigma(B_u)^2 - \sigma(B_s)^2]du \int_s^u [\sigma(B_v) - \sigma(B_s)]dB_v.\end{aligned}$$

As previously, by the stochastic version of Fubini theorem and thanks to Lemma 3.7, we show that each term tends in L^2 to zero, as $\varepsilon \to 0$.

Step 8. Finally, if $i_0 > j$, we illustrate the proof for $m = 3$, $\delta = (2,1)$ and $j = 1$. We can write

$$\begin{aligned}\mathcal{Y}_\varepsilon^{((2,1),1)}(t) &= \frac{1}{\varepsilon^2}\int_0^t ds \int_s^{s+\varepsilon} [\sigma(B_u)^2 - \sigma(B_s)^2]du \int_s^u \sigma(B_v)dB_v \\ &= \frac{1}{\varepsilon^2}\int_0^t ds \int_s^{s+\varepsilon} du \int_s^u 2(\sigma\sigma')(B_w)dB_w \int_s^u \sigma(B_v)dB_v \\ &+\frac{1}{\varepsilon^2}\int_0^t ds \int_s^{s+\varepsilon} du \int_s^u (\sigma\sigma')'(B_w)dw \int_s^u \sigma(B_v)dB_v.\end{aligned}$$

By using Itô's formula we split again the first term in two terms and we can show that each term tends in L^2 to zero, as $\varepsilon \to 0$ using the same tools. ∎

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References

[1] Alòs, E., Mazet, O., Nualart, D. (2000) *Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than* $\frac{1}{2}$ Stoch. Proc. Appl., **86**, 121-139.

[2] Alòs, E., León, J.A., Nualart, D. (2001) *Stratonovich stochastic calculus for fractional Brownian motion with Hurst parameter lesser than* $\frac{1}{2}$ Taiwanese J. Math, **5**, 609-632.

[3] Cheredito, P., Nualart, D. (2003) *Symmetric integration with respect to fractional Brownian motion* Preprint Barcelona.

[4] Coutin, L., Qian, Z. (2000) *Stochastic differential equations for fractional Brownian motion* C. R. Acad. Sci. Paris, **330**, Serie I, 1-6.

[5] Föllmer, H. (1981) *Calcul d'Itô sans probabilités*, Séminaire de Probabilités XV 1979/80, Lect. Notes in Math. **850**, 143-150, Springer-Verlag.

[6] Giraitis, L., Surgailis, D.,(1985) *CLT and other limit theorems for functionals of Gaussian processes.* Z. Wahrsch. verw. Gebiete, **70**, 191-212.

[7] Gradinaru, M., Russo, F., Vallois, P., (2001) *Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index* $H \geq \frac{1}{4}$. To appear in Ann. Probab., **31**.

[8] Gradinaru, M., Nourdin, I., Russo, F., Vallois, P. (2002) *m-order integrals and Itô's formula for non-semimartingale processes; the case of a fractional Brownian motion with any Hurst index.* Preprint IECN 2002-48.

[9] Guyon, X., León, J. (1989) *Convergence en loi des H-variations d'un processus gaussien stationnaire sur* $\mathbb{R}$, Ann. Inst. Henri Poincaré, **25**, 265-282.

[10] Istas, J., Lang, G. (1997) *Quadratic variations and estimation of the local Hölder index of a Gaussian process*, Ann. Inst. Henri Poincaré, **33**, 407-436.

[11] Itô, K. (1951) *Multiple Wiener integral*, J. Math. Soc. Japan **3**, 157-169.

[12] Jakubowski, A., Mémin, J., Pagès, G. (1989) *Convergence en loi des suites d'intégrales stochastiques sur l'espace* $\mathbb{D}^1$ *de Skorokhod*, Probab. Theory Related Fields, **81**, 111-137.

[13] Nualart, D. (1995) *The Malliavin calculus and related topics*, Springer, Berlin Heidelber New-York.

[14] Nualart, D., Peccati, G. (2003) *Convergence in law of multiple stochastic integrals*, Preprint Barcelona.

[15] Revuz, D., Yor, M. (1994) *Continuous martingales and Brownian motion*, 2nd edition, Springer-Verlag.

[16] Rogers, L.C.G. (1997) *Arbitrage with fractional Brownian motion*, Math. Finance, **7**, 95-105.

[17] Russo, F., Vallois, P. (1996) *Itô formula for* C^1-*functions of semimartingales.* Probab. Theory Relat. Fields **104**, 27-41.

[18] Taqqu, M.S. (1977) *Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exibit a long range dependence* Z. Wahrsch. verw. Gebiete **40**, 203-238.

[19] Taqqu, M.S. (1979) *Convergence of integrated processes of arbitrary Hermite rank.* Z. Wahrsch. verw. Gebiete **50**, 53-83.

[20] Zähle, M. (1998) *Integration with respect to fractal functions and stochastic calculus I.* Probab. Theory Relat. Fields **111**, 333-374.