Electronic Journal of Probability

Vol. 13 (2008), Paper no. 53, pages 1479–1526.

Journal URL
http://www.math.washington.edu/~ejpecp/

Large Deviations for One Dimensional Diffusions with a Strong Drift

Jochen Voss*
(Jochen.Voss@warwick.ac.uk)

Abstract

We derive a large deviation principle which describes the behaviour of a diffusion process with additive noise under the influence of a strong drift. Our main result is a large deviation theorem for the distribution of the end-point of a one-dimensional diffusion with drift ϑb where b is a drift function and ϑ a real number, when ϑ converges to ∞. It transpires that the problem is governed by a rate function which consists of two parts: one contribution comes from the Freidlin-Wentzell theorem whereas a second term reflects the cost for a Brownian motion to stay near a equilibrium point of the drift over long periods of time.

Key words: large deviations, diffusion processes, stochastic differential equations.

AMS 2000 Subject Classification: Primary 60F10,60H10.

Submitted to EJP on April 28, 2005, final version accepted November 15, 2006.

*Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom.

1 Introduction

The Freidlin-Wentzell theorem and its generalisations are well-known large deviation results. This theorem provides a large deviation principle (LDP) on the path space for solutions of the SDE $dX = b(X)dt + \sqrt{\varepsilon}\,dB$ when ε converges to 0. The related, but different, problem of the large deviation behaviour of a diffusion process under the influence of a strong drift is less studied. In this article we derive an LDP for the behaviour of the endpoint X_t^ϑ of solutions of the $\mathbb{R}$-valued stochastic differential equation

$$\begin{aligned} dX_s^\vartheta &= \vartheta b(X_s^\vartheta)ds + dB_s \quad \text{for all } s \in [0,t] \\ X_0^\vartheta &= z \in \mathbb{R} \end{aligned} \tag{1.1}$$

when the parameter ϑ converges to infinity.

For comparison with the Freidlin-Wentzell result one can convert the case of strong drift into the case of weak noise with the help of the following scaling argument: Define $\tilde{X}_s^\vartheta = X_{s/\vartheta}^\vartheta$ and $\tilde{B}_s = \sqrt{\vartheta}B_{s/\vartheta}$ for all $s \in [0,\vartheta t]$. Then the process $\tilde{X}^\vartheta$ is a solution of the SDE

$$\begin{aligned} d\tilde{X}_s^\vartheta &= b(\tilde{X}_s^\vartheta)ds + \frac{1}{\sqrt{\vartheta}}\,dB_s \quad \text{for all } s \in [0,\vartheta t] \\ \tilde{X}_0^\vartheta &= z \end{aligned}$$

and we have

$$P(X_t^\vartheta \in A) = P\big(\tilde{X}^\vartheta \in \{\omega \mid \omega_{\vartheta t} \in A\}\big).$$

The rescaled problem looks more similar to the situation from the Freidlin-Wentzell theory, but now the event in question depends on the parameter ϑ. Thus the Freidlin-Wentzell theorem still does not apply easily. Therefore a more sophisticated proof will be required.

The text is structured as follows: In section 2 we state our main result and two corollaries. Since the proof of the theorem is quite long we give an overview of the proof of our theorem in section 3. The proof itself is spread over sections 4, 5 and 6.

The result presented in this text was originally derived as part of my PhD-thesis [Vos04].

2 Results

Recall that a family $(X^\vartheta)_{\vartheta>0}$ of random variables with values in some topological space $\mathscr{X}$ satisfies the LDP with rate function $I\colon \mathscr{X} \to [0,\infty]$, if it satisfies the estimates

$$\liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log P(X^\vartheta \in O) \geq -\inf_{x\in O} I(x)$$

for every open set $O \subseteq \mathscr{X}$ and

$$\limsup_{\vartheta\to\infty} \frac{1}{\vartheta} \log P(X^\vartheta \in A) \leq -\inf_{x\in A} I(x)$$

for every closed set $A \subseteq \mathscr{X}$. The family $(X^\vartheta)_{\vartheta>0}$ satisfies the weak LDP if the upper bound holds for every compact (instead of closed) set $A \subseteq \mathscr{X}$. For details about the theory of large deviations we refer to [DZ98].

Our main result is the following theorem together with the corollaries 2 and 4.

Theorem 1. *Let $b\colon \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz C^2-function with* $\liminf_{|x|\to\infty} |b(x)| > 0$. *Assume that there is an $m \in \mathbb{R}$ with $b(x) = 0$ if and only if $x = m$ and with $b'(m) \neq 0$. Furthermore let $z \in \mathbb{R}$, $t > 0$ and for every $\vartheta > 0$ let X^ϑ be the solution of the SDE*

$$\begin{aligned} dX_s^\vartheta &= \vartheta b(X_s^\vartheta)ds + dB_s \quad \text{for } s \in [0,t], \text{ and} \\ X_0^\vartheta &= z. \end{aligned} \tag{2.1}$$

Then the family $(X_t^\vartheta)_{\vartheta>0}$ satisfies the weak LDP on $\mathbb{R}$ with rate function

$$J_t(x) = V_z^m(\Phi) - \Phi(z) + t(\Phi''(m))^- + V_m^x(\Phi) + \Phi(x) \tag{2.2}$$

for all $x \in \mathbb{R}$, where Φ satisfies $b = -\Phi'$, $V_a^b(\Phi)$ is the total variation of Φ between a and b, and $(\Phi''(m))^-$ denotes the negative part of $\Phi''(m)$, i.e. $(\Phi''(m))^- = 0$ if $\Phi''(m) \geq 0$ and $(\Phi''(m))^- = |\Phi''(m)|$ if $\Phi''(m) < 0$.

Note that the condition $b = -\Phi'$ defines Φ only up to a constant, but the rate function J_t does not depend on the choice of this constant.

In the theorem $V_a^b(\Phi)$ can be interpreted as the "cost" for the process of going from a to b. Using $b = -\Phi'$ we find

$$V_a^b(\Phi) = \Big|\int_a^b |b(x)|\,dx\Big|$$

for any $a, b \in \mathbb{R}$. The term $(\Phi''(m))^-$ can be interpreted as the "cost" of staying near m for a unit of time. This term only occurs, if the equilibrium point m is unstable.

Given the sign of $b'(m)$ the rate function from the theorem can be simplified because the drift b has only one zero. The following corollary describes the case of $b'(m) < 0$, which corresponds to attracting drift. In this case the weak LDP from the theorem can be strengthend to the full LDP.

Corollary 2. *Under the conditions of theorem 1 with $b'(m) < 0$ the following claims hold.*

a) For every $t > 0$ the family $(X_t^\vartheta)_{\vartheta>0}$ satisfies the weak LDP on $\mathbb{R}$ with rate function

$$J_t(x) = 2\big(\Phi(x) - \Phi(m)\big) \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

b) If b is monotonically decreasing, then the family $(X_t^\vartheta)_{\vartheta>0}$ satisfies the full LDP with rate function J_t.

In the situation of corollary 2 the rate function is independent of the interval length t and of the initial point z. This makes sense, because for strong drift we would expect the process to reach the equilibrium very quickly. Because we have $\liminf_{|x|\to\infty} |b(x)| > 0$ the potential Φ converges to $+\infty$ for $|x| \to \infty$ and J_t is a good rate function. In fact the rate function coincides with the rate function of the LDP for the stationary distribution as given in theorem 3 (This is an easy application of the Laplace principle, see e.g. [Vos04] for details).

Theorem 3. *Let $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ be differentiable and such that $\exp(-2\Phi(x))$ is a probability density on $\mathbb{R}^d$. Let Φ be bounded from below with $\Phi_* = \inf\{\,\Phi(x) \mid x \in \mathbb{R}^d\,\} > -\infty$. Finally let $b = -\operatorname{grad}\Phi$ be Lipschitz continuous.*

Then for every $\vartheta \geq 1$ the stochastic differential equation

$$dX^\vartheta = \vartheta\, b(X^\vartheta)dt + dW$$

has a stationary distribution μ_ϑ and for every measurable set $A \subseteq \mathbb{R}^d$ we have

$$\lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log \mu_\vartheta(A) = -\operatorname{ess\,inf}_{x\in A} 2\big(\Phi(x) - \Phi_*\big).$$

Proof. (of corollary 2) a) Since we assume that m is the only zero of the drift b, for $b'(m) < 0$ the point m is the minimum of Φ. In this case we have $V_z^m(\Phi) = \Phi(z) - \Phi(m)$, $V_m^x(\Phi) = \Phi(x) - \Phi(m)$ and $\Phi''(m) > 0$, so the rate function simplifies to the expression given in formula (2.3).

b) To strengthen the weak LDP to the full LDP we have to check exponential tightness, i.e. we have to show that for every $c > 0$ there is an $a > 0$ with

$$\limsup_{\vartheta\to\infty} \frac{1}{\vartheta} \log P\big(|X_t^\vartheta - m| > a\big) < -c$$

(for reference see lemma 1.2.18 from [DZ98]). We use a comparison argument to obtain this estimate.

Using the assumption $\liminf_{|x|\to\infty} |b(x)| > 0$ we find that $\exp(-2\vartheta\Phi)$ is integrable and SDE (2.1) has a stationary distribution with density proportional to $\exp(-2\vartheta\Phi)$. Let X^ϑ be a solution of (2.1) with start in z and Y^ϑ be a stationary solution, both with respect to the same Brownian motion. Then we get the deterministic differential equation

$$\frac{d}{dt}(X_t^\vartheta - Y_t^\vartheta) = \vartheta\big(b(X_t^\vartheta) - b(Y_t^\vartheta)\big)$$

for the difference between the processes. First assume $X_0^\vartheta - Y_0^\vartheta \geq 0$. Because for $X_t^\vartheta - Y_t^\vartheta = 0$ the right hand side vanishes, the process $X_t^\vartheta - Y_t^\vartheta$ can never change its sign and stays positive. Since b is decreasing we have $b(X_t^\vartheta) - b(Y_t^\vartheta) \leq 0$ and we can conclude

$$0 \leq X_t^\vartheta - Y_t^\vartheta \leq X_0^\vartheta - Y_0^\vartheta.$$

For the case $X_0^\vartheta - Y_0^\vartheta \leq 0$ we can interchange the roles of X and Y to obtain the estimate

$$0 \leq Y_t^\vartheta - X_t^\vartheta \leq Y_0^\vartheta - X_0^\vartheta.$$

Combining these two cases gives

$$|Y_t^\vartheta - X_t^\vartheta| \leq |Y_0^\vartheta - X_0^\vartheta| = |Y_0^\vartheta - z|.$$

Using

$$\begin{aligned}
|X_t^\vartheta - m| &\leq |X_t^\vartheta - Y_t^\vartheta| + |Y_t^\vartheta - m| \\
&\leq |z - Y_0^\vartheta| + |Y_t^\vartheta - m| \\
&\leq |z - m| + |Y_0^\vartheta - m| + |Y_t^\vartheta - m|
\end{aligned}$$

we can conclude

$$\begin{aligned}
P\big(|X_t^\vartheta - m| > a\big) &\leq P\big(|Y_0^\vartheta - m| + |Y_t^\vartheta - m| > a - |z - m|\big) \\
&\leq P\Big(|Y_0^\vartheta - m| > \frac{a - |z-m|}{2}\Big) \\
&\qquad\qquad + P\Big(|Y_t^\vartheta - m| > \frac{a - |z-m|}{2}\Big) \\
&= 2P\Big(|Y_0^\vartheta - m| > \frac{a - |z-m|}{2}\Big).
\end{aligned}$$

Now let $c > 0$. Then using theorem 3 we can find an $a > 0$ with

$$\lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log P\Big(|Y_0^\vartheta - m| > \frac{a - |z-m|}{2}\Big) \leq -c$$

and using the above estimate we get

$$\lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log P\big(|X_t^\vartheta - m| > a\big) \leq -c.$$

Since this is the required exponential tightness condition, the proof is complete. □

The case of repelling drift, i.e. of $b'(m) > 0$, is described in the following corollary.

Corollary 4. *Under the conditions of theorem 1 with $b'(m) > 0$, for every $t > 0$ the family $(X_t^\vartheta)_{\vartheta>0}$ satisfies the weak LDP on $\mathbb{R}$ with constant rate function*

$$J_t(x) = 2\big(\Phi(m) - \Phi(z)\big) - t\Phi''(m). \tag{2.4}$$

Proof. (of corollary 4) In the case $b'(m) > 0$ the point m is the maximum of Φ and because of $V_z^m(\Phi) = \Phi(m) - \Phi(z)$, $V_m^x(\Phi) = \Phi(m) - \Phi(x)$ and $\Phi''(m) < 0$ we get

$$\begin{aligned} J_t(x) &= \big(\Phi(m) - \Phi(z)\big) - \Phi(z) - t\Phi''(m) + \big(\Phi(m) - \Phi(x)\big) + \Phi(x) \\ &= 2\big(\Phi(m) - \Phi(z)\big) - t\Phi''(m) \end{aligned}$$

for all $x \in \mathbb{R}$. □

The corollary shows that in the case of repelling drift the rate function does not depend on x. In particular it is not a good rate function. Here it is impossible to strengthen the weak LDP to the full LDP because we have

$$\lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log P(X_t^\vartheta \in \mathbb{R}) = 0 \quad \neq \quad 2\big(\Phi(m) - \Phi(z)\big) - t\Phi''(m).$$

3 Overall Structure of the Proof

The remaining part of this text contains the proof of theorem 1. Since the proof is quite long, we use this section to give an overview of the proof. All the technical details are contained in sections 4, 5, and 6.

Let X^ϑ be a solution of the SDE (1.1). From the Girsanov formula we know the density of the distribution of X_t^ϑ w.r.t. the Wiener measure $\mathbb{W}$: assuming $X_0^\vartheta = 0$ and $b = -\nabla\Phi$ we get

$$P(X_t^\vartheta \in A) = \int 1_A(\omega_t) \exp\big(\vartheta F(\omega) - \vartheta^2 G(\omega)\big)\, d\mathbb{W}(\omega) \tag{3.1}$$

where

$$\begin{aligned} F(\omega) &= \Phi(\omega_0) - \Phi(\omega_t) + \frac{1}{2}\int_0^t \Phi''(\omega_s)\, ds \quad \text{and} \\ G(\omega) &= \frac{1}{2}\int_0^t b^2(\omega_s)\, ds. \end{aligned}$$

For large values of ϑ the $\vartheta^2 G$ term dominates over the ϑF term and we show that the only paths which contribute for the large deviations behaviour of X_t^ϑ are those, which correspond to very small values of G. These paths run quickly to the equilibrium point m of the drift b, stay close to this point for most of the time, and shortly before time t move quickly into the set A. Assuming for the moment $A = B(a,\delta)$ with a small $\delta > 0$, we get

$$P(X_t^\vartheta \approx a) \approx \exp\big(\vartheta(\Phi(0) - \Phi(a) + \frac{t}{2}\Phi''(m))\big) \int 1_{\{\omega_t \approx a\}} \exp\big(-\vartheta^2 G(\omega)\big)\, d\mathbb{W}(\omega)$$

and thus

$$\begin{aligned} &\lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log P(X_t^\vartheta \approx a) \\ &\qquad \approx \Phi(0) - \Phi(a) + \frac{t}{2}\Phi''(m) + \lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int 1_{\{\omega_t\approx a\}} \exp\big(-\frac{\vartheta^2}{2}\int_0^t b^2(\omega_s)\,ds\big)\, d\mathbb{W}(\omega). \end{aligned} \tag{3.2}$$

Lemma 26 in section 6 resolves the technical details which are hidden in the $\approx$-signs here and also gives the required upper and lower limits for (3.2).

To evaluate the integral on the right hand side of (3.2) we use the following result about upper and lower limits in Tauberian theorems of exponential type. The theorem is proved in [Vos04]. It is a generalisation of de Bruijn's theorem (see theorem 4.12.9 in [BGT87]).

Theorem 5. *Let $X \geq 0$ be a random variable and A an event with $P(A) > 0$. Define the upper and lower limits*

$$\bar r = \limsup_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} \log E(e^{-\lambda X} 1_A) \quad \text{and} \quad \underline{r} = \liminf_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} \log E(e^{-\lambda X} 1_A)$$

as well as

$$\bar s = \limsup_{\varepsilon\to 0} \varepsilon \log P(X \leq \varepsilon, A) \quad \text{and} \quad \underline{s} = \liminf_{\varepsilon\to 0} \varepsilon \log P(X \leq \varepsilon, A).$$

Then $-\bar r^2/4 = \bar s$ and for the lower limits we have the sharp estimates $-\underline{r}^2 \leq \underline{s} \leq -\underline{r}^2/4$.

Using theorem 5 we can reduce the original problem to the calculation of exponential rates like

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log P\Big(\int_0^t b^2(B_s)\,ds \leq \varepsilon, B_t \approx a\Big).$$

In section 4 we examine the situation that during a short time interval the process runs from 0 to m or from m to a respectively while still keeping $\int b^2(\omega_s)\,ds$ small. This will be used for the initial and the final section of the path. As indicated in section 1 we can rescale the problem in these domains and apply the known results for weak noise. The problem here is to identify the infimum of the rate function.

In section 5 we examine the situation that $\int b^2(\omega_s)\,ds$ is small over a long interval of time. This will be used to study the middle section of the path. We will use theorem 5 again to deduce the probability for this case from the known Laplace transform of $\int_0^t B_s^2\,ds$.

Finally, in section 6, we fit these two results together to complete the proof of theorem 1. This part of the proof is modelled after the proof of proposition 6 which we give below. We want to use $X_1, X_2, X_3 = \int b^2(B_s)\,ds$ where the integral is taken over the initial, middle, and final section of the path respectively. Since these random variables are not independent, we cannot directly apply proposition 6 but have to use an enhanced version of the proof. This is provided in lemma 27.

We give the full prove of proposition 6 here, because we will need the proposition itself in the proof of lemma 23, and also because we hope that reading the proof of proposition 6 might make it easier to follow the proof of lemma 27 below.

Proposition 6. *Let $X_1,\dots,X_n$ be independent, positive random variables with*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_k \leq \varepsilon\big) = -b_k^2 \quad \text{and} \quad \limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_k \leq \varepsilon\big) = -c_k^2$$

where $b_k, c_k \geq 0$ for $k = 1,\dots,n$. Then we have

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 + \cdots + X_n \leq \varepsilon\big) \geq -(b_1 + \cdots + b_n)^2$$

and

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 + \cdots + X_n \leq \varepsilon\big) \leq -(c_1 + \cdots + c_n)^2.$$

Proof. Let $\delta > 0$. Since the simplex

$$S_n^\varepsilon = \big\{(\varepsilon_1,\dots,\varepsilon_n) \in \mathbb{R}^n_{\geq 0} \,\big|\, \varepsilon_1 + \cdots + \varepsilon_n \leq \varepsilon\big\}$$

is compact and covered by the open sets $\big\{(\varepsilon_1,\dots,\varepsilon_n) \in \mathbb{R}^n \,\big|\, \varepsilon_j < \alpha_j\varepsilon \text{ for } j = 1,\dots,n\big\}$ for $\alpha_1,\dots,\alpha_n > 0$ with $\alpha_1 + \cdots + \alpha_n = 1 + \delta$, we can find a finite set

$$D_n^\delta \subseteq \big\{\alpha \in \mathbb{R}^n_{>0} \,\big|\, \alpha_1 + \cdots + \alpha_n = 1 + \delta\big\} \tag{3.3}$$

with

$$S_n^\varepsilon \subseteq \bigcup_{\alpha\in D_n^\delta} \big\{(\varepsilon_1,\dots,\varepsilon_n) \in \mathbb{R}^n_{\geq 0} \,\big|\, \varepsilon_j \leq \alpha_j\varepsilon \text{ for } j = 1,\dots,n\big\}$$

for all $\varepsilon > 0$. This gives

$$P\big(X_1 + \cdots + X_n \leq \varepsilon\big) \leq \sum_{\alpha\in D_n^\delta} P\big(X_1 \leq \alpha_1\varepsilon,\dots,X_k \leq \alpha_k\varepsilon\big).$$

and for the individual terms in the sum we can use the relation

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 \leq \alpha_1\varepsilon,\dots,X_k \leq \alpha_k\varepsilon\big) \\
&\qquad = \limsup_{\varepsilon\downarrow 0} \varepsilon \log \prod_{k=1}^n P\big(X_k \leq \alpha_k\varepsilon\big) = -\sum_{k=1}^n \frac{c_k^2}{\alpha_k}.
\end{aligned}$$

Let $a = \sum_{k=1}^n \alpha_k$, $p_k = \alpha_k/a$, and $d_k = c_k/p_k$ for $k = 1,\dots,n$. Applying Jensen's inequality to the random variable which takes value d_k with probability p_k gives

$$\frac{c_1^2}{\alpha_1} + \cdots + \frac{c_n^2}{\alpha_n} \geq \frac{(c_1 + \cdots + c_n)^2}{\sum_{k=1}^n \alpha_k} \tag{3.4}$$

where equality holds if and only if there is a $\lambda \in \mathbb{R}$ with $\lambda\alpha_k = c_k$ for $k = 1, \ldots, n$. Thus we get

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 \le \alpha_1\varepsilon, \ldots, X_n \le \alpha_n\varepsilon\big) \le -\frac{(c_1 + \cdots + c_n)^2}{1+\delta}$$

for every $\alpha \in D_n^\delta$. Using lemma 1.2.15 of [DZ98] we can conclude

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 + \cdots + X_n \le \varepsilon\big) \\
&\qquad \le \max_{\alpha\in D_n^\delta} \limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 \le \alpha_1\varepsilon, \ldots, X_n \le \alpha_n\varepsilon\big) \\
&\qquad \le -\frac{(c_1 + \cdots + c_n)^2}{1+\delta}
\end{aligned}$$

for every $\delta > 0$ and thus

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 + \cdots + X_n \le \varepsilon\big) \le -(c_1 + \cdots + c_n)^2.$$

From (3.4) we know that we should choose α_k proportional to b_k in order to get the optimal lower bound. This leads to the estimate

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_1 + \cdots + X_n \le \varepsilon\big) \\
&\qquad \ge \liminf_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_k \le \frac{b_k}{b_1 + \cdots + b_n}\varepsilon, k = 1, \ldots, n\big) \\
&\qquad = \liminf_{\varepsilon\downarrow 0} \varepsilon \log \prod_{k=1}^n P\big(X_k \le \frac{b_k}{b_1 + \cdots + b_n}\varepsilon\big) \\
&\qquad \ge \sum_{k=1}^n \frac{b_1 + \cdots + b_n}{b_k} \liminf_{\varepsilon\downarrow 0} \varepsilon \log P\big(X_k \le \varepsilon\big) \\
&\qquad = -\sum_{k=1}^n \frac{b_1 + \cdots + b_n}{b_k} b_k^2 \\
&\qquad = -(b_1 + \cdots + b_n)^2
\end{aligned}$$

which completes the proof. □

4 Reaching the Final Point

The results of this section help to estimate the probability that the path travels quickly between the equilibrium point of the drift and the final resp. initial point. Here Schilder's theorem (see theorem 5.2.1 in [DZ98]) can be applied and we will reduce the evaluation of the rate function to a variational problem.

The main result of this section is the following proposition which describes the large deviation behaviour of the event

$$\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\, ds \le \varepsilon$$

when $\varepsilon \downarrow 0$, where the final point $B_{t\varepsilon}$ stays in a fixed, compact set. Evaluating the rate for fixed $t > 0$ is difficult, but it transpires that there is an explicit representation for the limit of the rate as t tends to infinity.

Proposition 7. *Let P_z be the distribution of a Brownian motion with start in z and B be the canonical process. Let $b\colon \mathbb{R} \to \mathbb{R}$ be a C^2-function with $\liminf_{|x|\to\infty} |b(x)| > 0$. Assume that there is an $m \in \mathbb{R}$ with $b(x) = 0$ if and only if $x = m$ and with $b'(m) \neq 0$. Then for every pair of compact sets $K_1, K_2 \subseteq \mathbb{R}$ we have*

$$\limsup_{t\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{z\in K_1} P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds \le \varepsilon, B_{t\varepsilon} \in K_2\Big)$$
$$\le -\frac{1}{4} \inf_{z\in K_1} \inf_{a\in K_2} \Big(\Big|\int_z^m |b(x)|\,dx\Big| + \Big|\int_m^a |b(x)|\,dx\Big|\Big)^2$$

and for every $z \in \mathbb{R}$ and every open set $O \subseteq \mathbb{R}$ we have

$$\liminf_{t\to\infty} \liminf_{\varepsilon\downarrow 0} \varepsilon \log P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds \le \varepsilon, B_{t\varepsilon} \in O\Big)$$
$$\ge -\frac{1}{4} \inf_{a\in O} \Big(\Big|\int_z^m |b(x)|\,dx\Big| + \Big|\int_m^a |b(x)|\,dx\Big|\Big)^2.$$

The modulus of the integrals is taken to properly handle the cases $m < z$ and $a < m$. The proof of proposition 7 is based on the following two lemmas. Lemma 8 evaluates the infimum of the rate function from Schilder's theorem. Since the proof of lemma 8 is quite long, we defer the proof until the end of the section. We will write $C_0([0,t],\mathbb{R}) = \{\omega \in C([0,t],\mathbb{R}) \mid \omega_0 = 0\}$ as an abbreviation.

Lemma 8. *Let $v\colon \mathbb{R} \to [0,\infty)$ be a positive C^2-function with $\liminf_{|x|\to\infty} v(x) > 0$ and $m \in \mathbb{R}$ with $v(x) = 0$ if and only if $x = m$ and $v''(m) > 0$. For $a, z \in \mathbb{R}$ and $\beta \ge 0$ define*

$$M_t^{a,z,\beta} = \Big\{\omega \in C[0,t] \;\Big|\; \omega_0 = 0, \omega_t = a - z, \frac{1}{2}\int_0^t v(\omega_s + z)\,ds = \beta\Big\}$$

and

$$J(a,z) = \frac{1}{4}\Big(\Big|\int_z^m \sqrt{v(x)}\,dx\Big| + \Big|\int_m^a \sqrt{v(x)}\,dx\Big|\Big)^2.$$

Consider the rate function

$$I_t(\omega) = \begin{cases} \frac{1}{2}\int_0^t |\dot{\omega}|^2\,ds, & \text{if } \omega \text{ is absolutely continuous, and} \\ +\infty & \text{else.} \end{cases}$$

Let $K_1, K_2 \subseteq \mathbb{R}$ be compact sets with $m \notin K_1 \cap K_2$ and $B \subseteq \mathbb{R}_+$ be bounded with $0 \in B$. Then we have

$$\inf\Big\{I_t(\omega) \;\Big|\; \omega \in \bigcup_{\beta\in B} M_t^{a,z,\beta}\Big\} \longrightarrow \frac{1}{\sup B} J(a,z) \quad \text{for } t \to \infty, \tag{4.1}$$

uniformly over all $a \in K_2$ and $z \in K_1$.

Lemma 9. *Let $M_t^{a,z,\beta}$ be as in lemma 8. Then for every pair $K_1, K_2 \subseteq \mathbb{R}$ of compact sets the set*

$$M = \bigcup_{z\in K_1} \bigcup_{a\in K_2} \bigcup_{0\leq\beta\leq 1} M_t^{a,z,\beta}$$

is closed in $C_0([0,t],\mathbb{R})$.

Proof. By definition of the sets $M_t^{a,z,\beta}$ we have

$$M = \bigcup_{z\in K_1} \Big\{ \omega \in C[0,t] \,\Big|\, \omega_0 = 0, \omega_t + z \in K_2, \frac{1}{2}\int_0^t v(\omega_r + z)\,dr \leq 1 \Big\}.$$

Assume that $\omega \in C_0([0,t],\mathbb{R}) \setminus M$. Then either $\omega_t + z \notin K_2$ for all $z \in K_1$, i.e. ω_t lies outside the compact set $K_2 - K_1$, or

$$\frac{1}{2}\int_0^t v(\omega_r + z)\,dr > 1$$

for every $z \in K_2$, i.e.

$$\inf_{z\in K_2} \frac{1}{2}\int_0^t v(\omega_r + z)\,dr > 1$$

because K_2 is compact and v and the integral are continuous. In both cases we can find an $\varepsilon > 0$, such that the ball $B(\omega,\varepsilon)$ also lies in $C_0([0,t],\mathbb{R})\setminus M$. Thus M is the complement of an open set. □

With these preparations in place we can now give the proof for proposition 7.

Proof. (of proposition 7) We want to apply Schilder's theorem [DZ98, theorem 5.2.1] and to evaluate the rate function using lemma 8. Let $K_1, K_2 \subseteq \mathbb{R}$ be compact. Define the process $\tilde{B}$ by setting $\tilde{B}_r = (B_{r\varepsilon} - z)/\sqrt{\varepsilon}$ for every $r > 0$. Then $\tilde{B}$ is a Brownian motion with start in 0 and we get

$$\begin{aligned}
&P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds \leq \varepsilon, B_{t\varepsilon} \in K_2\Big) \\
&\overset{s=r\varepsilon}{=} \quad P_z\Big(\frac{1}{2}\int_0^{t} b^2(B_{r\varepsilon})\,dr \leq 1, B_{t\varepsilon} \in K_2,\Big) \\
&= P\Big(\frac{1}{2}\int_0^{t} b^2(\sqrt{\varepsilon}\tilde{B}_r + z)\,dr \leq 1, \sqrt{\varepsilon}\tilde{B}_t + z \in K_2\Big) \\
&= P\Big(\sqrt{\varepsilon}\tilde{B} \in \bigcup_{a\in K_2} \bigcup_{\beta\leq 1} M_t^{a,z,\beta}\Big)
\end{aligned}$$

and thus

$$\begin{aligned}
&\sup_{z\in K_1} P_z\Big(B_{t\varepsilon} \in K_2, \frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds \leq \varepsilon\Big) \\
&\qquad \leq P\Big(\sqrt{\varepsilon}\tilde{B} \in \bigcup_{z\in K_1} \bigcup_{a\in K_2} \bigcup_{\beta\leq 1} M_t^{a,z,\beta}\Big).
\end{aligned} \tag{4.2}$$

Since from lemma 9 we know that the set $\bigcup_{z\in K_1}\bigcup_{a\in K_2}\bigcup_{\beta\le 1} M_t^{a,z,\beta}$ is closed in the path space $\big(C_0[0,t],\|\cdot\|_\infty\big)$, we can apply Schilder's theorem to get

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0}\varepsilon\log\sup_{z\in K_1} P\Big(\sqrt{\varepsilon}\tilde{B}\in\bigcup_{z\in K_1}\bigcup_{a\in K_2}\bigcup_{\beta\le 1}M_t^{a,z,\beta}\Big)&\\
\le -\inf\Big\{I_t(\omega)\,\Big|\,\omega\in\bigcup_{z\in K_1}\bigcup_{a\in K_2}\bigcup_{\beta\le 1}M_t^{a,z,\beta}\Big\}&\\
= -\inf_{z\in K_1}\inf_{a\in K_2}\inf\big\{I_t(\omega)\,\big|\,\omega\in\bigcup_{\beta\le 1}M_t^{a,z,\beta}\big\}.&
\end{aligned}$$

First assume $m\in K_1\cap K_2$. Define the path ω by $\omega_s=0$ for all $s\in[0,t]$. Then clearly we have $\omega\in M_t^{m,m,0}$ for every t and since we find $I_t(\omega)=0$ we have

$$\inf\Big\{I_t(\omega)\,\Big|\,\omega\in\bigcup_{\beta\in B}M_t^{a,z,\beta}\Big\}=0$$

for all $t\ge 0$. On the other hand we have $J(m,m)=0$.

Otherwise the evaluation of the infimum is done in lemma 8. Using $v(x)=b^2(x)$ we get $v''(m)=2(b'(m))^2>0$ and for every $\eta>0$ we can find a $t_0>0$, such that

$$\inf_{\beta\le 1}\inf\big\{I_t(\omega)\,\big|\,\omega\in M_t^{a,z,\beta}\big\}\ge J(a,z)-\eta$$

for all $z\in K_1$, $a\in K_2$ and $t\ge t_0$. This gives

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0}\varepsilon\log\sup_{z\in K_1} P\Big(\sqrt{\varepsilon}\tilde{B}\in\bigcup_{z\in K_1}\bigcup_{a\in K_2}\bigcup_{\beta\le 1}M_t^{a,z,\beta}\Big)&\\
\le -\inf_{z\in K_1}\inf_{a\in K_2}\inf_{m\in N}J(a,z)+\eta&\\
= -\frac{1}{4}\inf_{z\in K_1}\inf_{a\in K_2}\Big(\Big|\int_z^m |b(x)|\,dx\Big|+\Big|\int_m^a |b(x)|\,dx\Big|\Big)^2+\eta&
\end{aligned}$$

for every $\eta>0$. Together with the relation (4.2) this proves the upper bound.

For the lower bound we follow the same procedure. Without loss of generality we can assume that O is bounded. Here we get

$$\begin{aligned}
P_z\Big(B_{t\varepsilon}\in O,\frac{1}{2}\int_0^{t\varepsilon}b^2(B_s)\,ds\le\varepsilon\Big)&\\
\ge P_z\Big(B_{t\varepsilon}\in O,\frac{1}{2}\int_0^{t\varepsilon}b^2(B_s)\,ds<\varepsilon\Big)&\\
= P\Big(\sqrt{\varepsilon}\tilde{B}\in\bigcup_{a\in O}\bigcup_{\beta<1}M_t^{a,z,\beta}\Big)&
\end{aligned}$$

where the set

$$\bigcup_{a\in O}\bigcup_{\beta<1}M_t^{a,z,\beta}=\Big\{\omega\in C[0,t]\,\Big|\,\omega_0=0,\omega_t\in O-z,\frac{1}{2}\int_0^t b^2(\omega_r+z)\,dr<1\Big\}$$

is open in $\big(C_0[0,t], \|\cdot\|_\infty\big)$. So we can use the lower bound from Schilder's theorem and lemma 8 to complete the proof. □

Corollary 10. *Under the assumptions of proposition 7 we have*

$$\begin{aligned}\lim_{\eta\downarrow 0}\liminf_{t\to\infty}\liminf_{\varepsilon\downarrow 0}\varepsilon\log\inf_{m-\eta\le z\le m+\eta}&P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds\le\varepsilon, B_{t\varepsilon}\in O\Big)\\ &\ge -\frac{1}{4}\inf_{a\in O}\Big(\int_m^a |b(x)|\,dx\Big)^2\end{aligned}$$

for every open set $O\subseteq\mathbb{R}$.

Proof. For $z\in\mathbb{R}$ define

$$M_t^z=\Big\{\omega\in C[0,t]\;\Big|\;\omega_0=0,\omega_t+z\in O,\frac{1}{2}\int_0^t b^2(\omega_s+z)\,ds<1\Big\}.$$

Let $\delta>0$. Choose an $\tilde\omega\in M_t^m$ with $I_t(\tilde\omega)<\inf\{I_t(\omega)\mid\omega\in M_t^m\}+\delta$. Because O is open and b and the integral are continuous we can find an $E>0$, such that for every $\eta<E$ the ball $B_\eta(\tilde\omega)\subseteq C_0([0,t],\mathbb{R})$ is contained in all of the sets M_t^z for $m-\eta<z<m+\eta$. This gives

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}\varepsilon\log&\inf_{m-\eta\le z\le m+\eta}P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds\le\varepsilon, B_{t\varepsilon}\in O\Big)\\ &=\liminf_{\varepsilon\downarrow 0}\varepsilon\log\inf_{m-\eta\le z\le m+\eta}P_z\Big(\sqrt{\varepsilon}B\in M_t^z\Big)\\ &\ge\liminf_{\varepsilon\downarrow 0}\varepsilon\log\inf_{m-\eta\le z\le m+\eta}P_z\Big(\sqrt{\varepsilon}B\in B_\eta(\tilde\omega)\Big).\end{aligned}$$

and using Schilder's theorem and the relation

$$-\inf\{I_t(\omega)\,\big|\,\omega\in B_\eta(\tilde\omega)\}\ge -I_t(\tilde\omega)>-\inf\{I_t(\omega)\,\big|\,\omega\in M_t^m\}-\delta$$

we find

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}\varepsilon\log&\inf_{m-\eta\le z\le m+\eta}P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds\le\varepsilon, B_{t\varepsilon}\in O\Big)\\ &\ge-\inf\{I_t(\omega)\,\big|\,\omega\in B_\eta(\tilde\omega)\}\\ &>-\inf\{I_t(\omega)\,\big|\,\omega\in M_t^m\}-\delta.\end{aligned}$$

Now we can evaluate the infimum on the right hand side as we did in proposition 7. We get

$$\begin{aligned}\liminf_{t\to\infty}\liminf_{\varepsilon\downarrow 0}\varepsilon\log\inf_{m-\eta\le z\le m+\eta}&P_z\Big(\frac{1}{2}\int_0^{t\varepsilon} b^2(B_s)\,ds\le\varepsilon, B_{t\varepsilon}\in O\Big)\\ &\ge-\frac{1}{4}\inf_{a\in O}\Big(\int_m^a|b(x)|\,dx\Big)^2-\delta\end{aligned}$$

for every $\eta<E$. Taking the limit $\delta\downarrow 0$ completes the proof. □

The only thing which remains to be done in this section is to give a proof for lemma 8. Before we do so we need some preparations. For the remaining part of this section we assume throughout that v is non-negative and two times continuously differentiable and that $a, z \in \mathbb{R}$ are fixed.

Notation: For $x, y \in \mathbb{R}$ we will write $[x, y]$ for the closed interval between x and y; in the case $x < y$ this is to be read as $[y, x]$ instead.

As a first step towards the proof of lemma 8 we get rid of the parameter β.

Lemma 11. *Let $\{0\} \subset B \subseteq \mathbb{R}_+$ be bounded. Assume that*

$$\lim_{t\to\infty} \inf\{I_t(\omega) \mid \omega \in M_t^{a,z,1}\} = J(a,z)$$

locally uniform in $a, z \in \mathbb{R}$. Then the relation (4.1) *holds.*

Proof. Let $\beta > 0$. For $\omega \in M_t^{a,z,\beta}$ define $\tilde{\omega}$ by

$$\tilde{\omega}_r = \omega_{r\beta} \quad \text{for all } r \in [0, t/\beta].$$

Then we have $\tilde{\omega}_0 = 0$, $\tilde{\omega}_{t/\beta} = \omega_t$, and

$$\frac{1}{2}\int_0^{t/\beta} v(\tilde{\omega}_r + z)\, dr \overset{s=r\beta}{=} \frac{1}{\beta}\frac{1}{2}\int_0^t v(\omega_s + z)\, ds.$$

Thus $\omega \mapsto \tilde{\omega}$ is a one-to-one mapping from $M_t^{a,z,\beta}$ onto $M_{t/\beta}^{a,z,1}$.
Because of

$$I_{t/\beta}(\tilde{\omega}) = \frac{1}{2}\int_0^{t/\beta} \dot{\tilde{\omega}}_r^2\, dr = \frac{\beta^2}{2}\int_0^{t/\beta} \dot{\omega}_{r\beta}^2\, dr \overset{s=r\beta}{=} \frac{\beta}{2}\int_0^t \dot{\omega}_s^2\, ds = \beta I_t(\omega)$$

we find

$$\inf\{I_t(\omega) \mid \omega \in M_t^{a,z,\beta}\} = \frac{1}{\beta}\inf\{I_{t/\beta}(\omega) \mid \omega \in M_{t/\beta}^{a,z,1}\}.$$

Now let $z \in K_1$ and $a \in K_2$. Since $m \notin K_1 \cap K_2$ every continuous path ω with $\omega_0 = 0$ and $\omega_t = a - z$ has

$$\frac{1}{2}\int_0^t v(\omega_s + z)\, ds > 0,$$

the set $M_t^{a,z,0}$ is empty and we find

$$\begin{aligned}\inf\Big\{I_t(\omega) \,\Big|\, \omega \in \bigcup_{\beta\in B} M_t^{a,z,\beta}\Big\} &= \inf_{\beta\in B\setminus\{0\}} \inf\{I_t(\omega) \mid \omega \in M_t^{a,z,\beta}\} \\ &= \inf_{\beta\in B\setminus\{0\}} \frac{1}{\beta}\inf\{I_{t/\beta}(\omega) \mid \omega \in M_{t/\beta}^{a,z,1}\}.\end{aligned}$$

Now let $K_1, K_2 \subseteq \mathbb{R}$ be compact. Let $\eta > 0$ and choose a $t_0 > 0$ with

$$\Big|\inf\{I_t(\omega) \mid \omega \in M_t^{a,z,1}\} - J(a,z)\Big| \le \eta \sup B$$

for all $t > t_0$, $z \in K_1$, and $a \in K_2$. Then for every $t > t_0 \sup B$ and every $\beta > 0$ we have

$$\begin{aligned}&\Big|\inf\Big\{I_t(\omega)\;\Big|\;\omega \in \bigcup_{\beta' \in B} M_t^{a,z,\beta'}\Big\} - \frac{1}{\beta} J(a,z)\Big| \\ &\quad = \Big|\frac{1}{\beta}\inf\{I_t(\omega) \mid \omega \in M_{t/\beta}^{a,z,1}\} - \frac{1}{\beta} J(a,z)\Big| \leq \frac{\eta \sup B}{\beta}\end{aligned}$$

Choosing $\beta = \sup B$ gives

$$\Big|\inf\Big\{I_t(\omega)\;\Big|\;\omega \in \bigcup_{\beta \in B} M_t^{a,z,\beta}\Big\} - \frac{1}{\sup B} J(a,z)\Big| \leq \eta$$

for all $t > t_0 \sup B$, $z \in K_1$, and $a \in K_2$. Because η was arbitrary, this completes the proof. □

Because of $I_t(\omega + z) = I_t(\omega)$ we can shift every path from $M_t^{a,z,1}$ by z and get

$$\inf\{I_t(\omega) \mid \omega \in M_t^{a,z,1}\} = \inf\Big\{I_t(\omega)\;\Big|\;\omega_0 = z, \omega_t = a, \frac{1}{2}\int_0^t v(\omega_s)\,ds = 1\Big\}.$$

For the moment assume that there is a path $\tilde{\omega}$ with $I_t(\tilde{\omega}) = \inf\{I_t(\omega) \mid \omega \in M_t^{a,z,1}\}$. Later we will show that such an $\tilde{\omega}$ in fact does exist. In order to evaluate the rate function I_t for this path $\tilde{\omega}$, we solve the Euler-Lagrange equations (see section 12 of [GF63]) for extremal values of I_t under the constraint

$$K(\omega) = \frac{1}{2}\int_0^t v(\omega_s)\,ds \stackrel{!}{=} 1$$

and with the boundary conditions

$$\omega_0 = z \quad \text{and} \quad \omega_t = a.$$

Because of $v \in C^2(\mathbb{R})$ we can use theorem 1 from section 12.1 of [GF63] to find that for every extremal point ω of I, under the given constraints, there is a constant λ, such that ω solves the equations

$$\ddot{\omega}_s = \lambda v'(\omega_s) \quad \text{for all } s \in (0,t], \text{ and} \quad \omega_0 = z \tag{4.3a}$$

$$\frac{1}{2}\int_0^t v(\omega_s)\,ds = 1 \tag{4.3b}$$

$$\omega_t = a. \tag{4.3c}$$

Existence of solutions: the autonomous second order equation (4.3a) describes the motion of a classical particle on the real line in the potential $-\lambda v$. The differential equation can be reduced to an autonomous first order equation in the plane with the usual trick: defining $x(s) = (\omega_s, \dot{\omega}_s)$ and $F(x_1, x_2) = \big(x_2, \lambda v'(x_1)\big)$ the equation becomes

$$\dot{x}(s) = F(x(s)) \quad \text{for all } s \in [0,t].$$

See e.g. section 5.3 of [BR89] for details. Because v' and thus F is locally Lipschitz continuous, for every pair $\omega_0 = z$, $\dot{\omega}_0 = v_0$ of initial conditions and every bounded region we find a unique solution of the ODE at least up to the boundary of that region (see theorem 8 in section 6.9 of [BR89]).

There are two degrees of freedom in (4.3a) because we can choose $\dot\omega_0$ and λ. In the following we will show, that the two additional conditions (4.3b) and (4.3c) guarantee the existence of a unique solution to the system (4.3).

For $\lambda = 0$ the only solution of (4.3a) and (4.3c) is given by $\omega_s = z + (a-z)s/t$ for $0 \le s \le t$ and consequently in this case we have

$$\frac{1}{2}\int_0^t v(\omega_s)\,ds = th(z,a)$$

with

$$h(z,a) = \begin{cases} \frac{1}{2(a-z)}\int_z^a v(x)\,dx, & \text{if } a \neq z, \text{ and} \\ \frac{1}{2}v(z) & \text{else.} \end{cases}$$

Since $m \neq K_1 \cap K_2$, $z \in K_1$, and $a \in K_2$ we have $h(z,a) > 0$ for every $z \in K_1, a \in K_2$ and because $K_1 \times K_2$ is compact we find $c = \inf_{(z,a)\in K_1\times K_2} h(z,a) > 0$. In the following assume $t > 1/c$. Then we know from (4.3b) that every solution of (4.3) has $\lambda \neq 0$.

The interpretation as the motion of a classical particle helps us to determine the behaviour of the solutions. We can use conservation of energy: Because of

$$\partial_s\big(\frac{1}{2}\dot\omega_s^2 - \lambda v(\omega_s)\big) = \dot\omega_s\ddot\omega_s - \lambda v'(\omega_s)\dot\omega_s = \dot\omega_s\big(\ddot\omega_s - \lambda v'(\omega_s)\big) \overset{(4.3a)}{=} 0$$

we have

$$\frac{1}{2}\dot\omega_s^2 - \lambda v(\omega_s) = \frac{1}{2}\dot\omega_0^2 - \lambda v(\omega_0) =: E \quad \text{for all } s \in [0,t]. \tag{4.4}$$

This conservation law describes the speed for any point of the path: the speed of the path at point ω_s is

$$|\dot\omega_s| = \sqrt{2(E + \lambda v(\omega_s))}. \tag{4.5}$$

Thus the rate function I_t can be expressed as a function of E and λ as follows.

$$\begin{aligned} I_t(\omega) &= \frac{1}{2}\int_0^t \dot\omega_s^2\,ds = \int_0^t E + \lambda v(\omega_s)\,ds \\ &= tE + 2\lambda, \end{aligned} \tag{4.6}$$

where λ and E are determined by equations (4.3b) and (4.3c).

Because of relation (4.4) we find that whenever ω is a solution of (4.3a) we have $E \ge -\lambda v(\omega_s)$ for all $s \in [0,t]$ and the path can only stop and turn at points x with $-\lambda v(x) = E$. Let $x \in \mathbb{R}$ be such a point and assume $v'(x) = 0$. Then η with $\eta_s = x$ for all $s \ge 0$ is the unique solution of (4.3a) with $\eta_0 = x$ and $\dot\eta_0 = 0$. Now assume that $\omega_s = x$ for some $s > 0$. Then $(\omega_{s-r})_{r\in[0,s]}$ is also a solution of (4.3a) with start in x and initial speed 0, so we have $\omega_{s-r} = \eta_r = x$ for all $r \in [0,s]$. This shows that a point $x \neq z$ with $E = -\lambda v(x)$ and $v'(x) = 0$ cannot be reached by a solution ω of (4.3a). Thus whenever a non-constant path reaches an $x \in \mathbb{R}$ with $E = -\lambda v(x)$ then we have $\ddot\omega_s = \lambda v'(\omega_s) \neq 0$ and the path always changes direction there. Figure 1 illustrates two different kinds of solution, one where ω_s moves monotonically and one where the path reaches a point b with $-\lambda v(b) = E$ and turns there.

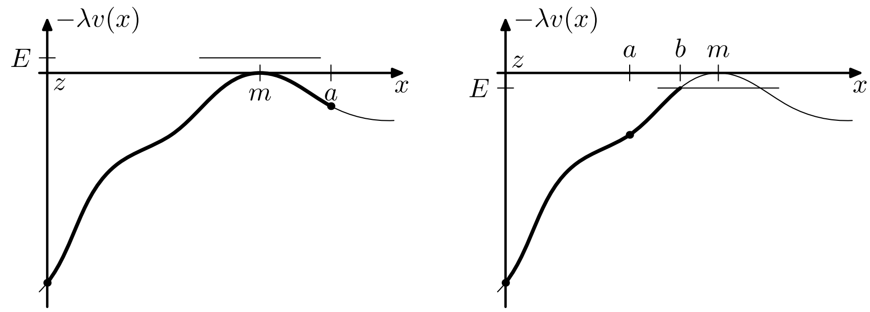

Figure 1: *This figure illustrates two types of solution for equation* (4.3a). *Here we only consider the case $\lambda > 0$. The curved line is the graph of the function $x \mapsto -\lambda v(x)$. The bold part of the lines corresponds to the points visited by the path ω. The thick dots are $\big(\omega_0, -\lambda v(\omega_0)\big)$ and $\big(\omega_t, -\lambda v(\omega_t)\big)$. Both solutions start at $z \in K_1$, head towards a neighbourhood of the zero m, and finally reach a point $a \in K_2$. The left hand image shows a free solution, i.e. one with $E > 0$, the right hand image shows a bound solution, i.e. one with $E \le 0$ where the path ω turns at the point b with $-\lambda v(b) = E$.*

Since the differential equation (4.3a) is autonomous and since a solution ω changes direction every time is reaches a point x with $-\lambda v(x) = E$, the path can reach at most two distinct points of these nature. In this case the solution oscillates between these points periodically. Thus every solution of (4.3a) changes direction only a finite number of times before time t.

In order to find the path which minimises the rate function I_t we need to keep track of the different possible traces of the path. For the remaining part of this section we use the following notation. The path $(\omega_s)_{0\le s\le t}$ is said to have trace $T = (x_0, x_1, \dots, x_n)$ when $\omega_0 = x_0$, $\omega_t = x_n$, and the path ω moves monotonically in either direction from x_{i-1} to x_i for $i = 1, \dots, n$ in order and changes direction only at the points $x_1, \dots, x_{n-1}$. We use the abbreviation

$$|T| = \sum_{i=1}^{n} |x_i - x_{i-1}|$$

for the length of the trace and sometimes identify T with the set $\bigcup_{i=1}^{n} [x_{i-1}, x_i]$ of covered points to write $\min T$, $\max T$, $v|_T$, or $\inf_{x\in T} v(x)$. For positive functions $f : \mathbb{R} \to \mathbb{R}$ we use the notation

$$\int_T f(x)\,dx := \sum_{i=1}^{n} \Big| \int_{x_{i-1}}^{x_i} f(x)\,dx \Big|.$$

The absolute values are taken to make the integral positive even when $x_i < x_{i-1}$. If a solution ω of (4.3a) has trace $T = (x_0, x_1, \dots, x_n)$, this then implies that $v(x_1) = \cdots = v(x_{n-1}) = -E/\lambda$ and each of the $x_1, \dots, x_{n-1}$ is either $\min T$ or $\max T$. Between the points x_i the path is strictly monotonic, i.e. after the start in z it oscillates zero or more times between $\min T$ and $\max T$ before it reaches a at time t. Using this notation we can formulate the following Lemma.

Lemma 12. *Let $\lambda, E \in \mathbb{R}$ and a trace $T = (x_0, \dots, x_n)$ be given. Then the following two conditions are equivalent.*

(j) The unique solution $\omega\colon [0,t] \to \mathbb{R}$ of

$$\ddot{\omega}_s = \lambda v'(\omega_s) \quad \text{for all } s \in [0,t]$$

with initial conditions $\omega_0 = z$ and $\dot{\omega}_0 = \operatorname{sgn}(x_1 - x_0)\sqrt{2(E+\lambda v(0))}$ has trace T and solves (4.3b) *and* (4.3c).

(ij) We have $x_0 = z$, $x_n = a$, $E = -\lambda v(x_i)$ for $i = 1,\dots,n-1$, as well as $E > -\lambda v(x)$ for all $\min T < x < \max T$, and the pair (λ, E) solves

$$\int_T \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx = \sqrt{8} \tag{4.7a}$$

and

$$\int_T \frac{1}{\sqrt{E+\lambda v(x)}}\,dx = \sqrt{2}t. \tag{4.7b}$$

Proof. Assume the conditions from (j). Then ω is a solution of (4.3a), there are times $t_0, t_1, \dots, t_n$ with $\omega_{t_i} = x_i$ for $i = 0,\dots,n$, and between the times t_i the process moves monotonically. For any integrable, positive function $g\colon \mathbb{R} \to \mathbb{R}$ substitution using (4.5) yields

$$\begin{aligned}
\int_0^t g(\omega_s)\,ds &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g(\omega_s)\,ds \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} g(x) \frac{dx}{\operatorname{sgn}(x_i - x_{i-1})\sqrt{2(E+\lambda v(x))}} \\
&= \int_T \frac{g(x)}{\sqrt{2(E+\lambda v(x))}}\,dx.
\end{aligned} \tag{4.8}$$

Applying (4.8) to the function $g = v$ gives

$$1 \overset{(4.3b)}{=} \frac{1}{2}\int_0^t v(\omega_s)\,ds \overset{(4.8)}{=} \frac{1}{\sqrt{8}} \int_T \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx.$$

This is equation (4.7a). Applying (4.8) to the constant function $g = 1$ gives

$$t = \int_0^t 1\,ds \overset{(4.8)}{=} \frac{1}{\sqrt{2}} \int_0^a \frac{1}{\sqrt{E+\lambda v(x)}}\,dx,$$

which is equation (4.7b).

Now assume condition (ij). For $i = 1,\dots,n$ define the function F_i by

$$F_i(x) = \frac{1}{\sqrt{2}} \Big| \int_{x_{i-1}}^x \frac{1}{\sqrt{E+\lambda v(x)}}\,dx \Big|$$

for all x between x_{i-1} and x_i. Then F_i is finite because of (4.7b), strictly monotonic (increasing if $x_i > x_{i-1}$ and decreasing else), and has $F_i(x_{i-1}) = 0$. Further define

$$t_k = \sum_{i=1}^{k} F_i(x_i).$$

Equation (4.7b) gives $t_n = t$. Because the functions F_i are monotonic they have inverse functions F_i^{-1} and we can define $\omega\colon [0,t] \to \mathbb{R}$ by

$$\omega(s) = F_i^{-1}(s - t_{i-1}) \quad \text{for all } s \in [t_{i-1}, t_i].$$

We will prove that ω satisfies all the conditions from (j).

Because we have $t_i - t_{i-1} = F_i(x_i)$ and thus $F_i^{-1}(t_i - t_{i-1}) = x_i = F_{i+1}^{-1}(t_i - t_i)$ the function ω is well-defined on the connection points at times t_i and is continuous. This also shows $\omega_{t_i} = x_i$ for $i = 0, 1, \ldots, n$ and especially $\omega_0 = x_0 = z$ and $\omega_t = x_n = a$.

Because the F_i are differentiable at all points x strictly between x_{i-1} and x_i, the function ω is differentiable on the intervals (t_{i-1}, t_i) with derivative

$$\dot\omega_s = \frac{1}{F_i'(\omega_s)} = \operatorname{sgn}(x_i - x_{i-1})\sqrt{2(E + \lambda v(\omega_s))}.$$

Because ω is continuous and the limits $\lim_{s\to t_i} \dot\omega_s$ exist, we see that ω is even differentiable on $[0,t]$ with $\dot\omega_0 = \operatorname{sgn}(x_1 - x_0)\sqrt{2(E + \lambda v(0))}$ and $\dot\omega_{t_i} = 0$ for $i = 1, \ldots, n-1$.

Using the same kind of argument again, we find

$$\ddot\omega_s = \frac{\operatorname{sgn}(x_i - x_{i-1})}{2\sqrt{2(E - \lambda v(\omega_s))}} 2\lambda v'(\omega_s) \operatorname{sgn}(x_i - x_{i-1})\sqrt{2(E - \lambda v(\omega_s))} = \lambda v'(\omega_s),$$

first between the t_i and then on the whole interval $[0,t]$. Thus ω really solves the differential equation from (j).

Using the substitution

$$\frac{1}{2}\int_0^t v(\omega_s)\,ds = \frac{1}{\sqrt{8}} \int_T \frac{v(x)}{\sqrt{E + \lambda v(x)}}\,dx$$

as in the first part, we also get back (4.3b) from (4.7a). $\square$

Now we have reduced the problem of minimising $I_t(\omega)$ over the solutions ω of the system (4.3) to the problem of minimising

$$I_t(E,\lambda) = tE + 2\lambda$$

over the solutions (E,λ) of the system (4.7).

For a trace T define

$$H_T = \big\{(E,\lambda) \,\big|\, E \geq -\inf_{x\in T} \lambda v(x)\big\} \subseteq \mathbb{R}^2$$

and furthermore define the functions $f, g\colon H_t \to [0,\infty]$ by

$$f(E,\lambda) = \int_T \frac{1}{\sqrt{E + \lambda v(x)}}\,dx$$

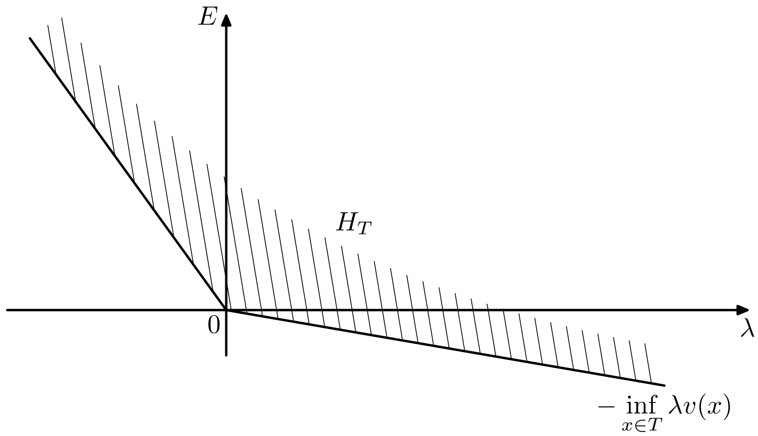

Figure 2: *This figure illustrates the domain H_T of the functions f and g. The domain is unbounded in directions $\lambda \to \infty$ and $E \to \infty$. It is bounded from below by $\lambda \mapsto -\inf_{x\in T} \lambda v(x)$, which is equal to $-\lambda \sup_{x\in T} v(x)$ for $\lambda \le 0$ and to $-\lambda \inf_{x\in T} v(x)$ for $\lambda \ge 0$.*

and

$$g(E,\lambda) = \int_T \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx.$$

Figure 2 illustrates the domain H_T. Both functions are finite in the interior of the domain, but can be infinite at the boundary. The equations (4.7) are equivalent to $f(E_\lambda,\lambda) = \sqrt{2}t$ and $g(E,\lambda) = \sqrt{8}$. For paths which change direction at some point we will find solutions (E,λ) of (4.7), which lay on the boundary of H_T. For paths which go straight from z to a we will find solutions (E,λ) in the interior of H_T.

Lemma 13. *Let $t > 0$ and T be a trace from $z \in \mathbb{R}$ to $a \in \mathbb{R}$ such that $v|_T$ is not constant. Then there is at most one solution (E,λ) of* (4.7).

Proof. For $E > -\inf_{x\in T} \lambda v(x)$ we can choose an E_* between $-\inf_{x\in T} \lambda v(x)$ and E. Then $v(x)/(E_*+\lambda v(x))^{3/2}$ is an integrable upper bound of $v(x)/(e+\lambda v(x))^{3/2}$ for all e in a $(E-E_*)$-Neighbourhood of E. So we can use the theorem about interchanging the Lebesgue-integral with derivatives to get

$$\frac{\partial}{\partial E} g(E,\lambda) = -\frac{1}{2}\int_T \frac{v(x)}{\big(E+\lambda v(x)\big)^{3/2}}\,dx < 0.$$

So for every λ the map $E \mapsto g(E,\lambda)$ is strictly decreasing and there can be at most one E_λ with $g\big(E_\lambda,\lambda\big) = \sqrt{8}$.

With the help of the implicit function theorem we can calculate the derivative of E_λ. Interchanging the integral with the derivative as above we get

$$\begin{aligned}\frac{\partial}{\partial\lambda}E_\lambda &= -\frac{\frac{\partial}{\partial\lambda}g(E_\lambda,\lambda)}{\frac{\partial}{\partial E}g(E_\lambda,\lambda)}\\ &= -\frac{(-\frac{1}{2})\int_T v^2(x)\big(E_\lambda+\lambda v(x)\big)^{-3/2}\,dx}{(-\frac{1}{2})\int_T v(x)\big(E_\lambda+\lambda v(x)\big)^{-3/2}\,dx}\\ &= -\frac{\int_T v^2(x)\,d\mu(x)}{\int_T v(x)\,d\mu(x)}\end{aligned}$$

where μ is the probability measure, with density

$$\frac{d\mu}{dx} = \frac{1}{Z}\big(E_\lambda+\lambda v(x)\big)^{-3/2}$$

and the normalisation constant is

$$Z = \int_T \big(E_\lambda+\lambda v(y)\big)^{-3/2}\,dy.$$

Furthermore for $(E,\lambda)\in(H_T)^\circ$ we have

$$\frac{\partial}{\partial E}f(E,\lambda) = -\frac{1}{2}\int_T \big(E+\lambda v(x)\big)^{-3/2}\,dx = -\frac{Z}{2}$$

and thus

$$\begin{aligned}\frac{\partial}{\partial\lambda}\big(f(E_\lambda,\lambda)\big) &= \frac{\partial f}{\partial E}(E_\lambda,\lambda)\frac{\partial}{\partial\lambda}E_\lambda + \frac{\partial f}{\partial\lambda}(E_\lambda,\lambda)\\ &= \frac{Z}{2}\frac{\int_T v^2(x)\,d\mu(x)}{\int_T v(x)\,d\mu(x)} - \frac{Z}{2}\int_T v(x)\,d\mu(x)\\ &= \frac{Z}{2}\frac{\int_T v^2(x)\,d\mu(x) - \big(\int_T v(x)\,d\mu(x)\big)^2}{\int_T v(x)\,d\mu(x)}\\ &\geq 0.\end{aligned}$$

Equality would only hold for the case of constant $v|_T$. So the map $\lambda\mapsto f(E_\lambda,\lambda)$ is strictly increasing and there can be at most one λ with $f\left(E_\lambda,\lambda\right)=\sqrt{2}t$. This completes the proof. ☐

Lemma 14. *Let T a trace with $m\in T$ and $t\geq 2|T|/\int_T v(x)\,dx$. Then equation (4.7) has a solution (E,λ) with with $E,\lambda>0$.*

Proof. Define $\lambda^* = (\int_T \sqrt{v(x)}\,dx)^2/8$ and assume $0<\lambda\leq\lambda^*$. Then we have

$$g(0,\lambda) = \int_T \frac{v(x)}{\sqrt{\lambda v(x)}}\,dx = \frac{1}{\sqrt{\lambda}}\int_T \sqrt{v(x)}\,dx \geq \sqrt{8}.$$

and the dominated convergence theorem gives

$$\lim_{E\to\infty} g(E,\lambda) = 0.$$

Thus for all $0 < \lambda \leq \lambda^*$ there exists an $E_\lambda \geq 0$ with $g(E_\lambda, \lambda) = \sqrt{8}$.

Because of $g(0,\lambda^*) = \sqrt{8}$ we have $E_{\lambda^*} = 0$. Fatou's lemma then gives

$$\liminf_{\lambda\uparrow\lambda^*} f(E_\lambda,\lambda) \geq \int_T \frac{1}{\sqrt{\lambda^* v(x)}}\, dx.$$

Because v is positive and $v(m) = 0$, we have $v'(m) = 0$ and $v''(m) \geq 0$. Then by Taylor's theorem there exists a $c > 0$ and a closed interval $I \subseteq \mathbb{R}$ with $m \in I \subseteq T$, such that $v(x) \leq c^2(x-m)^2$ for all $x \in I$. Therefore we find

$$\int_T \frac{1}{\sqrt{v(x)}}\, dx \geq \int_I \frac{1}{\sqrt{c^2(x-m)^2}}\, dx = \int_I \frac{1}{c|x-m|}\, dx = +\infty$$

and thus $\lambda \mapsto f(E_\lambda, \lambda)$ is a continuous function with

$$\lim_{\lambda\uparrow\lambda^*} f(E_\lambda,\lambda) = +\infty.$$

On the other hand because of $g(E_0, 0) = \sqrt{8}$ we have $E_0 = (\int_T v(x)\, dx)^2/8$. So for $\lambda = 0$ we get

$$f(E_0, 0) = \int_T \frac{1}{\sqrt{E_0}}\, dx = \frac{\sqrt{8}}{\int_T v(x)\, dx}|T|.$$

Together this shows that for all

$$t \geq \frac{2|T|}{\int_T v(x)\, dx}$$

there exists a solution (E_λ, λ) with $f(E_\lambda, \lambda) = \sqrt{2}t$. $\square$

Lemma 15. *There are numbers $\varepsilon, c_1, c_2 > 0$ such that the following holds: For every trace T starting in K_1, ending in K_2, and visiting the ball $B_\varepsilon(m)$ there is a non-empty, closed interval $A \subseteq \mathbb{R}$, such that $A \subseteq T$, $|A| = \varepsilon$ and we have $c_1 \leq v(x) \leq c_2$ for every $x \in A$.*

Proof. Because $m \notin K_1 \cap K_2$ either K_1 or K_2 has a positive distance from m. Let ε be one third of this distance. Define $A' = \{x \in \mathbb{R} \mid \varepsilon \leq |x-m| \leq 2\varepsilon\}$ and let $c_1 = \inf\{v(x) \mid x \in A'\}$ and $c_2 = \sup\{v(x) \mid x \in A'\}$.

Each trace starting in K_1, ending in K_2, and visiting the ball $B_\varepsilon(m)$ either crosses $[m-2\varepsilon, m-\varepsilon]$ or $[m+\varepsilon, m+2\varepsilon]$. Let A be the crossed interval. Then clearly $|A| = \varepsilon$ and and because of $A \subseteq A'$ the estimates for v hold on A. $\square$

Lemma 16. *For every $\eta > 0$ there is a $t_1 > 0$, such that whenever $t \geq t_1$, T is a trace from $z \in K_1$ to $a \in K_2$ with $m \in [z,a]$ and (E,λ) solves* (4.7)*, then we have*

$$\left| I_t(E,\lambda) - \frac{1}{4}\Big(\int_T \sqrt{v(x)}\, dx\Big)^2 \right| \leq \eta.$$

Proof. This case is illustrated in the left hand image of figure 1. Because of $m \in [z,a]$, any path from z to a must visit m and thus we find $E > -\lambda v(m) = 0$. Thus the only possible trace in this case is $T = (z,a)$, because the process could only turn at points x where $-\lambda v(x) = E$.

Now let $\eta > 0$. Define $L = \sup\{|a-z| \bigm| z \in K_1, a \in K_2\}$. Then we get

$$\sqrt{2}t = \int_T \frac{1}{\sqrt{E+\lambda v(x)}}\,dx \le \int_z^a \frac{1}{\sqrt{E}}\,dx \le \frac{L}{\sqrt{E}}$$

and thus

$$E \le \frac{L^2}{2t^2}.$$

So we can find a $t_1 > 0$ with

$$Et < \eta \tag{4.9}$$

whenever $t \ge t_1$.

Choosing A, c_1, and c_2 as in lemma 15 we get

$$\sqrt{8} = \int_T \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx \ge \int_A \frac{c_1}{\sqrt{E+\lambda c_2}}\,dx = \frac{c_1|A|}{\sqrt{E+\lambda c_2}}$$

and thus

$$\lambda \ge \frac{c_1^2|A|^2 - E}{8c_2} \ge \frac{c_1^2|A|^2 - L^2/2t^2}{8c_2}.$$

So we can choose a small $c_3 > 0$ and increase t_1 to achieve $\lambda > c_3$ whenever $t \ge t_1$.

Because of

$$\lim_{E\downarrow 0} \int_T \frac{v(x)}{\sqrt{E+v(x)}}\,dx = \int_T \sqrt{v(x)}\,dx$$

we can find a $c_4 > 0$ with

$$\int_T \frac{v(x)}{\sqrt{E+v(x)}}\,dx \ge \sqrt{1-\eta/J(z,a)} \int_T \sqrt{v(x)}\,dx$$

for all $E \le c_4$. Increase t_1 until

$$\frac{L^2}{2t^2 c_3} < c_4$$

and thus

$$\begin{aligned}
\sqrt{8} &= \int_T \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx \\
&\ge \frac{1}{\sqrt{\lambda}} \int_T \frac{v(x)}{\sqrt{L^2/2t^2\lambda + v(x)}}\,dx \\
&\ge \frac{1}{\sqrt{\lambda}} \int_T \frac{v(x)}{\sqrt{c_4 + v(x)}}\,dx \\
&\ge \frac{1}{\sqrt{\lambda}} \sqrt{1-\eta/J(z,a)} \int_T \sqrt{v(x)}\,dx
\end{aligned}$$

for all $t \geq t_1$. Solving this for λ we get

$$2\lambda \geq (1-\eta/J(z,a))J(z,a) = J(z,a)-\eta. \tag{4.10}$$

Because E is positive we also find

$$\sqrt{8} = \int_T \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx \leq \frac{1}{\sqrt{\lambda}}\int_T \sqrt{v(x)}\,dx$$

and thus

$$2\lambda \leq J(z,a). \tag{4.11}$$

For the rate function I_t equation (4.10) gives

$$I_t(E,\lambda) = Et + 2\lambda \geq J(z,a) - \eta$$

and equations (4.9) and (4.11) give

$$I_t(E,\lambda) = Et + 2\lambda \leq J(z,a) + \eta$$

for all $t > t_1$. □

Lemma 17. *For every $\eta > 0$ there is a $t_2 > 0$, such that whenever $t \geq t_2$, T is a trace from $z \in K_1$ to $a \in K_2$ with $m \notin [z,a]$, and (E,λ) solves (4.7), then we have*

$$\left| I_t(E,\lambda) - \frac{1}{4}\Big(\int_T \sqrt{v(x)}\,dx\Big)^2 \right| \leq \eta.$$

Proof. This case is illustrated in the right hand image of figure 1. Because the path has to change direction we will have $E < 0$ in this case. Without loss of generality we can assume that $m < a,z$. We call a value $b \in \mathbb{R}$ admissible if it lies in the interval $(m, \min(a,z))$ and if additionally $v(x) > v(b)$ for all $x > b$ holds. For admissible values b consider the trace $T = (z,b,a)$ and define

$$h_{z,a}(b) = 2\frac{\int_{(z,b,a)} \frac{1}{\sqrt{v(x)-v(b)}}\,dx}{\int_{(z,b,a)} \frac{v(x)}{\sqrt{v(x)-v(b)}}\,dx}.$$

Using Taylor approximation as in lemma 14, one sees that for $b \to m$ the numerator converges to $+\infty$ and by dominated convergence the denominator converges to $\int_{(0,m,a)} \sqrt{v(x)}\,dx$. So h is a continuous function with $h_{z,a}(b) \to \infty$ for $b \to m$.

Let ε, c_1, and c_2 and A be as in lemma 15. We would like to find a $b \in B_\varepsilon(m)$ with $h_{z,a}(b) = t$, so we need an upper bound on

$$\inf_{b\in(m,m+\varepsilon)} h_{a,z}(b) \tag{4.12}$$

which is uniform in a and z. We find

$$h_{z,a}(b) \leq 2\frac{\sup_{z\in K_1, a\in K_2} \int_{(z,b,a)} \frac{1}{\sqrt{v(x)-v(b)}}\,dx}{\int_A \frac{c_1}{\sqrt{c_2}}\,dx}. \tag{4.13}$$

Because $v''(m) > 0$ and $\liminf_{|x|\to\infty} v(x) > 0$, we can decrease ε to ensure that $v'(x) \geq v''(m)(x-m)/2$ for all $x \in [m, m+\varepsilon]$ and $v(x) \geq v(m+\varepsilon)$ for all $x \geq m+\varepsilon$. Using Taylor's theorem again we get

$$v(x) - v(b) = v'(\xi)(x-b) \geq \frac{v''(m)(b-m)}{2}(x-b)$$

for some $\xi \in [b, x]$ for all $x \in [m, m+\varepsilon]$. Thus we can conclude

$$\begin{aligned}
&\int_{(z,b,a)} \frac{1}{\sqrt{v(x)-v(b)}}\,dx \\
&\quad\leq 2\int_b^{m+\varepsilon} \frac{1}{\sqrt{\frac{v''(m)(b-m)}{2}(x-b)}}\,dx \\
&\qquad + \int_{m+\varepsilon}^{z} \frac{1}{\sqrt{v(m+\varepsilon)-v(b)}}\,dx + \int_{m+\varepsilon}^{a} \frac{1}{\sqrt{v(m+\varepsilon)-v(b)}}\,dx \\
&\quad\leq 2\sqrt{\frac{2}{v''(m)(b-m)}}\sqrt{m+\varepsilon-b} \\
&\qquad + 2\frac{1}{\sqrt{v(m+\varepsilon)-v(b)}} \sup\{|x-m| \mid x \in K_1 \cup K_2\}.
\end{aligned} \tag{4.14}$$

The right hand side of (4.14) is independent of a and z. So we can take the infimum over all $b \in (m, m+\varepsilon)$ and use (4.13) to get the uniform upper bound on (4.12). Call this bound t_2.

Now let $t > t_2$. Then for every $z \in K_1$ and $a \in K_2$ we can find a $b \in (m, m+\varepsilon)$ with $h_{z,a}(b) = t$. Further define $\lambda > 0$ by

$$\sqrt{\lambda} = \frac{1}{\sqrt{8}} \int_{(z,b,a)} \frac{v(x)}{\sqrt{v(x)-v(b)}}\,dx$$

and E by

$$E = -\lambda v(b).$$

Then for the trace $T = (z, b, a)$ these values E and λ solve

$$E + \lambda v(b) = 0,$$

$$\int_{(z,b,a)} \frac{v(x)}{\sqrt{E+\lambda v(x)}}\,dx = \frac{1}{\sqrt{\lambda}} \int_{(z,b,a)} \frac{v(x)}{\sqrt{v(x)-v(b)}}\,dx = \sqrt{8}$$

and

$$\int_{(z,b,a)} \frac{1}{\sqrt{E+\lambda v(x)}}\,dx = \frac{1}{\sqrt{\lambda}} \int_{(z,b,a)} \frac{1}{\sqrt{v(x)-v(b)}}\,dx = \sqrt{2}t.$$

For $t \to \infty$ we have $b \to m$ uniformly in a and z,

$$\lambda \to \frac{1}{8}\Big(\int_{(z,m,a)} \frac{v(x)}{\sqrt{v(x)-v(m)}}\,dx\Big)^2 = \frac{1}{8}\Big(\int_{(z,m,a)} \sqrt{v(x)}\,dx\Big)^2,$$

and again $E \to 0$ (this time from below). This gives

$$I_t(E,\lambda) = \frac{1}{2}\int_T \sqrt{2(E+\lambda v(x))}\,dx \to \frac{1}{4}\Big(\int_{(z,m,a)} \sqrt{v(x)}\,dx\Big)^2$$

which proves the lemma. □

With all these preparations in place we are now ready to calculate the asymptotic lower bound from lemma 8.

Proof. (of lemma 8) Because of lemma 11 we can restrict ourselves to the case $\beta = 1$, i.e. we have to prove

$$\lim_{t\to\infty} \inf\{I_t(\omega) \bigm| \omega \in M_t^{a,z,1}\} = J(a,z)$$

locally uniformly in $a,z \in \mathbb{R}$.

Let $K_1, K_2 \subseteq \mathbb{R}$ be compact with $0 \notin K_1 \cap K_2$ and $\eta > 0$. Furthermore let $z \in K_1$ and $a \in K_2$.

Assume first the case $m \in [z,a]$. From lemma 16 we get a $t_0 > 0$, such that for every $t > t_0$ there exists a solution (E,λ) of (4.7) for the trace $T = (z,a)$ with $\big|I_t(E,\lambda) - J(a,z)\big| \leq \eta$. This t_0 only depends on K_1 and K_2, but not on z and a.

Now assume the case $m \notin [z,a]$. From lemma 17 we again get a $t_0 > 0$, such that for every $t > t_0$ there exists a solution (E,λ) of (4.7) for a trace $T = (z,x_1,a)$ with $\big|I_t(E,\lambda) - J(a,z)\big| \leq \eta$ and t_0 only depends on K_1 and K_2, but not on z and a.

In either case we can use lemma 12 to conclude, that there exists an ω, which solves (4.3a), (4.3b), and (4.3c). Because of (4.6) this path has

$$\big|I_t(\omega) - J(a,z)\big| \leq \eta.$$

Let $c = \inf\{I_t(\omega) \bigm| \omega \in M_t^{a,z,1}\}$. Because the path ω constructed just now is both, in $M_t^{a,z,1}$ and absolutely continuous, we have $c < \infty$. Let $M_n = M_t^{a,z,1} \cap \{\,\omega \mid I_t(\omega) < c + 1/n\,\}$. Because $M_t^{a,z,1}$ is closed and I_t is a good rate function, the sets M_n are compact, non-empty, and satisfy $M_n \supseteq M_{n+1}$ for every $n \in \mathbb{N}$. So the intersection $M = \bigcap_{n\in\mathbb{N}} M_n$ is again non-empty. Because every $\tilde\omega \in M$ has $I_t(\tilde\omega) = c$, we see that there in fact exists a path $\tilde\omega$ for which the infimum is attained. From the Euler-Lagrange method we know that $\tilde\omega$ also solves equations (4.3a), (4.3b), and (4.3c). From lemmas 12 and 13 we know that the solution is unique, so $\tilde\omega$ must coincide with our path ω constructed above and we get

$$\Big|\inf\{I_t(\omega) \bigm| \omega \in M_t^{a,z,1}\} - J(a,z)\Big| \leq \eta$$

for all $z \in K_1$, $a \in K_2$ and $t \geq t_0$. Since $\eta > 0$ was arbitrary this completes the proof of lemma 8. □

5 Staying Near the Equilibrium

In this section we study the event that for some drift function b the integral $\frac{1}{2}\int_0^t b^2(B_s)\,ds$ is small. In contrast to the previous section, here we are considering long time intervals but have no conditions on the final point. The main result of this section are the following two propositions.

Proposition 18. *Let $b\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $b(0) = 0$, $b'(0) \neq 0$ and $\liminf_{|x|\to\infty} |b(x)| > 0$. Then for every $\eta > 0$ we have*

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \varepsilon \log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon,\ \sup_{0\le s\le t} |B_s| \le \eta\Big) \\
&\qquad = \lim_{\varepsilon\downarrow 0} \varepsilon \log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon\Big) = -\frac{|b'(0)|^2 t^2}{16}.
\end{aligned}$$

Proposition 19. *Let $b\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $b(0) = 0$, $b'(0) \neq 0$ and $\liminf_{|x|\to\infty} |b(x)| > 0$. Then for every $\eta > 0$ we have*

$$\begin{aligned}
&\lim_{\zeta\downarrow 0} \liminf_{\varepsilon\downarrow 0} \varepsilon \log \inf_{-\zeta<z<\zeta} P_z\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon,\ \sup_{0\le s\le t} |B_s| \le \eta\Big) \\
&\qquad = -\frac{|b'(0)|^2 t^2}{16}
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{y\in\mathbb{R}} P_y\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon,\ \sup_{0\le s\le t} |B_s| \le \eta\Big) \\
&\qquad = -\frac{|b'(0)|^2 t^2}{16}.
\end{aligned}$$

The rest of this section is devoted to the proof of these two propositions. The main idea of the proof is to use Taylor approximation around the zero of b to reduce the problem to the case of linear b. We start by proving a result for the case $b(x) = x$.

Lemma 20. *Let B be a one-dimensional Brownian Motion. Then*

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log P_x\left(\int_0^t B_s^2\,ds \le \varepsilon, B_t \in A\right) = -\frac{\big(t + x^2 + \operatorname{ess\,inf}_{z\in A} z^2\big)^2}{8}$$

for every $x \in \mathbb{R}$ and every set A with $P(B_t \in A) > 0$ and in particular

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log P\left(\int_0^t B_s^2\,ds \le \varepsilon\right) = -\frac{t^2}{8}.$$

Proof. Formula (1–1.9.7) from [BS96] gives

$$\begin{aligned}
&\int 1_A(\omega_t) \exp\Big(-\frac{\vartheta^2}{2}\int_0^t \omega_s^2\,ds\Big)\,d\mathbb{W}_x(\omega) \\
&\qquad = \int_A \frac{\sqrt{\vartheta}}{\sqrt{2\pi\sinh(t\vartheta)}} \exp\Big(-\frac{(x^2+z^2)\vartheta\cosh(t\vartheta) - 2xz\vartheta}{2\sinh(t\vartheta)}\Big)\,dz.
\end{aligned}$$

By definition of cosh and sinh there are constants $0 < c_1 < c_2$ with

$$c_1 e^{-t\vartheta/2} \le \frac{1}{\sqrt{2\pi\sinh(t\vartheta)}} \le c_2 e^{-t\vartheta/2} \quad \text{for all } \vartheta > 1. \tag{5.1}$$

(The value 1 is arbitrary, any positive number would do.) Also we can use the relation $|2xy| \leq x^2 + y^2$ to get

$$\frac{(x^2+z^2)}{2}\frac{\cosh(\gamma t)-1}{\sinh(\gamma t)} \leq \frac{(x^2+z^2)\cosh(\gamma t)-2xz}{2\sinh(\gamma t)} \leq \frac{(x^2+z^2)}{2}\frac{\cosh(\gamma t)+1}{\sinh(\gamma t)}$$

for all $x, z \in \mathbb{R}$. Now let $\eta > 0$. Because of

$$\frac{\cosh(\vartheta t) \pm 1}{\sinh(\vartheta t)} = \frac{e^{\vartheta t} + e^{-\vartheta t} \pm 1}{e^{\vartheta t} - e^{-\vartheta t}} \longrightarrow 1 \quad \text{for } \vartheta \to \infty.$$

we can then find a $\vartheta_0 > 0$, such that whenever $\vartheta > \vartheta_0$ the estimate

$$\frac{x^2+z^2}{2}(1-\eta) \leq \frac{(x^2+z^2)\cosh(\vartheta t)-2xz}{2\sinh(\vartheta t)} \leq \frac{x^2+z^2}{2}(1+\eta)$$

holds for all $x, z \in \mathbb{R}$. Thus we can conclude

$$\begin{aligned}
&\lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log E_x\Big(\exp\big(-\frac{\vartheta^2}{2}\int_0^1 B_s^2\,ds\big)1_A(B_t)\Big) \\
&\quad= \lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log \sqrt{\vartheta} \int_A \frac{1}{\sqrt{2\pi\sinh(t\vartheta)}} \exp\Big(-\vartheta\, \frac{(x^2+z^2)\cosh(t\vartheta)-2xz}{2\sinh(t\vartheta)}\Big)\,dz \\
&\quad= \lim_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int_A e^{-t\vartheta/2} \exp\Big(-\vartheta\,\frac{x^2+z^2}{2}\Big)\,dz \\
&\quad= -\frac{1}{2}\operatorname{ess\,inf}_{z\in A}\big(t + x^2 + z^2\big).
\end{aligned}$$

The exponential Tauber theorem [BGT87, theorem 4.12.9] now gives the first equality of the claim. The second claim follows by taking $x = 0$ and $A = \mathbb{R}$. ☐

We will also need a version of lemma 20 which holds uniformly in the initial condition x. This is given in the following lemma.

Lemma 21. *Let B be a one-dimensional Brownian Motion and $A \subseteq \mathbb{R}$ closed. Then*

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in A} P_x\Big(\int_0^t B_s^2\,ds \leq \varepsilon\Big) = -\inf_{x\in A} \frac{(t+x^2)^2}{8}.$$

Proof. Let $x, y \in A$ with $0 < |x| < |y|$. Then the symmetry of Brownian motion and Anderson's inequality [And55, corollary 5] applied to the processes $X = B + |x|$ and $Y = B + |y|$ gives

$$P_x\Big(\int_0^t B_s^2\,ds \leq \varepsilon\Big) \geq P_y\Big(\int_0^t B_s^2\,ds \leq \varepsilon\Big). \tag{5.2}$$

Now choose $x \in A$ with $|x| = \inf\{\,|y| \mid y \in A\}$. Then the estimate (5.2) becomes

$$P_x\Big(\int_0^t B_s^2\,ds \leq \varepsilon\Big) = \sup_{y\in A} P_y\Big(\int_0^t B_s^2\,ds \leq \varepsilon\Big)$$

and the claim follows with lemma 20. ☐

The following lemma gives a set of conditions under which dominated terms can be neglected when calculating large deviation rate functions. The proof is elementary and we omit it here.

Lemma 22. *Let $f, g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be two functions and assume that either one of the two conditions $\limsup_{\varepsilon\downarrow 0} \varepsilon \log g(\varepsilon) \leq \liminf_{\varepsilon\downarrow 0} \varepsilon \log f(\varepsilon)$ or $\limsup_{\varepsilon\downarrow 0} \varepsilon \log g(\varepsilon) < \liminf_{\varepsilon\downarrow 0} \varepsilon \log\big(f(\varepsilon) + g(\varepsilon)\big)$ holds. Then we have*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log\big(f(\varepsilon)+g(\varepsilon)\big) = \liminf_{\varepsilon\downarrow 0} \varepsilon \log f(\varepsilon)$$

and

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log\big(f(\varepsilon)+g(\varepsilon)\big) = \limsup_{\varepsilon\downarrow 0} \varepsilon \log f(\varepsilon).$$

In order to make the Taylor approximation work we need upper bounds on the probability that the process leaves a neighbourhood of the zero of b. This is given by the following lemma.

Lemma 23. *Let B be a Brownian motion, $a, t > 0$, and $v\colon \mathbb{R} \to \mathbb{R}$ be a function with $v(x) \geq x^2 \wedge a^2$ for every $x \in \mathbb{R}$. Then we have*

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in\mathbb{R}} P_x\Big(\int_0^t v(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t} |B_s| > a\Big) \leq -\frac{1}{8}\Big(t + \frac{1}{2}a^2\Big)^2.$$

Proof. We need to find an upper bound on the exponential rate for the probability of the event

$$A^\varepsilon = \Big\{\int_0^t v(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s<t} |B_s| > a\Big\},$$

which is uniform in the initial point $B_0 = x$. First define two interlaced sequences of stopping times $(S_j)_{j\in\mathbb{N}}$ and $(T_j)_{j\in\mathbb{N}_0}$ by letting $T_0 = 0$ and

$$S_j = \inf\big\{s > T_{j-1} \bigm| |B_s| \geq a\big\}$$
$$T_j = \inf\big\{s > S_j \bigm| |B_s| = a/2\big\}$$

for all $j \in \mathbb{N}$. If the initial point $B_0 = x$ has $|x| > a$ we have $S_0 = 0$ and $|B_{S_0}| > a$. Except for this we have $|B_{S_j}| = a$. For $s \in [S_j, T_j]$ we have $|B_s| \geq a/2$ and thus $v(B_s) \geq a^2/4$. Outside these intervals we have $|B_s| < a$ and thus $v(B_s) \geq B_s^2$. Therefore we can conclude

$$\Big\{\int_{S_j}^{T_j} v(B_s)\,ds \leq \varepsilon\Big\} \subseteq \Big\{\int_{S_j}^{T_j} a^2/4\,ds \leq \varepsilon\Big\} = \Big\{T_j - S_j \leq 4\varepsilon/a^2\Big\}$$

and for $d > 0$ also

$$\begin{aligned}\Big\{\int_{T_{j-1}}^{S_j} v(B_s)\,ds \leq \varepsilon, S_j - T_{j-1} \geq d\Big\} &\subseteq \Big\{\int_{T_{j-1}}^{S_j} B_s^2\,ds \leq \varepsilon, S_j - T_{j-1} \geq d\Big\} \\ &\subseteq \Big\{\int_{T_{j-1}}^{T_{j-1}+d} B_s^2\,ds \leq \varepsilon\Big\}.\end{aligned}$$

As an abbreviation define $J = \lceil 2t/a^2 \rceil + 1$ where $\lceil x \rceil = \min\{ n \in \mathbb{N} \mid n \geq x \}$. We want to split the set A^ε into the two parts

$$A^\varepsilon = \big(A^\varepsilon \cap \{T_J \leq t\}\big) \cup \big(A^\varepsilon \cap \{T_J > t\}\big).$$

The first part corresponds to the case that there are at least J excursions up to the level $|B_s| = a$ and then back to $|B_s| = a/2$ before time t. For this case we will get an upper bound on the probability from the fact that the process has to move very fast during the intervals $[S_j, T_j]$. The second part corresponds to the case that there are at most $J-1$ such excursions. This case is more difficult, because we have to take the intervals between the excursions into account.

First consider the case $T_J \leq t$. Here we have

$$A^\varepsilon \cap \{T_J \leq t\} \subseteq \Big\{ \sum_{j=1}^J \int_{S_j}^{T_j} v(B_s)\, ds \leq \varepsilon \Big\} \subseteq \Big\{ \sum_{j=1}^J (T_j - S_j) \leq 4\varepsilon/a^2 \Big\}.$$

Using the strong Markov property for Brownian motion and the reflection principle we find

$$\begin{aligned} P_x\big(T_j - S_j \leq \varepsilon\big) &\leq P\big(\sup_{0 \leq s \leq \varepsilon} B_s > a/2 \big) \\ &= 2P\big(B_\varepsilon > a/2\big) \\ &= 2P\big(\sqrt{\varepsilon} B_1 > a/2\big) \end{aligned}$$

for all $x \in \mathbb{R}$. The basic large deviation result for the standard normal distribution on $\mathbb{R}$ now gives

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \sup_{x \in \mathbb{R}} P_x\big(T_j - S_j \leq \varepsilon\big) \leq -\frac{1}{2}\big(a/2\big)^2 = -\frac{a^2}{8}.$$

In this situation we can apply proposition 6 to get

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \varepsilon \log \sup_{x \in \mathbb{R}} P_x\big(A^\varepsilon \cap \{T_J \leq t\}\big) \\ &\qquad \leq \limsup_{\varepsilon \downarrow 0} \varepsilon \log P_x\Big(\sum_{j=1}^J (T_j - S_j) \leq 4\varepsilon/a^2 \Big) \\ &\qquad = \frac{a^2}{4} \limsup_{\varepsilon \downarrow 0} \varepsilon \log P_x\Big(\sum_{j=1}^J (T_j - S_j) \leq \varepsilon \Big) \\ &\qquad \leq -\frac{a^2}{4} \Big(\sum_{j=1}^J \frac{a}{\sqrt{8}} \Big)^2 \leq -\frac{1}{8}\big(t + \frac{1}{2}a^2\big)^2. \end{aligned} \tag{5.3}$$

Now consider the case $T_J > t$. Choose $n \in \mathbb{N}$ with $n > 2J$ and $\varepsilon > 0$ with $4\varepsilon/a^2 < t/n$. Define $\Delta t = t/n$, the intervals $I_1 = [0, \Delta t]$ and $I_k = \big((k-1)\Delta t, k\Delta t\big]$ for $k = 2, \ldots, n$, the index set

$$Q = \Big\{ (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell \;\Big|\; \ell \in \{1, \ldots, J\}, 1 \leq k_1 \leq \cdots \leq k_\ell \leq n \Big\},$$

and the event

$$A^\varepsilon_{(k_1,\ldots,k_\ell)} = A^\varepsilon \cap \big\{ S_j \in I_{k_j} \text{ for } j = 1, \ldots, \ell \text{ and } S_{\ell+1} > t \big\}.$$

Then we have

$$A^{\varepsilon} \cap \{T_J > t\} = \bigcup_{q \in Q} A_q^{\varepsilon}.$$

Choose $(k_1, \dots, k_\ell) \in Q$. As we have seen above the condition $\int_{S_j}^{T_j} v(B_s)\,ds \le \varepsilon$ implies $T_j - S_j \le 4\varepsilon/a^2 \le \Delta t$. Thus on A_q^{ε} we have

$$S_j - T_{j-1} \ge \max\big((k_j - k_{j-1} - 2)\Delta t, 0\big) =: d_{j-1} \tag{5.4}$$

for $j = 1, \dots, \ell - 1$, where we use the convention $k_0 = 0$. If $k_\ell < n$ then we use 5.4 also for $j = \ell$ and we have

$$t - T_\ell \ge \max\big((n - k_\ell - 2)\Delta t, 0\big) =: d_\ell.$$

For $k_\ell = n$ it will turn out that we need to treat the right endpoint of the interval specially, here we define $d_{\ell-1} = \max\big((n - k_{\ell-1} - 3)\Delta t, 0\big)$.

Let $\delta > 0$ and define $D_{2\ell+1}^{\delta}$ as in (3.3). For $\alpha \in D_{2\ell+1}^{\delta}$ further define

$$\begin{aligned} A_{(k_1,\dots,k_\ell)}^{\alpha\varepsilon} = \Big\{ & \int_{T_0}^{S_1} v(B_s)\,ds \le \alpha_1 \varepsilon, \int_{S_1}^{T_1} v(B_s)\,ds \le \alpha_2 \varepsilon, S_1 \in I_{k_1}, \\ & \vdots \\ & \int_{T_{\ell-1}}^{S_\ell} v(B_s)\,ds \le \alpha_{2\ell-1} \varepsilon, \int_{S_\ell}^{T_\ell} v(B_s)\,ds \le \alpha_{2\ell} \varepsilon, S_\ell \in I_{k_\ell}, \\ & \int_{T_\ell}^{t} v(B_s)\,ds \le \alpha_{2\ell+1} \varepsilon, S_{\ell+1} > t \Big\} \end{aligned}$$

if $k_\ell < n$ and

$$\begin{aligned} A_{(k_1,\dots,k_\ell)}^{\alpha\varepsilon} = \Big\{ & \int_{T_0}^{S_1} v(B_s)\,ds \le \alpha_1 \varepsilon, \int_{S_1}^{T_1} v(B_s)\,ds \le \alpha_2 \varepsilon, S_1 \in I_{k_1}, \\ & \vdots \\ & \int_{T_{\ell-1}}^{S_\ell} v(B_s)\,ds \le \alpha_{2\ell-1} \varepsilon, S_\ell \in I_n, S_{\ell+1} > t \Big\} \end{aligned}$$

else. Then we have

$$A^{\varepsilon} \cap \{T_J > t\} = \bigcup_{q \in Q} A_q^{\varepsilon} \subseteq \bigcup_{q \in Q} \bigcup_{\alpha \in D_{2\ell+1}^{\delta}} A_q^{\alpha\varepsilon}.$$

Assume first the case $k_\ell < n$. Then we get

$$
\begin{aligned}
P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le P_x\Big(&\int_{T_0}^{T_0+d_0} B_s^2\,ds \le \alpha_1\varepsilon, T_1 - S_1 \le 4\alpha_2\varepsilon/a^2, S_1 \in I_{k_1},\\
&\vdots\\
&\int_{T_{\ell-1}}^{T_{\ell-1}+d_{\ell-1}} B_s^2\,ds \le \alpha_{2\ell-1}\varepsilon, T_\ell - S_\ell \le 4\alpha_{2\ell}\varepsilon/a^2, S_\ell \in I_{k_\ell},\\
&\int_{T_\ell}^{T_\ell+d_\ell} B_s^2\,ds \le \alpha_{2\ell+1}\varepsilon, S_{\ell+1} > t\Big).
\end{aligned}
$$

Now we use the strong Markov property of Brownian motion for the stopping times S_j and T_j. Because $|B_{T_j}| = a/2$ and $|B_{S_j}| = a$ are deterministic and the Brownian motion is symmetric we get

$$
\begin{aligned}
P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le P_x\Big(&\int_{T_0}^{T_0+d_0} B_s^2\,ds \le \alpha_1\varepsilon, T_1 - S_1 \le 4\alpha_2\varepsilon/a^2, S_1 \in I_{k_1},\\
&\vdots\\
&\int_{T_{\ell-1}}^{T_{\ell-1}+d_{\ell-1}} B_s^2\,ds \le \alpha_{2\ell-1}\varepsilon, T_\ell - S_\ell \le 4\alpha_{2\ell}\varepsilon/a^2, S_\ell \in I_{k_\ell}\Big)\\
&P_{\frac{a}{2}}\Big(\int_0^{d_\ell} B_s^2\,ds \le \alpha_{2\ell+1}\varepsilon\Big)\\
\le\ P_x\Big(&\int_{T_0}^{T_0+d_0} B_s^2\,ds \le \alpha_1\varepsilon, T_1 - S_1 \le 4\alpha_2\varepsilon/a^2, S_1 \in I_{k_1},\\
&\vdots\\
&\int_{T_{\ell-1}}^{T_{\ell-1}+d_{\ell-1}} B_s^2\,ds \le \alpha_{2\ell-1}\varepsilon, S_\ell \in I_{k_\ell}\Big)\\
&P_0\Big(\sup_{0\le s\le 4\alpha_{2\ell}\varepsilon/a^2} B_s > a/2\Big)\\
&P_{\frac{a}{2}}\Big(\int_0^{d_\ell} B_s^2\,ds \le \alpha_{2\ell+1}\varepsilon\Big).
\end{aligned}
$$

Repeating these two steps for $j = \ell - 1, \dots, 0$ finally gives

$$
\begin{aligned}
P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le P_x\Big(&\int_0^{d_0} B_s^2\,ds \le \alpha_1\varepsilon\Big)\\
&\prod_{j=1}^{\ell} P_{\frac{a}{2}}\Big(\int_0^{d_j} B_s^2\,ds \le \alpha_{2j+1}\varepsilon\Big)\\
&\prod_{j=1}^{\ell} P_0\Big(\sup_{0\le s\le 4\alpha_{2j}\varepsilon/a^2} B_s > a/2\Big).
\end{aligned}
$$

In order to use inequality (3.4) we have to calculate the individual rates for the factors on the right-hand side. Using lemma 21 we get

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in\mathbb{R}} P_x\Big(\int_0^d B_s^2\,ds \le \varepsilon\Big) = -\frac{1}{8}d^2. \tag{5.5}$$

Using the reflection principle and the basic scaling property of Brownian motion we find

$$\begin{aligned} P_0\Big(\sup_{0\le s\le 4\varepsilon/a^2} B_s > a/2\Big) &= 2P\big(B_{4\varepsilon/a^2} > a/2\big) \\ &= 2P\big(\sqrt{4\varepsilon/a^2}B_1 > a/2\big) = 2P\big(\sqrt{\varepsilon}B_1 > a^2/4\big). \end{aligned}$$

The large deviation principle for the standard normal distribution on $\mathbb{R}$ now gives

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log P_0\Big(\sup_{0\le s\le 4\varepsilon/a^2} B_s > a/2\Big) = -\frac{1}{2}\big(a^2/4\big)^2 = -\frac{1}{8}\Big(\frac{a^2}{2}\Big)^2. \tag{5.6}$$

Now we can apply inequality (3.4) to get the combined rate. The result is

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in\mathbb{R}} P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le -\frac{1}{1+\delta}\frac{1}{8}\Big(\sum_{j=0}^{\ell} d_j + n_1\frac{a^2}{4} + \ell\frac{a^2}{2}\Big)^2,$$

where $n_1 = \Big|\{j = 1,\dots,\ell \mid d_j > 0\}\Big|$. Because each of the intervals $[S_j, T_j]$ can have a non-empty intersection with at most two of the n intervals I_k we have $\sum_{j=0}^{\ell} d_j \ge n - 2J$ and thus $n_1 \ge 1$. So we find

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in\mathbb{R}} P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le -\frac{1}{1+\delta}\frac{1}{8}\Big(\frac{n-2J}{n}t + \frac{a^2}{4} + \ell\frac{a^2}{2}\Big)^2 \tag{5.7}$$

for all $\alpha \in D^{\delta}_{2\ell+1}$ and all $\delta > 0$.

Now assume $k_\ell = n$. This case is similar, but needs an additional argument to take care of the case $t \in [S_\ell, T_\ell)$. Here we can no longer use (5.6) for the interval $[S_\ell, T_\ell)$. To work around this we define a stopping time R by

$$R = \inf\big\{s \ge \max(T_{\ell-1}, (n-2)\Delta t) \,\big|\, |B_s| = a/2\big\}.$$

Given the event $A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}$ the process cannot have $|B_s| > a/2$ for a period of time of length Δt and using the special definition of $d_{\ell-1}$ for this case we get $T_\ell - 1 + d_{\ell-1} \le R \le S_\ell$.

Similar to the other case we get then

$$\begin{aligned} P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le P_x\Big(&\int_{T_0}^{T_0+d_0} B_s^2\,ds \le \alpha_1\varepsilon, T_1 - S_1 \le 4\alpha_2\varepsilon/a^2, S_1 \in I_{k_1}, \\ &\vdots \\ &\int_{T_{\ell-2}}^{T_{\ell-2}+d_{\ell-2}} B_s^2\,ds \le \alpha_{2\ell-3}\varepsilon, \\ &\qquad T_{\ell-1} - S_{\ell-1} \le 4\alpha_{2\ell-2}\varepsilon/a^2, S_{\ell-1} \in I_{k_{\ell-1}}, \\ &\int_{T_{\ell-1}}^{T_{\ell-1}+d_{\ell-1}} B_s^2\,ds \le \alpha_{2\ell-1}\varepsilon, S_\ell - R \le 4\alpha_{2\ell}\varepsilon/a^2, S_\ell \in I_n\Big). \end{aligned}$$

Using the strong Markov property for the stopping time R first gives

$$\begin{aligned}
P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le P_x\Big(&\int_{T_0}^{T_0+d_0} B_s^2\,ds \le \alpha_1\varepsilon, T_1 - S_1 \le 4\alpha_2\varepsilon/a^2, S_1 \in I_{k_1},\\
&\vdots\\
&\int_{T_{\ell-2}}^{T_{\ell-2}+d_{\ell-2}} B_s^2\,ds \le \alpha_{2\ell-3}\varepsilon,\\
&\qquad\qquad T_{\ell-1} - S_{\ell-1} \le 4\alpha_{2\ell-2}\varepsilon/a^2, S_{\ell-1} \in I_{k_{\ell-1}},\\
&\int_{T_{\ell-1}}^{T_{\ell-1}+d_{\ell-1}} B_s^2\,ds \le \alpha_{2\ell-1}\varepsilon\Big)\\
P_0\Big(&\sup_{0\le s\le 4\alpha_{2\ell}\varepsilon/a^2} B_s > a/2\Big).
\end{aligned}$$

Now we can continue splitting of terms as in the first case to get

$$\begin{aligned}
P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) \le P_x\Big(&\int_0^{d_0} B_s^2\,ds \le \alpha_1\varepsilon\Big)\\
&\prod_{j=1}^{\ell-1} P_{\frac{a}{2}}\Big(\int_0^{d_j} B_s^2\,ds \le \alpha_{2j+1}\varepsilon\Big)\\
&\prod_{j=1}^{\ell} P_0\Big(\sup_{0\le s\le 4\alpha_{2j}\varepsilon/a^2} B_s > a/2\Big).
\end{aligned}$$

Using equations (5.5), (5.6) and inequality (3.4) as in the first case we get

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in\mathbb{R}} P_x\big(A^{\alpha\varepsilon}_{(k_1,\dots,k_\ell)}\big) &\le -\frac{1}{1+\delta}\Big(\sum_{j=0}^{\ell-1} d_j + n_1\frac{a^2}{4} + \ell\frac{a^2}{2}\Big)^2\\
&\le -\frac{1}{1+\delta}\frac{1}{8}\Big(\frac{n-2J-1}{n}t + \ell\frac{a^2}{2}\Big)^2
\end{aligned} \tag{5.8}$$

for all $\alpha \in D^\delta_{2\ell+1}$ and all $\delta > 0$. Note that in this case $n_1 = 0$ is possible, this occurs in the case $\ell = 1$ and $S_1 \in I_n$, because I_n was the interval we treated specially.

To estimate the upper exponential rate of $A^\varepsilon \cap \{T_J > t\}$ we need to compare all the rates from (5.7) and (5.8). We get

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{x\in\mathbb{R}} P_x\big(A^\varepsilon \cap \{T_J > t\}\big)\\
&\quad = \max_{q\in Q} \max_{\alpha\in D^\delta_{2\ell+1}} \limsup_{\varepsilon\downarrow 0} \varepsilon \log P_x\big(A^{\alpha\varepsilon}_q\big)\\
&\quad \le -\frac{1}{1+\delta}\frac{1}{8}\Big(\frac{n-2J-1}{n}t + \frac{a^2}{2}\Big)^2
\end{aligned}$$

for all $\delta > 0$ and large enough n, where the largest bound came from the case $\ell = 1$, $k_1 = n$. Letting first $\delta \downarrow 0$ and then $n \to \infty$ shows

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log P_x \big(A^\varepsilon \cap \{T_J > t\} \big) \leq \frac{1}{8} \big(t + \frac{a^2}{2} \big)^2. \tag{5.9}$$

This gives the upper bound for $P(A^\varepsilon)$. Using the estimates (5.3) and (5.9) we find

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \sup_{x \in \mathbb{R}} P_x(A^\varepsilon) \leq \frac{1}{8} \big(t + \frac{a^2}{2} \big)^2.$$

This completes the proof of the lemma 23. ☐

Lemma 24. *For every $a > 0$ and every $x \in (-a/\sqrt{2}, +a/\sqrt{2})$ we have*

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \varepsilon \log P_x \Big(\int_0^t B_s^2 \, ds \leq \varepsilon, \sup_{0 \leq s \leq t} |B_s| \leq a \Big) \\ &\qquad = \lim_{\varepsilon \downarrow 0} \varepsilon \log P_x \Big(\int_0^t B_s^2 \, ds \leq \varepsilon \Big) = -\frac{\big(t + x^2\big)^2}{8}. \end{aligned}$$

Proof. The second equality is proved in lemma 20. Applying lemma 23 to the function $v(x) = x^2$ we see that

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \varepsilon \log P_x \Big(\int_0^t B_s^2 \, ds \leq \varepsilon, \sup_{0 \leq s \leq t} |B_s| > a \Big) \\ &\qquad \leq -\frac{1}{8} \Big(t + \frac{1}{2} a^2 \Big)^2 \\ &\qquad < \liminf_{\varepsilon \downarrow 0} \varepsilon \log P_x \Big(\int_0^t B_s^2 \, ds \leq \varepsilon \Big). \end{aligned}$$

Thus we can use lemma 22 to prove the first equality. ☐

Now we can combine the results of the previous lemmas to give the proofs of proposition 18.

Proof. (of proposition 18) Choose some $0 < \delta < |b'(0)|$. Using the Taylor formula $b(x) = b'(0)x + o(x)$ we find an $a > 0$ with

$$\big(|b'(0)| + \delta\big)^2 x^2 \geq b^2(x) \geq \big(|b'(0)| - \delta\big)^2 x^2 \quad \text{for all } x \in [-a, a]. \tag{5.10}$$

Without loss of generality we may assume that a is smaller than η and also small enough to permit $|b(x)| \geq a\big(|b'(0)| - \delta\big)$ for all $x \in \mathbb{R}$ with $|x| > a$.

We have to calculate the exponential rates of

$$\begin{aligned} P\Big(\frac{1}{2} \int_0^t b^2(B_s) \, ds \leq \varepsilon \Big) &= P\Big(\frac{1}{2} \int_0^t b^2(B_s) \, ds \leq \varepsilon, \sup_{0 \leq s \leq t} |B_s| \leq a \Big) \\ &\qquad + P\Big(\frac{1}{2} \int_0^t b^2(B_s) \, ds \leq \varepsilon, \sup_{0 \leq s \leq t} |B_s| > a \Big). \end{aligned} \tag{5.11}$$

Whenever $\sup_{0\le s\le t}|B_s|\le a$ we can approximate $b(x)$ by $b'(0)x$ as in (5.10). This gives

$$\begin{aligned}
&P\Big(\frac{1}{2}\int_0^t \big(|b'(0)|+\delta\big)^2 B_s^2\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|\le a\Big)\\
&\quad\le P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|\le a\Big)\\
&\quad\le P\Big(\frac{1}{2}\int_0^t \big(|b'(0)|-\delta\big)^2 B_s^2\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|\le a\Big).
\end{aligned}$$

Both bounds of this estimate can be handled using

$$\lim_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\int_0^t cB_s^2\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|\le a\Big)=-c\frac{t^2}{8},$$

which is a consequence of lemma 24.

For the lower bound this gives

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon\Big)\\
&\quad\ge\liminf_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|\le a\Big)\\
&\quad\ge-\frac{\big(|b'(0)|+\delta\big)^2}{16}t^2
\end{aligned}$$

whenever $\delta>0$. For the upper bound we find

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|\le a\Big)\le-\frac{\big(|b'(0)|-\delta\big)^2}{16}t^2. \tag{5.12}$$

Define $v(x)=b^2(x)/\big(|b'(0)|-\delta\big)^2$. Then by our choice of a we have $v(x)\ge x^2\wedge a^2$ and lemma 23 gives

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|>\eta\Big)\\
&\quad\le\limsup_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon,\ \sup_{0\le s\le t}|B_s|>a\Big)\\
&\quad\le-\frac{1}{8}\big(t+\frac{1}{2}a^2\big)^2\frac{\big(|b'(0)|-\delta\big)^2}{2}\\
&\quad<-\frac{\big(|b'(0)|-\delta\big)^2}{16}t^2.
\end{aligned}\tag{5.13}$$

Using only the last three lines of equation (5.13) we see that the upper bound for (5.11) is dominated by (5.12) and from lemma 22 we get

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds\le\varepsilon\Big)\le-\frac{\big(|b'(0)|-\delta\big)^2}{16}t^2$$

for all $\delta > 0$. Letting $\delta \downarrow 0$ completes the proof of

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon\Big) = -\frac{|b'(0)|^2 t^2}{16}.$$

Utilising lemma 22 again, but this time with the full equation (5.13) also proves the first equality of the proposition's claim. □

In order to prove proposition 19 we need an additional coupling argument.

Lemma 25. *Given $x, y \in \mathbb{R}$ with $|x| \ge |y|$ we can choose two Brownian motions B^x and B^y on a common probability space with $B_0^x = x$, $B_0^y = y$, and $|B_t^x| \ge |B_t^y|$ for all $t \ge 0$.*

Proof. Let B^x be any Brownian motion with start in x and B be another one on the same probability space, but with start in y. Define the stopping time T by

$$T = \inf\big\{t \ge 0 \,\big|\, |B_t^x| = |B_t|\big\}$$

and the random variable σ by $\sigma = 1$ if $B_T^x = B_T$ and $\sigma = -1$ else. Then the process B^y defined by

$$B_t^y = \begin{cases} B_t & \text{if } t \le T, \text{ and} \\ B_T + \sigma(B_t^x - B_T^x) & \text{if } t > T \end{cases}$$

is a Brownian motion with $|B_t^y| < |B_t^x|$ for $t < T$ and either $B_t^y = B_t^x$ or $B_t^y = -B_t^x$ for $t \ge T$. This proves the claim. □

Proof. (of proposition 19) We start by proving the claim about the lim inf. Using proposition 18 we find

$$\begin{aligned} &\liminf_{\varepsilon \downarrow 0} \varepsilon \log \inf_{-\zeta < z < \zeta} P_z\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon,\ \sup_{0 \le s \le t} |B_s| \le \eta\Big) \\ &\le \lim_{\varepsilon \downarrow 0} \varepsilon \log P_0\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon,\ \sup_{0 \le s \le t} |B_s| \le \eta\Big) \\ &= -\frac{|b'(0)|^2 t^2}{16} \end{aligned}$$

for every $\zeta > 0$.

Now let $\kappa > 0$ and choose a $\delta > 0$ with

$$-\big(|b'(0)| + \delta\big)^2 \frac{(t + \delta^2)^2}{16} > -\frac{|b'(0)|^2 t^2}{16} - \kappa.$$

As in the proof of proposition 18 we can use Taylor approximation to find an $a > 0$ with

$$b^2(x) \le \big(|b'(0)| + \delta\big)^2 x^2$$

for all $x \in [-a, a]$. Without loss of generality we may assume $a \le \min(2\delta, \eta)$.

Let $\zeta < a/2$ and $z \in [-\zeta, +\zeta]$. Then we can use lemma 25 to choose two Brownian motions B^ζ and B^z with $B_0^\zeta = \zeta$, $B_0^z = z$, and $|B_t^\zeta| \geq |B_t^z|$ for all $t \geq 0$. We find

$$\begin{aligned}
&P\Big(\frac{1}{2}\int_0^t b^2(B_s^z)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s^z| \leq \eta\Big)\\
&\qquad \geq P\Big(\frac{1}{2}\int_0^t b^2(B_s^z)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s^z| \leq a\Big)\\
&\qquad \geq P\Big(\frac{1}{2}\int_0^t \big(|b'(0)|+\delta\big)^2 (B_s^z)^2\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s^z| \leq a\Big)\\
&\qquad \geq P\Big(\frac{1}{2}\int_0^t \big(|b'(0)|+\delta\big)^2 (B_s^\zeta)^2\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s^\zeta| \leq a\Big)
\end{aligned}$$

for every $z \in [-\zeta, +\zeta]$, and thus

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \varepsilon \log \inf_{-\zeta<z<\zeta} P_z\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s| \leq \eta\Big)\\
&\qquad \geq \liminf_{\varepsilon\downarrow 0} \varepsilon \log P_\zeta\Big(\frac{1}{2}\int_0^t \big(|b'(0)|+\delta\big)^2 B_s^2\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s| \leq a\Big)\\
&\qquad = \frac{1}{2}\big(|b'(0)|+\delta\big)^2 \liminf_{\varepsilon\downarrow 0} \varepsilon \log P_\zeta\Big(\int_0^t B_s^2\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s| \leq a\Big).
\end{aligned}$$

Because $\zeta < a/2 < \delta$ we can use lemma 24 to get

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \varepsilon \log \inf_{-\zeta<z<\zeta} P_z\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s| \leq \eta\Big)\\
&\qquad \geq -\frac{1}{2}\big(|b'(0)|+\delta\big)^2 \frac{(t+\zeta^2)^2}{8}\\
&\qquad \geq -\frac{1}{2}\big(|b'(0)|+\delta\big)^2 \frac{(t+\delta^2)^2}{8}\\
&\qquad > -\frac{|b'(0)|^2 t^2}{16} - \kappa
\end{aligned}$$

for all sufficiently small $\kappa > 0$. Letting $\zeta \downarrow 0$ completes the proof of the first claim.

For the second claim first note that

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{y\in\mathbb{R}} P_y\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s| \leq \eta\Big)\\
&\qquad \geq \limsup_{\varepsilon\downarrow 0} \varepsilon \log P_0\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t}|B_s| \leq \eta\Big)\\
&\qquad = -\frac{|b'(0)|^2 t^2}{16},
\end{aligned}$$

again by proposition 18.

Let $\kappa > 0$ and choose $\delta > 0$ with

$$-\big(|b'(0)| - \delta\big)^2 \frac{t^2}{16} < -\frac{|b'(0)|^2 t^2}{16} + \kappa.$$

Using Taylor approximation we can find an $a > 0$ with

$$b^2(x) \geq \big(|b'(0)| - \delta\big)^2 x^2$$

for all $x \in [-a, a]$ and by choosing a small enough we can find a smooth, antisymmetric, monotone function $\varphi\colon \mathbb{R} \to \mathbb{R}$ with $|b(x)| \geq |\varphi(x)|$ for all $x \in \mathbb{R}$ and $\varphi'(0) = |b'(0)| - \delta$.

Using the coupling argument and proposition 18 again, we get

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{y\in\mathbb{R}} P_y\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t} |B_s| \leq \eta\Big) \\
&\quad\leq \limsup_{\varepsilon\downarrow 0} \varepsilon \log \sup_{y\in\mathbb{R}} P_y\Big(\frac{1}{2}\int_0^t \varphi^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t} |B_s| \leq \eta\Big) \\
&\quad\leq \limsup_{\varepsilon\downarrow 0} \varepsilon \log P_0\Big(\frac{1}{2}\int_0^t \varphi^2(B_s)\,ds \leq \varepsilon,\ \sup_{0\leq s\leq t} |B_s| \leq \eta\Big) \\
&\quad= -\frac{\Big(|b'(0)| - \delta\Big)^2 t^2}{16} \\
&\quad< -\frac{|b'(0)|^2 t^2}{16} + \kappa
\end{aligned}$$

for all $\kappa > 0$. Taking the limit $\kappa \downarrow 0$ completes the proof of proposition 19. □

6 The LDP for the Endpoint

In this section we use the results of the previous section to complete the proof of theorem 1.

Notation. To avoid complicated and hard to read expressions in small print we sometimes write (A) for the indicator function of the event A during this section.

Lemma 26. *Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a C^2-function with bounded Φ'' and let $b = -\Phi'$. Assume that there is an $m \in \mathbb{R}$ with $b(x) = 0$ if and only if $x = m$ and $\liminf_{|x|\to\infty} |b(x)| > 0$. Further assume that there is a rate function $I\colon \mathbb{R} \to [0, \infty]$ with*

$$\liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log E\big(\exp(-\frac{\vartheta^2}{2}\int_0^t b^2(\omega_s)\,ds) 1_O(B_t)\big) \geq -\inf_{x\in O} I(x)$$

for every open set $O \subseteq \mathbb{R}$ and

$$\limsup_{\vartheta\to\infty} \frac{1}{\vartheta} \log E\big(\exp(-\frac{\vartheta^2}{2}\int_0^t b^2(\omega_s)\,ds) 1_K(B_t)\big) \leq -\inf_{x\in K} I(x)$$

for every compact set $K \subseteq \mathbb{R}$. For $\vartheta > 0$ let X^ϑ be a solution of the SDE (1.1) *with start in $X_0^\vartheta = 0$. Then for $\vartheta \to \infty$ the family $(X_t^\vartheta)_\vartheta$ satisfies the weak LDP with rate function J, where J is defined by*

$$J(x) = \Phi(x) - \Phi(0) - \frac{1}{2} t \Phi''(m) + I(x).$$

Proof. First let O be open, $x \in O$ and $\delta > 0$. Then we can find an η with $0 < \eta < \delta$, $B_\eta(x) \subseteq O$, and $|\Phi(y) - \Phi(x)| \le \delta$ for all $y \in B_\eta(x)$. Define

$$F^*(x) = \Phi(0) - \Phi(x) + \frac{1}{2} t \Phi''(m).$$

Let F and G be as in (3.1). Then we find

$$\begin{aligned}
&\liminf_{\vartheta \to \infty} \frac{1}{\vartheta} \log P(X_t^\vartheta \in O) \\
&\quad \ge \liminf_{\vartheta \to \infty} \frac{1}{\vartheta} \log P(X_t^\vartheta \in B_\eta(x)) \\
&\quad = \liminf_{\vartheta \to \infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(\vartheta F(\omega) - \vartheta^2 G(\omega)\bigr)\, d\mathbb{W}(\omega) \\
&\quad \ge \liminf_{\vartheta \to \infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(\vartheta(F^*(x) - 2\delta) - \vartheta^2 G(\omega)\bigr) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \bigl(|F(\omega) - F^*(x)| \le 2\delta\bigr)\, d\mathbb{W}(\omega) \\
&\quad = F^*(x) - 2\delta + \liminf_{\vartheta \to \infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(-\vartheta^2 G(\omega)\bigr) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \bigl(|F(\omega) - F^*(x)| \le 2\delta\bigr)\, d\mathbb{W}(\omega).
\end{aligned}$$

By definition of $F^*(x)$ we have

$$\begin{aligned}
\bigl|F(\omega) - F^*(x)\bigr| &= \Bigl|\Phi(0) - \Phi(\omega_t) + \frac{1}{2} \int_0^t \Phi''(\omega_s)\, ds \\
&\qquad\qquad - \Phi(0) + \Phi(x) - \frac{1}{2} t \Phi''(m)\Bigr| \\
&\le \bigl|\Phi(x) - \Phi(\omega_t)\bigr| + \frac{1}{2} \int_0^t \bigl|\Phi''(\omega_s) - \Phi''(m)\bigr|\, ds.
\end{aligned}$$

Thus whenever $\omega_t \in B_\eta(x)$ and $\bigl|F(\omega) - F^*(x)\bigr| \ge 2\delta$ we find

$$\frac{1}{2} \int_0^t \bigl|\Phi''(\omega_s) - \Phi''(m)\bigr|\, ds \ge 2\delta - \delta = \delta.$$

Because Φ'' is bounded the above estimate implies that we can find an $\varepsilon > 0$ with

$$\Bigl|\bigl\{s \in [0,t] \bigm| |\omega_s - m| \ge \delta/t\bigr\}\Bigr| > \varepsilon$$

for all paths ω with $\omega_t \in B_\eta(x)$ and $\left|F(\omega) - F^*(x)\right| \geq 2\delta$. Because m is the only zero of b and because $\liminf_{|x|\to\infty} |b(x)| > 0$ we have

$$\inf\{b^2(x) \mid |x-m| \geq \delta/t\} > 0,$$

i.e. we can find a $g > 0$ with $G(\omega) > g$ for all paths ω with $\omega_t \in B_\eta(x)$ and $\left|F(\omega) - F^*(x)\right| \geq 2\delta$. Together this gives

$$\begin{aligned}
&\limsup_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(-\vartheta^2 G(\omega)\bigr)(|F(\omega) - F^*(x)| > 2\delta)\, d\mathbb{W}(\omega) \\
&\qquad \leq \limsup_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int \exp\bigl(-\vartheta^2 g\bigr)\, d\mathbb{W}(\omega) \\
&\qquad = -\infty.
\end{aligned}$$

So we can use lemma 22 to conclude

$$\begin{aligned}
&\liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(-\vartheta^2 G(\omega)\bigr)\, d\mathbb{W}(\omega) \\
&\qquad = \liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(-\vartheta^2 G(\omega)\bigr)(|F(\omega) - F^*(x)| \leq 2\delta)\, d\mathbb{W}(\omega)
\end{aligned}$$

and get

$$\begin{aligned}
&\liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log P(X_t^\vartheta \in O) \\
&\qquad \geq F^*(x) - 2\delta + \liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log \int 1_{B_\eta(x)}(\omega_t) \exp\bigl(-\vartheta^2 G(\omega)\bigr)\, d\mathbb{W}(\omega) \\
&\qquad \geq F^*(x) - 2\delta - \inf_{y\in B_\eta(x)} I(y) \\
&\qquad \geq F^*(x) - 2\delta - I(x)
\end{aligned}$$

for all $\delta > 0$. Letting $\delta \downarrow 0$ gives

$$\liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log P(X_t^\vartheta \in O) \geq F^*(x) - I(x)$$

and taking the supremum over all $x \in O$ on the right hand side proves the lower bound.

Now let $K \subseteq \mathbb{R}$ be compact and $\delta > 0$. For each $x \in K$ we can find an $\eta > 0$ with $|\Phi(y) - \Phi(x)| \leq \delta$ whenever $y \in B_\eta(x)$. Because I is lower semi-continuous we can assume $I(y) \geq I(x) - \delta$ for every $y \in \overline{B}_\eta(x)$ by choosing η small enough. Using the compactness of K we can cover K with a finite number of such balls: there are $x_1, \ldots, x_n \in K$ and $0 < \eta_1, \ldots, \eta_n < \delta$ with

$$K \subseteq \bigcup_{k=1}^n B_{\eta_k}(x_k)$$

and the above assumption on Φ and I hold for each k. For $k=1,\dots,n$ consider $F^*(x_k)$ as defined above. This time we find

$$\begin{aligned}
&\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log P(X_t^\vartheta\in K)\\
&\qquad\le \limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\sum_{k=1}^{n}P(X_t^\vartheta\in B_{\eta_k}(x_k))\\
&\qquad= \max_{k=1,\dots,n}\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\int 1_{B_{\eta_k}(x_k)}(\omega_t)\exp\big(\vartheta F(\omega)-\vartheta^2 G(\omega)\big)\,d\mathbb{W}(\omega).
\end{aligned}$$

Because F is bounded on $\{\omega_t\in B_{\eta_k}(x_k)\}$ we can use lemma 22 as above to conclude

$$\begin{aligned}
&\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\int 1_{B_{\eta_k}(x_k)}(\omega_t)\exp\big(\vartheta F(\omega)-\vartheta^2 G(\omega)\big)\,d\mathbb{W}(\omega)\\
&\qquad= \limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\int 1_{B_{\eta_k}(x_k)}(\omega_t)\exp\big(\vartheta F(\omega)-\vartheta^2 G(\omega)\big)\\
&\qquad\qquad\qquad\qquad(|F(\omega)-F^*(x_k)|\le 2\delta)\,d\mathbb{W}(\omega)
\end{aligned}$$

for $k=1,\dots,n$. This gives

$$\begin{aligned}
&\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log P(X_t^\vartheta\in K)\\
&\qquad\le \max_{k=1,\dots,n}\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\int 1_{B_{\eta_k}(x_k)}(\omega_t)\exp\big(\vartheta F(\omega)-\vartheta^2 G(\omega)\big)\\
&\qquad\qquad\qquad\qquad(|F(\omega)-F^*(x_k)|\le 2\delta)\,d\mathbb{W}(\omega)\\
&\qquad\le \max_{k=1,\dots,n}\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\int 1_{B_{\eta_k}(x_k)}(\omega_t)\exp\big(\vartheta(F^*(x_k)+2\delta)-\vartheta^2 G(\omega)\big)\\
&\qquad\qquad\qquad\qquad\big(|F(\omega)-F^*(x_k)|\le 2\delta\big)\,d\mathbb{W}(\omega)\\
&\qquad\le \max_{k=1,\dots,n}F^*(x_k)+2\delta\\
&\qquad\qquad\qquad+\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log\int 1_{\bar{B}_{\eta_k}(x_k)}(\omega_t)\exp\big(-\vartheta^2 G(\omega)\big)\,d\mathbb{W}(\omega).
\end{aligned}$$

Now we can use the upper bound on the rate of the integral and our choice of η_k to get

$$\begin{aligned}
&\limsup_{\vartheta\to\infty}\frac{1}{\vartheta}\log P(X_t^\vartheta\in K)\\
&\qquad\le \max_{k=1,\dots,n}F^*(x_k)+2\delta-\inf_{y\in\bar{B}_\delta(x_k)}I(y)\\
&\qquad\le \max_{k=1,\dots,n}F^*(x_k)+2\delta-I(x_k)+\delta.
\end{aligned}$$

and letting $\delta\downarrow 0$ completes the proof. □

The following lemma is a generalisation of lemma 20. It helps to determine the rate function I which is needed to apply lemma 26.

Lemma 27. *Let $b\colon \mathbb{R} \to \mathbb{R}$ be a C^2-function with $\liminf_{|x|\to\infty} |b(x)| > 0$. Assume that there is an $m \in \mathbb{R}$ with $b(x) = 0$ if and only if $x = m$ and with $b'(m) \neq 0$. Then for any compact set $K \subseteq \mathbb{R}$ we have*

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon, B_t \in K\Big) \\
&\qquad \le -\frac{1}{4}\inf_{a\in K}\Big(\Big|\int_0^m |b(x)|\,dx\Big| + \frac{1}{2}|b'(m)|t + \Big|\int_m^a |b(x)|\,dx\Big|\Big)^2
\end{aligned}$$

and for any open set $O \subseteq \mathbb{R}$ we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon \log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon, B_t \in O\Big) \\
&\qquad \ge -\frac{1}{4}\inf_{a\in O}\Big(\Big|\int_0^m |b(x)|\,dx\Big| + \frac{1}{2}|b'(m)|t + \Big|\int_m^a |b(x)|\,dx\Big|\Big)^2.
\end{aligned}$$

Proof. As an abbreviation define $v(x) = b^2(x)/2$ for all $x \in \mathbb{R}$. For the proof of the upper bound choose a compact set K, let $\delta, \tau > 0$ and choose D_3^δ as in (3.3). Then for $\varepsilon < t/2\tau$ we have

$$\begin{aligned}
&\Big\{\int_0^t v(B_s)\,ds \le \varepsilon, B_t \in K\Big\} \\
&\qquad \subseteq \bigcup_{\alpha\in D_3^\delta} \Big\{\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_1\varepsilon, \int_{\varepsilon\tau}^{t-\varepsilon\tau} v(B_s)\,ds \le \alpha_2\varepsilon, \\
&\qquad\qquad\qquad \int_{t-\varepsilon\tau}^{t} v(B_s)\,ds \le \alpha_3\varepsilon, B_t \in K\Big\}.
\end{aligned}$$

Writing (A) for the indicator function of A and using the strong Markov property of Brownian motion this gives

$$\begin{aligned}
&P\Big(\int_0^t v(B_s)\,ds \le \varepsilon, B_t \in K\Big) \\
&\quad \le \sum_{\alpha\in D_3^\delta} E\Big(\big(\textstyle\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_1\varepsilon\big)\big(\int_{\varepsilon\tau}^{t-\varepsilon\tau} v(B_s)\,ds \le \alpha_2\varepsilon\big) \\
&\qquad\qquad E\big(\big(\textstyle\int_{t-\varepsilon\tau}^{t} v(B_s)\,ds \le \alpha_3\varepsilon, B_t \in K\big) \,\big|\, \mathscr{F}_{t-\varepsilon\tau}\big)\Big) \\
&\quad = \sum_{\alpha\in D_3^\delta} E\Big(\big(\textstyle\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_1\varepsilon\big)\big(\int_{\varepsilon\tau}^{t-\varepsilon\tau} v(B_s)\,ds \le \alpha_2\varepsilon\big) \\
&\qquad\qquad E_{B_{t-\varepsilon\tau}}\big(\big(\textstyle\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_3\varepsilon, B_{\varepsilon\tau} \in K\big)\big)\Big) \\
&\quad =: \sum_{\alpha\in D_3^\delta} p(\alpha,\varepsilon)
\end{aligned}$$

Now let $\alpha \in D_3^\delta$ be fixed and $a > 0$. We split the corresponding event further by distinguishing the two cases $\{\sup_{\varepsilon\tau \le s \le t-\varepsilon\tau} |B_s - m| > a\}$ and $\{\sup_{\varepsilon\tau \le s \le t-\varepsilon\tau} |B_s - m| \le a\}$. Since omitting some conditions makes the probability only larger, we get

$$p(\alpha,\varepsilon) \le p_1(\alpha,\varepsilon) + p_2(\alpha,\varepsilon)$$

with

$$p_1(\alpha,\varepsilon) = \sup_{y\in\mathbb{R}} P_y\Big(\int_0^{t-2\varepsilon\tau} v(B_s)\,ds \le \alpha_2\varepsilon, \sup_{0\le s\le t-2\varepsilon\tau} |B_s - m| > a\Big)$$

and

$$\begin{aligned}
p_2(\alpha,\varepsilon) = P\Big(&\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_1\varepsilon, |B_{\varepsilon\tau} - m| \le a\Big) \\
&\sup_{y\in\mathbb{R}} P_y\Big(\int_0^{t-2\varepsilon\tau} v(B_s)\,ds \le \alpha_2\varepsilon, \sup_{0\le s\le t-2\varepsilon\tau} |B_s - m| \le a\Big) \\
&\sup_{|z-m|\le a} P_z\Big(\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_3\varepsilon, B_{\varepsilon\tau} \in K\Big).
\end{aligned}$$

To calculate the rate for the sum $p_1(\alpha,\varepsilon) + p_2(\alpha,\varepsilon)$ we have to calculate the rates of the individual terms. Let $\eta > 0$. For p_1 we can use lemma 23 to get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log p_1(\alpha,\varepsilon) \\
&\quad \le \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{|y-m|<a/2} P_y\Big(\int_0^{t-\eta} v(B_s)\,ds \le \alpha_2\varepsilon, \sup_{0\le s\le t-\eta} |B_s - m| > a\Big), \\
&\quad \le -\frac{1}{8\alpha_2}\Big(t - \eta + \frac{1}{2}a^2\Big)^2.
\end{aligned}$$

Since for fixed η this rate become arbitrarily negative when a becomes large, we can choose a large enough that the rate of $p_1(\alpha,\varepsilon) + p_2(\alpha,\varepsilon)$ is dominated by p_2.

To treat the p_2-term we apply inequality (3.4) as we did in the proof of proposition 6. From proposition 7 we know the individual rates

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon \log P\Big(&\int_0^{\varepsilon\tau} v(B_s)\,ds \le \varepsilon, |B_{\varepsilon\tau} - m| \le a\Big) \\
&\le -\frac{1}{4}\Big(\int_0^m |b(x)|\,dx\Big)^2 r_1^2(\tau)
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{|z-m|\le a} P_z\Big(&\int_0^{\varepsilon\tau} v(B_s)\,ds \le \varepsilon, B_{\varepsilon\tau} \in K\Big) \\
&\le -\frac{1}{4}\inf_{a\in K}\Big(\int_m^a |b(x)|\,dx\Big)^2 r_2^2(\tau)
\end{aligned}$$

where $\lim_{\tau\to\infty} r_1(\tau) = \lim_{\tau\to\infty} r_2(\tau) = 1$, and proposition 19 gives

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in\mathbb{R}} P_y\Big(\int_0^{t-2\varepsilon\tau} v(B_s)\,ds \le \varepsilon, \sup_{0\le s\le t-2\varepsilon\tau} |B_s - m| \le a\Big)\\
&\qquad \le \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{|y-m|<a/2} P_y\Big(\int_0^{t-\eta} v(B_s)\,ds \le \varepsilon, \sup_{0\le s\le t-\eta} |B_s - m| \le a\Big)\\
&\qquad \le -\frac{|b'(m)|^2(t-\eta)^2}{16}.
\end{aligned}$$

Using inequality (3.4) we get the combined rate

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log p_2(\alpha,\varepsilon)\\
&\qquad \le -\frac{1}{1+\delta}\Big(\frac{1}{2}\Big|\int_0^m |b(x)|\,dx\Big| r_1(\tau)\\
&\qquad\qquad\qquad + \frac{1}{4}|b'(m)|(t-\eta) + \frac{1}{2}\inf_{a\in K}\Big|\int_m^a |b(x)|\,dx\Big| r_2(\tau)\Big)^2
\end{aligned}$$

for all $\alpha \in D_3^\delta$.

The rate for the sum over all $\alpha \in D_3^\delta$ is the maximum of the individual rates. The result is

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\int_0^t v(B_s)\,ds \le \varepsilon, B_t \in K\Big)\\
&\qquad \le -\frac{1}{1+\delta}\Big(\frac{1}{2}\Big|\int_0^m |b(x)|\,dx\Big| r_1(\tau)\\
&\qquad\qquad\qquad + \frac{1}{4}|b'(m)|(t-\eta) + \frac{1}{2}\inf_{a\in K}\Big|\int_m^a |b(x)|\,dx\Big| r_2(\tau)\Big)^2
\end{aligned}$$

for all $\eta > 0$, $\delta > 0$, and $\tau > 0$. Letting finally $\tau \to \infty$, $\delta \downarrow 0$, and $\eta \downarrow 0$ gives

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon, B_t \in K\Big)\\
&\qquad \le -\frac{1}{4}\Big(\frac{1}{2}\Big|\int_0^m |b(x)|\,dx\Big| + \frac{1}{2}|b'(m)|t + \inf_{a\in K}\Big|\int_m^a |b(x)|\,dx\Big|\Big)^2.
\end{aligned}$$

This proves the upper bound.

For the lower bound: Let $\zeta, \eta, \tau > 0$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then for $\varepsilon < t/2\tau$

we have

$$
\begin{aligned}
&\Big\{\int_0^t \nu(B_s)\,ds \le \varepsilon, B_t \in O\Big\} \\
&\qquad \supseteq \Big\{\int_0^{\varepsilon\tau} \nu(B_s)\,ds \le \alpha_1\varepsilon, |B_{\varepsilon\tau} - m| < \zeta\Big\} \\
&\qquad\qquad \cap \Big\{\int_{\varepsilon\tau}^{t-\varepsilon\tau} \nu(B_s)\,ds \le \alpha_2\varepsilon, |B_{t-\varepsilon\tau} - m| < \eta\Big\} \\
&\qquad\qquad \cap \Big\{\int_{t-\varepsilon\tau}^{t} \nu(B_s)\,ds \le \alpha_3\varepsilon, B_t \in O\Big\}
\end{aligned}
$$

and thus we get

$$
\begin{aligned}
&P\Big(\int_0^t \nu(B_s)\,ds \le \varepsilon, B_t \in O\Big) \\
&\quad \ge E\Big((\int_0^{\varepsilon\tau} \nu(B_s)\,ds \le \alpha_1\varepsilon, |B_{\varepsilon\tau} - m| < \zeta\Big\}) \\
&\qquad\quad (\int_{\varepsilon\tau}^{t-\varepsilon\tau} \nu(B_s)\,ds \le \alpha_2\varepsilon, |B_{t-\varepsilon\tau} - m| < \eta) \\
&\qquad\quad E\Big((\int_{t-\varepsilon\tau}^{t} \nu(B_s)\,ds \le \alpha_3\varepsilon, B_t \in O) \Big| \mathscr{F}_{t-\varepsilon\tau}\Big)\Big) \\
&\quad \ge E\Big((\int_0^{\varepsilon\tau} \nu(B_s)\,ds \le \alpha_1\varepsilon, |B_{\varepsilon\tau} - m| < \zeta) \\
&\qquad\quad E\Big((\int_{\varepsilon\tau}^{t-\varepsilon\tau} \nu(B_s)\,ds \le \alpha_2\varepsilon, |B_{t-\varepsilon\tau} - m| < \eta) \Big| \mathscr{F}_{\varepsilon\tau}\Big)\Big) \\
&\qquad\quad \inf_{m-\eta<y<m+\eta} P_y\Big(\int_0^{\varepsilon\tau} \nu(B_s)\,ds \le \alpha_3\varepsilon, B_{\varepsilon\tau} \in O\Big) \\
&\quad \ge P_0\Big(\int_0^{\varepsilon\tau} \nu(B_s)\,ds \le \alpha_1\varepsilon, B_{\varepsilon\tau} \in (m-\zeta, m+\zeta)\Big) \\
&\qquad\quad \inf_{m-\zeta<z<m+\zeta} P_z\Big(\int_0^{t-2\varepsilon\tau} \nu(B_s)\,ds \le \alpha_2\varepsilon, |B_{t-2\varepsilon\tau} - m| < \eta\Big) \\
&\qquad\quad \inf_{m-\eta<y<m+\eta} P_y\Big(\int_0^{\varepsilon\tau} \nu(B_s)\,ds \le \alpha_3\varepsilon, B_{\varepsilon\tau} \in O\Big).
\end{aligned}
$$

First take lower exponential rates for $\varepsilon \downarrow 0$. The lower exponential rate of the left-hand side is greater or equal to the sum of the lower rates of the right-hand side. This inequality holds for all $\eta, \tau > 0$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Then let $\tau \to \infty$. We treat the three terms on the right hand side individually. First term: from

Lemma 5.1 we know

$$\begin{aligned}
\lim_{\tau\to\infty}\liminf_{\varepsilon\downarrow 0}\varepsilon\log P_0\Big(\int_0^{\varepsilon\tau} v(B_s)\,ds \le \alpha_1\varepsilon, B_{\varepsilon\tau}\in(m-\zeta,m+\zeta)\Big) \\
\ge -\frac{1}{\alpha_1}\frac{1}{4}\inf_{m-\zeta<a<m+\zeta}\Big(\Big|\int_0^m |b(x)|\,dx\Big|+\Big|\int_m^a |b(x)|\,dx\Big|\Big)^2 \\
= -\frac{1}{\alpha_1}\frac{1}{4}\Big(\Big|\int_0^m |b(x)|\,dx\Big|\Big)^2 r_1(\zeta)
\end{aligned}$$

where $\lim_{\zeta\downarrow 0} r_1(\zeta)=1$.

Second term: we can make the probability smaller by replacing $t-2\varepsilon\tau$ with t. Then the term is no longer τ-dependent and using proposition 19 we get

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\varepsilon\log\inf_{m-\zeta<z<m+\zeta} P_z\Big(\int_0^{t-2\varepsilon\tau} v(B_s)\,ds\le\alpha_2\varepsilon, |B_{t-2\varepsilon\tau}-m|<\eta\Big)\Big) \\
\ge -\frac{1}{\alpha_2}\frac{|b'(m)|^2}{16}t^2 r_2(\zeta)
\end{aligned}$$

where $\lim_{\zeta\downarrow 0} r_2(\zeta)=1$.

Third term: using corollary 10 we get

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\varepsilon\log\inf_{m-\eta<y<m+\eta} P_y\Big(\int_0^{\varepsilon\tau} v(B_s)\,ds\le\alpha_3\varepsilon, B_{\varepsilon\tau}\in O\Big) \\
\ge -\frac{1}{\alpha_3}\frac{1}{4}\inf_{a\in O}\Big(\int_m^a |b(x)|\,dx\Big)^2 r_3(\eta)
\end{aligned}$$

where $\lim_{\eta\downarrow 0} r_3(\eta)=1$.

Combining the three rates we get

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\varepsilon\log P\Big(B_t\in O, \int_0^t v(B_s)\,ds\le\varepsilon\Big) \\
\ge -\frac{1}{\alpha_1}\frac{1}{4}\Big(\Big|\int_0^m |b(x)|\,dx\Big|\Big)^2 r_1(\zeta) \\
-\frac{1}{\alpha_2}\frac{|b'(m)|^2}{16}t^2 r_2(\zeta) \\
-\frac{1}{\alpha_3}\frac{1}{4}\inf_{a\in O}\Big(\Big|\int_m^a |b(x)|\,dx\Big|\Big)^2 r_3(\eta).
\end{aligned}$$

and letting first $\zeta \downarrow 0$ and then $\eta \downarrow 0$ yields

$$\begin{aligned}
\liminf_{\varepsilon \downarrow 0} \varepsilon \log P\Big(B_t \in O, \int_0^t v(B_s)\,ds \le \varepsilon\Big)& \\
\ge -\frac{1}{\alpha_1}\Big(\frac{1}{2}\int_0^m |b(x)|\,dx\Big)^2& \\
-\frac{1}{\alpha_2}\Big(\frac{|b'(m)|}{4}t\Big)^2& \\
-\frac{1}{\alpha_3}\Big(\frac{1}{2}\inf_{a\in O}\int_m^a |b(x)|\,dx\Big)^2&
\end{aligned}$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Choosing optimal α_1, α_2, and α_3 as in (3.4) we get

$$\begin{aligned}
&\liminf_{\varepsilon \downarrow 0} \varepsilon \log P\Big(B_t \in O, \frac{1}{2}\int_0^t b^2(B_s)\,ds \le \varepsilon\Big) \\
&\quad \ge -\Big(\frac{1}{2}\Big|\int_0^m |b(x)|\,dx\Big| + \frac{|b'(m)|}{4}t + \frac{1}{2}\inf_{a\in O}\Big|\int_m^a |b(x)|\,dx\Big|\Big)^2 \\
&\quad = -\frac{1}{4}\Big(\Big|\int_0^m |b(x)|\,dx\Big| + \frac{|b'(m)|}{2}t + \inf_{a\in O}\Big|\int_m^a |b(x)|\,dx\Big|\Big)^2.
\end{aligned}$$

This completes the proof. □

Proof. (of theorem 1) Since the rate function J_t is invariant under space shifts we can without loss of generality assume $z = 0$ by replacing Φ with the shifted function $\Phi(\cdot + z)$ and starting the SDE in 0. Since most of the work was already done, the proof consists only of three steps.

First define

$$\begin{aligned}
H(x) &= \frac{1}{4}\Big(\Big|\int_0^m |b(y)|\,dy\Big| + \frac{1}{2}|b'(m)|t + \Big|\int_{[m,x]} |b(y)|\,dy\Big|\Big)^2 \\
&= \frac{1}{4}\Big(V_0^m(\Phi) + \frac{1}{2}|b'(m)|t + V_m^x(\Phi)\Big)^2
\end{aligned}$$

and $v(x) = b^2(x)/2$ for all $y \in \mathbb{R}$. From lemma 27 we know that for every compact set $K \subseteq \mathbb{R}$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\int_0^t v(B_s)\,ds \le \varepsilon, B_t \in K\Big) \le -\inf_{a\in K} H(a)$$

and for every open set $O \subseteq \mathbb{R}$ we have

$$\liminf_{\varepsilon\to 0} \varepsilon \log P\Big(\int_0^t v(B_s)\,ds \le \varepsilon, B_t \in O\Big) \ge -\inf_{a\in O} H(a).$$

Second, let

$$I(x) = 2\sqrt{H(x)} = V_0^m(\Phi) + \frac{1}{2}|b'(m)|t + V_m^x(\Phi)$$

for all $x \in \mathbb{R}$. Then for every set $A \subseteq \mathbb{R}$ we find

$$-2\sqrt{\left|-\inf_{x\in A} H(x)\right|} = -2\sqrt{\inf_{x\in A} H(x)} = -\inf_{x\in A} I(x)$$

and the Tauberian theorem 5 allows us to conclude

$$\limsup_{\vartheta\to\infty} \frac{1}{\vartheta} \log E\big(\exp(-\vartheta^2 \int_0^t v(\omega_s)\,ds) 1_K(B_t)\big) \leq -\inf_{x\in K} I(x)$$

for every compact set $K \subseteq \mathbb{R}$ and

$$\liminf_{\vartheta\to\infty} \frac{1}{\vartheta} \log E\big(\exp(-\vartheta^2 \int_0^t v(\omega_s)\,ds) 1_O(B_t)\big) \geq -\inf_{x\in O} I(x)$$

for every open set $O \subseteq \mathbb{R}$.

Finally we can use lemma 26 to conclude that the family $(X_t^\vartheta)_{\vartheta>0}$ satisfies the weak LDP with rate function

$$\begin{aligned} J_t(x) &= \Phi(x) - \Phi(0) - \frac{1}{2} t \Phi''(m) + I(x) \\ &= \Phi(x) - \Phi(0) + V_0^m(\Phi) + t(\Phi''(m))^- + V_m^x(\Phi). \end{aligned}$$

This completes the proof of theorem 1. □

References

[And55] T. W. Anderson. The integral of symmetric unimodular functions over a symmetric convex set and some probability inequalities. *Proceedings of the American Mathematical Society*, 6:170–176, 1955. MR0069229

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*. Cambridge University Press, 1987. MR0898871

[BR89] Garrett Birkhoff and Gian-Carlo Rota. *Ordinary Differential Equations*. John Wiley & Sons, fourth edition, 1989. MR0972977

[BS96] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian Motion — Facts and Formulae*. Probability and its Applications. Birkhäuser, Basel, 1996.

[DZ98] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*, volume 38 of *Applications of Mathematics*. Springer, second edition, 1998. MR1619036

[GF63] I. M. Gelfand and S. V. Fomin. *Calculus of Variations*. Prentice-Hall, 1963. MR0160139

[Vos04] Jochen Voss. *Some Large Deviation Results for Diffusion Processes*. PhD thesis, University of Kaiserslautern, Germany, 2004.
`http://kluedo.ub.uni-kl.de/volltexte/2004/1763/`.