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On Lévy Processes Conditioned to Stay Positive.

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Abstract: We construct the law of Lévy processes conditioned to stay positive under general hypotheses. We obtain a Williams type path decomposition at the minimum of these processes. This result is then applied to prove the weak convergence of the law of Lévy processes conditioned to stay positive as their initial state tends to 0. We describe an absolute continuity relationship between the limit law and the measure of the excursions away from 0 of the underlying Lévy process reflected at its minimum. Then, when the Lévy process creeps upwards, we study the lower tail at 0 of the law of the height of this excursion.

Key words: Lévy process conditioned to stay positive, path decomposition, weak convergence, excursion measure, creeping.

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1 Introduction

In [6] it was shown how, given the measure $\mathbb{P}$ of a Lévy process satisfying some weak assumptions, one can construct for each $x > 0$ a measure $\mathbb{P}_x^\uparrow$ corresponding, in the sense of Doob's theory of h-transforms, to conditioning the process starting at x to stay positive. Using a different construction Bertoin [3] had shown the existence of a measure $\mathbb{P}_0^\uparrow$ under which the process starts at 0 and stays positive. A natural question is whether $\mathbb{P}_x^\uparrow$ converges to $\mathbb{P}^\uparrow := \mathbb{P}_0^\uparrow$ as $x \downarrow 0$. Recently, it has been proved by Tanaka [15] that this convergence in law holds in the sense of finite dimensional distributions, under very general hypotheses, but here we are interested in convergence in law on Skohorod's space of càdlàg trajectories. Since it was also shown in [6], extending a famous result for the 3-dimensional Bessel process due to Williams, that under $\mathbb{P}_x^\uparrow$ the post-minimum process is independent of the pre-minimum process and has law $\mathbb{P}^\uparrow$, this essentially amounts to showing that the pre-minimum process vanishes as $x \downarrow 0$. Such a result has been verified in the case of spectrally negative processes in [2], and for stable processes and for processes which creep downwards in [6].

In the third section of this paper, we give a simple proof of this result for a general Lévy process. This proof does not use the description of the law of the pre-minimum process which is given in [6], but is based on knowledge of the distribution of the all-time minimum under $\mathbb{P}_x^\uparrow$, which was also established in [6]. However these results in [6], and also some results from [9] which we need, were established under what can now be seen to be unnecessary assumptions, such as the existence of an absolutely continuous semigroup, regularity of both half-lines, etc. We therefore devote the second section to a self-contained account of Lévy processes conditioned to stay positive. In it we give improved proofs of the basic existence and decomposition results (Proposition 1 and Theorem 1) under the sole assumption that our Lévy process is not Compound Poisson.

As a consequence of our convergence reult Theorem 2, we are able to extend the description of the excursion measure of the process reflected at its minimum (see Proposition 15, p. 202 of [1] for the spectrally negative case) to the general case. This description has recently been used in [8] to perform some semi-explicit calculations for the reflected process when X has jumps of one sign only, and our result suggests the possibility that such calculations could be performed in other cases. However a key fact in the calculations in [8] is that, in the spectrally negative case, the harmonic function h used in conditioning to stay positive and the excursion measure $\underline{n}$ of the reflected process satisfy

$$\underline{n}(H > x)h(x) = 1,$$

where H denotes the height of a generic excursion. This leads us to wonder whether this relation could hold **asymptotically as** $x \downarrow 0$ in other cases. Intuitively it seems obvious that this should be the case when the reflected excursion **creeps upwards**. This is what we establish in Theorem 3.

2 The process conditioned to stay positive

2.1 Notation and Preliminaries

Let $\mathcal{D}$ denote the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\delta\}$ with lifetime $\zeta(\omega) = \inf\{s : \omega_s = \delta\}$, where δ is a cemetery point. $\mathcal{D}$ will be equipped with the Skorokhod topology, with its Borel σ-algebra $\mathcal{F}$, and the usual filtration $(\mathcal{F}_s, s \geq 0)$. Let $X = (X_t, t \geq 0)$ be the coordinate process defined on the space $\mathcal{D}$. We write $\overline{X}$ and $\underline{X}$ for the supremum and infimum processes, defined for all $t < \zeta$ by

$$
\begin{aligned}
\overline{X}_t &= \sup\{X_s : 0 \leq s \leq t\}, \\
\underline{X}_t &= \inf\{X_s : 0 \leq s \leq t\}.
\end{aligned}
$$

We write τ_A for the entrance time into a Borel set A, and m for the time at which the absolute infimum is attained:

$$
\tau_A = \inf\{s > 0 : X_s \in A\}, \tag{2.1}
$$

$$
m = \sup\{s < \zeta : X_s \wedge X_{s-} = \underline{X}_s\}, \tag{2.2}
$$

with the usual convention that $\inf\{\emptyset\} = +\infty$ and $\sup\{\emptyset\} = 0$. (Note that the definition of m here reduces to that in [6] when 0 is regular for both half-lines, which was assumed in [6], because then there can be no jump at the minimum.)

For each $x \in \mathbb{R}$ we denote by $\mathbb{P}_x$ the law of a Lévy process starting from x, and write $\mathbb{P}_0 = \mathbb{P}$. We assume throughout the sequel that $(X, \mathbb{P})$ is not a compound Poisson process. It is well known that the reflected process $X - \underline{X}$ is Markov. Note that the state 0 is regular for $(-\infty, 0)$ under $\mathbb{P}$, if and only if it is regular for $\{0\}$ for the reflected process. In this case, we will simply say that 0 is regular downwards and if 0 is regular for $(0,\infty)$ under $\mathbb{P}$, we will say that 0 is regular upwards.

Let $\underline{L}$ be the local time of the reflected process $X - \underline{X}$ at 0 and let $\underline{n}$ be the measure of its excursions away from 0. If 0 is regular downwards then, up to a multiplicative constant, $\underline{L}$ is the unique additive functional of the reflected process whose set of increasing points is $\{t : (X - \underline{X})_t = 0\}$ and $\underline{n}$ is the corresponding Itô measure of excursions; we refer to [1], Chap. IV, sections 2-4 for a proper definition of $\underline{L}$ and $\underline{n}$. If 0 is not regular downwards then the set $\{t : (X - \underline{X})_t = 0\}$ is discrete and we define the local time $\underline{L}$ as the counting process of this set, i.e. $\underline{L}$ is a jump process whose jumps have size 1 and occur at each zero of $X - \underline{X}$. Then, the measure $\underline{n}$ is the probability law of the process X under the law $\mathbb{P}$, killed at its first passage time in the negative halfline, i.e. $\tau_{(-\infty,0)}$, (see the definition of $\mathbb{Q}_0$ below).

Let us first consider the function h defined for all $x \geq 0$ by:

$$
h(x) := \mathbb{E}\left(\int_{[0,\infty)} \mathbb{I}_{\{\underline{X}_t \geq -x\}} \, d\underline{L}_t\right). \tag{2.3}
$$

It follows from (2.3) (or (2.5) below) and general properties of Lévy processes that h is *finite, continuous, increasing* and that $h-h(0)$ is *subadditive* on $[0,\infty)$. Moreover, $h(0)=0$ if 0 is regular downwards and $h(0)=1$ if not (in the latter case, the counting measure $d\underline{L}_t$ gives mass 1 to the point $t=0$).

Let $\mathbf{e}$ be an exponential time with parameter 1, which is independent of $(X,\mathbb{P})$. The following identity follows from Maisonneuve's exit formula of excursion theory when 0 is regular downwards and is obtained by direct calculations in the other case. For all $\varepsilon>0$,

$$\mathbb{P}_x(\tau_{(-\infty,0)}>\mathbf{e}/\varepsilon)=\mathbb{P}(\underline{X}_{\mathbf{e}/\varepsilon}\ge -x)=\mathbb{E}\left(\int_{[0,\infty)}e^{-\epsilon t}\mathbb{I}_{\{\underline{X}_t\ge -x\}}\,d\underline{L}_t\right)\underline{n}(\mathbf{e}/\varepsilon<\zeta)\,, \tag{2.4}$$

so that, by monotone convergence, for all $x\ge 0$:

$$h(x)=\lim_{\varepsilon\to 0}\frac{\mathbb{P}_x(\tau_{(-\infty,0)}>\mathbf{e}/\varepsilon)}{\underline{n}(\mathbf{e}/\varepsilon<\zeta)}\,. \tag{2.5}$$

In the next lemma, we show that h is excessive or invariant for the process $(X,\mathbb{P}_x)$, $x>0$ killed at time $\tau_{(-\infty,0)}$. This result has been proved in the context of potential theory by Silverstein [13] Th. 2, where it is assumed that the semigroup is absolutely continuous, 0 is regular for $(-\infty,0)$, and $(X,\mathbb{P})$ does not drift to $-\infty$; see also Tanaka [15], Th. 2 and Th. 3. Here, we give a different proof which uses the representation of h stated in (2.5). For $x>0$, we denote by $\mathbb{Q}_x$ the law of the killed process, i.e. for $\Lambda\in\mathcal{F}_t$:

$$\mathbb{Q}_x(\Lambda,t<\zeta)=\mathbb{P}_x(\Lambda,\,t<\tau_{(-\infty,0)})\,,$$

and by (q_t) its semigroup. Recall that $\mathbb{Q}_0$ and $q_t(0,dy)$ are well defined when 0 is not regular downwards, and in this case we have $\mathbb{Q}_0=\underline{n}$.

Lemma 1 *If $(X,\mathbb{P})$ drifts towards $-\infty$ then h is excessive for (q_t), i.e. for all $x\ge 0$ and $t\ge 0$, $\mathbb{E}^{\mathbb{Q}}_x(h(X_t)\mathbb{I}_{\{t<\zeta\}})\le h(x)$. If $(X,\mathbb{P})$ does not drift to $-\infty$, then h is invariant for (q_t), i.e. for all $x\ge 0$ and $t\ge 0$, $\mathbb{E}^{\mathbb{Q}}_x(h(X_t)\mathbb{I}_{\{t<\zeta\}})=h(x)$.*

Proof: From (2.5), monotone convergence and the Markov property, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_x(h(X_t)\mathbb{I}_{\{t<\zeta\}}) &= \lim_{\varepsilon\to 0}\mathbb{E}_x\left(\frac{\mathbb{P}_{X_t}(\tau_{(-\infty,0)}>\mathbf{e}/\varepsilon)\mathbb{I}_{\{t\le\tau_{(-\infty,0)}\}}}{\underline{n}(\mathbf{e}/\varepsilon<\zeta)}\right)\\ &= \lim_{\varepsilon\to 0}\mathbb{E}_x\left(\frac{\mathbb{I}_{\{\tau_{(-\infty,0)}>t+\mathbf{e}/\varepsilon\}}}{\underline{n}(\mathbf{e}/\varepsilon<\zeta)}\right)=\lim_{\varepsilon\to 0}e^{\varepsilon t}\left(\frac{\mathbb{P}_x(\tau_{(-\infty,0)}>\mathbf{e}/\varepsilon)}{\underline{n}(\mathbf{e}/\varepsilon<\zeta)}\right.\\ &\qquad\left.-\int_0^t\varepsilon e^{-\varepsilon u}\frac{\mathbb{P}_x(\tau_{(-\infty,0)}>u)}{\underline{n}(\mathbf{e}/\varepsilon<\zeta)}\,du\right)\\ &= h(x)-\frac{1}{\underline{n}(\zeta)}\int_0^t\mathbb{P}_x(\tau_{(-\infty,0)}>u)\,du\,,\end{aligned} \tag{2.6}$$

where $\underline{n}(\zeta) = \int_0^\infty \underline{n}(\zeta > t)\,dt$. It is known that for $x > 0$, $\mathbb{E}_x(\tau_{(-\infty,0)}) < \infty$ if and only if X drifts towards $-\infty$, see [1], Prop. VI.17. Hence, since moreover for $x > 0$, $0 < h(x) < +\infty$, then (2.5) shows that $\underline{n}(\zeta) < +\infty$ if and only if X drifts towards $-\infty$. Consequently, from (2.6), if X drifts towards $-\infty$, then $\mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_{\{t<\zeta\}}) \le h(x)$, for all $t \ge 0$ and $x \ge 0$, whereas if $(X, \mathbb{P})$ does not drift to $-\infty$, then $\underline{n}(\zeta) = +\infty$ and (2.6) shows that $\mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_{\{t<\zeta\}}) = h(x)$, for all $t \ge 0$ and $x \ge 0$. ■

2.2 Definition and path decomposition

We now define the Lévy process $(X, \mathbb{P}_x)$ conditioned to stay positive. This notion has now a long history, see [3], [6] , [9], [15] and the references contained in those papers.

Write $(p_t, t \ge 0)$ for the semigroup of $(X, \mathbb{P})$ and recall that $(q_t, t \ge 0)$ is the semigroup (in $(0, \infty)$ or in $[0, \infty)$) of the process $(X, \mathbb{Q}_x)$. Then we introduce the new semigroup

$$p_t^{\uparrow}(x, dy) := \frac{h(y)}{h(x)} q_t(x, dy),\ \ x > 0, y > 0,\ \ t \ge 0\,. \tag{2.7}$$

From Lemma 1, $(p_t^{\uparrow})$ is sub-Markov when $(X, \mathbb{P})$ drifts towards $-\infty$ and it is Markov in the other cases. For $x > 0$ we denote by $\mathbb{P}_x^{\uparrow}$ the law of the strong Markov process started at x and whose semigroup in $(0, \infty)$ is $(p_t^{\uparrow})$. When $(p_t^{\uparrow})$ is sub-Markov, $(X, \mathbb{P}_x^{\uparrow})$ has state space $(0, \infty) \cup \{\delta\}$ and this process has finite lifetime. In any case, for $\Lambda \in \mathcal{F}_t$, we have

$$\mathbb{P}_x^{\uparrow}(\Lambda, t < \zeta) = \frac{1}{h(x)} \mathbb{E}_x^{\mathbb{Q}}(h(X_t)\mathbb{I}_\Lambda \mathbb{I}_{\{t<\zeta\}})\,. \tag{2.8}$$

Note that when 0 is not regular downwards then definitions (2.7) and (2.8) also make sense for $x = 0$. We show in the next proposition that $\mathbb{P}_x^{\uparrow}$ is the limit as $\varepsilon \downarrow 0$ of the law of the process under $\mathbb{P}_x$ conditioned to stay positive up to an independent exponential time with parameter ε, so we will refer to $(X, \mathbb{P}_x^{\uparrow})$ as the process "conditioned to stay positive". Note that the following result has been shown in [6] Th. 1 under the same assumptions that Silverstein [13] required for his Th. 2, but here we only assume that X is not a compound Poisson process.

Proposition 1 *Let* $\mathbf{e}$ *be an exponential time with parameter* 1 *which is independent of* $(X, \mathbb{P})$.

For any $x > 0$, *and any* $(\mathcal{F}_t)$ *stopping time* T *and for all* $\Lambda \in \mathcal{F}_T$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_x(\Lambda, T < \mathbf{e}/\varepsilon \,|\, X_s > 0,\ 0 \le s \le \mathbf{e}/\varepsilon) = \mathbb{P}_x^{\uparrow}(\Lambda, T < \zeta)\,.$$

This result also holds for $x = 0$ *when* 0 *is not regular downwards.*

Proof: According to the Markov property and the lack-of-memory property of the exponential law, we have

$$\mathbb{P}_x(\Lambda,\, T < \mathbf{e}/\varepsilon \,|\, X_s > 0,\, 0 \le s \le \mathbf{e}/\varepsilon) =$$
$$\mathbb{E}_x\left(\mathbb{I}_\Lambda \mathbb{I}_{\{T<(\mathbf{e}/\varepsilon)\wedge\tau_{(-\infty,0)}\}} \frac{\mathbb{P}_{X_T}(\tau_{(-\infty,0)} \ge \mathbf{e}/\varepsilon)}{\mathbb{P}_x(\tau_{(-\infty,0)} \ge \mathbf{e}/\varepsilon)}\right) . \tag{2.9}$$

Let $\varepsilon_0 > 0$. From (2.3) and (2.4), for all $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{I}_{\{T<(\mathbf{e}/\varepsilon)\wedge\tau_{(-\infty,0)}\}} \frac{\mathbb{P}_{X_T}(\tau_{(-\infty,0)} \ge \mathbf{e}/\varepsilon)}{\mathbb{P}_x(\tau_{(-\infty,0)} \ge \mathbf{e}/\varepsilon)} \le$$
$$\mathbb{I}_{\{T<\tau_{(-\infty,0)}\}}\mathbb{E}\left(\int_{[0,\infty)} e^{-\varepsilon_0 t}\mathbb{I}_{\{\underline{X}_t \ge -x\}}\, d\underline{L}_t\right)^{-1} h(X_T)\,, \quad \text{a.s.} \tag{2.10}$$

Recall that h is excessive for the semigroup (q_t), hence the inequality of Lemma 1 also holds at any stopping time, i.e. $\mathbb{E}_x^{\mathbb{Q}}(h(X_T)\mathbb{I}_{\{T<\zeta\}}) \le h(x)$. Since h is finite, the expectation of the right hand side of (2.10) is finite so that we may apply Lebesgue's theorem of dominated convergence in the right hand side of (2.9) when ε goes to 0. We conclude by using the representation of h in (2.5) and the definition of $\mathbb{P}_x^{\uparrow}$ in (2.8). ■

Remark 1 *In the discrete time setting, that is for random walks, another characterization of the harmonic function h has been given by Bertoin and Doney [4], Lemma* 1. *Using similar arguments, one can show that a continuous time equivalent holds. For Lévy processes, such that* $\limsup_t X_t = +\infty$, *this result is*

$$\lim_{n\to+\infty} \frac{\mathbb{P}_x(\tau_{[n,\infty)} < \tau_{(-\infty,0)})}{\mathbb{P}_y(\tau_{[n,\infty)} < \tau_{(-\infty,0)})} = \frac{h(x)}{h(y)}\,, \quad x, y > 0.$$

Then, as in discrete time, a consequence is the following equivalent definition of $(X, \mathbb{P}_x^{\uparrow})$:

$$\lim_{n\to+\infty} \mathbb{P}_x(\Lambda \,|\, \tau_{[n,\infty)} < \tau_{(-\infty,0)}) = \mathbb{P}_x^{\uparrow}(\Lambda)\,, \quad t > 0, \quad \Lambda \in \mathcal{F}_t\,.$$

Note that a similar conditioning has been studied by Hirano [10] in some special cases.

In the case where 0 is regular downwards, definition (2.7) does not make sense for $x = 0$, but in [3] it was shown that in all cases, the law of the process

$$((X - \underline{X})_{g_t+s},\, s \le t - g_t), \quad \text{where} \quad g_t = sup\{s \le t : (X - \underline{X})_s = 0\}$$

converges as $t \to \infty$ to a Markovian law under which X starts at 0 and has semigroup $p_t^{\uparrow}$. Similarly, under additional hypotheses, Tanaka [15], Th.7 proved that the process

$$(X - \underline{X})_{b_\lambda+s},\, s \le a_\lambda - b_\lambda), \quad \text{where} \quad \begin{cases} a_\lambda = inf\{t : (X - \underline{X})_t > \lambda\} \\ b_\lambda = sup\{t \le a_\lambda : (X - \underline{X})_t = 0\} \end{cases}$$

converges as $\lambda \to +\infty$ towards the same law. We will denote this law by $\mathbb{P}^\uparrow$. Thm 3 of [6] gives the entrance law of the process $(X, \mathbb{P}^\uparrow)$, (see (3.2) below). Note that Doney [7], extending a discrete time result from Tanaka [14], obtained a path construction of $(X, \mathbb{P}^\uparrow)$. Another path construction of $(X, \mathbb{P}^\uparrow)$ is contained in Bertoin [3]. These two constructions are quite different from each other but coincide in the Brownian case. Roughly speaking, we could say that Doney-Tanaka's construction is based on a rearrangement of the excursions away from 0 of the Lévy process reflected at its minimum whereas Bertoin's construction consists in sticking together the positive excursions away from 0 of the Lévy process itself.

The next theorem describes the decomposition of the process $(X, \mathbb{P}^\uparrow_x)$ at the time of its minimum. It is also well known in the literature under various hypotheses: in [6] Th. 5 and in [9] Prop. 4.7, Cor. 4.8, the state 0 is supposed to be regular both downwards and upwards, moreover in [6], the process does not drift towards $-\infty$ and its semigroup is absolutely continuous, and in [9], the process X creeps both downwards and upwards (see (4.1) for a definition of creeping). Here, we only assume that X is not a compound Poisson process.

Theorem 1 *Define the pre-minimum and post-minimum processes respectively as follows:* $(X_t\,,\,0 \le t < m)$ *and* $(X_{t+m} - U\,,\,0 \le t < \zeta - m)$*, where* $U := X_m \wedge X_{m-}$.

1. *Under* $\mathbb{P}^\uparrow_x$*,* $x > 0$*, the pre-minimum and post-minimum processes are independent. The process* $(X, \mathbb{P}^\uparrow_x)$ *reaches its absolute minimum* U *once only and its law is given by:*

$$\mathbb{P}^\uparrow_x(U \ge y) = \frac{h(x-y)}{h(x)} 1_{\{y \le x\}}\,. \tag{2.11}$$

2. *Under* $\mathbb{P}^\uparrow_x$*, the law of the post-minimum process is* $\mathbb{P}^\uparrow$*. In particular, it is strongly Markov and does not depend on* x*. The semigroup of* $(X, \mathbb{P}^\uparrow)$ *in* $(0,\infty)$ *is* $(p^\uparrow_t)$*. Moreover,* $X_0 = 0$*,* $\mathbb{P}^\uparrow$*-a.s. if and only if* 0 *is regular upwards.*

Proof: Denote by $\mathbb{P}^{\mathbf{e}/\varepsilon}_x$ the law of the process $(X, \mathbb{P}_x)$ killed at time $\mathbf{e}/\varepsilon$. Since $(X, \mathbb{P})$ is not a compound Poisson process, it almost surely reaches its minimum at a unique time on the interval $[0, \mathbf{e}/\varepsilon]$. Recall that by a result of Millar [11], pre-minimum and post-minimum processes are independent under $\mathbb{P}^{\mathbf{e}/\varepsilon}_x$ for all $\varepsilon > 0$. According to Proposition 1, the same properties hold under $\mathbb{P}^\uparrow_x$. Let $0 \le y \le x$. From Proposition 1 and (2.5):

$$\begin{aligned}
\mathbb{P}^\uparrow_x(U < y) &= \mathbb{P}^\uparrow_x(\tau_{[0,y)} < \zeta) = \lim_{\varepsilon\to 0} \mathbb{P}_x(\tau_{[0,y)} < \mathbf{e}/\varepsilon \,|\, \tau_{(-\infty,0)} > \mathbf{e}/\varepsilon) \\
&= \lim_{\varepsilon\to 0}\left(1 - \frac{\mathbb{P}_x(\tau_{[0,y)} \ge \mathbf{e}/\varepsilon,\, \tau_{(-\infty,0)} > \mathbf{e}/\varepsilon)}{\mathbb{P}_x(\tau_{(-\infty,0)} > \mathbf{e}/\varepsilon)}\right) \\
&= 1 - \lim_{\varepsilon\to 0} \frac{\mathbb{P}_{x-y}(\tau_{(-\infty,0)} \ge \mathbf{e}/\varepsilon)}{\mathbb{P}_x(\tau_{(-\infty,0)} > \mathbf{e}/\varepsilon)} = 1 - \frac{h(x-y)}{h(x)}\,,
\end{aligned}$$

and the first part of the theorem is proved.

From the independence mentioned above, the law of the post-minimum process under $\mathbb{P}_x^{\mathbf{e}/\varepsilon}(\,\cdot\,|\,U>0)$ is the same as the law of the post-minimum process under $\mathbb{P}_x^{\mathbf{e}/\varepsilon}$. Then, from Proposition 1 or from [3], Corollary 3.2, the law of the post-minimum processes under $\mathbb{P}_x^{\uparrow}$ is the limit of the law of the post-minimum process under $\mathbb{P}_x^{\mathbf{e}/\varepsilon}$, as $\varepsilon\to 0$. But in [3], Corollary 3.2, it has been proved that this limit law is that of a strong Markov process with semigroup $(p_t^{\uparrow})$. Moreover, from [11], the process $(X,\mathbb{P}_x^{\mathbf{e}/\varepsilon})$ leaves its minimum continuously, (that is $\mathbb{P}_x^{\mathbf{e}/\varepsilon}(X_m>X_{m-})=0$) if and only if 0 is regular upwards. Then we conclude using Proposition 1. ■

When $(X,\mathbb{P})$ has no negative jumps and 0 is not regular upwards, the initial law of $(X,\mathbb{P}^{\uparrow})$ has been computed in [5]. It is given by:

$$\mathbb{P}^{\uparrow}(X_0\in dx)=\frac{x\,\pi(dx)}{\int_0^\infty u\,\pi(du)},\quad x\ge 0\,, \tag{2.12}$$

where π is the Lévy measure of $(X,\mathbb{P})$. It seems more difficult to obtain an explicit formula which only involves π in the general case.

3 The convergence result

We can now state our convergence result. Recall that it has been proved in [6] Th. 6 in the special cases where $(X,\mathbb{P})$ is either a Lévy process which creeps downwards (see the next section for the definition of the creeping of a stochastic process) or a stable process. Note also that from Bertoin [1], Prop. VII.14 this convergence also holds when $(X,\mathbb{P})$ has no positive jumps. Then Tanaka [15], Ths 4 and 5 proved the finite dimensional convergence in the very general case.

Theorem 2 *Assume that* 0 *is regular upwards. Then the family* $(\mathbb{P}_x^{\uparrow},x>0)$ *converges on the Skorokhod space to* $\mathbb{P}^{\uparrow}$*. Moreover the semigroup* $(p_t^{\uparrow},t\ge 0)$ *satisfies the Feller property on the space* $\mathcal{C}_0([0,\infty))$ *of continuous functions vanishing at infinity.*

If 0 *is not regular upwards, then for any* $\varepsilon>0$ *, the process* $(X\circ\theta_\varepsilon,\mathbb{P}_x^{\uparrow})$ *converges weakly towards* $(X\circ\theta_\varepsilon,\mathbb{P}^{\uparrow})$*, as* x *tends to* 0*.*

Proof: Let $(\Omega,\mathcal{F},P)$ be a probability space on which we can define a family of processes $(Y^{(x)})_{x>0}$ such that each process $Y^{(x)}$ has law $\mathbb{P}_x^{\uparrow}$. Let also Z be a process with law $\mathbb{P}^{\uparrow}$ which is independent of the family $(Y^{(x)})$. Let m_x be the unique hitting time of the minimum of $Y^{(x)}$ and define, for all $x>0$, the process $Z^{(x)}$ by:

$$Z_t^{(x)}=\begin{cases} Y_t^{(x)} & t<m_x\\ Z_{t-m_x}+Y_{m_x}^{(x)} & t\ge m_x\,.\end{cases}$$

By the preceding theorem, under P, $Z^{(x)}$ has law $\mathbb{P}_x^\uparrow$.

Now first assume that 0 is regular upwards, so that $\lim_{t\downarrow 0} Z_t = 0$, almost surely. We are going to show that the family of processes $Z^{(x)}$ converges in probability towards the process Z as $x \downarrow 0$ for the norm of the J_1-Skorohod topology on the space $\mathcal{D}([0,1])$. Let (x_n) be a decreasing sequence of real numbers which tends to 0. For $\omega \in \mathcal{D}([0,1])$, we easily see that the path $Z^{(x_n)}(\omega)$ tends to $Z(\omega)$ as n goes to ∞ in the Skohorod's topology, if both $m_{x_n}(\omega)$ and $\overline{Z}^{(x_n)}_{m_{x_n}}(\omega)$ tend to 0. Hence, it suffices to prove that both m_x and $\overline{Z}^{(x)}_{m_x}$ converge in probability to 0 as $x \to 0$. In the canonical notation (i.e. with $(m, \mathbb{P}_x^\uparrow) = (m_x, P)$, where m is defined in (2.2) and $(X, \mathbb{P}_x^\uparrow) = (Z^{(x)}, P)$), we have to show that for any fixed $\varepsilon > 0, \eta > 0$,

$$\lim_{x\downarrow 0} \mathbb{P}_x^\uparrow(m > \varepsilon) = 0 \quad \text{and} \quad \lim_{x\downarrow 0} \mathbb{P}_x^\uparrow(\overline{X}_m > \eta) = 0. \tag{3.1}$$

First, applying the Markov property at time ε gives

$$\begin{aligned}
\mathbb{P}_x^\uparrow(m > \varepsilon) &= \int_{0<y\le x}\int_{z>y} \mathbb{P}_x^\uparrow(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy, \varepsilon < \zeta)\mathbb{P}_z^\uparrow(U < y) \\
&= \int_{0<y\le x}\int_{z>y} \mathbb{Q}_x(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy, \varepsilon < \zeta)\frac{h(z)}{h(x)}\mathbb{P}_z^\uparrow(U < y) \\
&= \int_{0<y\le x}\int_{z>y} \mathbb{P}_x(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy)\frac{h(z) - h(z-y)}{h(x)},
\end{aligned}$$

where we have used the result of Theorem 1 and the fact that $\mathbb{Q}_x$ and $\mathbb{P}_x$ agree on $\mathcal{F}_\varepsilon \cap (\underline{X}_\varepsilon > 0)$. Put $\tilde{h} := h - h(0)$, and recall from Section 2 that $\tilde{h}$ is increasing and subadditive, hence we have $h(z) - h(z-y) \le \tilde{h}(y)$, and so

$$\begin{aligned}
\mathbb{P}_x^\uparrow(m &> \varepsilon) \le \frac{1}{h(x)}\int_{0<y\le x}\int_{z>y} \mathbb{P}_x(X_\varepsilon \in dz, \underline{X}_\varepsilon \in dy)\tilde{h}(y) \\
&= \frac{1}{h(x)}\int_{0<y\le x} \mathbb{P}_x(\underline{X}_\varepsilon \in dy)\tilde{h}(y) \le \frac{\tilde{h}(x)}{h(x)}\mathbb{P}_x(\underline{X}_\varepsilon > 0)\,.
\end{aligned}$$

When 0 is not regular downwards $h(0) = 1$ (see Section 2), hence $\frac{\tilde{h}(x)}{h(x)} \to 0$ as $x \to 0$, and we obtain the result in that case. When 0 is regular, $\frac{\tilde{h}(x)}{h(x)} = 1$, but in this case, we clearly have $\mathbb{P}_x(\underline{X}_\varepsilon > 0) \to 0$ as $x \to 0$, so the result is also true.

For the second claim in (3.1), we apply the strong Markov property at time $\tau := \tau_{(\eta,\infty)}$, with $x < \eta$, to get

$$\begin{aligned}
\mathbb{P}_x^\uparrow(\overline{X}_m &> \eta) = \int_{z\ge\eta}\int_{0<y\le x} \mathbb{P}_x^\uparrow(X_\tau \in dz, \underline{X}_\tau \in dy, \tau < \zeta)\mathbb{P}_z^\uparrow(U < y) \\
&= \int_{z\ge\eta}\int_{0<y\le x} \mathbb{P}_x^\uparrow(X_\tau \in dz, \underline{X}_\tau \in dy, \tau < \zeta)\frac{h(z) - h(z-y)}{h(z)}.
\end{aligned}$$

We now apply the simple bound

$$\frac{h(z) - h(z-y)}{h(z)} \le \frac{\tilde{h}(y)}{h(z)} \le \frac{\tilde{h}(x)}{h(\eta)} \text{ for } 0 < y \le x \text{ and } z \ge \eta$$

to deduce that

$$\mathbb{P}_x^{\uparrow}(\overline{X}_m > \eta) \le \frac{\tilde{h}(x)}{h(\eta)} \to 0 \text{ as } x \downarrow 0\,.$$

Then, the weak convergence of $(\mathbb{P}_x^{\uparrow})$ towards $\mathbb{P}^{\uparrow}$ is proved. When 0 is regular upwards, the Feller property of the semigroup $(p_t^{\uparrow},\, t \ge 0)$ on the space $\mathcal{C}_0([0,\infty))$ follows from its definition in (2.7), the properties of Lévy processes and the weak convergence at 0 of $(\mathbb{P}_x^{\uparrow})$.

Finally when 0 is not regular upwards, (3.1) still holds but we can check that, at time $t = 0$, the family of processes $Z^{(x)}$ does not converge in probability towards 0. However following the above arguments we can still prove that for any $\varepsilon > 0$, $(Z^{(x)} \circ \theta_\varepsilon)$ converges in probability towards $Z \circ \theta_\varepsilon$ as $x \downarrow 0$. ■

The following absolute continuity relation between the measure $\underline{n}$ of the process of the excursions away from 0 of $X - \underline{X}$ and $\mathbb{P}^{\uparrow}$ has been shown in [6]: for $t > 0$ and $A \in \mathcal{F}_t$

$$\underline{n}(A, t < \zeta) = k\mathbb{E}^{\uparrow}(h(X_t)^{-1}A), \tag{3.2}$$

where $k > 0$ is a constant which depends only on the normalization of the local time $\underline{L}$. Relation (3.2) was established in [6] Th. 3 under the additional hypotheses mentioned before Theorem 1 above, but we can easily check that it still holds under the sole assumption that X is not a Poisson process. Then a consequence of Theorem 2 is:

Corollary 1 *Assume that* 0 *is regular upwards. For any* $t > 0$ *and for any* $\mathcal{F}_t$ *-measurable, continuous and bounded functional* F,

$$\underline{n}(F, t < \zeta) = k \lim_{x \to 0} \mathbb{E}_x^{\uparrow}(h(X_t)^{-1}F).$$

Another application of Theorem 2 is to the asymptotic behavior of the semigroup $q_t(x, dy)$, $t > 0$, $y > 0$, when x goes towards 0. Let us denote by $j_t(dx)$, $t \ge 0$, $x \ge 0$ the the entrance law of the excursion measure $\underline{n}$, that is the Borel function which is defined for any $t \ge 0$ as follows:

$$\underline{n}(f(X_t), t < \zeta) = \int_0^\infty f(x) j_t(dx)\,,$$

where f is any positive or bounded Borel function f.

Corollary 2 *The asymptotic behavior of* $q_t(x, dy)$ *is given by:*

$$\int_0^\infty f(y) q_t(x, dy) \sim_{x \to 0} h(x) \int_0^\infty f(y) j_t(dy)\,,$$

for $t > 0$ *and for every continuous and bounded function* f.

Note that when 0 is not regular downwards, the measure $\underline{n}$ is nothing but $\mathbb{Q}_0$ and $h(0) = 1$, so in that case Corollaries 1 and 2 are straightforward. In the other case, they are direct consequences of Theorem 2, (2.7) and (3.2), so their proofs are omitted.

4 On the asymptotic behaviour of the function h

We end this paper by a study of the asymptotic behaviour of the function h at 0 in terms of the lower tail of the height of the generic reflected excursion. We define the height of a path ω with finite lifetime $\zeta(\omega)$ as follows:

$$H(\omega) := \sup_{0\le t\le\zeta} \omega_t \,.$$

The equality $\underline{n}(H > x)h(x) = 1$ is proved in [1], Proposition VII.15 in the spectrally negative case, i.e. when $(X, \mathbb{P})$ has no positive jumps. However, this relation does not hold in general.

Example: First note that, using Corollary 1, one has in any case

$$\begin{aligned}\underline{n}(H > x) = \underline{n}(\tau_{[x,\infty)} < \infty) &= \lim_{y\downarrow 0} \frac{1}{h(y)}\mathbb{Q}_y(\tau_{[x,\infty)} < \tau_{(-\infty,0]}) \\ &= \lim_{y\downarrow 0} \frac{1}{h(y)}\mathbb{P}_y(\overline{X}(\tau_{(-\infty,0]}) > x)\,.\end{aligned}$$

Moreover, Th. VII.8 in [1] implies that when the process has no negative jumps,

$$\mathbb{P}_y(\overline{X}(\tau_{(-\infty,0]}) > x) = \frac{\hat{h}(x+y)}{\hat{h}(x)}\,,$$

where $\hat{h}$ is the harmonic function defined as in Section 2 with respect to the dual process $\hat{X} \overset{\text{(def)}}{=} -X$. Then in this case, one has $h(y) = y$, so that from above, $\underline{n}(H > x) = \hat{h}'(x)/\hat{h}(x)$. We conclude that the expression $\underline{n}(H > x)h(x)$ can be equal to a constant only when $\hat{h}$ is of the form $\hat{h}(x) = cx^\gamma$, for some positive constants c and γ, but this is possible only if $(X, \mathbb{P})$ is stable. Note that the equality $\underline{n}(H > x)h(x) = 1$ has recently been noticed independently by Rivero [12] for any stable Lévy process, in a work on the more general setting of Markov self-similar processes.

In the spectrally positive case discussed above, we can check that the expression $\underline{n}(H > x)h(x)$ tends to a constant as x goes to 0 if the process is also in the domain of attraction of a stable process, i.e. if there exists $\alpha \in (0, 2]$ such that

$((t^{-1/\alpha}X_{st},\ s\geq 0),\mathbb{P})$ converges weakly to a stable process with index α, as t goes to 0. This raised the question of finding some other conditions under which this property holds. The creeping of $(X,\mathbb{P})$ is one such condition.

In the rest of the paper, we suppose that $(X,\mathbb{P})$ does not drift towards $-\infty$. We say that the process $(X,\mathbb{P})$ creeps (upwards) across the level $x>0$ if

$$\mathbb{P}(X(\tau_{[x,\infty)})=x)>0\,. \tag{4.1}$$

It is well known that if X creeps across a positive level $x>0$, then it creeps across all positive levels. Moreover this can happen if and only if

$$\lim_{x\downarrow 0}\mathbb{P}(X(\tau_{[x,\infty)})=x)=1\,. \tag{4.2}$$

We refer for instance to [1], chap. VI for a proof of this equivalence.

Define for all $x>0$, $\sigma_x=\sup\{t:X_t\leq x\}$. Then it is easy to see from proposition 1 that when $\mathbb{P}_x(\limsup_{t\to+\infty}X_t=+\infty)=1$, we have $\mathbb{P}^{\uparrow}(\lim_{t\to+\infty}X_t=+\infty)=1$, hence $\mathbb{P}^{\uparrow}(\sigma_x<\infty)=1$, for any $x\geq 0$. We need the following lemma for the proof of the next theorem.

Lemma 2 *If* $(X,\mathbb{P})$ *creeps upwards, then* $\lim_{x\downarrow 0}\mathbb{P}^{\uparrow}(X(\sigma_x)=x)=1$.

Proof: In the case where, 0 is regular for both $(-\infty,0)$ and $(0,\infty)$, it is a direct consequence of (4.2) and Th. 4.2 (i) in [9].

However the regularity of 0 is not needed. Indeed, the main argument of the proof is the identity

$$(X(\sigma_x),\mathbb{P}^{\uparrow})=(X(\tau_{[x,\infty)})+[X(g_x)-X(\tau_{[x,\infty)}-)],\mathbb{P})\,, \tag{4.3}$$

where $g_x=\sup\{t\leq\tau_{[x,\infty)}:\overline{X}_t=X_t\vee X_{t-}\}$. This identity can be checked through Doney-Tanaka's construction of Lévy processes conditioned to stay positive (see Doney [7]). Although the author in [7] also supposes the regularity of 0, it is easily checked that this construction (and thus identity (4.3)) still holds in the very general case.

Then observe that $X(\tau_{[x,\infty)})\geq x$ and $X(g_x)-X(\tau_{[x,\infty)}-)\geq 0$, $\mathbb{P}$-a.s. Moreover since $(X,\mathbb{P})$ creeps upwards, it is not a compound Poisson process, hence it cannot reach a positive level for the first time by a jump. In particular, on the event $\{X(\tau_{[x,\infty)})=x\}$, the process is $\mathbb{P}$-a.s. continuous at time $\tau_{[x,\infty)}$ and on this event, $X(g_x)=X(\tau_{[x,\infty)}-)=X(\tau_{[x,\infty)})$ so that

$$\{X(\tau_{[x,\infty)})=x\}=\{X(\tau_{[x,\infty)})+[X(g_x)-X(\tau_{[x,\infty)}-)]=x\}\,,\quad \mathbb{P}\text{-a.s.}$$

We conclude with (4.3) and (4.2). ∎

In the following result, without loss of generality, we make the convention that the normalization constant k in relation (3.2) is 1.

Theorem 3 *If* $(X,\mathbb{P})$ *creeps upwards, then* $\underline{n}(H>x)h(x)\to 1$*, as* $x\to 0$.

Proof: Recall that under the hypothesis of the theorem, the lifetime of $(X,\mathbb{P}^\uparrow)$ is almost surely infinite. Then fix $x>0$. Since $\{H>x\}=\{\tau_{(x,\infty)}<\zeta\}$, from the identity (3.2) applied at the stopping time $\tau_{(x,\infty)}$, we have

$$h(x)\,\underline{n}(H>x)=h(x)\,\underline{n}(\tau_{(x,\infty)}<\zeta)\le\underline{n}(h(X(\tau_{(x,\infty)})),\,\tau_{(x,\infty)}<\zeta)=1\,. \qquad (4.4)$$

From the Markov property applied at time $\tau_{(0,x]}$, and since h is increasing, we have for any $0<x\le y$,

$$\begin{aligned}\mathbb{P}^\uparrow_y(X(\sigma_x)=x) &= \mathbb{E}^\uparrow_y\left(\mathbb{P}^\uparrow_{X(\tau_{(0,x]})}(X_{\sigma_x}=x)1\!\!1_{\{\tau_{(0,x]}<\infty\}}\right)\\ &= \frac{1}{h(y)}\mathbb{E}^{\mathbb{Q}}_y\left(h(X(\tau_{(0,x]}))\mathbb{P}^\uparrow_{X(\tau_{(0,x]})}(X_{\sigma_x}=x)1\!\!1_{\{\tau_{(0,x]}<\zeta\}}\right)\\ &\le \frac{h(x)}{h(y)}\,. \end{aligned} \qquad (4.5)$$

On the other hand, from the Markov property at time $\tau_{(x,\infty)}$ and (3.2), we have under $\mathbb{P}^\uparrow$:

$$\begin{aligned}\mathbb{P}^\uparrow(X(\sigma_x)=x) &= \mathbb{E}^\uparrow\left(\mathbb{P}^\uparrow_{X(\tau_{(x,\infty)})}(X_{\sigma_x}=x)\right)\\ &= \underline{n}h(X(\tau_{(x,\infty)}))\mathbb{P}^\uparrow_{X(\tau_{(x,\infty)})}(X_{\sigma_x}=x)1\!\!1_{\{\tau_{(x,\infty)}<\zeta\}}). \end{aligned} \qquad (4.6)$$

But since $X(\tau_{(x,\infty)})\ge x$, inequality (4.5) gives

$$\mathbb{P}^\uparrow_{X(\tau_{(x,\infty)})}(X_{\sigma_x}=x)\le h(x)/h(X(\tau_{(x,\infty)})),\ a.s.$$

Hence, from (4.6) we have

$$\mathbb{P}^\uparrow(X(\sigma_x)=x)\le h(x)\,\underline{n}(\tau_{(x,\infty)}<\zeta)\,. \qquad (4.7)$$

Finally, we deduce the result from (4.4), (4.7) and Lemma 2. ■

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