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The spatial Λ-coalescent

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Abstract

This paper extends the notion of the Λ-coalescent of Pitman (1999) to the spatial setting. The partition elements of the spatial Λ-coalescent migrate in a (finite) geographical space and may only coalesce if located at the same site of the space. We characterize the Λ-coalescents that come down from infinity, in an analogous way to Schweinsberg (2000). Surprisingly, all spatial coalescents that come down from infinity, also come down from infinity in a uniform way. This enables us to study space-time asymptotics of spatial Λ-coalescents on large tori in $d \geq 3$ dimensions. Some of our results generalize and strengthen the corresponding results in Greven et al. (2005) concerning the spatial Kingman coalescent.

Key words: coalescent, Λ-coalescent, structured coalescent, limit theorems, coalescing random walks

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1 Introduction

The Λ-*coalescent*, sometimes also called the coalescent with multiple collisions, is a Markov process Π whose state space is the set of partitions of the positive integers. The *standard* Λ-*coalescent* Π starts at the partition of the positive integers into singletons, and its restriction to $[n] := \{1, \ldots, n\}$, denoted by Π_n, is the Λ-coalescent starting with n initial partition elements. The measure Λ, which is a finite measure on $[0, 1]$, dictates the rate of coalescence events, as well as how many of the (exchangeable) partition elements, which we will also refer to as *blocks*, may coalesce into one at any such event. More precisely, if we define for $2 \le k \le b$, $k, b \in \mathbf{N}$ integers,

$$\lambda_{b,k} := \int_{[0,1]} x^{k-2}(1-x)^{b-k} d\Lambda(x), \tag{1}$$

then the parameter $\lambda_{b,k} \ge 0$ is the rate at which any collection of k blocks coalesces into one new block when the current configuration has b blocks.

The Λ-coalescent was introduced by Pitman [23], and also studied by Schweinsberg [26]. It was obtained as a limit of genealogical trees in Cannings models by Sagitov [25]. The well-known Kingman coalescent [20] corresponds to the Λ-coalescent with $\Lambda(dx) = \delta_0(dx)$, the unit atomic measure at 0. For this coalescent, each pair of current partition elements coalesces at unit rate, independently of other pairs. Papers [1] and [15] are devoted to stochastic coalescents where again only pairs of partitions are allowed to coalesce, but the coalescence rate is not uniform over all pairs. The survey [1] gives many pointers to the literature. The Λ-coalescent generalizes the Kingman coalescent in the sense that now any number of partition elements may merge into one at a coalescence event, but the rate of coalescence for any k-tuple of partition elements depends still only on k. The first example of such a Λ-coalescent (other than the Kingman coalescent) was studied by Bolthausen and Sznitman [11], who were interested in the special case where $\Lambda(dx)$ is Lebesgue measure on $[0, 1]$ in connection with spin glasses. Bertoin and Le Gall [7] observed a correspondence of this particular coalescent to the genealogy of continuous state branching processes (CSBP). More recently, Birkner et al. [10] extended this correspondence to stable CSBP's to Λ-coalescents, where Λ is given by a Beta-distribution. Berestycki et al. [6] use this correspondence to study fine small time properties of the corresponding coalescents.

A further generalization of the Λ-coalescents, known as the *coalescents with simultaneous multiple collisions*, was originally studied by Möhle and Sagitov [21] and Schweinsberg [27]. Further connections to bridge processes and generalized Fleming-Viot processes were discovered by Bertoin and Le Gall [8], and to asymptotics of genealogies during selective sweeps, by Durrett and Schweinsberg [13].

Our first goal, in Section 2, is to extend the notion of the Λ-coalescent to the spatial setting. Here, partition elements migrate in a geographical space and may only coalesce while sharing the same location. Earlier works on variants of spatial coalescents, sometimes also referred to as structured coalescents, have all assumed Kingman coalescent-like behavior, and include Notohara (1990) [22], Herbots (1997) [19], and more recently Barton et al. [3] in the case of finite initial configurations, and Greven et al. [16] with infinite initial states. A related model has been studied by Zähle et al. [28] on two-dimensional tori. Finally, spatial coalescents are related to coalescing random walks, the difference being that for coalescing random walks blocks coalesce *instantaneously* when they enter the same site. Coalescing random walks have been studied extensively, in particular as dual processes to the voter model, see for example [12].

In most of this paper we assume that Λ is a finite measure on $[0,1]$ without an atom at 0 or at 1, such that $\Lambda([0,1]) > 0$. At the end of Section 2 we comment on how atoms at 0 or 1 would change the behavior of the coalescent. We extend the definition made in (1) by setting $\lambda_{b,k} = 0$ for $b = 1$ or $b = 0$, $k \in \mathbf{N}$. Define in addition

$$\lambda_b := \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k}, \tag{2}$$

and

$$\gamma_b := \sum_{k=2}^{b} \binom{b}{k} (k-1) \lambda_{b,k}. \tag{3}$$

Note that λ_b is the total rate of coalescence when the configuration has b blocks, and that γ_b is the total rate of decrease in the number of blocks when the configuration has b blocks. From the above definitions, one may already observe (see also proof of Theorem 1) that the Λ-coalescent can be derived from a Poisson point process on $\mathbf{R}_+ \times [0,1]$ ($\mathbf{R}_+ := [0,\infty)$) with intensity measure $dt x^{-2} d\Lambda(x)$ provided that Λ has no atom at 0 : If (t,x) is an atom of this Poisson point process, then at time t, we mark each block independently with probability x, and subsequently merge all marked blocks into one.

Now consider a finite graph $\mathcal{G}$, and denote by $|\mathcal{G}|$ the number of its vertices. Call the vertices of $\mathcal{G}$ *sites*. Consider a process started from a finite configuration of n blocks on sites in $\mathcal{G}$ where we allow only two types of transitions, referred to as *coalescence* and *migration* respectively:

(i) at each site blocks coalesce according to the Λ-coalescent,

(ii) the location process of each block is an independent continuous Markov chain on $\mathcal{G}$ with jump rate 1 and transition probabilities $p(g_i, g_j), g_i, g_j \in \mathcal{G}$.

The original Λ-coalescent of [23] and [26] corresponds to the setting where $|\mathcal{G}| = 1$, so migrations are impossible. The spatial Λ-coalescent started from a finite configuration $\{(1,i_1),\dots,(n,i_n)\}$ is a well-defined strong Markov process (chain) with state space being the set of all partitions of $[n] = \{1,\dots,n\}$ labeled by their location in $\mathcal{G}$. This will be stated precisely in Theorem 1 of Section 2 which is devoted to the construction of spatial Λ-coalescents Π^ℓ with general (possibly infinite) initial states.

After constructing the general spatial Λ-coalescent, we turn to characterizing those that come down from infinity in Section 3. Schweinsberg [26] shows that if

$$\sum_{b\ge 2} \frac{1}{\gamma_b} < \infty \tag{4}$$

holds, then the (non-spatial) Λ-coalescent started with infinitely many blocks at time 0 immediately comes down from infinity, that is, the number of its blocks at all times $t > 0$ is finite with probability 1; otherwise, the Λ-coalescent stays infinite forever, meaning that it contains infinitely many blocks at all times $t > 0$ with probability 1. Interestingly, Bertoin and LeGall [9] have recently found that (4) is equivalent to the almost sure extinction of the associated CSBP.

The goal of Section 3 is to show that the spatial Λ-coalescent inherits this property of either coming down from infinity or staying infinite, from its nonspatial counterpart. More precisely, let $(\Pi^\ell(t))_{t\geq 0}$ be the Λ-coalescent constructed in Theorem 1, and denote by $\#\Pi(t)$ its size at time t, i.e. the total number of blocks in $\Pi^\ell(t)$, with any label. In Lemma 8 and Proposition 11 we show that condition (4) implies $P[\#\Pi(t) < \infty, \forall t > 0] = 1$, even if the initial configuration $\Pi(0)$ contains infinitely many blocks. In this case we say that the spatial Λ-coalescent *comes down from infinity.* In Proposition 11 we also show via a coupling to the non-spatial coalescent that if (4) does not hold, provided $\#\Pi(0) = \infty$ and Λ has no atom at 1, then $P[\#\Pi(t) = \infty, \forall t > 0] = 1$. In this case we say that the spatial Λ-coalescent *stays infinite.* These results extend to the spatial coalescent for which the migration mechanism may be more general, for example non-exponential or depending on the past and in a restricted sense even on the future of the coalescence mechanism. We include a comment at the end of Section 3.

In Section 4 we continue to study the time T_n in more detail, which is the time until there are on average two blocks per site if there are initially n blocks per site. In particular, in Theorem 12 we obtain an upper bound on its expectation that is not only uniform in n but also, somewhat surprisingly, in the structure (size) of $\mathcal{G}$. In this case, we say that the coalescent *comes down from infinity uniformly.* The argument of Theorem 12 relies on the independence of the coalescence and migration mechanisms. Given Theorem 12, the approximation method of [16] section 7 applies verbatim and enables construction of the spatial Λ-coalescent on the whole lattice $\mathbf{Z}^d$ so that even if started with infinitely many partition elements at each site of $\mathbf{Z}^d$, at all positive times the configuration is locally finite, almost surely.

Our final goal, in Section 5, is to study space-time asymptotic properties of Λ-coalescents that come down from infinity uniformly on large finite tori at time scales on the order of the volume. In [16], this asymptotic behavior was studied for the spatial Kingman coalescent where $\Lambda = \gamma\delta_0$ for some $\gamma > 0$. It is interesting that on appropriate space-time scales, the scaling limit is again (as in [16]) the Kingman coalescent, with only its starting configuration depending on the specific properties of the underlying Λ-coalescent. We obtain functional limit theorems for the partition structure and for the number of partitions, in Theorems 13 and 19 respectively.

2 Construction of the coalescent

The construction of the spatial coalescent on an appropriate state space follows quite standard steps. The construction below is inspired by those in Evans and Pitman [15], Pitman [23], and Berestycki [5].

Let $\mathcal{P}$ be the set of partitions on $\mathbf{N}$, which can be identified with the set of equivalence relations on $\mathbf{N}$. Any $\pi \in \mathcal{P}$ can be represented uniquely by $\pi = (A_1, A_2, A_3, \dots)$ where $A_j \subset \mathbf{N}$ for $j \geq 1$ are called the the *blocks* of π, indexed according to the increasing ordering of the set $\{\min A_j : j \geq 1\}$ that contains the smallest element of each block. So in particular $\min A_{n-1} < \min A_n$, for any $n \geq 2$. Likewise, we define for any $n \in \mathbf{N}$, $\mathcal{P}_n$ as the set of partitions of $[n]$, and for $\pi \in \mathcal{P}_n$ we have $\pi = (A_1, A_2, \dots, A_n)$ in an analogous way. We will write $A \in \pi$ if $A \subset \mathbf{N}$ is a block of π, and

$$A_i \sim_\pi A_j$$

if $A_i, A_j \subset \mathbf{N}$ and $A_i \cup A_j \subset A$ for some (unique) $A \in \pi$. If the number of blocks of π, denoted by $\#\pi$, is finite, then set $A_j = \emptyset$ for all $i > \#\pi$.

For concreteness in the rest of the paper, let $|\mathcal{G}| = \upsilon$ for υ a positive integer and let the vertices of $\mathcal{G}$ be $\{g_1, \dots, g_\upsilon\}$. The spatial coalescent takes values in the set $\mathcal{P}^\ell$ of partitions on $\mathbf{N}$, indexed as described above, and labelled by $\mathcal{G}$, so

$$\mathcal{P}^\ell := \{(A_j, \zeta_j) : A_j \in \pi, \zeta_j \in \mathcal{G}, \pi \in \mathcal{P}, j \geq 1\}.$$

Similarly, the coalescent started from n blocks takes values in $\mathcal{P}_n^\ell := \{(A_j, \zeta_j) : A_j \in \pi, \zeta_j \in \mathcal{G}, \pi \in \mathcal{P}_n, 1 \leq j \leq n\}$. Here, the $\zeta_j \in \mathcal{G}$ is the *label* (or location) of $A_j \in \pi$, $j \geq 1$. Set $\zeta_j = \partial \notin \mathcal{G}$ if $A_j = \emptyset$. For any element $\pi \in \mathcal{P}^\ell$ or $\pi \in \mathcal{P}_n^\ell$ with $n \geq m$ define $\pi|_m \in \mathcal{P}_m^\ell$ as the labeled partition induced by π on $\mathcal{P}_m^\ell$. We equip $\mathcal{P}^\ell$ with the metric

$$d(\pi, \pi') = \sup_{m \in \mathbf{N}} 2^{-m} 1_{\{\pi|_m \neq \pi'|_m\}}; \quad \pi, \pi' \in \mathcal{P}^\ell, \tag{5}$$

and likewise $\mathcal{P}_n^\ell$ with the metric

$$d_n(\pi, \pi') = \sup_{m \leq n} 2^{-m} 1_{\{\pi|_m \neq \pi'|_m\}}; \quad \pi, \pi' \in \mathcal{P}_n^\ell.$$

Since $|\mathcal{G}|$ is finite, it is easy to see that $d(\pi, \pi') = \sup_n d_n(\pi|_n, \pi'|_n)$, and that $(\mathcal{P}_n^\ell, d_n)$ and $(\mathcal{P}^\ell, d)$ are *both* compact metric spaces.

Note that $\mathcal{P}^\ell$ can be interpreted as a subspace of the infinite product space $\mathcal{P}_\Pi^\ell := (\mathcal{P}_1^\ell, \mathcal{P}_2^\ell, \mathcal{P}_3^\ell, \dots)$ endowed with the metric $d(\pi, \pi') = \sup_n d_n(\pi_n, \pi'_n)$ for $\pi = (\pi_1, \pi_2, \dots)$, $\pi' = (\pi'_1, \pi'_2, \dots) \in \mathcal{P}_\Pi^\ell$, by identifying $\pi \in \mathcal{P}^\ell$ with $(\pi|_1, \pi|_2, \pi|_3, \dots)$. Note that this metric induces the product topology on $\mathcal{P}_\Pi^\ell$ and that an element $(\pi_1, \pi_2, \dots)$ of $\mathcal{P}_\Pi^\ell$ is also an element of $\mathcal{P}^\ell$ if and only if it fulfills the following *consistency relationship,*

$$\pi_{n+1}|_n = \pi_n \quad \text{for all } n \in \mathbf{N}. \tag{6}$$

In the rest of the paper, whenever Π^ℓ is a spatial coalescent process, we denote by Π the partition (without the labels of the blocks) of Π^ℓ, and by

$$(\#\Pi(t))_{t \geq 0}$$

the corresponding total number of blocks process. Thus $\#\Pi(t)$ is the number (finite or infinite) of blocks in $\Pi(t)$, or equivalently, in $\Pi^\ell(t)$.

With the above notation we are finally able to construct the spatial Λ-coalescent started from potentially infinitely many blocks, as stated in the following theorem. Recall the migration mechanism stated in the introduction: each block performs an independent continuous Markov chain on $\mathcal{G}$ with jump rate 1 and transition kernel $p(\cdot, \cdot)$.

Theorem 1 *Assume that Λ has no atom at 0. Let $\mathcal{G}$ be a finite graph with vertex set $\{g_1, g_2, \dots, g_v\}$. Then, for each $\pi \in \mathcal{P}^\ell$, there exists a càdlàg Feller and strong Markov process Π^ℓ on $\mathcal{P}^\ell$, called the* spatial Λ-coalescent, *such that $\Pi^\ell(0) = \pi$ and*

(i) blocks with the same label coalesce according to a (non-spatial) Λ-coalescent,

(ii) independently, each block of label $g_i \in \mathcal{G}$ changes its label to $g_j \in \mathcal{G}$ at rate $p(g_i, g_j)$ as mentioned in introduction.

This process also satisfies

(iii) $(\Pi^\ell(t)|_n)_{t\geq 0}$ *is a spatial* Λ*-coalescent started from* $\Pi^\ell(0)|_n$*,*

and its law is characterized by (iii) and the initial configuration π*.*

Proof. In order to define a càdlàg Markov process Π^ℓ with values in $\mathcal{P}^\ell_\prod$ such that $\Pi^\ell_n := \Pi^\ell|_n$ is a spatial coalescent starting at $\Pi^\ell(0)|_n \in \mathcal{P}^\ell_n$ for any $\Pi^\ell(0) \in \mathcal{P}^\ell$, we will make use of suitably chosen Poisson point processes.

For each $i \in [v]$ let N_i be an independent Poisson point process on $\mathbf{R}_+ \times [0,1] \times \{0,1\}^{\mathbf{N}}$ with intensity measure $dt\, x^{-2}\Lambda(dx)P_x(d\xi)$, where $\xi = (\xi_1, \xi_2, \dots)$ is a random vector whose entries ξ_j are i.i.d. Bernoulli(x) under P_x, defined on some probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

Let δ_n denote the Kronecker delta measure with unit atom at n. Let M be another independent Poisson point process on the same probability space Ω with values in $\mathbf{R}_+ \times \mathbf{N} \times \mathcal{G}^v$ and intensity measure given by $dt \sum_{k=1}^\infty \delta_k(dm)P^v(ds_1, \dots, ds_v)$, where P^v is the joint law of independent $\mathcal{G}$-valued random variables $S_1, \dots, S_v$, such that $P(S_{g_i} = g_j) = p(g_i, g_j)$, $g_i, g_j \in \mathcal{G}$.

Using the above random objects define a spatial Λ-coalescent with n initial blocks, Π^ℓ_n, on Ω for each $n \geq 1$ as follows: At any atom (t, x, ξ) of N_i, all blocks $A_j(t-)$ with $\zeta_j(t-) = g_i$ and $\xi_j = 1$ coalesce together into a new labeled block $(\bigcup_{j,\xi_j=1,\zeta_j(t-)=i} A_j(t-), g_i)$; at any atom $(t, m, (s_1, \dots, s_v))$ of M we set $\zeta_m(t) = s_{\zeta_m(t-)}$ provided $m \leq \#\Pi_n(t-)$, otherwise nothing changes. For all other $t \geq 0$ we set $\Pi^\ell_n(t) = \Pi^\ell_n(t-)$. Note that coalescence causes immediate reindexing (or reordering) of blocks that have neither participated in coalescence nor in migration, and that this reindexing operation decreases each index by a non-negative amount.

Since the sum of the above defined jump rates of Π^ℓ_n is finite it follows immediately that Π^ℓ_n is a well defined càdlàg Markov process on Ω for each $n \geq 1$ therefore inducing a càdlàg Markov process Π^ℓ on $\mathcal{P}^\ell_\prod$. It is important to note that each Π^ℓ_n so constructed is a Λ-coalescent started from $\Pi^\ell(0)|_n$. Since $\Pi^\ell_{n+1}(0)|_n = \Pi^\ell_n(0)$ and since clearly the consistency condition (6) is preserved under each transition of Π^ℓ_{n+1} in the construction (this is not always a transition for Π^ℓ_n), we have $\Pi^\ell_{n+1}(t)|_n = \Pi^\ell_n(t)$ for all $t \geq 0$. Therefore, $(\Pi^\ell(t))_{t\geq 0}$ constructed by $\Pi^\ell(t)|_n := \Pi^\ell_n(t), n \geq 1, t \geq 0$ is well-defined. It follows that Π^ℓ is a càdlàg Markov process with values in $\mathcal{P}^\ell$, which clearly satisfies properties (i)-(iii), and uniqueness in distribution follows similarly.

In order to verify that the semigroup $T_t\varphi(\pi) := E[\varphi(\Pi^\ell(t))|\Pi^\ell(0) = \pi]$ is a Feller-Dynkin semigroup it now suffices to check the following two properties (see [24] III (6.5)-(6.7)): (i) For any continuous (bounded) real valued function φ on $\mathcal{P}^\ell$ and all $\pi \in \mathcal{P}^\ell$ we have

$$\lim_{t\to 0+} T_t\varphi(\pi) = \varphi(\pi),$$

and (ii) for any continuous (bounded) real valued function φ on $\mathcal{P}^\ell$ and all $t > 0$, $\pi \mapsto T_t\varphi(\pi)$ is continuous (and bounded).

Note that (i) is an immediate consequence of the right-continuity of the paths and continuity with respect to (5). One can easily argue for (ii): if $\Pi^{\ell,k}$ is the spatial coalescent started from π^k and Π^ℓ is the spatial coalescent started from π such that $\lim_{k\to\infty} \pi^k = \pi \in \mathcal{P}^\ell$, then, due to the definition of the metric (5) on $\mathcal{P}^\ell$, there exists for all $k \in \mathbf{N}$ an $m = m(k)$ such that $\pi^k|_m = \pi|_m$, with the property $m(k) \to \infty$ as $k \to \infty$. This implies that one can construct a coupling of $\Pi^{\ell,k}$ and Π^ℓ (using the same Poisson point processes for all) such that $\Pi^{\ell,k}(t)|_m = \Pi^\ell(t)|_m$ for all $t \geq 0$. Hence $d(\Pi^{\ell,k}(t), \Pi^\ell(t)) \leq 2^{-(m(k)+1)}$ for all $t \geq 0$ and, since $m(k) \to \infty$, the second property follows due to the continuity of φ. Given that T_t is a Feller-Dynkin semigroup the strong Markov property holds. $\square$

Remark. A variation of the above construction could be repeated for the cases where Λ has an atom at 0, i.e. where $\Lambda(\cdot) = a\delta_0(\cdot) + \Lambda^+(\cdot)$, for some $a > 0$ and Λ^+ which does not have an atom at 0. This would correspond to superimposing Kingman coalescent type transitions and the Poisson process (Λ^+ driven) induced coalescent events. The spatial Λ-coalescent corresponding to the atomic part $a\delta_0$ is a time-change of the spatial Kingman coalescent and it comes down from infinity uniformly due to Proposition 3.4 in [16]. Superposition of coalescence driven by Λ^+ can only decrease the number of particles, so any such Λ-coalescent comes down from infinity uniformly. Also note that an atom of Λ at 1 implies complete collapse in finite time, even if the coalescent corresponding to the measure $\Lambda(\cdot \cap [0,1))$ stays infinite. See [23] for further discussion of atoms. $\diamond$

Remark. We stated Theorem 1 for $|\mathcal{G}| < \infty$. The same Poisson point process construction, even in the cases where the underlying graph $\mathcal{G}$ is an infinite and bounded degree (or a locally finite) graph, and where the starting configuration $\pi \in \mathcal{P}_n^\ell$ is finite, clearly works. This fact will be useful in Section 5 where we consider $\mathcal{G} = \mathbf{Z}^d$. To extend the construction to infinite graphs and infinite initial configurations requires a little more work. As mentioned in the introduction, given Theorem 12, one can conveniently approximate the whole lattice $\mathbf{Z}^d$ by large tori to construct a locally finite (at all positive times) version of the coalescent, even if started with infinitely many partition elements at each site of the lattice. $\diamond$

3 Coming down from infinity

In this section, we first obtain estimates on the coalescence rates and the rates of decrease in the number of blocks, both in the non-spatial and the spatial setting. Several of these estimates will be applied to showing that the spatial Λ-coalescent comes down from infinity if and only if (4) holds.

It is easy to see, using definitions (1)-(3), that

$$\lambda_b = \int_{[0,1]} \frac{1-(1-x)^b - bx(1-x)^{b-1}}{x^2} d\Lambda(x) \ , \ \gamma_b = \int_{[0,1]} \frac{bx - 1 + (1-x)^b}{x^2} d\Lambda(x). \qquad (7)$$

The following lemma is listing some facts, which are based on (7) and some simple computations.

Lemma 2 *We have the following estimates:*

(i) $\lambda_{b+1} - \lambda_b = \int_{[0,1]} b(1-x)^{b-1} d\Lambda(x)$ *for* $b \geq 2$, *in particular* $\lambda_b \leq \lambda_{b+1} \leq 3\lambda_b$,

(ii) $\gamma_{b+1} - \gamma_b = \int_{[0,1]} (1-(1-x)^b) x^{-1} d\Lambda(x) \geq 0$.

Proof. (i) Note that $-(1-x)^{b+1} - (b+1)x(1-x)^b + (1-x)^b + bx(1-x)^{b-1} = (1-x)^{b-1}(-(1-x)^2 - (b+1)x(1-x) + (1-x) + bx)$, and that the term in the parentheses equals bx^2. Combined with (7), this gives the initial statement of the lemma. The first inequality $\lambda_{b+1} \geq \lambda_b$ is immediate.

The second inequality follows again from (7), by integrating the following inequality with respect to Λ

$$b(1-x)^{b-1} \le b(b-1)x^{-2}(1-x)^{b-2} \le 2x^{-2}\left(1-(1-x)^b - bx(1-x)^{b-1}\right),\ x \in [0,1],$$

which is easy to check, for example, via the Binomial Theorem.

(ii) The stated property of the sequence γ was already noted and used by Schweinsberg, cf. [26] Lemma 3. For completeness we include a brief argument: From (7)

$$\begin{aligned}\gamma_{b+1}-\gamma_b &= \int_{[0,1]} (x+(1-x)^{b+1}-(1-x)^b)x^{-2}d\Lambda(x) \\ &= \int_{[0,1]} (1-(1-x)^b)x^{-1}d\Lambda(x) \ge 0.\end{aligned}$$

□

The following two lemmas and a corollary are auxiliary results, often implicitly observed in [23] or [26], and are of interest to anyone studying fine properties of Λ-coalescents. Fix $a \in (0,1)$. Let Λ^a be the restriction of Λ to $[0,a]$, namely

$$\Lambda^a([0,x]) = \Lambda([0,a]\cap[0,x]),\ x \in [0,1].$$

Let λ_b^a, γ_b^a be defined in (1)-(3) using Λ^a as the underlying measure instead of Λ.

Lemma 3 *(i) For each fixed a, such that $\Lambda^a((0,1)) > 0$, there exists a constant $C_1 = C_1(\Lambda, a) \in (0,\infty)$ such that for all large b,*

$$\lambda_b^a \le \lambda_b \le C_1\lambda_b^a.$$

(ii) There exists an $a < 1$ and $C_2 = C_2(\Lambda, a) \in (0,\infty)$ such that for all large b,

$$\gamma_b^a \le \gamma_b \le C_2\gamma_b^a.$$

(iii) If $\int_{[0,1]} \frac{1}{x}\, d\Lambda(x) = \infty$, in particular if (4) holds, then for each fixed a, the inequalities in (ii) hold with a constant $C_3 = C_3(\Lambda, a) \in (0,\infty)$.

Remark. For any fixed Λ let

$$\eta_b := \sum_{k=2}^{b} \binom{b}{k} k\, \lambda_{b,k}.$$

Then it is easy to see that $\gamma_b/\eta_b \to 1$ as $b \to \infty$, so statements (ii) and (iii) above extend to the corresponding η_b and η_b^a. ◇

Proof. For each $a \in (0,1)$, the first inequalities in both (i) and (ii) are trivial consequences of Λ^a being the restriction of Λ, the identities in (7), and the fact that $1-(1-x)^b - bx(1-x)^{b-1}$ and $bx - 1 + (1-x)^b$ are both non-negative on $[0,1]$.

The second inequality in (i) is easy as well, since $1-(1-x)^b - bx(1-x)^{b-1}$ is bounded by 1, which implies

$$\lambda_b \le \lambda_b^a + \frac{1}{a^2}\Lambda([a,1]). \tag{8}$$

Then either $\lambda_b \to \infty$, in which case (8) implies $\lambda_b^a \to \infty$ as $b \to \infty$, so that for all large b, $\frac{1}{a^2}\Lambda([a,1]) \leq \lambda_b^a$, or λ_b stays finite, in which case the upper bound is trivial.

The proof of the second inequality in (ii) is similar. First note that $bx - 1 + (1-x)^b \leq bx$ so that

$$\gamma_b \leq \gamma_b^a + \frac{b}{a}\Lambda([a,1]). \tag{9}$$

Now it is easy to see by Lemma 2(ii) that $\gamma_{b+1} - \gamma_b$ is non-decreasing in b so that $\gamma_b \geq (\gamma_3 - \gamma_2)(b-2)$ for each b. For a chosen sufficiently close to 1, $\frac{1}{a}\Lambda([a,1]) < (\gamma_3 - \gamma_2)/3$ (recall Λ is a finite measure). Hence, (9) implies $\gamma_b^a \geq (\gamma_3 - \gamma_2)b/2$ for all b large enough and (9) then also implies the upper bound in (ii) since $\frac{b}{a}\Lambda([a,1]) < (\gamma_3 - \gamma_2)\frac{b}{3} < \gamma_b^a$.

Part (iii) follows immediately from the argument for (ii), and the following fact (already noticed by Pitman [23], Lemma 25),

$$\int_{[0,1]} \frac{1}{x}\, d\Lambda(x) = \lim_{b\to\infty} \frac{\gamma_b}{b}. \tag{10}$$

In particular, (4) must imply that the left hand side in (10) is infinite. $\square$

Let the symbol $\asymp$ stand for "asymptotically equivalent behavior" in the sense that $a_m \asymp b_m$ (as $m \to \infty$) if there exist two finite positive constants c, C such that

$$c\, a_m \leq b_m \leq C\, a_m, \;\; m \geq 1.$$

Lemma 4 *We have*

(i)

$$\lambda_b \asymp b^2\Lambda[0,1/b] + \int_{[1/b,1]} \frac{1}{x^2} d\Lambda(x) - b\int_{[1/b,\log(2b/(1-e^{-1}))/b]} \frac{(1-x)^{(b-1)}}{x} d\Lambda(x),$$

(ii)

$$\gamma_b \asymp b^2\Lambda[0,1/b] + b\int_{[1/b,1]} \frac{1}{x} d\Lambda(x).$$

Proof. (i) To show the first claim, use expression (7) to get for $b \geq 2$,

$$\begin{aligned}\lambda_b &= \int_{[0,1/b]} \frac{1-(1-x)^b - bx(1-x)^{b-1}}{x^2} d\Lambda(x) \\ &\quad + \int_{[1/b,1]} \frac{1-(1-x)^b - bx(1-x)^{b-1}}{x^2} d\Lambda(x).\end{aligned}$$

Then note that

$$\frac{\int_{[0,1/b]} \frac{1-(1-x)^b - bx(1-x)^{b-1}}{x^2} d\Lambda(x)}{b(b-1)\Lambda([0,1/b])/2} \to 1, \text{ as } b \to \infty,$$

and also that

$$(1-e^{-1})\int_{[1/b,1]} \frac{1}{x^2}\, d\Lambda(x) \leq \int_{[1/b,1]} \frac{1-(1-x)^b}{x^2}\, d\Lambda(x) \leq \int_{[1/b,1]} \frac{1}{x^2}\, d\Lambda(x).$$

A calculus fact, $1-x \le e^{-x}$, $x \in [0,1]$ implies that if $x \ge \log(2b/(1-e^{-1}))/b$, then $(1-x)^b \le \frac{(1-e^{-1})}{2b}$. This in turn implies that

$$\begin{aligned}
& \int_{[\log(2b/(1-e^{-1}))/b,1]} \frac{b(1-x)^{b-1}}{x}\, d\Lambda(x) \\
\le\ & \int_{[\log(2b/(1-e^{-1}))/b,1]} \frac{b(1-e^{-1})}{2(b-1)x}\, d\Lambda(x) \\
\le\ & \frac{b(1-e^{-1})}{2(b-1)} \int_{[\log(2b/(1-e^{-1}))/b,1]} \frac{1}{x^2}\, d\Lambda(x) \\
\le\ & \frac{b}{2(b-1)} \int_{[1/b,1]} \frac{1-(1-x)^b}{x^2}\, d\Lambda(x),
\end{aligned}$$

so that $-\int_{[\log(2b/(1-e^{-1}))/b,1]} \frac{b(1-x)^{b-1}}{x}\, d\Lambda(x)$ can be ignored in the asymptotics, and the remaining term

$$\int_{[1/b,\log(2b/(1-e^{-1}))/b]} \frac{b(1-x)^{b-1}}{x}\, d\Lambda(x),$$

appears in the asymptotic expression for λ_b. (ii) Since $\gamma_b \asymp \eta_b$, see the above remark, it suffices to show the second statement for η_b instead. As in (7),

$$\eta_b = b\int_{[0,1/b]} \frac{1-(1-x)^{b-1}}{x}\, d\Lambda(x) + b\int_{[1/b,1]} \frac{1-(1-x)^{b-1}}{x}\, d\Lambda(x),$$

and since it is easy to see that

$$\frac{\int_{[0,1/b]} \frac{1-(1-x)^{b-1}}{x}\, d\Lambda(x)}{b\Lambda([0,1/b])} \to 1, \text{ as } b \to \infty,$$

while

$$\int_{[1/b,1]} \frac{1-(1-x)^{b-1}}{x}\, d\Lambda(x) \asymp \int_{[1/b,1]} \frac{1}{x}\, d\Lambda(x),$$

the claim on the asymptotics of γ_b (i.e., η_b) follows. □

Corollary 5 *(i) If $\lambda_b \to \infty$, as $b \to \infty$ then $\lim_{b\to\infty} \lambda_{b+1}/\lambda_b = 1$,*

(ii) Since $\gamma_b \to \infty$, as $b \to \infty$ we obtain that $\lim_{b\to\infty} \gamma_{b+1}/\gamma_b = 1$.

Proof. (i) By the Binomial Formula, for $x \in [0,1]$,

$$\frac{b-1}{2} bx^2(1-x)^{b-1} \le 1-(1-x)^b - bx(1-x)^{b-1},$$

so that

$$\begin{aligned}
& \int_{[0,\log(2(b-1))/(b-1)]} b(1-x)^{b-1}\, d\Lambda(x) \qquad (11) \\
\le\ & \frac{2}{b-1} \int_{[0,\log(2(b-1))/(b-1)]} \frac{1-(1-x)^b - bx(1-x)^{b-1}}{x^2} d\Lambda(x).
\end{aligned}$$

Since $\int_{[\log(2(b-1))/(b-1),1]} b(1-x)^{b-1}\, d\Lambda(x) \leq \frac{b}{2(b-1)}\Lambda([0,1])$, the conclusion follows by Lemma 2(i), (7), (11) and the fact that $\lambda_b \to \infty$.
(ii) Perhaps the easiest way to see that $\gamma_b \to \infty$ whenever $\Lambda([0,1]) > 0$ is by using the identity (10). The statement then follows immediately from Lemma 2(ii), Lemma 4(ii), and the fact that

$$\int_{[0,1]} \frac{1-(1-x)^b}{x} d\Lambda(x) \leq b\Lambda([0,1/b]) + \int_{[1/b,1]} \frac{1}{x}\, d\Lambda(x).$$

□

Lemma 6 *There exists a finite number $\rho \geq 1$ such that for any Λ, and all $b \geq 3, m \geq 2$ such that $b/m \geq 2$ we have*

$$\lambda_b \leq m^\rho \lambda_{\lceil b/m \rceil}$$

Proof. In this lemma we consider the identities (7) for all real $b \geq 1$. It suffices to show that for some $c \in (0,\infty)$

$$\lambda_b \leq c\lambda_{b/2} \tag{12}$$

for all $b \geq b_0$ where $b_0 > 2$ is some finite integer. Indeed, if $m \in (2^k, 2^{k+1}]$ for some k then (note $c^{\log_2 m} = m^{\log_2 c}$

$$\lambda_b \leq c^{k+1}\lambda_{b/2^{k+1}} \leq cm^{\log_2 c}\lambda_{\lceil b/m \rceil},$$

and now one can take $\rho > c + \log_2 c$ to get the statement of the lemma. Define the function

$$g(\beta, x) := 1 - (1-x)^\beta - \beta x(1-x)^{\beta-1}.$$

Due to representation (7) for λ_b it then suffices to study

$$f_b(x) := \frac{1-(1-x)^b - bx(1-x)^{b-1}}{1-(1-x)^{b/2} - \frac{b}{2}x(1-x)^{\frac{b}{2}-1}} = \frac{g(b,x)}{g(b/2,x)},$$

and show

$$\sup_{x\in[0,1]} f_b(x) \leq c,$$

uniformly in all $b \geq b_0$. Note that $f_b(0+) = \frac{b(b-1)}{b/2(b/2-1)} = 4\frac{b-1}{b-2} \leq 8$ if $b \geq 3$ and that $f_b(1) = 1$. The derivative $f_b'(x)$ can be written as a ratio $fn_b(x)/fd_b(x)$ where $fd_b(x) \geq 0$ and where $fn_b(x)$ equals

$$x(1-x)^{b/2-2}[b(b-1)(1-x)^{b/2} - (b/2-1)\frac{b}{2} - \frac{b}{2}(\frac{3b}{2}-1)(1-x)^b - b(\frac{b}{2})^2 x(1-x)^{b-1}].$$

Therefore $fn_b(x) < 0$ whenever $b(b-1)(1-x)^{b/2} < (\frac{b}{2}-1)\frac{b}{2}$ and in particular whenever $x > \frac{2}{b}\log 8$ for all $b \geq 4$. So it suffices to show that

$$\sup_{x\in[0,\frac{2}{b}\log 8]} f_b(x) \leq c.$$

For this note that $g(b/2,x) \geq \frac{b/2(b/2-1)}{2} x^2(1-x)^{\frac{b}{2}-2}$ for any $x \in [0,1]$, and that (by expanding the binomial terms and noting $x/(1-x) \leq \frac{4}{b}\log 8$ whenever $x < \frac{2}{b}\log 8$ and $b \geq 10$)

$$\begin{aligned} \frac{g(b,x)}{\binom{b}{2}x^2(1-x)^{b-2}} &= \sum_{l=2}^{b} \frac{2(b-2)!}{l!(b-l)!}\left(\frac{x}{1-x}\right)^{l-2} \\ &\leq 2\cdot\sum_{l=2}^{b}\left[\frac{(l-2)!(b-2)!}{l!(b-l)!b^{l-2}}\right]\frac{(\log 8^4)^{l-2}}{(l-2)!} \leq 2\cdot 8^4, \end{aligned}$$

since the term in square brackets is bounded by 1. □

Now we turn to the spatial setting. Recall that the vertex set of $\mathcal{G}$ is $\{g_1,\ldots,g_v\}$. Denote by $\lambda(b_1,b_2,\ldots,b_v)$ the total rate of coalescence for the configuration with b_i blocks at site g_i,

$$\lambda(b_1,b_2,\ldots,b_v) := \sum_{i=1}^{v}\sum_{k=2}^{b_i}\binom{b_i}{k}\lambda_{b_i,k} = \sum_{i=1}^{v}\lambda_{b_i}.$$

Similarly, let

$$\gamma(b_1,b_2,\ldots,b_v) := \sum_{i=1}^{v}\gamma_{b_i}.$$

Denote by $\lfloor x\rfloor$ the integer part of the real number x and let $\lceil x\rceil := -\lfloor -x\rfloor$.

The following two lemmas will be useful for the proof of the characterization result given in Proposition 11.

Lemma 7 *For all $v \geq 1, b_i \geq 0, i = 1,\ldots,v$ integers with $\sum_{i=1}^{v} b_i > v$,*

(i) $\gamma_{\sum_{i=1}^{v} b_i} \geq \gamma(b_1,b_2,\ldots,b_v) \geq v\gamma_{\lfloor\sum_{i=1}^{v} b_i/v\rfloor}$,

(ii) $v^{1+\rho}\lambda_{\lceil\sum_{i=1}^{v} b_i/v\rceil} \geq \lambda(b_1,b_2,\ldots,b_v) \geq \lambda_{\lceil\sum_{i=1}^{v} b_i/v\rceil}$.

Proof. (i) In order to verify the first inequality we observe that for $x \in [0,1]$,

$$v-1 \geq \left(\sum_{i=1}^{v}(1-x)^{b_i}\right) - (1-x)^{\sum_{i=1}^{v} b_i} \tag{13}$$

since one can simply check that equality holds for $x = 0$ and that $x \mapsto (\sum_{i=1}^{v}(1-x)^{b_i}) - (1-x)^{\sum_{i=1}^{v} b_i}$ is a decreasing function on $[0,1]$. Inequality (13) implies that

$$\left(\sum_{i=1}^{v} b_i\right)x - 1 + (1-x)^{\sum_{i=1}^{v} b_i} \geq \sum_{i=1}^{v}(b_i x - 1 + (1-x)^{b_i})$$

for all $x \in [0,1]$. The first inequality in (i) now follows from this and from (7), since

$$\gamma(b_1,b_2,\ldots,b_v) = \sum_{i=1}^{v}\gamma_{b_i} = \int_{[0,1]} x^{-2}\sum_{i=1}^{v}(b_i x - 1 + (1-x)^{b_i})d\Lambda(x), \tag{14}$$

for $b_i \geq 0$ (if we set $0^0 = 1$). The second inequality of (i) is immediate if $\sum_{i=1}^{\upsilon} b_i < 2\upsilon$. Otherwise, we note that

$$\sum_{i=1}^{\upsilon}(1-x)^{b_i} \geq \upsilon(1-x)^{\sum_{i=1}^{\upsilon} b_i/\upsilon} \tag{15}$$

by Jensen's Inequality since the function $y \mapsto a^y$ is convex for every $a > 0$. Therefore, (14) is bounded below by

$$\upsilon \int_{[0,1]} x^{-2}\left(\beta x - 1 + (1-x)^{\beta}\right) d\Lambda(x),$$

where $\beta = \sum_{i=1}^{\upsilon} b_i/\upsilon$. If β is an integer then the last expression is just $\upsilon\gamma_\beta$. Now note that the function $\beta \mapsto \beta x - 1 + (1-x)^{\beta}$ is increasing (for $\beta \geq 1$) and this implies the second inequality in (i).

(ii) Use Lemma 6 to conclude

$$\lambda(b_1, b_2, \ldots, b_\upsilon) \leq \upsilon^{1+\rho}\lambda_{\lceil \sum_{i=1}^{\upsilon} b_i/\upsilon \rceil}.$$

The second inequality of (ii) is a simple consequence of the fact that there exists a $1 \leq j \leq \upsilon$ such that $b_j \geq \lceil \sum_{i=1}^{\upsilon} b_i/\upsilon \rceil$ and Lemma 2(i). $\square$

Now consider the coalescent $(\Pi_{n\upsilon}^{\ell}(t))_{t\geq 0}$ such that its initial configuration $\Pi_{n\upsilon}^{\ell}(0)$ has n blocks at each site of $\mathcal{G}$. Let

$$T_n := \inf\{t > 0 : \#\Pi_{n\upsilon}(t) \leq 2\upsilon\}. \tag{16}$$

Lemma 8 *If condition (4) holds then* $\sup_n E[T_n] \leq \sum_{b=2}^{\infty} \frac{3\upsilon^{\rho+1}}{\gamma_b} < \infty$.

Proof. The argument is an adaptation of the argument by Schweinsberg [26], Lemma 6, to our situation. In fact we will even use similar notation. For $n \in \mathbf{N}$ define $R_0 := 0$ and stopping times (with respect to the filtration generated by $\Pi_{n\upsilon}^{\ell}$) given by

$$\begin{aligned} R_i &:= 1_{\{\#\Pi_{n\upsilon}(R_{i-1})>2\upsilon\}} \inf\{t > R_{i-1} : \#\Pi_{n\upsilon}(t) < \#\Pi_{n\upsilon}(R_{i-1})\} \\ &\quad +1_{\{\#\Pi_{n\upsilon}(R_{i-1})\leq 2\upsilon\}} R_{i-1},\ i \geq 1. \end{aligned}$$

In words, R_i is the time of the ith coalescence as long as the number of blocks before this coalescence exceeds 2υ, otherwise R_i is set equal to the previous coalescence time. Since there are no more than 2υ blocks left after $(n-2)\upsilon$ coalescence events, note that

$$T_n = R_{(n-2)\upsilon}.$$

Of course, it is also possible that $T_n = R_i$ for $i < (n-2)\upsilon$, but the above identity holds almost surely as $R_{(n-2)\upsilon} = R_i$ in this case. Let

$$L_i = R_i - R_{i-1},\ J_i = \#\Pi_{n\upsilon}(R_{i-1}) - \#\Pi_{n\upsilon}(R_i),$$

and note that there exists some finite random number ξ_i such that $R_{i-1} = T_0^i < T_1^i < T_2^i < \ldots < T_{\xi_i}^i < R_i$, where $T_1^i, T_2^i, \ldots$ are the successive times of migration jumps of various blocks from site to site in between the $i-1$th and ith coalescence time. Let $B_i(t)$ be the number of blocks located at site $g_i \in \mathcal{G}$ at time t. Since the total number of blocks does not change at the jump times T_j^i

for $j = 1, \dots, \xi_i$ we have due to Lemma 7 (ii) that $\lambda(B_1(T_j^i), \dots, B_\upsilon(T_j^i)) \geq \lambda_{\lceil \sum_{e=1}^{\upsilon} B_e(T_j^i)/\upsilon \rceil} = \lambda_{\lceil \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rceil}$. This implies (by coupling of exponentials in a straightforward way) that

$$E[L_i|\Pi_{n\upsilon}(R_{i-1})] \leq \frac{1}{\lambda_{\lceil \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rceil}}. \tag{17}$$

Also note that for all i with $\Pi_{n\upsilon}(R_{i-1}) > 2\upsilon$,

$$\begin{aligned} E[J_i|\Pi_{n\upsilon}(R_{i-1})] &= E\left[\sum_{j=0}^{\infty} \frac{\gamma(B_1(T_j^i), \dots, B_\upsilon(T_j^i))}{\lambda(B_1(T_j^i), \dots, B_\upsilon(T_j^i))} 1_{\{\xi_i = j\}} \middle| \Pi_{n\upsilon}(R_{i-1})\right] \\ &\geq \frac{\upsilon \gamma_{\lfloor \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rfloor}}{\upsilon^{1+\rho} \lambda_{\lceil \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rceil}} P(1_{\{\xi_i < \infty\}} | \Pi_{n\upsilon}(R_{i-1})) \\ &= \frac{1}{\upsilon^\rho} \frac{\gamma_{\lfloor \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rfloor}}{\lambda_{\lceil \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rceil}}, \end{aligned} \tag{18}$$

where the first equality is a direct consequence of definitions (2) and (3), and the fact that J_i is the decrease in the number of blocks at the ith coalescence time R_i. The middle inequality is due to Lemma 7 (i) and (ii). From (17) and (18) and the fact that $L_i = 0$ if $J_i = 0$ we get the important relation

$$E[L_i|\Pi_{n\upsilon}(R_{i-1})] \leq \frac{\upsilon^\rho}{\gamma_{\lfloor \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rfloor}} E[J_i|\Pi_{n\upsilon}(R_{i-1})] \tag{19}$$

for $i \geq 1$. Now

$$\begin{aligned} E[T_n] &= E[\sum_{i=1}^{\upsilon(n-2)} L_i] = \sum_{i=1}^{\upsilon(n-2)} E\left[E\left[L_i|\Pi_{n\upsilon}(R_{i-1})\right]\right] \\ &\leq \sum_{i=1}^{\upsilon(n-2)} E\left[\frac{\upsilon^\rho}{\gamma_{\lfloor \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rfloor}} E\left[J_i|\Pi_{n\upsilon}(R_{i-1})\right]\right] \\ &= E\left[\sum_{i=1}^{\upsilon(n-2)} \frac{\upsilon^\rho}{\gamma_{\lfloor \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rfloor}} J_i\right] = E\left[\sum_{i=1}^{\upsilon(n-2)} \sum_{j=0}^{J_i - 1} \frac{\upsilon^\rho}{\gamma_{\lfloor \sum_{e=1}^{\upsilon} B_e(R_{i-1})/\upsilon \rfloor}}\right] \\ &\leq \upsilon^\rho E\left[\sum_{b=2}^{n} \frac{\upsilon}{\gamma_b} + \frac{2\upsilon}{\gamma_2}\right] \leq \sum_{b=2}^{n} \frac{3\upsilon^{\rho+1}}{\gamma_b}, \end{aligned}$$

where we have used Lemma 9 below. □

Lemma 9 *For a fixed υ, let m, n be positive integers such that $m \in [n\upsilon, (n+1)\upsilon)$. For any $k \geq 1$ and $j_1, \dots, j_k \geq 1$ such that $\sum_{i=1}^{k-1} j_i < m - 2\upsilon$ and $\sum_{i=1}^{k} j_i \in [m - 2\upsilon, m - 1]$ one has*

$$\sum_{i=1}^{k} \frac{j_i}{\gamma_{\lfloor (m - \sum_{\ell=1}^{i-1} j_\ell)/\upsilon \rfloor}} \leq \frac{m - n\upsilon}{\gamma_n} + \sum_{b=2}^{n-1} \frac{\upsilon}{\gamma_b} + \frac{2\upsilon}{\gamma_2}. \tag{20}$$

Proof. Statement (20) can be proved for each fixed v by induction in n. The base cases $n = 2$ with $m > 2v$ and $\sum_{i=1}^{k} j_i = m - 1$ explain the extra summands $2v/\gamma_2$. Here one also uses the fact that $(\gamma_b)_{b=2}^{\infty}$ is an increasing sequence (cf. Lemma 2 (ii)). $\square$

Let us now recall the construction in Theorem 1, and define

$$T_n^{(2)} := T_n \text{ from definition (16)},$$

and

$$T_\infty = T_\infty^{(2)} = \sup_n T_n^{(2)} = \inf\{t > 0 : \#\Pi(t) \leq 2v\},$$

and furthermore define

$$T_n^{(k)} := \inf\{t > 0 : \#\Pi_{nv}(t) \leq kv\}, \ T_\infty^{(k)} := \sup_n T_n^{(k)}, \ k \geq 3. \tag{21}$$

Note that by monotone convergence $T_n^{(k)} \nearrow T_\infty^{(k)}$ we have

$$E[T_\infty^{(k)}] = \lim_{n\to\infty} E[T_n^{(k)}], \ k \geq 2.$$

Corollary 10 *If condition (4) holds then for each $k \geq 2$, $\sup_n E[T_n^{(k)}] \leq \sum_{b=k}^{\infty} \frac{v^{\rho+1}}{\gamma_b} + \frac{kv^{\rho+1}}{\gamma_k} < \infty$, and in particular*

$$\lim_{k\to\infty} \sup_n E[T_n^{(k)}] = 0.$$

Proof. The upper bound on $E[T_n^{(k)}]$ can be shown as in the proof of Lemma 8. The second claim above now follows by relation (10) and the observation following it. $\square$

We can now establish the following analogues to Proposition 23 of Pitman [23] and Proposition 5 of Schweinsberg [26] in the spatial setting.

Proposition 11 *Assume that Λ has no atom at 1. Then the spatial Λ-coalescent either comes down from infinity or it stays infinite. Furthermore, it stays infinite if and only if $E[T_\infty] = \infty$.*

Proof. Define $T := \inf\{t \geq 0 : \#\Pi(t) < \infty\}$. The first statement could be shown following Pitman [23] Proposition 23 by observing that $P[0 < T < \infty] > 0$ leads to a contradiction. We choose a different approach, based on Corollary 10 and coupling with non-spatial coalescents.

Suppose that (4) holds. Then $E[T_\infty] < \infty$, by Lemma 8, implying $T_\infty < \infty$ almost surely. Also note that the Λ-coalescent comes down from infinity due to Corollary 10, since for any $t > 0$, and any $k \geq 2$,

$$P[T > t] \leq P[T_\infty^{(k)} > t] \leq \frac{E[T_\infty^{(k)}]}{t}.$$

This verifies that $P[T \in \{0, \infty\}] = P[T = 0] = 1$, again by Corollary 10.

If (4) does not hold, we will show next by a coupling argument that, provided $\#\Pi(0) = \infty$, we have $P[T \in \{0, \infty\}] = P[T = \infty] = 1$. This implies of course that $P[T_\infty = \infty] = 1$ and $E[T_\infty] = \infty$. So assume that $\#\Pi(0) = \infty$, i.e. that there exists at least one site g in $\mathcal{G}$ such that $\Pi^\ell(0)$ contains infinitely many blocks with label g. Then the spatial coalescent Π^ℓ is stochastically bounded below by a coalescing system $\tilde{\Pi}^\ell$, in which any block that attempts to migrate is assigned to a "cemetery site" ∂ instead. More precisely, the evolution of the process $\tilde{\Pi}^\ell$ at each site is independent of the evolution at any other site, and its transition mechanism is specified by:

(i) blocks coalesce according to a Λ-coalescent,

(ii') each block vanishes (moves to ∂) at rate 1.

By adapting the construction of Π^ℓ in Theorem 1, one can easily construct a coupling $(\Pi^\ell(t), \tilde{\Pi}^\ell(t))_{t\geq 0}$ on the same probability space, so that at each time t, and for each site g of $\mathcal{G}$, the number of blocks in $\Pi^\ell(t)$ located at g is larger than (or equal to) the number of blocks in $\tilde{\Pi}^\ell(t)$ located at g. We will show that in any given time interval $[0,t]$, at each site of $\mathcal{G}$ that initially contained infinitely many blocks, there are infinitely many blocks remaining in $\tilde{\Pi}^\ell$ (even though there are infinitely many blocks that do vanish to ∂ by time t). Therefore, $\infty = \#\tilde{\Pi}(t) \leq \#\Pi(t)$ so that Π^ℓ stays infinite.

To show that $P[\#\tilde{\Pi}(t) = \infty] = 1$ for each $t > 0$, it will be convenient to construct a coupling of $\tilde{\Pi}^\ell(t)$ with a new random object $\Pi^1(t)$. Since there is no interaction among the sites of $\mathcal{G}$ in $\tilde{\Pi}$, it suffices to consider the nonspatial case where $|\mathcal{G}| = 1$. Introduce an auxiliary family $(X_j)_{j\geq 1}$ of independent exponential random variables with parameter 1. Take a (non-spatial) Λ-coalescent $(\Pi^1(s))_{s\in[0,t]}$ such that $\Pi^1(0) = \tilde{\Pi}(0)$, and in addition augment the state space for Π^1 to accommodate a mark for each block. Initially all blocks start with an empty mark. At any $s \leq t$, any block $A \in \Pi^1(s)$ is marked by ∂ if $\{X_{\min A} \leq s\}$. In this way, if an already marked block A coalesces with a family $A_1, A_2, \ldots$ of blocks, such that $\min A \leq \min_j(\min A_j)$, the new block $A \cup \cup_j A_j$ automatically inherits the mark ∂. Note as well that if a marked block A coalesces with at least one unmarked block containing a smaller element than $\min A$, the new block will be unmarked.

The number $\#^u\Pi^1(t)$ of all *unmarked* blocks in $\Pi^1(t)$ is stochastically smaller than the number $\#\tilde{\Pi}(t)$. To see this, note the difference between $\tilde{\Pi}(t)$ and $\Pi^1(t)$: a marked block in $\Pi^1(s)$ is not removed from the population immediately (unlike in $\tilde{\Pi}$) so it may coalesce (and "gather") additional blocks with higher indexed elements during $[s,t]$ resulting in a smaller number of unmarked partition elements in $\Pi^1(0)$ than in $\tilde{\Pi}(t)$.

Another random object $\Pi^2(t)$, equal in distribution to $\Pi^1(t)$, can be constructed as follows: run a (non-spatial) Λ-coalescent $(\Pi^2(s))_{s\in[0,t]}$, and attach to each block $A \in \Pi^2(t)$ a mark ∂ with probability e^{-t}. Let $\#^u\Pi^2(t)$ be the number of all unmarked blocks in $\Pi^2(t)$. Since (4) does not hold, due to the corresponding result in [26], $P[\#\Pi^2(t) = \infty] = 1$. Since $\Pi^1(t)$ and $\Pi^2(t)$ have the same distribution, then $\#^u\Pi^1(t)$ and $\#^u\Pi^2(t)$ have the same distribution and by the above construction we conclude immediately that

$$1 = P[\#^u\Pi^2(t) = \infty] = P[\#^u\Pi^1(t) = \infty].$$

Recalling that $\#^u\Pi^1(t)$ is stochastically bounded above by $\tilde{\Pi}(t)$ for all $t \geq 0$ completes the proof. $\square$

Remark. It is intuitively clear that in the case in which the Λ-coalescent Π^ℓ stays infinite, there are infinitely many blocks in Π^ℓ at all positive times at all sites, a proof of this fact is left to an interested reader. $\diamond$

Remark. Note that the upper bounds in Lemma 8 and Corollary 10 neither depend on the structure of $\mathcal{G}$ nor on the underlying migration mechanism, as long as there is enough independence between the migration and coalescence. Of course it is easy to construct examples where the migration "anticipates" coalescent events and where the particles shift from sites where a coalescent event happens in the very near future to sites without a coalescent event in the near future, and where for this reason the size of the system stays large for a very long time. In view of this, it seems difficult to control the behavior of the system when the migration is general. However, after a careful look at the proof the reader will see that in fact similar estimates would hold with any migration mechanism for which one has (see the calculation leading to (18))

$$E(J_i|B_1(T_j^i),\dots,B_v(T_j^i),\xi_i=j,\Pi_{nv}(R_{i-1}))\geq c_j^i\frac{\gamma(B_1(T_j^i),\dots,B_v(T_j^i))}{\lambda(B_1(T_j^i),\dots,B_v(T_j^i))},$$

where $E(\sum_j c_j^i 1_{\{\xi_i=j\}}|\Pi_{nv}(R_{i-1}))>c$ for some deterministic $c>0$. $\diamond$

4 Uniform asymptotics

The result of this section is essentially based on the lower bound in Lemma 7 (i) and the observation that given X an exponential (rate λ) random variable, and another non-negative independent random variable T, one has

$$P(X<T)=\frac{1}{\lambda}E(X\wedge T).$$

Hence, one could quickly generalize the argument comprising (22)-(23) to more complicated migration mechanisms that are independent of the future of the coalescent mechanism.

To make the proof more transparent, we will assume the setting of Theorem 1, where each block changes its label (i.e. migrates) at rate 1, independently of the coalescent mechanism. Recall the setting of Lemma 8 and Corollary 10, and in particular the first time $T_n^{(k)}$ for the spatial Λ-coalescent to have kv or fewer blocks when started with n blocks at each site.

Theorem 12 *If (4) holds, then for $k\geq 2$*

$$\sup_n E[T_n^{(k)}]\leq\sum_{b=k}^{\infty}\frac{1}{\gamma_b}+\frac{k}{\gamma_k}.$$

Proof. We use the same notation as in the proof of Lemma 8, but this time the calculations are finer. First, fix an $i\geq 1$ (note the subscripts i are omitted in a number of places below for notational convenience). Recall that $\xi_i:=\max\{k:T_k^i<R_i\}$ is the number of migration events in between the $(i-1)$st and ith coalescence time and that T_j^i are the jump times and define

$$a:=\sum_{e=1}^{v}B_e(R_{i-1})=\sum_{e=1}^{v}B_e(T_j^i)\text{ for all }j\leq\xi_i.$$

Also set $\lambda_j^i=\lambda_j=\lambda(B_1(T_j^i),\dots,B_v(T_j^i))$ for all $j\leq\xi_i$. We extend the definition of T_j^i and λ_j^i to all $j\in\mathbf{N}$ by considering a coupled process Π_{nv}^i which agrees with Π_{nv} on $[R_{i-1},R_i)$ but for which no coalescence takes place. As before, T_j^i is the time of the jth migration event and λ_j is the coalescence rate associated with the configuration at time T_j^i for all $j \in \mathbf{N}$.

Using the first line of (18) and Lemma 7 (i) as well as further conditioning on $(\lambda_l)_{l\in\mathbf{N}_0}$ we obtain

$$\begin{aligned}
E[J_i|\Pi_{nv}(R_{i-1})] &\geq \upsilon\gamma_{\lfloor \frac{a}{\upsilon}\rfloor} E\left[\sum_{j=0}^{\infty} \lambda_j^{-1} 1_{\{\xi_i = j\}} \middle| \Pi_{nv}(R_{i-1})\right] \\
&= \upsilon\gamma_{\lfloor \frac{a}{\upsilon}\rfloor} E\left[\sum_{j=0}^{\infty} \lambda_j^{-1} P\left[\xi_i = j | (\lambda_l)_{l\in\mathbf{N}_0}, \Pi_{nv}(R_{i-1})\right] \middle| \Pi_{nv}(R_{i-1})\right] \\
&= \upsilon\gamma_{\lfloor \frac{a}{\upsilon}\rfloor} E\left[\sum_{j=0}^{\infty} \lambda_j^{-1} \left(\prod_{l=0}^{j-1} \frac{a}{\lambda_l + a}\right) \frac{\lambda_j}{\lambda_j + a} \middle| \Pi_{nv}(R_{i-1})\right]. \qquad (22)
\end{aligned}$$

For the next computation define an auxiliary i.i.d. sequence $(X_j)_{j\geq 0}$ of exponential random variables with parameter a, as well as a sequence $(Y_j)_{j\geq 0}$ of independent random variables where each Y_j has an exponential (rate λ_j) distribution. Note that $W_j := X_j \wedge Y_j$ are exponential random variables with rate $a + \lambda_j$ that are independent of $Z_j = 1_{\{X_j > Y_j\}}$.

Observe that conditioned on $((\lambda_j)_{j\in\mathbf{N}_0}, \Pi_{nv}(R_{i-1}))$ the X_j correspond to the waiting time until the next migration and the Y_j to the waiting time until coalescence as long as $\sum_{l=1}^{j-1} Z_l = 0$. So the event $\{Z_0 = \cdots = Z_{j-1} = 0\} = \{\xi_i \geq j\}$ is independent of W_j. This implies that

$$\begin{aligned}
E\left[L_i|\Pi_{nv}(R_{i-1})\right] &= E\left[E\left[\sum_{j=0}^{\xi_i} W_j \middle| (\lambda_l)_{l\in\mathbf{N}_0}, \Pi_{nv}(R_{i-1})\right] \middle| \Pi_{nv}(R_{i-1})\right] \\
&= E\left[E\left[\sum_{j=0}^{\infty} W_j 1_{\{\xi_i \geq j\}} \middle| (\lambda_l)_{l\in\mathbf{N}_0}, \Pi_{nv}(R_{i-1})\right] \middle| \Pi_{nv}(R_{i-1})\right] \\
&= E\Bigg[\sum_{j=0}^{\infty} E\left[W_j | (\lambda_l)_{l\in\mathbf{N}_0}, \Pi_{nv}(R_{i-1})\right] \\
&\qquad \cdot E\left[1_{\{\xi_i \geq j\}} \middle| (\lambda_l)_{l\in\mathbf{N}_0}, \Pi_{nv}(R_{i-1})\right] \Bigg| \Pi_{nv}(R_{i-1})\Bigg] \\
&= E\left[\sum_{j=0}^{\infty} \frac{1}{\lambda_j + a} \left(\prod_{l=0}^{j-1} \frac{a}{\lambda_l + a}\right) \middle| \Pi_{nv}(R_{i-1})\right]. \qquad (23)
\end{aligned}$$

Comparing now the terms in (22) and (23) we find that

$$E\left[L_i|\Pi_{nv}(R_{i-1})\right] \leq \frac{1}{\upsilon\gamma_{\lfloor \frac{a}{\upsilon}\rfloor}} E[J_i|\Pi_{nv}(R_{i-1})], \qquad (24)$$

where we gained a factor of $\upsilon^{\rho+1}$ in the denominator with respect to the analogous relation (19) in the proof of Lemma 8. The rest of the proof proceeds now as the proof of Lemma 8 and

Corollary 10 and hence we obtain

$$E[T_n^{(k)}] \leq \sum_{b=k}^{n} \frac{1}{\gamma_b} + \frac{k}{\gamma_k}.$$

□

Definition. We will say that the spatial Λ-coalescent *comes down from infinity uniformly* if

$$\lim_{k\to\infty} \sup_n ET_n^{(k)} = 0.$$

⋄

In particular, by Proposition 11 and Theorem 12 any coalescent with independent Markovian migration mechanism that comes down from infinity also comes down from infinity uniformly.

Example. Let $\alpha \in (0,2)$. The Beta$(2-\alpha,\alpha)$-coalescent, where $\Lambda \overset{d}{=}$ Beta$(2-\alpha,\alpha)$ has density $x^{1-\alpha}(1-x)^{\alpha-1}/\Gamma(2-\alpha)\Gamma(\alpha)$ is of special interest in [10]. As already noted in [26], for $\alpha \in (0,1]$ this (non-spatial) coalescent stays infinite, and for $\alpha \in (1,2)$ it comes down from infinity. By the previous theorem the spatial Beta$(2-\alpha,\alpha)$-coalescent comes down from infinity uniformly. An interesting consequence follows by the results of the next section. ⋄

5 Asymptotics on large tori

In this section we further restrict the setting in the following way:

- the graph $\mathcal{G}$ is a d-dimensional torus $T^N = [-N,N]^d \cap \mathbf{Z}^d$ for some $N \in \mathbf{N}$, where $d \geq 3$ is fixed,
- the migration corresponds to a random walk on the torus, meaning that the kernel $p(x,y), x,y \in \mathcal{G}$ is given as $p(x,y) \equiv \sum_{\{z:(z-y) \bmod N=0\}} \tilde{p}(z-x)$, where $\tilde{p}$ is purely d-dimensional distribution such that $\sum_x |x|^{d+2}\tilde{p}(x) < \infty$, and all particles move independently of each other and of the coalescence,
- the Λ-coalescent comes down from infinity (uniformly), i.e., condition (4) holds.

We are concerned here with convergence of the Λ-coalescent partition structure on T^N, if time is rescaled by the volume $(2N+1)^d$ of T^N, to that of a time-changed *non-spatial* Kingman coalescent as $N \to \infty$. The main results are presented in Theorem 13 and Theorem 19: Theorem 13 states convergence of the partition structure in a functional sense for arbitrary finite initial configurations. Theorem 19 states convergence of the number of partition elements in a functional sense if the initial number of partition elements is infinite.

Remark. There are strong reasons to believe that the behavior of the above system in recurrent dimensions $d=1$ and $d=2$ is mutually different and different from the behavior in transient dimensions $d \geq 3$.
Cox [12] verifies that coalescing random walks (which formally correspond to spatial coalescents with infinite collapse rate on all k-tuples, i.e. instantaneous collapse of all the individuals on the same site) satisfy an analogue of Proposition 18, under appropriate scaling. However, the limit corresponding to the one in Proposition 18 is different, and can be interpreted in terms of

Arratia [2] coalescing Brownian motions on the unit circle.
In the same paper [12] Theorem 6, Cox proves that for coalescing random walks on two-dimensional tori, under appropriate scaling, the analogue of Proposition 14 holds for sparse particles. We therefore believe it would not be difficult to extend Proposition 14 to $d = 2$, however a non-trivial analogue of Theorem 13 seems much more difficult since $\Pi^{\mathbf{Z}^2}(\infty)$ is the trivial partition with everything coalesced into one block, due to recurrence. Work in progress [18] is in part devoted to the asymptotic analysis of the spatial Kingman coalescent on two-dimensional tori. $\diamond$

We write $\mathcal{P}^{N,\ell}$ if we want to emphasize that the partitions are labeled by T^N. Let

$$\Pi^{N,\ell}_\pi \quad \text{and} \quad \Pi^{N,\ell}$$

denote the Λ-coalescent started from a partition $\pi \in \mathcal{P}^{N,\ell}$, and the Λ-coalescent started from any partition that contains infinitely many equivalence classes labeled by (located at) each site of T^N, respectively. In order to determine the large space-time asymptotics for $\Pi^{N,\ell}$, at time scales on the order of the volume $(2N+1)^d$ of T^N, we imitate a "bootstrapping" argument from [16].

Remark. Observe that in [16], only the singular $\Lambda = \delta_0$ case was studied in this context. However, the structure of the argument concerning large space-time asymptotics carries over due to the cascading property for general (spatial) Λ-coalescents, in particular due to the fact that any two partition elements $\pi_1, \pi_2 \in \Pi^{N,\ell}(0)$ coalesce at rate

$$\lambda_{2,2} = \Lambda([0,1]) \tag{25}$$

while they are at the same site, and that they do not coalesce otherwise. $\diamond$

We will need the following notation: for a marked partition $\pi \in \mathcal{P}^\ell_n$ (or $\pi \in \mathcal{P}^\ell$), and two real numbers $a < b \in \mathbf{R}$, write

$$\pi \in [[a,b]],$$

if $\forall i,j$ with $i \neq j$, such that $(A_i, \zeta_i), (A_j, \zeta_j) \in \pi$ we have $|\zeta_i - \zeta_j| \in [a,b]$. In words, $\pi \in [[a,b]]$ if and only if all the mutual distances for pairs of different partition elements of π are contained in $[a,b]$.

The following theorem states that, viewed on the right timescale $t(2N+1)^d$, and after some initial collapse of a finite starting configuration, the partitions of the Λ-coalescent on the tori T^N with N large behave like those of a (non-spatial) time-changed Kingman coalescent. To make this statement more precise, we introduce the following notation.

Let $G = \sum_{k=0}^\infty \tilde{p}_k(0)$ where $\tilde{p}_k$ denotes the k-step transition probability of a $\tilde{p}$ random walk. Note that this random walk is transient on $\mathbf{Z}^d$, so that $G < \infty$. Let $\Pi^{\mathbf{Z}^d,\ell}_\pi$ be the Λ-coalescent on $\mathcal{G} = \mathbf{Z}^d$ with migration given by the random walk kernel $\tilde{p}$, started from partition $\pi \in \mathcal{P}^\ell$ with $\#\pi < \infty$. The transience of $\tilde{p}$ also implies existence of non-trivial limit partitions

$$\Pi^{\mathbf{Z}^d}_\pi(\infty) = \lim_{t\to\infty} \Pi^{\mathbf{Z}^d}_\pi(t),$$

in the sense that if $\#\pi \geq 2$ then $\#\Pi^{\mathbf{Z}^d}_\pi(\infty) \geq 2$ with positive probability.

We define K_π as the *non-spatial* Kingman coalescent started in the partition $\pi \in \mathcal{P}$ or $\pi \in \mathcal{P}_n$. This means that K_π is the Λ_K-coalescent for $\Lambda_K = \delta_0$ and $|\mathcal{G}| = 1$ with initial configuration $K_\pi(0) = \pi$.

Denote by $D(\mathbf{R}_+, E)$ the càdlàg paths on $\mathbf{R}_+$ with values in some metric space E, and equip the space $D(\mathbf{R}_+, E)$ with the usual Skorokhod topology. Also let " $\Rightarrow$ " indicate convergence in distribution. Set

$$\kappa = \frac{2}{G + 2/\lambda_{2,2}}. \tag{26}$$

Recall that $\Pi^{N,\ell}$ starts from a configuration containing infinitely many blocks, namely the partition $\Pi(0) = \{\{j\} : j \in \mathbf{N}\}$. The theorem below concerns the behavior of only finitely many blocks. Recall that $\Pi^{N,\ell}(0)|_n$ is the restriction of the labeled partition $\Pi^{N,\ell}(0)$ to $[n]$. In the theorem below we use the abbreviation $\Pi_n^{N,\ell} := \Pi^{N,\ell}_{\Pi^{N,\ell}(0)}|_n$. Again, Π_n^N is the process of partitions corresponding to $\Pi_n^{N,\ell}$.

Theorem 13 *Assume that for each fixed $n \geq 1$ and all large N we have $\Pi^{N,\ell}(0)|_n = \Pi^{N+1,\ell}(0)|_n$. Then for each n, we obtain as $N \to \infty$, the following convergence of the (unlabeled) partition processes:*

$$(\Pi_n^N(t(2N+1)^d))_{t\geq 0} \Rightarrow (K_{\Pi_n^{\mathbf{Z}^d}(\infty)}(\kappa t))_{t \geq 0},$$

where convergence is with respect to the Skorokhod topology on $D(\mathbf{R}_+, \mathcal{P}_n)$, and both $\Pi_n^{N,\ell}$ and $\Pi_n^{\mathbf{Z}^d,\ell}$ are started from the same initial configuration $\Pi^{N,\ell}(0)|_n \in \mathcal{P}_n^\ell$.

Remark. The statement is a generalization of Proposition 7.2 in [16], which deals with the case of spatial Kingman coalescents, rather than Λ-coalescents and only states convergence of the marginals. Nevertheless, the first part of the argument is analogous, and we will change it only slightly in preparation for Proposition 18 and Theorem 19. $\diamond$

As the first step we will state a result for the case in which the initial configuration is sparse on the torus, so that no coalescence involving more than two particles may be seen in the limit. The general case, stated in Theorem 13, will then follow easily.

Proposition 14 *Let $a_N \to \infty$ be such that $a_N/N \to 0$. Fix $n \in \mathbf{N}$, and let $\pi^{N,\ell} \in \mathcal{P}^{N,\ell}$ be such that $\#\pi^{N,\ell} = n \geq 2$, $\pi^{N,\ell} \in [[a_N, \sqrt{d}N]]$, and such that its corresponding (unlabeled) partition π^N equals a constant partition $\pi_0 \in \mathcal{P}$ for all N. Then as $N \to \infty$, we have the following convergence in distribution of the (unlabeled) partition processes:*

$$(\Pi^N_{\pi^{N,\ell}}(t(2N+1)^d))_{t \geq 0} \Rightarrow (K_{\pi_0}(\kappa t))_{t\geq 0},$$

where the convergence is in the space $D(\mathbf{R}_+, \mathcal{P})$.

Proof. To simplify the notation we refer to the ith block of π_0 as $\{i\}$, for $i = 1, \ldots, n$. In order to show the convergence on the space $D(\mathbf{R}_+, \mathcal{P})$ we will prove that the joint distribution of inter-coalescence times converges, when appropriately rescaled, to the joint distribution of inter-coalescence times of $K(\kappa\,\cdot)$, and that, at each coalescence time, any pair of remaining blocks is equally likely to coalesce next, see also [17] for a similar argument.

We set $\tau_0^N = 0$. Since there are at most $n-1$ coalescence times in general, we then define recursively stopping times for $k = 1, \dots, n-1$,

$$\tau_k^N := \inf\{t \geq \tau_{k-1}^N : \#\Pi_{\pi^{N,\ell}}^N(t) \neq \#\Pi_{\pi^{N,\ell}}^N(\tau_{k-1}^N)\},$$

as long as $\#\Pi_{\pi^{N,\ell}}^N(\tau_{k-1}^N) > 1$. Also define inter-coalescence times $\sigma_k^N := \tau_k^N - \tau_{k-1}^N$, $k \leq n-1$. Let us first observe that for $n = 2$

$$P[\sigma_1^N/(2N+1)^d < t] = P[\tau_1^N/(2N+1)^d < t] \to e^{-\kappa t} \tag{27}$$

uniformly in $t \in [0, T]$ for any $T < \infty$, by Lemma 7.3 in [16]. Indeed, as remarked at the beginning of this section, the spatial Λ-coalescent restricted to two-particles is identical in law to the spatial $\lambda_{2,2}\delta_0(\cdot)$-coalescent from [16].

Let U_k be independent exponential random variables with parameters $\kappa\binom{n-(k-1)}{2}$ for $k < n-1$. We wish to show the convergence in distribution of the random vector

$$(\sigma_1^N/(2N+1)^d, \dots, \sigma_{n-1}^N/(2N+1)^d) \Rightarrow (U_1, \dots, U_{n-1}) \tag{28}$$

as $N \to \infty$. The statement is clear by (27) if $n = 2$. In order to show (28) for $n \geq 2$, the first step is to see that, we may exclude the possibility of coalescence of more than two particles at any given time with probability tending to 1 as $N \to \infty$.

Let $\tau^N(i,j)$ be the time of the coalescence which merges the block $A^{(i)}$ containing i and the block $A^{(j)}$ containing j, and for each i denote by $\zeta^{(i)}$ the label associated with the block $A^{(i)}$. Then, we have for any $0 < T < \infty$, and any distinct $i, j, k \in [n]$,

$$\int_0^{T(2N+1)^d} P\left[\tau_1^N = \tau^N(i,j) \in du, |\zeta^{(i)} - \zeta^{(k)}| \leq a_N\right] \to 0, \tag{29}$$

uniformly over all partitions $\pi^{N,\ell} \in [[a_N, \sqrt{d}N]]$, as $N \to \infty$. The statement (29) is analogous to (3.7) in Cox [12], and follows with exactly the same calculation. Likewise, a statement analogous to (3.8) in [12] holds, saying that uniformly over all $\pi^{N,\ell} \in [[a_N, \sqrt{d}N]]$

$$\int_0^{T(2N+1)^d} P\left[\tau_1^N = \tau^N(i,j) \in du, |\zeta^{(k)} - \zeta^{(l)}| \leq a_N\right] \to 0, \tag{30}$$

as $N \to \infty$ for $i, j, k, l \in [n]$ distinct.

Now fix $T < \infty$, $\epsilon > 0$, and let $n > 2$. Relation (29) implies that for N large enough,

$$P\left[\#\Pi_{\pi^{N,\ell}}^N(\tau_1^N) \neq n-1, \tau_1^N < T(2N+1)^d\right] < \epsilon, \tag{31}$$

and together with (30) it implies that

$$P\left[\Pi_{\pi^{N,\ell}}^{N,\ell}(\tau_1^N) \notin [[a_N, \sqrt{d}N]], \tau_1^N < T(2N+1)^d\right] < \epsilon.$$

A simple induction (using the strong Markov property and uniformity of (29) and (30) in $t \in [0, T]$) yields the following statement: for each $k < n-1$, and any fixed $\varepsilon > 0$, if N is large enough then

$$P\left[\#\Pi_{\pi^{N,\ell}}^N(\tau_k^N) \neq n-k, \tau_k^N < T(2N+1)^d\right] < \epsilon,$$

and

$$P\left[\Pi^{N,\ell}_{\pi^{N,\ell}}(\tau^N_k)\notin[[a_N,\sqrt{d}N]],\tau^N_k<T(2N+1)^d\right]<\epsilon,$$

for $k\le n-2$. From this we get that, for any fixed $\varepsilon>0$, if N is large enough,

$$P\left[\#\Pi^N_{\pi^{N,\ell}}(\tau^N_k)=n-k \text{ for each } k \text{ with } \tau^N_k<T(2N+1)^d\right]>1-\varepsilon, \tag{32}$$

and

$$P\left[\Pi^N_{\pi^{N,\ell}}(\tau^N_k)\in[[a_N,\sqrt{d}N]] \text{ for each } k \text{ with } \tau^N_k<T(2N+1)^d\right]>1-\epsilon. \tag{33}$$

Moreover, on the event

$$\{\tau^N_k<T(2N+1)^d\}\cap\{\#\Pi^N_{\pi^{N,\ell}}(\tau^N_k)=n-k\}\cap\{\Pi^N_{\pi^{N,\ell}}(\tau^N_k)\in[[a_N,\sqrt{d}N]]\}$$

we have as in (3.1) of [12] that

$$|P[\sigma^N_{k+1}/(2N+1)^d>u|\mathcal{F}_{\tau^N_k}]-e^{-\kappa\binom{n-k}{2}u}|<\varepsilon_N, \tag{34}$$

where ε_N depends on N only, and where $\varepsilon_N\to 0$, as $N\to\infty$.
In order to arrive at (28), we show that σ^N_k is asymptotically independent of $\sigma^N_{k-1},\dots,\sigma^N_1$ for all $k=2,\dots,n-1$. So consider for any fixed $0\le t_1,\dots,t_k$, where $\sum_{i=1}^k t_i<T$, the event

$$A^N_k:=\left\{\frac{\sigma^N_k}{(2N+1)^d}<t_k,\frac{\sigma^N_{k-1}}{(2N+1)^d}<t_{k-1},\dots,\frac{\sigma^N_1}{(2N+1)^d}<t_1\right\}.$$

In particular, on this event we have that $\tau^N_i<T(2N+1)^d$ is satisfied for $i=1,\dots,k$. We obtain

$$\begin{aligned}
&P\left[A^N_k\right]\\
=\;&E\left[P\left[\frac{\sigma^N_k}{(2N+1)^d}<t_k\middle|\mathcal{F}_{\tau^N_{k-1}}\right]1_{A^N_{k-1}}\right]\\
=\;&E\left[\left(P\left[\frac{\sigma^N_k}{(2N+1)^d}<t_k\middle|\mathcal{F}_{\tau^N_{k-1}}\right]-\left(1-e^{-\binom{n-(k-1)}{2}\kappa t_k}\right)\right)1_{A^N_{k-1}}\right]\\
&+(1-e^{-\binom{n-(k-1)}{2}\kappa t_k})P\left[A^N_{k-1}\right].
\end{aligned}$$

Now use (32), (33), and (34) to get

$$\lim_{N\to\infty}P\left[A^N_k\right]=(1-e^{-\binom{n-(k-1)}{2}\kappa t_k})\lim_{N\to\infty}P\left[A^N_{k-1}\right].$$

By iterating the argument we obtain asymptotic independence. This in turn implies that $(\#\Pi^N_{\pi^{N,\ell}}(t(2N+1)^d)_{t\ge0}\Rightarrow(\#K_{\pi_0}(\kappa t))_{t\ge0}$ in the Skorokhod topology, since by (32), as $N\to\infty$,

$$P\left[\#\Pi^N_{\pi^{N,\ell}}(t)=n-\sum_{k=1}^{n-1}1_{\{\tau^N_k<t\}} \text{ for all } t<(2N+1)^dT\right]\to 1,$$

so that the convergence of the jump times τ^N_k in $n-\sum_{k=1}^{n-1}1_{\{\tau^N_k<t\}}$ implies convergence in the Skorokhod topology, see for example Proposition 6.5 in Chapter 3 of [14].

Finally, (2.8) in [12] states that

$$\lim_{N\to\infty} \sup_{t\geq(\log N)N^2} \sup_{x\in T^N} (2N+1)^d |\tilde{p}_t(0,x) - (2N+1)^{-d}| = 0.$$

where $\tilde{p}_t(0,x) = P(X(t) = x|X(0) = 0)$ and X is a random walk on T^N. This implies that the positions of partition elements in $\Pi^{N,\ell}_{\pi^{N,\ell}}(\tau^N_k + (\log N)N^2)$ (note that $(\tau^N_k + (\log N)N^2)/(2N+1)^d \approx \tau^N_k/(2N+1)^d$ are approximately uniformly and independently distributed on the torus. Due to (28), with probability tending to 1 as $N \to \infty$, we also have

$$\#\Pi^N_{\pi^{N,\ell}}(\tau^N_k + (\log N)N^2) = \#\Pi^N_{\pi^{N,\ell}}(\tau^N_k).$$

Therefore, at time τ^N_{k+1}, each pair of partition elements of $\#\Pi^N_{\pi^{N,\ell}}(\tau^N_k)$ is approximately equally likely to coalesce, as is the case in the Kingman coalescent. This completes the proof of convergence on the space $D(\mathbf{R}_+, \mathcal{P})$. □

Proof of Theorem 13. Fix $n \in \mathbf{N}$. We will first show that, as $N \to \infty$, $\Pi^N_n(N^{3/2}) = \Pi^{\mathbf{Z}^d}_n(\infty)$ (note this is only a statement about the partition structure, not the locations), and that $\Pi^{N,\ell}_n(N^{3/2}) \in [[N^{3/4}/\log N, \sqrt{d}N]]$, with probability arbitrarily close to 1. The statement of the theorem will then follow by Proposition 14 if we continue running the process from time $N^{3/2}$ onwards, and use the strong Markov property, noting that $N^{3/2} = o((2N+1)^d)$.

Let $||x||_\infty = \max_{i=1,\dots,d} x_i$ for $x \in \mathbf{Z}^d$ denote the maximum norm on $\mathbf{Z}^d$. Define the stopping time

$$\tau^N := \inf\{t > 0 : \max\{||\zeta||_\infty : (A,\zeta) \in \Pi^{N,\ell}_n(t)\} \geq N\}.$$

Before time τ^N none of the blocks has reached the boundary of $[-N,N]^d$, so we may couple $\Pi^{N,\ell}_n$ and $\Pi^{\mathbf{Z}^d,\ell}_n$ in a natural way such that $\Pi^{N,\ell}_n(t) = \Pi^{\mathbf{Z}^d,\ell}_n(t)$ for $t \leq \tau^N$.

Note that by the functional CLT, any random walk X on $\mathbf{Z}^d$ with random walk kernel $\tilde{p}$ started at $X(0) \leq \frac{N}{2}$ satisfies

$$\lim_{N\to\infty} P\left[\sup_{0\leq t\leq N^{3/2}} ||X(t)||_\infty < N\right] = 1.$$

Since for N large enough, $\max\{\zeta : (A,\zeta) \in \Pi^{N,\ell}(0)|_n\} \leq \frac{N}{2}$ and since the coalescent has at most n blocks independently performing random walks, we immediately obtain

$$\lim_{N\to\infty} P\left[\tau^N > N^{3/2}\right] = 1.$$

In particular, we have

$$\lim_{N\to\infty} P\left[\Pi^{N,\ell}_n(N^{3/2}) = \Pi^{\mathbf{Z}^d,\ell}_n(N^{3/2})\right] = 1. \tag{35}$$

To see that the blocks remaining at time $N^{3/2}$ are at a mutual distance of $N^{3/4}/\log N$ with high probability, more precisely that

$$\lim_{N\to\infty} P[\Pi^{N,\ell}_n(N^{3/2}) \in [[N^{3/4}/\log N, \sqrt{d}N]]\,] = 1, \tag{36}$$

if suffices to observe that again by the functional CLT,

$$\lim_{N\to\infty} P\left[|X^1(N^{3/2}) - X^2(N^{3/2})| < N^{\frac{3}{4}}/\log N\right] = 0,$$

where X^1 and X^2 are two independent $\tilde{p}$-random walks on $\mathbf{Z}^d$ started at $X^1(0) = X^2(0) = 0$. Due to (36), and the fact that $P[X^1(t) = X^2(t)$ for some $t \geq 0 | X_0^1 - X_0^2 = x] \to 0$ as $|x| \to \infty$ we have,

$$\lim_{N\to\infty} P\left[\Pi_n^{\mathbf{Z}^d,\ell}(N^{3/2}) = \Pi_n^{\mathbf{Z}^d,\ell}(\infty)\right] = 1. \tag{37}$$

Now (35) and (37) imply

$$\lim_{N\to\infty} P\left[\Pi_n^{N,\ell}(N^{3/2}) = \Pi_n^{\mathbf{Z}^d,\ell}(\infty)\right] = 1. \tag{38}$$

□

We will also show a uniform convergence to the Kingman coalescent, on the same time scale, in the sense of the number of blocks, cf. Theorem 19 below. One starts with a bound on the mean number of partition elements left in the coalescent Π^N at a fixed time, say 1. The following useful monotonicity property carries over from the spatial Kingman coalescent setting to the spatial Λ-coalescent setting:

Suppose that the blocks of $\Pi^{N,\ell}(0)$ are initially divided into classes $\Pi^{N,1,\ell}(0)$, $\Pi^{N,2,\ell}(0), \ldots$ (in any prescribed deterministic way) and let $(\cup_j \Pi^{N,j,\ell}(t))_{t\geq 0}$ denote the united Λ-coalescent where only blocks of the same class are allowed to coalesce.

Lemma 15 *For each $t > 0$,*

$$E[\#\Pi^N(t)] \leq \sum_j E[\#\Pi^{N,j}(t)].$$

Proof. We can couple $\Pi^{N,\ell}$ and $\cup_j \Pi^{N,j,\ell}$, applying a modification of the Poisson point processes (from the construction of $\Pi^{N,\ell}$) to all the Λ-coalescents corresponding to different classes. By induction it suffices to construct jointly the coalescence and migration transition mechanisms for both processes $\Pi^{N,\ell}$ and $\cup_j \Pi^{N,j,\ell}$ so that if the number of blocks of $\Pi^{N,\ell}$ is bounded by the number of blocks of $\cup_j \Pi^{N,j,\ell}$ strictly before any (coupled) migration or coalescence time, the same will remain true after the transition takes place. To construct this coupling let $\Pi_0^{N,\ell} = \cup_j \Pi_0^{N,j,\ell}$ and assume a reservoir of independent Poisson point processes N_i, $i \in [v]$ on $\mathbf{R}_+ \times [0,1]$ with intensity measure $dt\, x^{-2}\Lambda(dx)$.

Initially link each block in $\Pi_0^{N,\ell}$ with its copy in $\cup_j \Pi_0^{N,j,\ell}$. Let the label processes of blocks in $\Pi_0^{N,\ell}$ evolve as stated in Theorem 1. In between the coalescence transition times, for any block $A' \in \cup_j \Pi_0^{N,j,\ell}$ which is linked to a block $A \in \Pi_0^{N,\ell}$, match the evolution of the label process of A' with the evolution of the label process of A.

Given (t, x), a point in the Poisson process N_i, generate a sequence of i.i.d. Bernoulli(x) random variables. To the kth linked pair (according to some prescribed and arbitrary ordering) of blocks with label i at time $t-$ assign the kth Bernoulli variable. Recall that if two blocks are linked, their label processes are equal so saying that a linked pair has label i makes sense. Note that there may be unlinked blocks in $\cup_j \Pi_{t-}^{N,j,\ell}$ (but not in $\Pi_{t-}^{N,\ell}$), and to each such block assign a different and unique Bernoulli variable above. To form $\Pi_t^{N,\ell}$ coalesce all blocks of $\Pi_{t-}^{N,\ell}$ with label i and Bernoulli outcome 1, and to form $\cup_j \Pi_t^{N,j,\ell}$, for each j, coalesce all blocks of $\cup_j \Pi_{t-}^{N,j,\ell}$ in class j with label i and Bernoulli outcome 1.

Note that if at this transition two blocks A_1 and A_2 of $\Pi^{N,\ell}_{t-}$ coalesce, then the corresponding blocks A'_1 and A'_2 of $\cup_j \Pi^{N,j,\ell}_{t-}$ linked to A_1 and A_2, respectively, coalesce only if they consist of elements of the same class. In case there exist two or more blocks $A'_1, A'_2, \ldots$ in $\cup_j \Pi^{N,j,\ell}$, belonging to different classes, and linked to $A_1, A_2, \ldots$ that have just coalesced into a new block A within $\Pi^{N,\ell}$ at time t, then at time t instantaneously relink A'_1 to A and unlink $A'_2, \ldots$. It is easy to continue the evolution of labels of unlinked blocks of $\cup_j \Pi^{N,j,\ell}$ using extra randomness so that each process $\Pi^{N,j,\ell}$ is distributed as the Λ-spatial coalescent.

Since in the above construction, at all times, each block of $\Pi^{N,\ell}$ is linked to a block of $\cup_j \Pi^{N,j,\ell}$ located at the same site, clearly $\#\Pi^N(t) \le \sum_j \#\Pi^{N,j}(t)$, and in particular the bound in expectation holds. □

The following lemma is taken from [16] and is similar to Theorem 1 in [4] and the proposition in Section 4 of [12].

Lemma 16 *There is a finite constant c_d such that uniformly in $N \in \mathbf{N}$, and in the sequences $(\Pi^N(0))_{N\in\mathbf{N}}$ satisfying $\#\Pi^N(0) \ge (2N+1)^d$,*

$$\mathbf{E}\left[\#\Pi^N(t)\right] \le c_d \max\left\{1, \frac{\#\Pi^N(0)}{t}\right\}.$$

Proof. All we need to do is translate the notation and explain the small differences in the argument.

Our $\lambda_{2,2}$ is γ in [16]. The migration walk $\tilde{p}$ is from the same class as in [16]. There are only two statements in the argument of [16], Lemmas 7.4 and 7.5 that depend on the structure of the underlying coalescent. One is relation (7.50) at the beginning of the argument of Lemma 7.4. Take $A_0 \in \Pi^N(0)$ and note that, similar to (7.44) in [16],

$$\#\Pi^N(t) \le \#\Pi^N(0) - \sum_{\Pi^N(0)\ni A \ne A_0} 1_{\{A_0 \sim_{\Pi^N(t)} A\}},$$

so that

$$E\left[\#\Pi^N(t)\right] \le E\left[\#\Pi^N(0)\right] - \sum_{\Pi^N(0)\ni A\ne A_0} P[A_0 \sim_{\Pi^N(t)} A],$$

leading to (7.46) of Lemma 7.4 in [16], and therefore to relation (7.50) since the remaining calculations concern the behavior of two partition elements (not the joint behavior of several partition elements).

The other statement concerns (7.58) in the proof of Lemma 7.5: here, the torus is cut up into boxes and (7.58) states that the expected number of blocks is bounded by the expected number of blocks in a coalescent in which only blocks that start in the same initial box may coalesce. This holds in our setting due to Lemma 15.

Given (7.50) and (7.58), the remaining arguments are the same as those in the proof of Lemmas 7.4 and 7.5 of [16]. □

The next lemma says that the number of the partition elements at time $\varepsilon(2N+1)^d$ is tight in N.

Lemma 17 *Fix $0 < \epsilon, \epsilon' < 1$. Then there exists a constant $M^0 = M^0(\varepsilon, \varepsilon')$ such that, for all $M \geq M^0$,*

$$\limsup_{N\to\infty} P[\#\Pi^N(\epsilon(2N+1)^d) > M] \leq \epsilon'.$$

Proof. Assume $1 < \frac{\epsilon(2N+1)^d}{2}$. Due to Theorem 12, for $k \in \mathbf{N}$,

$$\begin{aligned}\sup_N P[\#\Pi^N(1) > k(2N+1)^d] &= \sup_N P[T_\infty^{N,(k)} > 1] \\ &= \sup_N \sup_n E[T_n^{N,(k)}] \leq \left(\sum_{b=k}^{\infty} \frac{1}{\gamma_b} + \frac{k}{\gamma_k}\right).\end{aligned}$$

Due to (4) (more precisely observation (10)), the right hand side converges to zero as $k \to \infty$. Therefore, we may choose $M_0 \geq 1$ large enough so that $\sum_{b=M_0}^{\infty} \frac{1}{\gamma_b} + M_0/\gamma_{M_0} < \varepsilon'/2$ and also that $M_0 > \frac{4c_d}{\epsilon\epsilon'}$. Then for all $M \geq M_0$,

$$P[\#\Pi^N(1) > M(2N+1)^d] \leq \frac{\epsilon'}{2}. \tag{39}$$

Now take $M \geq M_0$ and define the event $A_M^N := \{\#\Pi^N(1) \leq M(2N+1)^d\}$. We then have by Lemma 16 that

$$E[\#\Pi^N(\epsilon(2N+1)^d)|A_M^N] \leq c_d \max\{1, \frac{M(2N+1)^d}{\epsilon(2N+1)^d - 1}\} \leq c_d \max\{1, \frac{2M}{\epsilon}\}.$$

Note that on A_M^N we may have $\#\Pi^N(1) \geq (2N+1)^d$ and we can apply Lemma 16 directly, otherwise couple the coalescent $(\Pi^N(t), t \geq 1)$ with another coalescent $\tilde{\Pi}^N(t), t \geq 1)$ such that $\tilde{\Pi}^N$ almost surely dominates $\Pi^N(t)$ at all times, at all sites, and such that $\#\Pi^N(0) = (2N+1)^d$, and apply Lemma 16 to $\tilde{\Pi}^N$. It follows that

$$P[\#\Pi^N(\epsilon(2N+1)^d) > M^2|A_M^N] \leq \frac{1}{M} c_d \max\{1, \frac{2}{\epsilon}\}.$$

By conditioning on whether A_M^N or its complement occurs, using (39)

$$P[\#\Pi^N(\epsilon(2N+1)^d) > M^2] \leq \frac{1}{M} c_d \max\left\{1, \frac{2}{\epsilon}\right\} + 1 \cdot \frac{\epsilon'}{2}.$$

Since $\frac{2c_d}{M\epsilon} < \frac{\epsilon'}{2}$ we arrive at

$$\sup_N P[\#\Pi^N(\epsilon(2N+1)^d) > M^2] \leq \epsilon',$$

for $M \geq M_0$, which gives the statement of the lemma with $M^0 = (M_0)^2$. □

As a consequence, we obtain the following asymptotics for the number of partitions in Π^N, a spatial Λ-coalescent started from a partition having infinitely many equivalence classes labeled by (located at) each site of T^N.

Proposition 18 *Let $(K(t))_{t\geq 0}$ be the (non-spatial) Kingman coalescent started from the partition $K(0)=\{\{i\}, i\in \mathbf{N}\}$, and let κ be defined in (26). Then, for each fixed $t>0$, we have*

$$\#\Pi^N(t(2N+1)^d) \Rightarrow \#K(\kappa t),$$

as $N\to\infty$, where the above convergence is in distribution.

Proof. We start with a lower bound on the asymptotic distribution of $\#\Pi^N(t(2N+1)^d)$. Let $a_N = N^{3/2}$ so that $a_N\to\infty$ and also $\sqrt{a_N}/N\to 0$. For any fixed M one can find N_0 large enough so that for all $N\geq N_0$, $\Pi^{N,\ell}(0)$ contains at least M blocks (say $A_{i_1},\dots,A_{i_M}$), having mutual distances larger than $\sqrt{a_N}$. Let $\tilde{\Pi}^{N,\ell}$ be the $\Pi^{N,\ell}$ coalescent restricted to $\{A_{i_1},\dots,A_{i_M}\}$. Then clearly

$$\#\Pi^N(t(2N+1)^d) \geq \#\tilde{\Pi}^N(t(2N+1)^d). \tag{40}$$

As a consequence of (the proof of) Theorem 13, for any $t>0$, as $N\to\infty$,

$$P[\#\tilde{\Pi}^N(t(2N+1)^d)=k] \to P[\#K_M(\kappa t)=k],\ k=1,\dots,M,$$

where $K_M(\cdot)$ is the Kingman coalescent started from partition $\{\{1\},\dots,\{M\}\}$. By (40), for $k=1,\dots,M$,

$$\begin{aligned}\liminf_{N\to\infty} P[\#\Pi^N(t(2N+1)^d)\geq k] &\geq \liminf_{N\to\infty} P[\#\tilde{\Pi}^N(t(2N+1)^d)\geq k]\\ &= P[\#K_M(\kappa t)\geq k].\end{aligned}$$

Taking $M\to\infty$ on both sides and using the well-known coming down (or entrance law) property for $K(\cdot)$, we get for each $k\geq 1$, and each $t>0$,

$$\liminf_{N\to\infty} P[\#\Pi^N(t(2N+1)^d)\geq k] \geq P[K(\kappa t)\geq k]. \tag{41}$$

To get the upper bound corresponding to (41), we use Lemma 17. Namely, fix $\varepsilon,\varepsilon'\in(0,1\wedge t/2)$, and find the corresponding $M^0=M^0(\varepsilon/2,\varepsilon')$. Running the configuration $\Pi^{N,\ell}(\varepsilon(2N+1)^d/2)$ for an additional $N^{3/2} << \varepsilon(2N+1)^d/2$ units of time will result in $\Pi^{N,\ell}(\varepsilon(2N+1)^d/2+N^{3/2})$. On the event $\{\#\Pi^N(\varepsilon(2N+1)^d/2)\leq M^0\}$, that has probability greater than $1-\varepsilon'$, we have $\{\#\Pi^N(\varepsilon(2N+1)^d/2+N^{3/2})\leq M^0\}$, and moreover due to (36), for N sufficiently large, all the (fewer than M^0) partition elements of $\Pi^{N,\ell}(\varepsilon(2N+1)^d/2+N^{3/2})$ are at mutual distances larger than $N^{3/4}/\log N$ with probability close to 1. More precisely, if we let C_{N,ε,M^0} be the event that

$$\Pi^{N,\ell}(\varepsilon(2N+1)^d/2+N^{3/2}) \in [[N^{3/4}/\log N, \sqrt{d}N]] \text{ and}$$
$$\#\Pi^N(\varepsilon(2N+1)^d/2+N^{3/2}) \leq M^0$$

then, for all sufficiently large N,

$$P[C_{N,\varepsilon,M^0}] \geq 1-2\varepsilon'. \tag{42}$$

Again by the proof of Theorem 13, on C_{N,ε,M^0} we have for $k=1,\dots,M^0$, almost surely

$$\begin{aligned}|P[\#\Pi^N(t(2N+1)^d)\geq k|\mathcal{F}_{\varepsilon(2N+1)^d/2+N^{3/2}}]\\ - P[\#K_{\#\Pi^N(\varepsilon(2N+1)^d/2+N^{3/2})}(\kappa(t-\frac{\epsilon}{2}))\geq k]| \leq \varepsilon_N,\end{aligned}$$

further implying,

$$|P[\#\Pi^N(t(2N+1)^d) \geq k, C_{N,\varepsilon,M^0}] - E[P[\#K_{\#\Pi^N(\varepsilon(2N+1)^d/2+N^{3/2})}(\kappa t - \varepsilon/2) \geq k] 1_{C_{N,\varepsilon,M^0}}]| \leq \varepsilon_N, \tag{43}$$

where $\varepsilon_N \to 0$ as $N \to \infty$. Now use

$$\begin{aligned} P[\#K_{\#\Pi^N(\varepsilon(2N+1)^d/2+N^{3/2})}(\kappa t) \geq k] 1_{C_{N,\varepsilon,M^0}} &\leq P[\#K_{M^0}(\kappa t - \varepsilon/2) \geq k] 1_{C_{N,\varepsilon,M^0}} \\ &\leq P[\#K(\kappa t - \varepsilon/2) \geq k] 1_{C_{N,\varepsilon,M^0}} \end{aligned}$$

together with the fact that C^c_{N,ε,M^0} happens with probability smaller than $2\varepsilon'$ to obtain from (43) that

$$\limsup_{N\to\infty} P[\#\Pi^N(t(2N+1)^d) \geq k] \leq 4\varepsilon' + P[\#K(\kappa t - \varepsilon/2) \geq k].$$

The last statement is true for any $\varepsilon', \varepsilon > 0$, and this combined with (41) gives

$$\lim_{N\to\infty} P(\#\Pi^N(t(2N+1)^d) \geq k) = P(K(\kappa t) \geq k), \; k \geq 1. \tag{44}$$

□

Remark. Before continuing, note an interesting consequence of (41):
If $t_N = o((2N+1)^d)$ then

$$\lim_{N\to\infty} P[\#\Pi^N(t_N) \geq k] \geq \lim_{t\searrow 0} \liminf_{N\to\infty} P[\#\Pi^N(t(2N+1)^d) \geq k] = 1, \; k \geq 1, \tag{45}$$

or equivalently, $\#\Pi^N(t_N) \to \infty$ in probability as $N \to \infty$. ◇

An even stronger form of convergence is true. It holds in any setting where Proposition 18 and Theorem 13 hold, in particular in the setting of [16], although there it does not appear explicitly. Its analogue is important for the diffusive clustering analysis in the two-dimensional setting of [18].

Theorem 19 *Let $(K(t))_{t\geq 0}$ be as in Proposition 18. Then for each fixed $a > 0$, we have*

$$(\#\Pi^N(t(2N+1)^d))_{t\geq a} \Rightarrow (\#K(\kappa t))_{t\geq a},$$

as $N \to \infty$, where the convergence is with respect to the Skorokhod topology on càdlàg processes.

Proof. As a consequence of (42) we have for any fixed $a > 0$,

$$\lim_{N\to\infty} P[\Pi^{N,\ell}(a(2N+1)^d) \in [[N^{3/4}/\log N, \sqrt{d}N]]] = 1. \tag{46}$$

Together with the convergence of marginals in Proposition 18, and Theorem 13, this yields the current statement. □

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