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Weighted power variations of iterated Brownian motion

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Abstract

We characterize the asymptotic behaviour of the weighted power variation processes associated with iterated Brownian motion. We prove weak convergence results in the sense of finite dimensional distributions, and show that the laws of the limiting objects can always be expressed in terms of three independent Brownian motions X, Y and B, as well as of the local times of Y. In particular, our results involve "weighted" versions of Kesten and Spitzer's *Brownian motion in random scenery*. Our findings extend the theory initiated by Khoshnevisan and Lewis (1999), and should be compared with the recent result by Nourdin and Réveillac (2008), concerning the weighted power variations of fractional Brownian motion with Hurst index $H = 1/4$.

Key words: Brownian motion; Brownian motion in random scenery; Iterated Brownian motion; Limit theorems; Weighted power variations.

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1 Introduction and main results

The characterization of the single-path behaviour of a given stochastic process is often based on the study of its power variations. A quite extensive literature has been developed on the subject, see e.g. [5; 16] (as well as the forthcoming discussion) for references concerning the power variations of Gaussian and Gaussian-related processes, and [1] (and the references therein) for applications of power variation techniques to the continuous-time modeling of financial markets. Recall that, for a given real $\kappa > 1$ and a given real-valued stochastic process Z, the κ-power variation of Z, with respect to a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_N = 1\}$ of $[0,1]$ ($N \geqslant 2$ is some integer), is defined to be the sum

$$\sum_{k=1}^{N} |Z_{t_k} - Z_{t_{k-1}}|^{\kappa}. \tag{1}$$

For the sake of simplicity, from now on we shall only consider the case where π is a dyadic partition, that is, $N = 2^n$ and $t_k = k2^{-n}$, for some integer $n \geqslant 2$ and for $k \in \{0, \ldots, 2^n\}$.

The aim of this paper is to study the asymptotic behaviour, for every integer $\kappa \geqslant 2$ and for $n \to \infty$, of the (dyadic) κ-power variations associated with a remarkable non-Gaussian and self-similar process with stationary increments, known as *iterated Brownian motion* (in the sequel, I.B.M.). Formal definitions are given below: here, we shall only observe that I.B.M. is a self-similar process of order $\frac{1}{4}$, realized as the composition of two independent Brownian motions. As such, I.B.M. can be seen as a non-Gaussian counterpart to Gaussian processes with the same order of self-similarity, whose power variations (and related functionals) have been recently the object of an intense study. In this respect, the study of the single-path behaviour of I.B.M. is specifically relevant, when one considers functionals that are obtained from (1) by dropping the absolute value (when κ is odd), and by introducing some weights. More precisely, in what follows we shall focus on the asymptotic behaviour of *weighted variations* of the type

$$\sum_{k=1}^{2^n} f(Z_{(k-1)2^{-n}})\big(Z_{k2^{-n}} - Z_{(k-1)2^{-n}}\big)^{\kappa}, \quad \kappa = 2, 3, 4, \ldots, \tag{2}$$

or

$$\sum_{k=1}^{2^n} \frac{1}{2}[f(Z_{(k-1)2^{-n}}) + f(Z_{k2^{-n}})]\big(Z_{k2^{-n}} - Z_{(k-1)2^{-n}}\big)^{\kappa}, \quad \kappa = 2, 3, 4, \ldots, \tag{3}$$

for a real function $f : \mathbb{R} \to \mathbb{R}$ satisfying some suitable regularity conditions.

Before dwelling on I.B.M., let us recall some recent results concerning (2), when $Z = B$ is a *fractional Brownian motion* (fBm) of Hurst index $H \in (0,1)$ (see e.g. [18] for definitions) and, for instance, $\kappa \geqslant 2$ is an *even* integer. Recall that, in particular, B is a continuous Gaussian process, with an order of self-similarity equal to H. In what follows, f denotes a smooth enough function such that f and its derivatives have subexponential growth. Also, here and for the rest of the paper, μ_q, $q \geqslant 1$, stands for the qth moment of a standard Gaussian random variable, that is, $\mu_q = 0$ if q is odd, and

$$\mu_q = \frac{q!}{2^{q/2}(q/2)!}, \quad \text{if } q \text{ is even.} \tag{4}$$

We have (see [15; 16; 17]) as $n \to \infty$:

1. When $H > \frac{3}{4}$,

$$2^{n-2Hn}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[\big(2^{nH}(B_{k2^{-n}}-B_{(k-1)2^{-n}})\big)^{\kappa}-\mu_\kappa\big] \xrightarrow{\mathrm{L}^2}\mu_{\kappa-2}\binom{\kappa}{2}\int_0^1 f(B_s)dZ_s^{(2)}, \tag{5}$$

where $Z^{(2)}$ denotes the Rosenblatt process canonically constructed from B (see [16] for more details.)

2. When $H = \frac{3}{4}$,

$$\frac{2^{-\frac{n}{2}}}{\sqrt{n}}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\Big[\big(2^{\frac{3n}{4}}(B_{k2^{-n}}-B_{(k-1)2^{-n}})\big)^{\kappa}-\mu_\kappa\Big] \xrightarrow{\mathrm{Law}}\sigma_{\frac{3}{4},\kappa}\int_0^1 f(B_s)dW_s, \tag{6}$$

where $\sigma_{\frac{3}{4},\kappa}$ is an explicit constant depending only on κ, and W is a standard Brownian motion independent of B.

3. When $\frac{1}{4} < H < \frac{3}{4}$,

$$2^{-\frac{n}{2}}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[\big(2^{nH}(B_{k2^{-n}}-B_{(k-1)2^{-n}})\big)^{\kappa}-\mu_\kappa\big] \xrightarrow{\mathrm{Law}}\sigma_{H,\kappa}\int_0^1 f(B_s)dW_s, \tag{7}$$

for an explicit constant $\sigma_{H,\kappa}$ depending only on H and κ, and where W denotes a standard Brownian motion independent of B.

4. When $H = \frac{1}{4}$,

$$2^{-\frac{n}{2}}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[\big(2^{nH}(B_{k2^{-n}}-B_{(k-1)2^{-n}})\big)^{\kappa}-\mu_\kappa\big] \xrightarrow{\mathrm{Law}}\frac{1}{4}\mu_{\kappa-2}\binom{\kappa}{2}\int_0^1 f''(B_s)ds + \sigma_{\frac{1}{4},\kappa}\int_0^1 f(B_s)dW_s, \tag{8}$$

where $\sigma_{\frac{1}{4},\kappa}$ is an explicit constant depending only on κ, and W is a standard Brownian motion independent of B.

5. When $H < \frac{1}{4}$,

$$2^{2Hn-n}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[\big(2^{nH}(B_{k2^{-n}}-B_{(k-1)2^{-n}})\big)^{\kappa}-\mu_\kappa\big] \xrightarrow{\mathrm{L}^2}\frac{1}{4}\mu_{\kappa-2}\binom{\kappa}{2}\int_0^1 f''(B_s)ds. \tag{9}$$

In the current paper, we focus on the iterated Brownian motion, which is a continuous non-Gaussian self-similar process of order $\frac{1}{4}$. More precisely, let X be a two-sided Brownian motion, and let Y be a standard (one-sided) Brownian motion independent of X. In what follows, we shall denote by Z the *iterated Brownian motion (I.B.M.) associated with X and Y*, that is,

$$Z(t) = X\big(Y(t)\big), \quad t \geqslant 0. \tag{10}$$

The process Z appearing in (10) has been first (to the best of our knowledge) introduced in [2], and then further studied in a number of papers – see for instance [12] for a comprehensive account up to 1999, and [6; 13; 14; 19] for more recent references on the subject. Such a process can be regarded as the realization of a Brownian motion on a random fractal (represented by the path of the underlying motion Y). Note that Z is self-similar of order $\frac{1}{4}$, Z has stationary increments, and Z is neither a Dirichlet process nor a semimartingale or a Markov process in its own filtration. A crucial question is therefore how one can define a stochastic calculus with respect to Z. This issue has been tackled by Khoshnevisan and Lewis in the ground-breaking paper [11] (see also [12]), where the authors develop a Stratonovich-type stochastic calculus with respect to Z, by extensively using techniques based on the properties of some special arrays of Brownian stopping times, as well as on excursion-theoretic arguments. Khoshnevisan and Lewis' approach can be roughly summarized as follows. Since the paths of Z are too irregular, one cannot hope to effectively define stochastic integrals as limits of Riemann sums with respect to a deterministic partition of the time axis. However, a winning idea is to approach deterministic partitions by means of random partitions defined in terms of hitting times of the underlying Brownian motion Y. In this way, one can bypass the random "time-deformation" forced by (10), and perform asymptotic procedures by separating the roles of X and Y in the overall definition of Z. Later in this section, by adopting the same terminology introduced in [12], we will show that the role of Y is specifically encoded by the so-called "intrinsic skeletal structure" of Z.

By inspection of the techniques developed in [11], one sees that a central role in the definition of a stochastic calculus with respect to Z is played by the asymptotic behavior of the quadratic, cubic and quartic variations associated with Z. Our aim in this paper is to complete the results of [12], by proving asymptotic results involving *weighted* power variations of Z of arbitrary order, where the weighting is realized by means of a well-chosen real-valued function of Z. Our techniques involve some new results concerning the weak convergence of non-linear functionals of Gaussian processes, recently proved in [20]. As explained above, our results should be compared with the recent findings, concerning power variations of Gaussian processes, contained in [15; 16; 17].

Following Khoshnevisan and Lewis [11; 12], we start by introducing the so-called *intrinsic skeletal structure* of the I.B.M. Z appearing in (10). This structure is defined through a sequence of collections of stopping times (with respect to the natural filtration of Y), noted

$$\mathscr{T}_n = \{T_{k,n} : k \geqslant 0\}, \quad n \geqslant 1, \tag{11}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geqslant 1$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geqslant 1$, the stopping times $T_{k,n}$, appearing in (11), are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf\left\{s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\}\right\}, \quad k \geqslant 1,$$

where, as usual, $A \setminus B = A \cap B^c$ (B^c is the complement of B). Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y, and that, for every $n \geqslant 1$, the discrete stochastic process

$$\mathscr{Y}_n = \{Y(T_{k,n}) : k \geqslant 0\}$$

defines a simple random walk over $\mathscr{D}_n$. The intrinsic skeletal structure of Z is then defined to be the sequence

$$\text{I.S.S.} = \{\mathscr{D}_n, \mathscr{T}_n, \mathscr{Y}_n : n \geqslant 1\},$$

describing the random scattering of the paths of Y about the points of the partitions $\{\mathscr{D}_n\}$. As shown in [11], the I.S.S. of Z provides an appropriate sequence of (random) partitions upon which one can build a stochastic calculus with respect to Z. It can be shown that, as n tends to infinity, the collection $\{T_{k,n} : k \geqslant 0\}$ approximates the common dyadic partition $\{k2^{-n} : k \geqslant 0\}$ of order n (see [11, Lemma 2.2] for a precise statement). Inspired again by [11], we shall use the I.S.S. of Z in order to define and study weighted power variations, which are the main object of this paper. To this end, recall that μ_κ is defined, via (4), as the κth moment of a centered standard Gaussian random variable. Then, the *weighted power variation* of the I.B.M. Z, associated with a real-valued function f, with an instant $t \in [0,1]$, and with integers $n \geqslant 1$ and $\kappa \geqslant 2$, is defined as follows:

$$V_n^{(\kappa)}(f,t) \;=\; \frac{1}{2}\sum_{k=1}^{\lfloor 2^n t\rfloor}\big(f\big(Z(T_{k,n})\big)+f\big(Z(T_{k-1,n})\big)\big)\left(\big(Z(T_{k,n})-Z(T_{k-1,n})\big)^\kappa-\mu_\kappa\,2^{-\kappa\frac{n}{4}}\right). \tag{12}$$

Note that, due to self-similarity and independence,

$$\mu_\kappa\,2^{-\kappa\frac{n}{4}} = E\big[\big(Z(T_{k,n})-Z(T_{k-1,n})\big)^\kappa\big] = E\big[\big(Z(T_{k,n})-Z(T_{k-1,n})\big)^\kappa \mid Y\big].$$

For each integer $n \geqslant 1$, $k \in \mathbb{Z}$ and $t \geqslant 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2},(j+1)2^{-n/2}]$ within the first $\lfloor 2^n t\rfloor$ steps of the random walk $\{Y(T_{k,n})\}_{k\geqslant 1}$ (see formulae (30) and (31) below for precise definitions). The following lemma plays a crucial role in the study of the asymptotic behavior of $V_n^{(\kappa)}(f,\cdot)$:

Lemma 1.1. *(See [11, Lemma 2.4]) Fix $t \in [0,1]$, $\kappa \geqslant 2$ and let $f : \mathbb{R} \to \mathbb{R}$ be any real function. Then*

$$V_n^{(\kappa)}(f,t) = \frac{1}{2}\sum_{j\in\mathbb{Z}}\Big(f(X_{(j-1)2^{-\frac{n}{2}}})+f(X_{j\,2^{-\frac{n}{2}}})\Big)\times \tag{13}$$
$$\left[\Big(X_{j2^{-\frac{n}{2}}}-X_{(j-1)2^{-\frac{n}{2}}}\Big)^\kappa-\mu_\kappa\,2^{-\kappa\frac{n}{4}}\right]\big(U_{j,n}(t)+(-1)^\kappa\,D_{j,n}(t)\big).$$

The main feature of the decomposition (13) is that it separates X from Y, providing a representation of $V_n^{(\kappa)}(f,t)$ which is amenable to analysis. Using Lemma 1.1 as a key ingredient, Khoshnevisan and Lewis [11] proved the following results, corresponding to the case where f is identically one in (12): as $n \to \infty$,

$$\frac{2^{-n/4}}{\sqrt{2}}\;V_n^{(2)}(1,\cdot)\;\overset{D[0,1]}{\Longrightarrow}\;\text{B.M.R.S.}\quad\text{and}\quad\frac{2^{n/4}}{\sqrt{96}}\;V_n^{(4)}(1,\cdot)\;\overset{D[0,1]}{\Longrightarrow}\;\text{B.M.R.S.}$$

Here, and for the rest of the paper, $\overset{D[0,1]}{\Longrightarrow}$ stands for the convergence in distribution in the Skorohod space $D[0,1]$, while 'B.M.R.S.' indicates Kesten and Spitzer's *Brownian Motion in Random Scenery* (see [10]). This object is defined as:

$$\text{B.M.R.S.} = \left\{\int_{\mathbb{R}} L_t^x(Y)\,dB_x\right\}_{t\in[0,1]}, \tag{14}$$

where B is a two-sided Brownian motion independent of X and Y, and $\{L_t^x(Y)\}_{x\in\mathbb{R},\,t\in[0,1]}$ is a jointly continuous version of the local time process of Y (the independence of X and B is

immaterial here, and will be used in the subsequent discussion). In [11] it is also proved that the asymptotic behavior of the cubic variation of Z is very different, and that in this case the limit is I.B.M. itself, namely:

$$\frac{2^{n/2}}{\sqrt{15}}\, V_n^{(3)}(1,\cdot) \quad \stackrel{D[0,1]}{\Longrightarrow} \quad \text{I.B.M.}$$

As anticipated, our aim in the present paper is to characterize the asymptotic behavior of $V_n^{(\kappa)}(f,t)$ in (12), $n \to \infty$, in the case of a general function f and of a general integer $\kappa \geqslant 2$. Our main result is the following:

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to C^2 with f' and f'' bounded, and $\kappa \geqslant 2$ be an integer. Then, as $n \to \infty$,*

1. *if κ is even, $\left\{X_x, 2^{(\kappa-3)\frac{n}{4}}\, V_n^{(\kappa)}(f,t)\right\}_{x\in\mathbb{R},\, t\in[0,1]}$ converges in the sense of finite dimensional distributions (f.d.d.) to*

$$\left\{X_x, \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{\mathbb{R}} f(X_z) L_t^z(Y) dB_z\right\}_{x\in\mathbb{R},\, t\in[0,1]}; \tag{15}$$

2. *if κ is odd, $\left\{X_x, 2^{(\kappa-1)\frac{n}{4}}\, V_n^{(\kappa)}(f,t)\right\}_{x\in\mathbb{R},\, t\in[0,1]}$ converges in the sense of f.d.d. to*

$$\left\{X_x, \int_0^{Y_t} f(X_z)\big(\mu_{\kappa+1}\, d^\circ X_z + \sqrt{\mu_{2\kappa} - \mu_{\kappa+1}^2}\, dB_z\big)\right\}_{x\in\mathbb{R},\, t\in[0,1]}. \tag{16}$$

Remark 1.3. 1. Here, and for the rest of the paper, we shall write $\int_0^{\cdot} f(X_z) d^\circ X_z$ to indicate the *Stratonovich* integral of $f(X)$ with respect to the Brownian motion X. On the other hand, $\int_0^t f(X_z) dB_z$ (resp. $\int_0^t f(X_z) L_t^z(Y) dB_z$) is well-defined (for each t) as the *Wiener-Itô* stochastic integral of the random mapping $z \mapsto f(X_z)$ (resp. $z \mapsto f(X_z) L_t^z(Y)$), with respect to the *independent* Brownian motion B. In particular, one uses the fact that the mapping $z \mapsto L_t^z(Y)$ has a.s. compact support.

2. We call the process

$$t \mapsto \int_{\mathbb{R}} f(X_z) L_t^z(Y) dB_z \tag{17}$$

appearing in (15) a *Weighted Browian Motion in Random Scenery* (W.B.M.R.S. – compare with (14)), the weighting being randomly determined by f and by the independent Brownian motion X.

3. The relations (15)-(16) can be reformulated in the sense of "stable convergence". For instance, (15) can be rephrased by saying that the finite dimensional distributions of

$$2^{(\kappa-3)\frac{n}{4}}\, V_n^{(\kappa)}(f,\cdot)$$

converge $\sigma(X)$-*stably* to those of

$$\sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{\mathbb{R}} f(X_z) L_\cdot^z(Y) dB_z$$

(see e.g. Jacod and Shiryayev [9] for an exhaustive discussion of stable convergence).

4. Of course, one recovers finite dimensional versions of the results by Khoshnevisan and Lewis by choosing f to be identically one in (15)-(16).

5. To keep the length of this paper within bounds, we defer to future analysis the rather technical investigation of the tightness of the processes $2^{(\kappa-3)\frac{n}{4}}\, V_n^{(\kappa)}(f,t)$ (κ even) and $2^{(\kappa-1)\frac{n}{4}}\, V_n^{(\kappa)}(f,t)$ (κ odd).

Another type of weighted power variations is given by the following definition: for $t \in [0,1]$, $f : \mathbb{R} \to \mathbb{R}$ and $\kappa \geqslant 2$, let

$$S_n^{(\kappa)}(f,t) = \sum_{k=0}^{\left\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\right\rfloor} f\big(Z(T_{2k+1,n})\big)\left[\big(Z(T_{2k+2,n}) - Z(T_{2k+1,n})\big)^{\kappa} + (-1)^{\kappa+1}\big(Z(T_{2k+1,n}) - Z(T_{2k,n})\big)^{\kappa}\right].$$

This type of variations have been introduced very recently by Swanson [21] (see, more precisely, relations (1.6) and (1.7) in [21]), and used in order to obtain a change of variables formula (in law) for the solution of the stochastic heat equation driven by a space/time white noise. Since our approach allows us to treat this type of *signed* weighted power variations, we will also prove the following result:

Theorem 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to C^2 with f' and f'' bounded, and $\kappa \geqslant 2$ be an integer. Then, as $n \to \infty$,*

1. *if κ is even, $\left\{X_x, 2^{(\kappa-1)\frac{n}{4}}\, S_n^{(\kappa)}(f,t)\right\}_{x\in\mathbb{R},\, t\in[0,1]}$ converges in the sense of f.d.d. to*

$$\left\{X_x, \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_0^{Y_t} f(X_z)\, dB_z\right\}_{x\in\mathbb{R},\, t\in[0,1]}; \tag{18}$$

2. *if κ is odd, $\left\{X_x, 2^{(\kappa-1)\frac{n}{4}}\, S_n^{(\kappa)}(f,t)\right\}_{x\in\mathbb{R},\, t\in[0,1]}$ converges in the sense of f.d.d. to*

$$\left\{X_x, \int_0^{Y_t} f(X_z)\big(\mu_{\kappa+1}\, d^\circ X_z + \sqrt{\mu_{2\kappa} - \mu_{\kappa+1}^2}\, dB_z\big)\right\}_{x\in\mathbb{R},\, t\in[0,1]}. \tag{19}$$

Remark 1.5. 1. See also Burdzy [3] for an alternative study of (non-weighted) signed variations of I.B.M. Note, however, that the approach of [3] is not based on the use of the I.S.S., but rather on thinning deterministic partitions of the time axis.

2. The limits and the rates of convergence in (16) and (19) are the same, while the limits and the rates of convergences in (15) and (18) are different.

The rest of the paper is organized as follows. In Section 2, we state and prove some ancillary results involving weighted sums of polynomial transformations of Brownian increments. Section 3 is devoted to the proof of Theorem 1.2, while in Section 4 we deal with Theorem 1.4.

2 Preliminaries

In order to prove Theorem 1.2 and Theorem 1.4, we shall need several asymptotic results, involving quantities that are solely related to the Brownian motion X. The aim of this section is to state and prove these results, that are of clear independent interest.

We let the notation of the Introduction prevail: in particular, X and B are two independent two-sided Brownian motions, and Y is a one-sided Brownian motion independent of X and B. For every $n \geqslant 1$, we also define the process $X^{(n)} = \{X_t^{(n)}\}_{t \geqslant 0}$ as

$$X_t^{(n)} = 2^{n/4} X_{t2^{-n/2}}. \tag{20}$$

Remark 2.1. In what follows, we will work with the dyadic partition of order $n/2$, instead of that of order n, since the former emerges very naturally in the proofs of Theorem 1.2 and Theorem 1.4, as given, respectively, in Section 3 and Section 4 below. Plainly, the results stated and proved in this section can be easily reformulated in terms of *any* sequence of partitions with equidistant points and with meshes converging to zero.

The following result plays an important role in this section. In the next statement, and for the rest of the paper, we will freely use the language of Wiener chaos and Hermite polynomials. The reader is referred e.g. to Chapter 1 in [18] for any unexplained definition or result.

Theorem 2.2. *(Peccati and Tudor [20]). Fix $d \geqslant 2$, fix d natural numbers $1 \leqslant n_1 \leqslant \ldots \leqslant n_d$ and, for every $k \geqslant 1$, let $\mathbf{F}^k = (F_1^k, \ldots, F_d^k)$ be a vector of d random variables such that, for every $j = 1, \ldots, d$, the sequence of F_j^k, $k \geqslant 1$, belongs to the n_jth Wiener chaos associated with X. Suppose that, for every $1 \leqslant i, j \leqslant d$, $\lim_{k\to\infty} E(F_i^k F_j^k) = \delta_{ij}$, where δ_{ij} is Kronecker symbol. Then, the following two conditions are equivalent:*

(i) The sequence $\mathbf{F}^k$, $k \geqslant 1$, converges in distribution to a standard centered Gaussian vector $\mathcal{N}(0, \mathbf{I}_d)$ ($\mathbf{I}_d$ is the $d \times d$ identity matrix),

(ii) For every $j = 1, \ldots, d$, the sequence F_j^k, $k \geqslant 1$, converges in distribution to a standard Gaussian random variable $\mathcal{N}(0, 1)$.

The forthcoming proposition is the key to the main results of this section.

Given a polynomial $\mathsf{P} : \mathbb{R} \to \mathbb{R}$, we say that P has *centered Hermite rank* $\geqslant 2$ whenever $E[G\,\mathsf{P}(G)] = 0$ (where G is a standard Gaussian random variable). Note that P has centered Hermite rank $\geqslant 2$ if, and only if, $\mathsf{P}(x) - E[\mathsf{P}(G)]$ has Hermite rank $\geqslant 2$, in the sense of Taqqu [22].

Proposition 2.3. *Let $\mathsf{P} : \mathbb{R} \to \mathbb{R}$ be a polynomial with centered Hermite rank $\geqslant 2$. Let $\alpha, \beta \in \mathbb{R}$ and denote by $\phi : \mathbb{N} \to \mathbb{R}$ the function defined by the relation: $\phi(i)$ equals α or β, according as i is even or odd. Fix an integer $N \geqslant 1$ and, for every $j = 1, ..., N$, set*

$$M_j^{(n)} = 2^{-n/4} \sum_{i=\lfloor (j-1)2^{n/2} \rfloor + 1}^{\lfloor j2^{n/2} \rfloor} \phi(i) \left\{ \mathsf{P}\left(X_i^{(n)} - X_{i-1}^{(n)} \right) - E[\mathsf{P}(G)] \right\},$$

$$M_{j+N}^{(n)} = 2^{-n/4} \sum_{i=\lfloor (j-1)2^{n/2} \rfloor + 1}^{\lfloor j2^{n/2} \rfloor} (-1)^i \left(X_i^{(n)} - X_{i-1}^{(n)} \right),$$

where $G \sim \mathcal{N}(0,1)$ is a standard Gaussian random variable. Then, as $n \to \infty$, the random vector

$$\left\{ M_1^{(n)}, ..., M_{2N}^{(n)}; \{X_t\}_{t \geqslant 0} \right\} \tag{21}$$

converges weakly in the space $\mathbb{R}^{2N} \times \mathrm{C}^0(\mathbb{R}_+)$ to

$$\left\{ \sqrt{\frac{\alpha^2+\beta^2}{2} \mathbf{Var}\big(\mathsf{P}(G)\big)} \left(\Delta B\left(1\right), ..., \Delta B\left(N\right)\right); \Delta B\left(N+1\right), ..., \Delta B\left(2N\right); \{X_t\}_{t \geqslant 0} \right\} \tag{22}$$

where $\Delta B\left(i\right) = B_i - B_{i-1}$, $i = 1, ..., N$.

Proof. For the sake of notational simplicity, we provide the proof only in the case where $\alpha = \sqrt{2}$ and $\beta = 0$, the extension to the general case being a straightfroward consequence of the independence of the Brownian increments. For every $h \in L^2\left(\mathbb{R}_+\right)$, we write $X\left(h\right) = \int_0^\infty h\left(s\right) dX_s$. To prove the result it is sufficient to show that, for every $\lambda = (\lambda_1, ..., \lambda_{2N+1}) \in \mathbb{R}^{2N+1}$ and every $h \in L^2\left(\mathbb{R}_+\right)$, the sequence of random variables

$$F_n = \sum_{j=1}^{2N} \lambda_j M_j^{(n)} + \lambda_{2N+1} X\left(h\right)$$

converges in law to

$$\sqrt{\mathbf{Var}\big(\mathsf{P}(G)\big)} \sum_{j=1}^{N} \lambda_j \Delta B\left(j\right) + \sum_{j=N+1}^{2N} \lambda_j \Delta B\left(j\right) + \lambda_{2N+1} X\left(h\right).$$

We start by observing that the fact that P has centered Hermite rank $\geqslant 2$ implies that P is such that

$$\mathsf{P}\left(X_i^{(n)} - X_{i-1}^{(n)}\right) - E\left[\mathsf{P}\left(G\right)\right] = \sum_{m=2}^{\kappa} b_m H_m\left(X_i^{(n)} - X_{i-1}^{(n)}\right), \quad \text{for some } \kappa \geqslant 2, \tag{23}$$

where H_m denotes the mth Hermite polynomial, and the coefficients b_m are real-valued and uniquely determined by (23). Moreover, one has that

$$\mathbf{Var}\big(\mathsf{P}(G)\big) = \sum_{m=2}^{\kappa} b_m^2\, \mathbb{E}\left[H_m\left(G\right)^2\right] = \sum_{m=2}^{\kappa} b_m^2\, m!. \tag{24}$$

We can now write

$$\begin{aligned} F_n &= \lambda_{2N+1} X(h) + \sqrt{2}\ 2^{-\frac{n}{4}} \sum_{j=1}^{N} \lambda_j \sum_{\substack{i=\lfloor (j-1)2^{n/2} \rfloor + 1 \\ i \text{ even}}}^{\lfloor j2^{n/2} \rfloor} \sum_{m=2}^{\kappa} b_m H_m\left(X_i^{(n)} - X_{i-1}^{(n)}\right) \\ &\qquad + 2^{-\frac{n}{4}} \sum_{j=1}^{N} \lambda_{N+j} \sum_{i=\lfloor (j-1)2^{n/2} \rfloor + 1}^{\lfloor j2^{n/2} \rfloor} (-1)^i \left(X_i^{(n)} - X_{i-1}^{(n)}\right) \\ &= \lambda_{2N+1} X(h) + \sum_{m=2}^{\kappa} b_m \sum_{j=1}^{N} \lambda_j\ \sqrt{2}\ 2^{-\frac{n}{4}} \sum_{\substack{i=\lfloor (j-1)2^{n/2} \rfloor + 1 \\ i \text{ even}}}^{\lfloor j2^{n/2} \rfloor} H_m\left(X_i^{(n)} - X_{i-1}^{(n)}\right) \\ &\qquad + 2^{-\frac{n}{4}} \sum_{j=1}^{N} \lambda_{N+j} \sum_{i=\lfloor (j-1)2^{n/2} \rfloor + 1}^{\lfloor j2^{n/2} \rfloor} (-1)^i \left(X_i^{(n)} - X_{i-1}^{(n)}\right). \end{aligned}$$

By using the independence of the Brownian increments, the Central Limit Theorem and Theorem 2.2, we deduce that the $\kappa + 1$ dimensional vector

$$\left\{ X(h); 2^{-\frac{n}{4}} \sum_{j=1}^{N} \lambda_{N+j} \sum_{i=\lfloor (j-1)2^{n/2} \rfloor + 1}^{\lfloor j2^{n/2} \rfloor} (-1)^i \left(X_i^{(n)} - X_{i-1}^{(n)}\right); \right.$$
$$\left. \sum_{j=1}^{N} \lambda_j\ \sqrt{2}\ 2^{-\frac{n}{4}} \sum_{\substack{i=\lfloor (j-1)2^{n/2} \rfloor + 1 \\ i \text{ even}}}^{\lfloor j2^{n/2} \rfloor} H_m\left(X_i^{(n)} - X_{i-1}^{(n)}\right) : m = 2, ..., \kappa \right\}$$

converges in law to

$$\left\{ \|h\|_2 \times G_0; \sum_{j=1}^{N} \lambda_{N+j}\, G_{j,1}; \sum_{j=1}^{N} \lambda_j \sqrt{m!} G_{j,m} : m = 2, ..., \kappa \right\},$$

where $\{G_0; G_{j,m} : j = 1, ..., N,\ \ m = 1, ..., \kappa\}$ is a collection of i.i.d. standard Gaussian random variables $\mathcal{N}(0,1)$. This implies that F_n converges in law, as $n \to \infty$, to

$$\begin{aligned} & \lambda_{2N+1} \|h\|_2 G_0 + \sum_{j=1}^{N} \lambda_{N+j} G_{j,1} + \sum_{j=1}^{N} \lambda_j \sum_{m=2}^{\kappa} b_m \sqrt{m!} G_{j,m} \\ \overset{\text{Law}}{=}\ & \lambda_{2N+1} X(h) + \sum_{j=N+1}^{2N} \lambda_j \Delta B(j) + \sum_{j=1}^{N} \lambda_j \sqrt{\mathbf{Var}\big(\mathsf{P}(G)\big)} \Delta B(j), \end{aligned}$$

where we have used (24). This proves our claim. □

Remark 2.4. It is immediately verified that the sequence of Brownian motions appearing in (20) is asymptotically independent of X (just compute the covariance function of the 2-dimensional Gaussian process $(X, X^{(n)})$). However, by inspection of the proof of Proposition 2.3, one sees that

this fact is immaterial in the proof of the asymptotic independence of the vector $(M_1^n, ..., M_{2N}^n)$ and X. Indeed, such a result *depends uniquely of the fact that the polynomial* $\mathsf{P} - E(\mathsf{P}(G))$ *has an Hermite rank of order strictly greater than one*. This is a consequence of Theorem 2.2. It follows that the statement of Proposition 2.3 still holds when the sequence $\{X^{(n)}\}$ is replaced by a sequence of Brownian motions $\{X^{(*,n)}\}$ of the type

$$X^{(*,n)}(t) = \int \psi_n(t,z) dX_z, \quad t \geqslant 0, \quad n \geqslant 1,$$

where, for each t, ψ_n is a square-integrable deterministic kernel.

Once again, we stress that d and d° denote, resp., the Wiener-Itô integral and the Stratonovich integral, see also Remark 1.3 (1).

Theorem 2.5. *Let the notation and assumptions of Proposition 2.3 prevail (in particular,* P *has centered Hermite rank* $\geqslant 2$*), and set*

$$\begin{aligned} J_t^{(n)}(f) &= 2^{-\frac{n}{4}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} \left(f(X_{(j-1)2^{-n/2}}) + f(X_{j2^{-n/2}}) \right) \\ &\quad \times \left[\phi(j) \left\{ \mathsf{P}\left(X_j^{(n)} - X_{j-1}^{(n)} \right) - E\left[\mathsf{P}(G) \right] \right\} + \gamma(-1)^j \left(X_j^{(n)} - X_{j-1}^{(n)} \right) \right], \quad t \geqslant 0, \end{aligned}$$

where $\gamma \in \mathbb{R}$ *and the real-valued function* f *is supposed to belong to* C^2 *with* f' *and* f'' *bounded. Then, as* $n \to +\infty$*, the two-dimensional process*

$$\left\{ J_t^{(n)}(f), X_t \right\}_{t \geqslant 0}$$

converges in the sense of f.d.d. to

$$\left\{ \sqrt{\gamma^2 + \frac{\alpha^2 + \beta^2}{2} \mathbf{Var}\big(\mathsf{P}(G)\big)} \int_0^t f(X_s) \, dB_s, X_t \right\}_{t \geqslant 0}. \tag{25}$$

Proof. Set $\sigma := \sqrt{\frac{\alpha^2+\beta^2}{2} \mathbf{Var}\big(\mathsf{P}(G)\big)}$. We have

$$J_t^{(m)}(f) = \widetilde{J}_t^{(m)}(f) + r_t^{(m)}(f) + s_t^{(m)}(f),$$

where

$$\begin{aligned}
\widetilde{J}_t^{(m)}(f) &= 2^{\frac{m}{4}} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} f(X_{(j-1)2^{-\frac{m}{2}}}) \Big[\phi(j) \Big\{ \mathsf{P}\left(X_j^{(m)} - X_{j-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right] \Big\} \\
&\qquad\qquad + \gamma(-1)^j \left(X_j^{(m)} - X_{j-1}^{(m)}\right) \Big] \\
r_t^{(m)}(f) &= 2^{-\frac{m}{2}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} f'(X_{(j-1)2^{-\frac{m}{2}}}) \left(X_j^{(m)} - X_{j-1}^{(m)}\right) \\
&\qquad \times \Big[\phi(j) \Big\{ \mathsf{P}\left(X_j^{(m)} - X_{j-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right] \Big\} + \gamma(-1)^j \left(X_j^{(m)} - X_{j-1}^{(m)}\right) \Big] \\
s_t^{(m)}(f) &= 2^{-\frac{3m}{4}} \frac{1}{4} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} f''(X_{\theta_{j,m}}) \left(X_j^{(m)} - X_{j-1}^{(m)}\right)^2 \\
&\qquad \times \Big[\phi(j) \Big\{ \mathsf{P}\left(X_j^{(m)} - X_{j-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right] \Big\} + \gamma(-1)^j \left(X_j^{(m)} - X_{j-1}^{(m)}\right) \Big],
\end{aligned}$$

for some $\theta_{j,m}$ between $(j-1)2^{-\frac{m}{2}}$ and $j2^{-\frac{m}{2}}$. We decompose

$$\begin{aligned}
r_t^{(m)}(f) &= 2^{-\frac{m}{2}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} \phi(j) f'(X_{(j-1)2^{-\frac{m}{2}}}) \left(X_j^{(m)} - X_{j-1}^{(m)}\right) \Big\{ \mathsf{P}\left(X_j^{(m)} - X_{j-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right] \Big\} \\
&\quad + \frac{\gamma}{2} 2^{-\frac{m}{2}} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} (-1)^j f'(X_{(j-1)2^{-\frac{m}{2}}}) \left(X_j^{(m)} - X_{j-1}^{(m)}\right)^2 = r_t^{(1,m)}(f) + r_t^{(2,m)}(f).
\end{aligned}$$

By independence of increments and because $E\left[G\big(\mathsf{P}(G) - E[\mathsf{P}(G)]\big)\right] = 0$ we have

$$E\big|r_t^{(1,m)}(f)\big|^2 = \frac{1}{4} 2^{-m} E\left[G^2\big(\mathsf{P}(G) - E[\mathsf{P}(G)]\big)^2\right] \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} \phi^2(j) E\big|f'(X_{(j-1)2^{-\frac{m}{2}}})\big|^2 = O(2^{-\frac{m}{2}}).$$

For $r_t^{(2,m)}(f)$, we can decompose

$$\begin{aligned}
r_t^{(2,m)}(f) &= \frac{\gamma}{2} 2^{-\frac{m}{2}} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} (-1)^j f'(X_{(j-1)2^{-\frac{m}{2}}}) \left[\left(X_j^{(m)} - X_{j-1}^{(m)}\right)^2 - 1 \right] \\
&\quad + \frac{\gamma}{2} 2^{-\frac{m}{2}} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} (-1)^j f'(X_{(j-1)2^{-\frac{m}{2}}}) = r_t^{(2,1,m)}(f) + r_t^{(2,2,m)}(f).
\end{aligned}$$

By independence of increments and because $E\big(G^2 - 1\big) = 0$, we have

$$E\big|r_t^{(2,1,m)}(f)\big|^2 = \frac{\gamma^2}{4} \mathbf{Var}(G^2) 2^{-m} \sum_{j=1}^{\lfloor 2^{\frac{m}{2}} t \rfloor} E\big|f'(X_{(j-1)2^{-\frac{m}{2}}})\big|^2 = O(2^{-\frac{m}{2}}).$$

For $r_t^{(2,2,m)}(f)$, remark that

$$2^{-\frac{m}{2}}\sum_{k=0}^{\lfloor 2^{\frac{m}{2}}t\rfloor}(-1)^j f'(X_{(j-1)2^{-\frac{m}{2}}}) = 2^{-\frac{m}{2}}\sum_{k=0}^{\lfloor\lfloor 2^{\frac{m}{2}}t\rfloor/2\rfloor}\left(f'(X_{(2k+1)2^{-\frac{m}{2}}}) - f'(X_{(2k)2^{-\frac{m}{2}}})\right) + O(2^{-\frac{m}{2}})$$

so that $E\left|r_t^{(2,2,m)}(f)\right| = O(2^{-\frac{m}{4}})$. Thus, we obtained that $E\left|r_t^{(m)}(f)\right| \longrightarrow 0$ as $m \to \infty$. Since we obviously have, using the boundedness of f'', that

$$E\left|s_t^{(m)}(f)\right| \longrightarrow 0 \quad \text{as } m \to \infty,$$

we see that the convergence result in Theorem 2.5 is equivalent to the convergence of the pair $\left\{\widetilde{J}_t^{(n)}(f), X_t\right\}_{t\geqslant 0}$ to the object in (25).

Now, for every $m \geqslant n$, one has that

$$\begin{aligned} J_t^{(m)}(f) &= 2^{-\frac{m}{4}}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{i=\lfloor(j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor} f(X_{(i-1)2^{-m/2}}) \\ &\qquad \times\left[\phi(i)\left\{\mathsf{P}\left(X_i^{(m)} - X_{i-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right]\right\} + \gamma(-1)^i\left(X_i^{(m)} - X_{i-1}^{(m)}\right)\right] \\ &= A_t^{(m,n)} + B_t^{(m,n)}, \end{aligned}$$

where

$$\begin{aligned} A_t^{(m,n)} &= 2^{-\frac{m}{4}}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} f(X_{(j-1)2^{-n/2}}) \\ &\qquad \times \sum_{i=\lfloor(j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor}\left[\phi(i)\left\{\mathsf{P}\left(X_i^{(m)} - X_{i-1}^{(m)}\right) - \mathbb{E}\left[\mathsf{P}(G)\right]\right\} + \gamma(-1)^i\left(X_i^{(m)} - X_{i-1}^{(m)}\right)\right] \\ B_t^{(m,n)} &= 2^{-\frac{m}{4}}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{i=\lfloor(j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor}\left[f(X_{(i-1)2^{-m/2}}) - f(X_{(j-1)2^{-n/2}})\right] \\ &\qquad \times\left[\phi(i)\left\{\mathsf{P}\left(X_i^{(m)} - X_{i-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right]\right\} + \gamma(-1)^i\left(X_i^{(m)} - X_{i-1}^{(m)}\right)\right]. \end{aligned}$$

We shall study $A^{(m,n)}$ and $B^{(m,n)}$ separately. By Proposition 2.3, we know that, as $m \to \infty$, the random element

$$\left\{X; 2^{-\frac{m}{4}}\sum_{i=\lfloor(j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor}\phi(i)\left\{\mathsf{P}\left(X_i^{(m)} - X_{i-1}^{(m)}\right) - E\left[\mathsf{P}(G)\right]\right\} : j = 1, ..., 2^{n/2};\right.$$
$$\left. 2^{-\frac{m}{4}}\sum_{i=\lfloor(j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor}(-1)^i\left(X_i^{(m)} - X_{i-1}^{(m)}\right) : j = 1, ..., 2^{n/2}\right\}$$

converges in law to

$$\begin{aligned}
&\left\{X; 2^{-\frac{n}{4}}\sigma\,\Delta B\,(j) : j=1,...,2^{n/2}; 2^{-\frac{n}{4}}\,\Delta B\left(j+2^{n/2}\right) : j=1,...,2^{n/2}\right\}\\
\overset{\text{Law}}{=}\ &\left\{X; \sigma\left(B\left(j2^{-n/2}\right)-B\left((j-1)\,2^{-n/2}\right)\right) : j=1,...,2^{n/2};\right.\\
&\qquad\left. B_2\left(j2^{-n/2}\right)-B_2\left((j-1)\,2^{-n/2}\right) : j=1,...,2^{n/2}\right\},
\end{aligned}$$

where B_2 denotes a standard Brownian motion, independent of X and B. Hence, as $m\to\infty$,

$$\left\{X; A^{(m,n)}\right\} \overset{\text{f.d.d.}}{\Longrightarrow} \left\{X; A^{(\infty,n)}\right\}$$

where

$$\begin{aligned}
A_t^{(\infty,n)} := \sigma &\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} f(X_{(j-1)2^{-n/2}})\left[B\left(j2^{-n/2}\right)-B\left((j-1)\,2^{-n/2}\right)\right]\\
&+\gamma\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} f(X_{(j-1)2^{-n/2}})\left[B_2\left(j2^{-n/2}\right)-B_2\left((j-1)\,2^{-n/2}\right)\right].
\end{aligned}$$

By letting $n\to\infty$, and by using the independence of X, B and B_2, one obtains that $A^{(\infty,n)}$ converges uniformly on compacts in probability (u.c.p.) towards

$$A_t^{(\infty,\infty)} \triangleq \int_0^t f\,(X_s)\left(\sigma\,dB(s)+\gamma\,dB_2(s)\right).$$

This proves that, by letting m and then n go to infinity $\left\{X; A^{(m,n)}\right\}$ converges in the sense of f.d.d. to

$$\left\{X; A^{(\infty,\infty)}\right\} \overset{\text{Law}}{=} \left\{X; \sqrt{\sigma^2+\gamma^2}\int_0^{\cdot} f\,(X_s)\,dB(s)\right\}.$$

To conclude the proof of the Theorem we shall show that, by letting m and then n go to infinity, $B_t^{(m,n)}$ converges to zero in L^2, for any fixed $t>0$. To see this, write $K=\max(|\alpha|,|\beta|)$ and observe that, for every $t>0$, the independence of the Brownian increments yields the chain of inequalities:

$$\begin{aligned}
E\left[\left|B_t^{(m,n)}\right|^2\right] &= 2^{-m/2}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor} E\left\{\left|f(X_{(i-1)2^{-m/2}})-f(X_{(j-1)2^{-n/2}})\right|^2\right\}\\
&\quad\times E\left\{\left|\phi\,(i)\left\{\mathsf{P}\left(X_i^{(m)}-X_{i-1}^{(m)}\right)-E\left[\mathsf{P}\,(G)\right]\right\}+\gamma(-1)^i\left(X_i^{(m)}-X_{i-1}^{(m)}\right)\right|^2\right\}\\
&\leqslant 2|f'|_\infty^2\left(K^2\mathbf{Var}\left(\mathsf{P}(G)\right)+\gamma^2\right)2^{-m/2}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor}\left[(i-1)\,2^{-\frac{m}{2}}-(j-1)\,2^{-\frac{n}{2}}\right]\\
&\leqslant 2|f'|_\infty^2\left(K^2\mathbf{Var}\left(\mathsf{P}(G)\right)+\gamma^2\right)2^{-\frac{m+n}{2}}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor}1\\
&\leqslant 2|f'|_\infty^2 t\left(K^2\mathbf{Var}\left(\mathsf{P}(G)\right)+\gamma^2\right)2^{-n/2}.
\end{aligned}$$

This shows that

$$\limsup_{m\to\infty} E\left[\left|B_t^{(m,n)}\right|^2\right] \leqslant \text{cst.}\, 2^{-n/2},$$

and the desired conclusion is obtained by letting $n \to \infty$. □

We now state several consequences of Theorem 2.5. The first one (see also Jacod [8]) is obtained by setting $\alpha = \beta = 1$ (that is, ϕ is identically one), $\gamma = 0$ and by recalling that, for an even integer $\kappa \geqslant 2$, the polynomial $P(x) = x^\kappa - \mu_\kappa$ has centered Hermite rank $\geqslant 2$.

Corollary 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to C^2 with f' and f'' bounded, and fix an even integer $\kappa \geqslant 2$. For $t \geqslant 0$, we set:*

$$J_t^{(n)}(f) = 2^{-\frac{n}{4}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} \left(f(X_{(j-1)2^{-n/2}}) + f(X_{j2^{-n/2}})\right) \left\{2^{\kappa\frac{n}{4}} \left(X_{j2^{-n/2}} - X_{(j-1)2^{-n/2}}\right)^\kappa - \mu_\kappa\right\}. \tag{26}$$

Then, as $n \to +\infty$, $\left\{J_t^{(n)}(f), X_t\right\}_{t\geqslant 0}$ converges in the sense of f.d.d. to

$$\left\{\sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_0^t f(X_s)\, dB_s, X_t\right\}_{t\geqslant 0}.$$

The next result derives from Theorem 2.5 in the case $\alpha = 1$, $\beta = -1$ and $\gamma = 0$.

Corollary 2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to C^2 with f' and f'' bounded, $\kappa \geqslant 2$ be an even integer, and set, for $t \geqslant 0$:*

$$J_t^{(n)}(f) = 2^{(\kappa-1)\frac{n}{4}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} \left(f(X_{(j-1)2^{-n/2}}) + f(X_{j2^{-n/2}})\right) (-1)^j \left(X_{j2^{-n/2}} - X_{(j-1)2^{-n/2}}\right)^\kappa. \tag{27}$$

Then, as $n \to +\infty$, the process $\left\{J_t^{(n)}(f), X_t\right\}_{t\geqslant 0}$ converges in the sense of f.d.d. to

$$\left\{\sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_0^t f(X_s)\, dB_s, X_t\right\}_{t\geqslant 0}. \tag{28}$$

Proof. It is not difficult to see that the convergence result in the statement is equivalent to the convergence of the pair $\left\{Z_t^{(n)}(f), X_t\right\}_{t\geqslant 0}$ to the object in (28), where

$$Z_t^{(n)}(f) = 2^{-\frac{n}{4}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} \left(f(X_{(j-1)2^{-n/2}}) + f(X_{j2^{-n/2}})\right) (-1)^j \left\{2^{\kappa\frac{n}{4}} \left(X_{j2^{-n/2}} - X_{(j-1)2^{-n/2}}\right)^\kappa - \mu_\kappa\right\},$$

so that the conclusion is a direct consequence of Theorem 2.5. Indeed, we have

$$2^{-\frac{n}{4}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} (-1)^j \left(f(X_{(j-1)2^{-n/2}}) + f(X_{j2^{-n/2}})\right) = 2^{-\frac{n}{4}} \frac{1}{2} \left(f(X_{2\lfloor 2^{n/2} t \rfloor 2^{-n/2}}) - f(X_0)\right)$$

which tends to zero in L^2, as $n \to \infty$. □

A slight modification of Corollary 2.7 yields:

Corollary 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to C^2 with f' and f'' bounded, let $\kappa \geqslant 2$ be an even integer, and set:*

$$\widetilde{J}_t^{(n)}(f) = 2^{(\kappa-1)\frac{n}{4}} \sum_{j=1}^{\left\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\right\rfloor} f(X_{(2j+1)2^{-\frac{n}{2}}}) \left[\left(X_{(2j+2)2^{-\frac{n}{2}}} - X_{(2j+1)2^{-\frac{n}{2}}} \right)^{\kappa} - \left(X_{(2j+1)2^{-\frac{n}{2}}} - X_{(2j)2^{-\frac{n}{2}}} \right)^{\kappa} \right], \quad t \geqslant 0.$$

Then, as $n \to +\infty$, the process $\left\{ \widetilde{J}_t^{(n)}(f), X_t \right\}_{t \geqslant 0}$ converges in the sense of f.d.d. to (28).

Proof. By separating the sum according to the eveness of j, one can write

$$\widetilde{J}_t^{(n)}(f) = J_t^{(n)}(f) - r_t^{(n)}(f) + s_t^{(n)}(f),$$

for $J_t^{(n)}(f)$ defined by (27) and

$$r_t^{(n)}(f) = \frac{2^{(\kappa-1)\frac{n}{4}}}{2} \sum_{j=1}^{\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\rfloor} \left(f(X_{(2j+2)2^{-n/2}}) - f(X_{(2j+1)2^{-n/2}}) \right) \left(X_{(2j+2)2^{-n/2}} - X_{(2j+1)2^{-n/2}} \right)^{\kappa}$$

$$s_t^{(n)}(f) = \frac{2^{(\kappa-1)\frac{n}{4}}}{2} \sum_{j=1}^{\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\rfloor} \left(f(X_{(2j)2^{-n/2}}) - f(X_{(2j+1)2^{-n/2}}) \right) \left(X_{(2j)2^{-n/2}} - X_{(2j+1)2^{-n/2}} \right)^{\kappa}.$$

We decompose

$$r_t^{(n)}(f) = \frac{2^{(\kappa-1)\frac{n}{4}}}{2} \sum_{j=1}^{\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\rfloor} f'(X_{(2j+1)2^{-n/2}}) \left(X_{(2j+2)2^{-n/2}} - X_{(2j+1)2^{-n/2}} \right)^{\kappa+1}$$

$$+ \frac{2^{(\kappa-1)\frac{n}{4}}}{2} \sum_{j=1}^{\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\rfloor} f''(X_{\theta_{j,n}}) \left(X_{(2j+2)2^{-n/2}} - X_{(2j+1)2^{-n/2}} \right)^{\kappa+2} = r_t^{(1,n)}(f) + r_t^{(2,n)}(f),$$

for some $\theta_{j,n}$ between $(2j+1)2^{-n/2}$ and $(2j+2)2^{-n/2}$. By independence of increments and because $E\left[G^{\kappa+1}\right] = 0$, we have

$$E\left|r_t^{(1,n)}(f)\right|^2 = E\left[G^{2\kappa+2}\right] 2^{-n} \frac{1}{4} \sum_{j=1}^{\lfloor \frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\rfloor} E\left|f'(X_{(2j+1)2^{-n/2}})\right|^2 = O(2^{-n/2}).$$

For $r_t^{(2,n)}(f)$, we have

$$E\left|r_t^{(2,n)}(f)\right| = O(2^{-n/4}).$$

Similarly, we prove that $E\left|s_t^{(n)}(f)\right|$ tends to zero as $n \to \infty$, so that the conclusion is a direct consequence of Corollary 2.7. □

The subsequent results focus on odd powers.

Corollary 2.9. *Let $f:\mathbb{R}\to\mathbb{R}$ belong to C^2 with f' and f'' bounded, $\kappa\geqslant 3$ be an odd integer, and define $J_t^{(n)}(f)$ according to (26) (remark however that $\mu_\kappa=0$). Then, as $n\to+\infty$, $\left\{J_t^{(n)}(f),X_t\right\}_{t\geqslant 0}$ converges in the sense of f.d.d. to*

$$\left\{\int_0^t f(X_s)\left(\mu_{\kappa+1}d^\circ X(s)+\sqrt{\mu_{2\kappa}-\mu_{\kappa+1}^2}dB_s\right),X_t\right\}_{t\geqslant 0}. \tag{29}$$

Proof. One can write:

$$\begin{aligned}
J_t^{(n)}(f) &= 2^{-\frac{n}{4}}\frac{1}{2}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\left(f(X_{(j-1)2^{-n/2}})+f(X_{j2^{-n/2}})\right)\\
&\qquad\times\left\{\left(X_j^{(n)}-X_{(j-1)}^{(n)}\right)^\kappa-\mu_{\kappa+1}\left(X_j^{(n)}-X_{(j-1)}^{(n)}\right)\right\}\\
&\quad+\frac{\mu_{\kappa+1}}{2}\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}\left(f(X_{(j-1)2^{-n/2}})+f(X_{j2^{-n/2}})\right)\left(X_{j2^{-n/2}}-X_{(j-1)2^{-n/2}}\right)\\
&= D_t^{(n)}+E_t^{(n)}.
\end{aligned}$$

Since $x^\kappa-\mu_{\kappa+1}x$ has centered Hermite rank $\geqslant 2$, one can deal with $D_t^{(n)}$ directly via Theorem 2.5. The conclusion is obtained by observing that $E_t^{(n)}$ converges u.c.p. to $\mu_{\kappa+1}\int_0^t f(X_s)\,d^\circ X_s$. □

A slight modification of Corollary 2.9 yields:

Corollary 2.10. *Let $f:\mathbb{R}\to\mathbb{R}$ belong to C^2 with f' and f'' bounded, $\kappa\geqslant 3$ be an odd integer, and set:*

$$\begin{aligned}
\widetilde{J}_t^{(n)}(f) &= 2^{(\kappa-1)\frac{n}{4}}\sum_{j=1}^{\left\lfloor\frac{1}{2}\left(2^{\frac{n}{2}}t-1\right)\right\rfloor}f(X_{(2j+1)2^{-\frac{n}{2}}})\left[\left(X_{(2j+2)2^{-\frac{n}{2}}}-X_{(2j+1)2^{-\frac{n}{2}}}\right)^\kappa\right.\\
&\qquad\left.+\left(X_{(2j+1)2^{-\frac{n}{2}}}-X_{(2j)2^{-\frac{n}{2}}}\right)^\kappa\right],\quad t\geqslant 0.
\end{aligned}$$

Then, as $n\to+\infty$, the process $\left\{\widetilde{J}_t^{(n)}(f),X_t\right\}_{t\geqslant 0}$ converges in the sense f.d.d. to (29).

Proof. . Follows the proof of Corollary 2.8. □

The next result can be proved analogously.

Corollary 2.11. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to C^2 with f' and f'' bounded, $\kappa \geqslant 3$ be an odd integer, and define $J_t^{(n)}(f)$ according to (27). Then, as $n \to +\infty$, the process $\left\{ J_t^{(n)}(f), X_t \right\}_{t \geqslant 0}$ converges in the sense of f.d.d. to*

$$\left\{ \sqrt{\mu_{2\kappa}} \int_0^t f(X_s)\, dB(s), X_t \right\}_{t \geqslant 0}.$$

Proof. One can write

$$\begin{aligned} J_t^{(n)}(f) &= 2^{-\frac{n}{4}} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} f(X_{(j-1)2^{-n/2}})(-1)^j \left\{ \left(X_j^{(n)} - X_{(j-1)}^{(n)} \right)^\kappa - \mu_{\kappa+1} \left(X_j^{(n)} - X_{(j-1)}^{(n)} \right) \right\} \\ &\quad + \mu_{\kappa+1} 2^{-\frac{n}{4}} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} f(X_{(j-1)2^{-n/2}})(-1)^j \left(X_j^{(n)} - X_{j-1}^{(n)} \right). \end{aligned}$$

Since $x^\kappa - \mu_{\kappa+1} x$ has centered Hermite rank $\geqslant 2$, Theorem 2.5 gives the desired conclusion. □

3 Proof of Theorem 1.2

Fix $t \in [0,1]$, and let, for any $n \in \mathbb{N}$ and $j \in \mathbb{Z}$,

$$\begin{aligned} U_{j,n}(t) = \sharp\big\{ & k = 0, \ldots, \lfloor 2^n t \rfloor - 1 : \\ & Y(T_{k,n}) = j2^{-n/2} \text{ and } Y(T_{k+1,n}) = (j+1)2^{-n/2} \big\} \end{aligned} \tag{30}$$

$$\begin{aligned} D_{j,n}(t) = \sharp\big\{ & k = 0, \ldots, \lfloor 2^n t \rfloor - 1 : \\ & Y(T_{k,n}) = (j+1)2^{-n/2} \text{ and } Y(T_{k+1,n}) = j2^{-n/2} \big\} \end{aligned} \tag{31}$$

denote the number of upcrossings and downcrossings of the interval $[j2^{-n/2}, (j+1)2^{-n/2}]$ within the first $\lfloor 2^n t \rfloor$ steps of the random walk $\{Y(T_{k,n}),\, k \in \mathbb{N}\}$, respectively. Also, set

$$\mathcal{L}_{j,n}(t) = 2^{-n/2} \big(U_{j,n}(t) + D_{j,n}(t) \big).$$

The following statement collects several useful estimates proved in [11].

Proposition 3.1. *1. For every $x \in \mathbb{R}$ and $t \in [0,1]$, we have*

$$E\big[|L_t^x(Y)| \big] \leqslant 2\, E\big[|L_1^0(Y)| \big] \sqrt{t} \exp\big(-\frac{x^2}{2t} \big).$$

2. For every fixed $t \in [0,1]$, we have $\sum_{j \in \mathbb{Z}} E\big[|\mathcal{L}_{j,n}(t)|^2 \big] = O(2^{n/2})$.

3. There exists a positive constant μ such that, for every $a, b \in \mathbb{R}$ with $ab \geqslant 0$ and $t \in [0,1]$,

$$E\big[|L_t^b(Y) - L_t^a(Y)|^2 \big] \leqslant \mu\, |b-a| \sqrt{t} \exp\left(-\frac{a^2}{2t} \right).$$

4. *There exists a random variable* $K \in \mathrm{L}^8$ *such that, for every* $j \in \mathbb{Z}$, *every* $n \geqslant 0$ *and every* $t \in [0,1]$, *one has that*

$$|\mathcal{L}_{j,n}(t) - L_t^{j2^{-n/2}}(Y)| \leqslant Kn2^{-n/4}\sqrt{L_t^{j2^{-n/2}}(Y)}.$$

Proof. The first point is proved in [11, Lemma 3.3]. The proof of the second point is obtained by simply mimicking the arguments displayed in the proof of [11, Lemma 3.7]. The third point corresponds to the content of [11, Lemma 3.4], while the fourth point is proved in [11, Lemma 3.6]. □

We will also need the following result:

Proposition 3.2. *Fix some integers* $n, N \geqslant 1$, *and let* κ *be an even integer. Then, as* $m \to \infty$, *the random element*

$$\left\{ X_x,\, 2^{-\frac{m}{4}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor + 1}^{\lfloor j2^{\frac{m-n}{2}} \rfloor} \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^{\kappa} - \mu_\kappa \right] \mathcal{L}_{i,m}(t) : j = -N, \dots, N \right\}_{x \in \mathbb{R},\, t \in [0,1]}$$

converges weakly in the sense of f.d.d. to

$$\left\{ X_x,\, \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{(j-1)2^{-n/2}}^{j2^{-n/2}} L_t^x(Y) dB_x : j = -N, \dots, N \right\}_{x \in \mathbb{R},\, t \in [0,1]}.$$

Proof. For every $m \geqslant k \geqslant n$, we can write

$$2^{-\frac{m}{4}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor + 1}^{\lfloor j2^{\frac{m-n}{2}} \rfloor} \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^{\kappa} - \mu_\kappa \right] \mathcal{L}_{i,m}(t) = A_{j,n,t}^{(m,k)} + B_{j,n,t}^{(m,k)} + C_{j,n,t}^{(m,k)}$$

with

$$\begin{aligned}
A_{j,n,t}^{(m,k)} &= 2^{-\frac{m}{4}} \sum_{i=\lfloor (j-1)2^{\frac{k-n}{2}} \rfloor + 1}^{\lfloor j2^{\frac{k-n}{2}} \rfloor} L_t^{i2^{-k/2}}(Y) \sum_{\ell=\lfloor (i-1)2^{\frac{m-k+n}{2}} \rfloor + 1}^{\lfloor i2^{\frac{m-k+n}{2}} \rfloor} \left[\left(X_\ell^{(m)} - X_{\ell-1}^{(m)} \right)^{\kappa} - \mu_\kappa \right] \\
B_{j,n,t}^{(m,k)} &= 2^{-\frac{m}{4}} \sum_{i=\lfloor (j-1)2^{\frac{k-n}{2}} \rfloor + 1}^{\lfloor j2^{\frac{k-n}{2}} \rfloor} \sum_{\ell=\lfloor (i-1)2^{\frac{m-k}{2}} \rfloor + 1}^{\lfloor i2^{\frac{m-k}{2}} \rfloor} \left[L_t^{\ell 2^{-m/2}}(Y) - L_t^{i2^{-k/2}}(Y) \right] \left[\left(X_\ell^{(m)} - X_{\ell-1}^{(m)} \right)^{\kappa} - \mu_\kappa \right] \\
C_{j,n,t}^{(m,k)} &= 2^{-\frac{m}{4}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor + 1}^{\lfloor j2^{\frac{m-n}{2}} \rfloor} \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^{\kappa} - \mu_\kappa \right] \left[L_t^{i2^{-m/2}}(Y) - \mathcal{L}_{i,m}(t) \right]
\end{aligned}$$

We shall study $A^{(m,k)}$, $B^{(m,k)}$ and $C^{(m,k)}$ separately. By Proposition 2.3, we know that, as $m \to \infty$, the random element

$$\left\{ X;\, 2^{-\frac{m}{4}} \sum_{\ell=\lfloor (i-1)2^{\frac{m-k+n}{2}} \rfloor + 1}^{\lfloor i2^{\frac{m-k+n}{2}} \rfloor} \left[\left(X_\ell^{(m)} - X_{\ell-1}^{(m)} \right)^{\kappa} - \mu_\kappa \right] : -N \leq j \leq N,\, \lfloor (j-1)\, 2^{\frac{k-n}{2}} \rfloor + 1 \leqslant i \leqslant \lfloor j2^{\frac{k-n}{2}} \rfloor \right\}$$

converges in law to

$$\left\{X; \sqrt{\mu_{2\kappa}-\mu_\kappa^2}\big(B_{i2^{-k/2}} - B_{(i-1)2^{-k/2}}\big) : -N \leq j \leq N,\, \lfloor (j-1)\, 2^{\frac{k-n}{2}} \rfloor + 1 \leq i \leq \lfloor j 2^{\frac{k-n}{2}} \rfloor \right\}.$$

Hence, as $m \to \infty$, using also the independence between X and Y, we have:

$$\left\{X; A_{j,n}^{(m,k)} : j = -N, \ldots, N\right\} \overset{\text{f.d.d.}}{\Longrightarrow} \left\{X; A_{j,n}^{(\infty,k)} : j = -N, \ldots, N\right\},$$

where

$$A_{j,n,t}^{(\infty,k)} \triangleq \sqrt{\mu_{2\kappa}-\mu_\kappa^2} \sum_{i=\lfloor (j-1)2^{\frac{k-n}{2}} \rfloor + 1}^{\lfloor j 2^{\frac{k-n}{2}} \rfloor} L_t^{i2^{-k/2}}(Y)\big(B_{i2^{-k/2}} - B_{(i-1)2^{-k/2}}\big).$$

By letting $k \to \infty$, one obtains that $A_{j,n,t}^{(\infty,k)}$ converges in probability towards

$$\sqrt{\mu_{2\kappa}-\mu_\kappa^2} \int_{(j-1)2^{-n/2}}^{j2^{-n/2}} L_t^x(Y) dB_x.$$

This proves, by letting m and then k go to infinity, that $\{X; A_{j,n}^{(m,k)} : j = -N, \ldots, N\}$ converges in the sense of f.d.d. to

$$\left\{X; \sqrt{\mu_{2\kappa}-\mu_\kappa^2} \int_{(j-1)2^{-n/2}}^{j2^{-n/2}} L_t^x(Y) dB_x : j = -N, \ldots, N\right\}.$$

To conclude the proof of the Proposition we shall show that, by letting m and then k go to infinity (for fixed j, n and $t > 0$), one has that $\left|B_{j,n,t}^{(m,k)}\right|$ and $\left|C_{j,n,t}^{(m,k)}\right|$ converge to zero in L^2. Let us first consider $C_{j,n,t}^{(m,k)}$. In what follows, $c_{j,n}$ denotes a constant that can be different from line to line. When $t \in [0,1]$ is fixed, we have, by the independence of Brownian increments and the first and the fourth points of Proposition 3.1:

$$\begin{aligned}
E\left[\left|C_{j,n,t}^{(m,k)}\right|^2\right] &= 2^{-\frac{m}{2}}\, \mathbf{Var}(G^\kappa) \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor + 1}^{\lfloor j 2^{\frac{m-n}{2}} \rfloor} E\left[\left|L_t^{i2^{-m/2}}(Y) - \mathcal{L}_{i,m}(t)\right|^2\right] \\
&\leqslant c_{j,n}\, 2^{-m}\, m^2 \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor + 1}^{\lfloor j 2^{\frac{m-n}{2}} \rfloor} E\left[\left|L_t^{i2^{-m/2}}(Y)\right|\right] \\
&\leqslant c_{j,n}\, 2^{-m/2}\, m^2.
\end{aligned}$$

Let us now consider $B_{j,n,t}^{(m,k)}$. We have, by the independence of Brownian increments and the

third point of Proposition 3.1:

$$\begin{aligned} E\left[\left|B_{j,n,t}^{(m,k)}\right|^2\right] &= 2^{-\frac{m}{2}}\,\mathbf{Var}(G^\kappa) \sum_{i=\lfloor (j-1)2^{\frac{k-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{k-n}{2}}\rfloor+1} \sum_{\ell=\lfloor (i-1)2^{\frac{m-k}{2}}\rfloor+1}^{\lfloor i2^{\frac{m-k}{2}}\rfloor} E\left[\left|L_t^{\ell 2^{-m/2}}(Y) - L_t^{i2^{-k/2}}(Y)\right|^2\right] \\ &\leqslant c_{j,n}\,2^{-m/2} \sum_{i=\lfloor (j-1)2^{\frac{k-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{k-n}{2}}\rfloor} \sum_{\ell=\lfloor (i-1)2^{\frac{m-k}{2}}\rfloor+1}^{\lfloor i2^{\frac{m-k}{2}}\rfloor} \left(i2^{-k/2} - \ell\,2^{-m/2}\right) \\ &\leqslant c_{j,n}\,2^{-k/2}. \end{aligned}$$

The desired conclusion follows immediately. □

The next result will be the key in the proof of the convergence (15):

Theorem 3.3. *For even $\kappa \geqslant 2$ and $t \in [0,1]$, set*

$$J_t^{(n)}(f) = 2^{-\frac{n}{4}}\,\frac{1}{2}\sum_{j\in\mathbb{Z}} \left(f(X_{(j-1)2^{-n/2}}) + f(X_{j\,2^{-n/2}})\right)\left[2^{\kappa\frac{n}{4}}\left(X_{j2^{-\frac{n}{2}}} - X_{(j-1)2^{-\frac{n}{2}}}\right)^\kappa - \mu_\kappa\right]\mathcal{L}_{j,n}(t),$$

where the real-valued function f belong to C^2 with f' and f'' bounded. Then, as $n \to +\infty$, the random element $\left\{X_x, J_t^{(n)}(f)\right\}_{x\in\mathbb{R},\,t\in[0,1]}$ converges in the sense of f.d.d. to

$$\left\{X_x, \sqrt{\mu_{2\kappa} - \mu_\kappa^2}\int_{\mathbb{R}} f(X_x) L_t^x(Y)\,dB_x\right\}_{x\in\mathbb{R},\,t\in[0,1]}. \tag{32}$$

Proof. By proceeding as in the beginning of the proof of Theorem 2.5, it is not difficult to see that the convergence result in the statement is equivalent to the convergence of the pair $\left\{X_x, \widetilde{J}_t^{(n)}(f)\right\}_{x\in\mathbb{R},t\in[0,1]}$ to the object in (32) where

$$\widetilde{J}_t^{(n)}(f) = 2^{-\frac{n}{4}}\sum_{j\in\mathbb{Z}} f(X_{(j-1)2^{-n/2}})\left[2^{\kappa\frac{n}{4}}\left(X_{j2^{-\frac{n}{2}}} - X_{(j-1)2^{-\frac{n}{2}}}\right)^\kappa - \mu_\kappa\right]\mathcal{L}_{j,n}(t).$$

For every $m \geqslant n$ and $p \geqslant 1$, one has that

$$\begin{aligned} \widetilde{J}_t^{(m)}(f) &= 2^{-\frac{m}{4}}\sum_{j\in\mathbb{Z}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor} f(X_{(i-1)2^{-m/2}})\left[\left(X_i^{(m)} - X_{i-1}^{(m)}\right)^\kappa - \mu_\kappa\right]\mathcal{L}_{i,m}(t) \\ &= A_t^{(m,n,p)} + B_t^{(m,n,p)} + C_t^{(m,n,p)}, \end{aligned}$$

where

$$A_t^{(m,n,p)} = 2^{-\frac{m}{4}} \sum_{|i|>p2^{m/2}} f(X_{(i-1)2^{-m/2}}) \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^\kappa - \mu_\kappa \right] \mathcal{L}_{i,m}(t),$$

$$B_t^{(m,n,p)} = 2^{-\frac{m}{4}} \sum_{|j|\leqslant p2^{n/2}} f(X_{(j-1)2^{-n/2}}) \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor +1}^{\lfloor j2^{\frac{m-n}{2}} \rfloor} \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^\kappa - \mu_\kappa \right] \mathcal{L}_{i,m}(t),$$

$$C_t^{(m,n,p)} = 2^{-\frac{m}{4}} \sum_{|j|\leqslant p2^{n/2}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor +1}^{\lfloor j2^{\frac{m-n}{2}} \rfloor} \left[f(X_{(i-1)2^{-m/2}}) - f(X_{(j-1)2^{-n/2}}) \right] \times \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^\kappa - \mu_\kappa \right] \mathcal{L}_{i,m}(t).$$

We shall study $A^{(m,n,p)}$, $B^{(m,n,p)}$ and $C^{(m,n,p)}$ separately. By Proposition 3.2, we know that, as $m \to \infty$, the random element

$$\left\{ X; 2^{-\frac{m}{4}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}} \rfloor +1}^{\lfloor j2^{\frac{m-n}{2}} \rfloor} \left[\left(X_i^{(m)} - X_{i-1}^{(m)} \right)^\kappa - \mu_\kappa \right] \mathcal{L}_{i,m}(t) : |j| \leqslant p2^{n/2} \right\}$$

converges in law to

$$\left\{ X; \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{(j-1)2^{-n/2}}^{j2^{-n/2}} L_t^x(Y) dB_x : |j| \leqslant p2^{n/2} \right\}.$$

Hence, as $m \to \infty$,

$$\{X; B^{(m,n,p)}\} \stackrel{\text{f.d.d.}}{\Longrightarrow} \{X; B^{(\infty,n,p)}\}$$

where

$$B_t^{(\infty,n,p)} = \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \sum_{|j|\leqslant p2^{n/2}} f(X_{(j-1)2^{-n/2}}) \int_{(j-1)2^{-n/2}}^{j2^{-n/2}} L_t^x(Y) dB_x.$$

By letting $n \to \infty$, one obtains that $B^{(\infty,n,p)}$ converges in probability towards

$$B_t^{(\infty,\infty,p)} = \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{-p}^{p} f(X_x) L_t^x(Y) dB_x.$$

Finally, by letting $p \to \infty$, one obtains, as limit, $\sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{\mathbb{R}} f(X_x) L_t^x(Y) dB_x$. This proves that, by letting m and then n and finally p go to infinity, $\{X; B^{(m,n,p)}\}$ converges in the sense of f.d.d. to $\{X; \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_{\mathbb{R}} f(X_x) L_t^x(Y) dB_x\}$.

To conclude the proof of the Theorem we shall show that, by letting m and then n and finally p go to infinity, $\left| A_t^{(m,n,p)} \right|$ and $\left| C_t^{(m,n,p)} \right|$ converge to zero in L^2. Let us first consider $C_t^{(m,n,p)}$.

When $t \in [0,1]$ is fixed, the independence of the Brownian increments yields that

$$\begin{aligned} E\left[\left|C_t^{(m,n,p)}\right|^2\right] &= 2^{-\frac{m}{2}}\,\mathbf{Var}(G^\kappa) \sum_{|j|\leqslant p2^{n/2}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor} E\left|f(X_{(i-1)2^{-m/2}}) - f(X_{(j-1)2^{-n/2}})\right|^2 \\ &\qquad \times E\left\{|\mathcal{L}_{i,m}(t)|^2\right\} \\ &\leqslant \mathbf{Var}(G^\kappa)|f'|_\infty^2\, 2^{-\frac{m+n}{2}} \sum_{j\in\mathbb{Z}} \sum_{i=\lfloor (j-1)2^{\frac{m-n}{2}}\rfloor+1}^{\lfloor j2^{\frac{m-n}{2}}\rfloor} E\left\{|\mathcal{L}_{i,m}(t)|^2\right\} \\ &= \mathbf{Var}(G^\kappa)|f'|_\infty^2\, 2^{-\frac{m+n}{2}} \sum_{i\in\mathbb{Z}} E\left\{|\mathcal{L}_{i,m}(t)|^2\right\}. \end{aligned}$$

The second point of Proposition 3.1 implies that $\sum_{i\in\mathbb{Z}} E\left[|\mathcal{L}_{i,m}(t)|^2\right] \leqslant \text{cst.}\, 2^{m/2}$ uniformly in $t \in [0,1]$. This shows that

$$\sup_{m,p}\; E\left[\left|C_t^{(m,n,p)}\right|^2\right] \leqslant \text{cst.} 2^{-n/2}.$$

Let us now consider $A_t^{(m,n,p)}$. We have

$$\begin{aligned} E\left[|A_t^{(m,n,p)}|^2\right] &= \mathbf{Var}(G^\kappa)\, 2^{-\frac{m}{2}} \sum_{|i|>p2^{m/2}} E\big|f(X_{(i-1)2^{-m/2}})\big|^2 E\left[|\mathcal{L}_{i,m}(t)|^2\right] \\ &\leqslant \mathbf{Var}(G^\kappa) \sup_{t\in[0,1]} E\big|f(X_t)\big|^2\, 2^{-\frac{m}{2}} \sum_{|i|>p2^{m/2}} E\left[|\mathcal{L}_{i,m}(t)|^2\right]. \end{aligned}$$

The fourth point in the statement of Proposition 3.1 yields

$$|\mathcal{L}_{i,m}(t)| \leqslant L_t^{i2^{-m/2}}(Y) + Km2^{-m/4}\sqrt{L_t^{i2^{-m/2}}(Y)}.$$

By using the first point in the statement of Proposition 3.1, we deduce that:

$$\begin{aligned} E\left[|A_t^{(m,n,p)}|^2\right] &\leqslant \text{cst.}\, 2^{-m/2} \sum_{i>p2^{m/2}} \exp\big(-\frac{i^2 2^{-m}}{2t}\big) \\ &\leqslant \text{cst.} \sum_{i>p2^{m/2}} \int_{(i-1)2^{-m/2}}^{i2^{-m/2}} \exp\big(-\frac{x^2}{2t}\big)dx \\ &= \text{cst.} \int_p^{+\infty} \exp\big(-\frac{x^2}{2t}\big)dx \leq \frac{\text{cst.}}{p}. \end{aligned}$$

The desired conclusion follows. □

We are finally in a position to prove Theorem 1.2:

Proof of Theorem 1.2

Proof of (15). By using an equality analogous to [11, p. 648, line 8] (observe that our definition of $V_n^{(\kappa)}(f,t)$ is slightly different than the one given in [11]), $2^{(\kappa-3)\frac{n}{4}} V_n^{(\kappa)}(f,t)$ equals

$$2^{-\frac{n}{4}} \frac{1}{2} \sum_{j\in\mathbb{Z}} \Big(f(X_{(j-1)2^{-n/2}}) + f(X_{j\,2^{-n/2}}) \Big) \Big[2^{\kappa\frac{n}{4}} \Big(X_{j2^{-n/2}} - X_{(j-1)2^{-n/2}} \Big)^{\kappa} - \mu_\kappa \Big] \mathcal{L}_{j,n}(t).$$

As a consequence, (15) derives immediately from Theorem 3.3.

Proof of (16). Based on Lemma 1.1, it is showed in [11], p. 658, that

$$V_n^{(\kappa)}(f,t) = \begin{cases} \frac{1}{2} \sum_{j=0}^{j^\star-1} \Big(f(X^+_{(j-1)2^{-n/2}}) + f(X^+_{j\,2^{-n/2}}) \Big) \Big(X^+_{j2^{-n/2}} - X^+_{(j-1)2^{-n/2}} \Big)^{\kappa} & \text{if } j^\star > 0 \\ 0 & \text{if } j^\star = 0 \\ \frac{1}{2} \sum_{j=0}^{|j^\star|-1} \Big(f(X^-_{(j-1)2^{-n/2}}) + f(X^-_{j\,2^{-n/2}}) \Big) \Big(X^-_{j2^{-n/2}} - X^-_{(j-1)2^{-n/2}} \Big)^{\kappa} & \text{if } j^\star < 0 \end{cases}$$

Here, X^+ (resp. X^-) represents X restricted to $[0,\infty)$ (resp. $(-\infty,0]$), and $j^\star$ is defined as follows:

$$j^\star = j^\star(n,t) = 2^{n/2}\, Y(T_{\lfloor 2^n t\rfloor, n}).$$

For $t \in [0,1]$, let

$$Y_n(t) = Y(T_{\lfloor 2^n t\rfloor, n}).$$

Also, for $t \geqslant 0$, set

$$J_n^{\pm}(f,t) = 2^{(\kappa-1)\frac{n}{4}} \frac{1}{2} \sum_{j=1}^{\lfloor 2^{n/2} t\rfloor} \Big(f(X^{\pm}_{(j-1)2^{-n/2}}) + f(X^{\pm}_{j\,2^{-n/2}}) \Big) \Big(X^{\pm}_{j2^{-n/2}} - X^{\pm}_{(j-1)2^{-n/2}} \Big)^{\kappa}$$

and, for $u \in \mathbb{R}$:

$$J_n(f,u) = \begin{cases} J_n^+(f,u), & \text{if } u \geqslant 0, \\ J_n^-(f,-u), & \text{if } u \geqslant 0. \end{cases}$$

Observe that

$$2^{(\kappa-1)\frac{n}{4}}\ V_n^{(\kappa)}(f,t) = J_n\big(f, Y_n(t)\big) \quad \text{(see also (4.3) in [11])}. \tag{33}$$

For every $s,t \in \mathbb{R}$ and $n \geqslant 1$, we shall prove

$$E\big|J_n(f,t) - J_n(f,s)\big|^2 \leqslant c_{f,\kappa} \Big(2^{-\frac{n}{2}} \big|\lfloor 2^{\frac{n}{2}} t\rfloor - \lfloor 2^{\frac{n}{2}} s\rfloor\big| + 2^{-n} \big|\lfloor 2^{\frac{n}{2}} t\rfloor - \lfloor 2^{\frac{n}{2}} s\rfloor\big|^2 \Big) \tag{34}$$

for a constant $c_{f,\kappa}$ depending only of f and κ. For simplicity, we only make the proof when $s,t \geqslant 0$, but the other cases can be handled in the same way. For $u \geqslant 0$, we can decompose

$$J_n(f,u) = J_n^{(a)}(f,u) + J_n^{(b)}(f,u)$$

where

$$\begin{aligned} J_n^{(a)}(f,u) &= 2^{(\kappa-1)\frac{n}{4}} \sum_{j=1}^{\lfloor 2^{\frac{n}{2}} u\rfloor} f\big(X_{(j-1)2^{-\frac{n}{2}}}\big) \big(X_{j2^{-\frac{n}{2}}} - X_{(j-1)2^{-\frac{n}{2}}}\big)^{\kappa} \\ J_n^{(b)}(f,u) &= \frac{1}{2}\, 2^{(\kappa-1)\frac{n}{4}} \sum_{j=1}^{\lfloor 2^{\frac{n}{2}} u\rfloor} f'\big(X_{\theta_{j,n}}\big) \big(X_{j2^{-\frac{n}{2}}} - X_{(j-1)2^{-\frac{n}{2}}}\big)^{\kappa+1} \end{aligned}$$

for some $\theta_{j,n}$ lying between $(j-1)2^{-\frac{n}{2}}$ and $j2^{-\frac{n}{2}}$. By independence, and because κ is odd, we can write, for $0 \leq s \leq t$:

$$\begin{aligned} E\big|J_n^{(a)}(f,t) - J_n^{(a)}(f,s)\big|^2 &= \mu_{2\kappa}\, 2^{-\frac{n}{2}} \sum_{j=\lfloor 2^{\frac{n}{2}} s\rfloor + 1}^{\lfloor 2^{\frac{n}{2}} t\rfloor} E\big|f\big(X_{(j-1)2^{-\frac{n}{2}}}\big)\big|^2 \\ &\leqslant c_{f,\kappa}\, 2^{-\frac{n}{2}} \big|\lfloor 2^{\frac{n}{2}} t\rfloor - \lfloor 2^{\frac{n}{2}} s\rfloor\big|. \end{aligned}$$

For $J_n^{(b)}(f,\cdot)$, we have by Cauchy-Schwarz inequality:

$$E\big|J_n^{(b)}(f,t) - J_n^{(b)}(f,s)\big|^2 \leqslant c_{f,\kappa} \left(2^{-\frac{n}{2}} \big|\lfloor 2^{\frac{n}{2}} t\rfloor - \lfloor 2^{\frac{n}{2}} s\rfloor\big|\right)^2.$$

The desired conclusion (34) follows. Since X and Y are independent, (34) yields that

$$E\left|J_n\big(f, Y_n(t)\big) - J_n\big(f, Y(t)\big)\right|^2$$

is bounded by

$$c_{f,\kappa} E\left[2^{-\frac{n}{2}} \big|\lfloor 2^{\frac{n}{2}} Y_n(t)\rfloor - \lfloor 2^{\frac{n}{2}} Y(t)\rfloor\big| + 2^{-n} \big|\lfloor 2^{\frac{n}{2}} Y_n(t)\rfloor - \lfloor 2^{\frac{n}{2}} Y(t)\rfloor\big|^2\right].$$

But this quantity tends to zero as $n \to \infty$, because $Y_n(t) \xrightarrow{\mathrm{L}^2} Y(t)$ (recall that $T_{\lfloor 2^n t\rfloor, n} \xrightarrow{\mathrm{L}^2} t$, see Lemma 2.2 in [11]). Combining this latter fact with the independence between $J_n(f,\cdot)$ and Y, and the convergence in the sense of f.d.d. given by Corollary 2.9, one obtains

$$J_n(f,\cdot) \longrightarrow \int_0^{\cdot} f(X_z)\big(\mu_{\kappa+1} d^\circ X_z + \sqrt{\mu_{2\kappa} - \mu_{\kappa+1}^2}\, dB_z\big),$$

where the convergence is in the sense of f.d.d., hence

$$J_n(f, Y_n) \longrightarrow \int_0^{Y} f(X_z)\big(\mu_{\kappa+1} d^\circ X_z + \sqrt{\mu_{2\kappa} - \mu_{\kappa+1}^2}\, dB_z\big),$$

where the convergence is once again in the sense of f.d.d.. In view of (33), this concludes the proof of (16).

4 Proof of Theorem 1.4

Let

$$\begin{aligned} UU_{2j+1,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^{n-1} t\rfloor - 1 : \quad Y(T_{2k,n}) = (2j)2^{-n/2}, \\ &\qquad\qquad Y(T_{2k+1,n}) = (2j+1)2^{-n/2},\, Y(T_{2k+2,n}) = (2j+2)2^{-n/2}\big\} \\ UD_{2j+1,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^{n-1} t\rfloor - 1 : \quad Y(T_{2k,n}) = (2j)2^{-n/2}, \\ &\qquad\qquad Y(T_{2k+1,n}) = (2j+1)2^{-n/2},\, Y(T_{2k+2,n}) = (2j)2^{-n/2}\big\} \\ DU_{2j+1,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^{n-1} t\rfloor - 1 : \quad Y(T_{2k,n}) = (2j+2)2^{-n/2}, \\ &\qquad\qquad Y(T_{2k+1,n}) = (2j+1)2^{-n/2},\, Y(T_{2k+2,n}) = (2j+2)2^{-n/2}\big\} \\ DD_{2j+1,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^{n-1} t\rfloor - 1 : \quad Y(T_{2k,n}) = (2j+2)2^{-n/2}, \\ &\qquad\qquad Y(T_{2k+1,n}) = (2j+1)2^{-n/2},\, Y(T_{2k+2,n}) = (2j)2^{-n/2}\big\}. \end{aligned}$$

denote the number of *double* upcrossings and/or downcrossings of the interval $[(2j)2^{-n/2}, (2j+2)2^{-n/2}]$ within the first $\lfloor 2^n t \rfloor$ steps of the random walk $\{Y(T_{k,n}),\, k \in \mathbb{N}\}$. Observe that

$$\begin{aligned} S_n^{(\kappa)}(f,t) &= \sum_{j\in\mathbb{Z}} f\big(X_{(2j+1)2^{-n/2}}\big)\Big[\big(X_{(2j+2)2^{-n/2}} - X_{(2j+1)2^{-n/2}}\big)^{\kappa} \\ &\qquad +(-1)^{\kappa+1}\big(X_{(2j+1)2^{-n/2}} - X_{(2j)2^{-n/2}}\big)^{\kappa}\Big]\big(UU_{2j+1,n}(t) - DD_{2j+1,n}(t)\big). \end{aligned} \tag{35}$$

The proof of the following lemma is easily obtained by observing that the double upcrossings and downcrossings of the interval $[(2j)2^{-n/2}, (2j+2)2^{-n/2}]$ alternate:

Lemma 4.1. *Let $t > 0$. For each $j \in \mathbb{Z}$,*

$$UU_{2j+1,n}(t) - DD_{2j+1,n}(t) = \begin{cases} \mathbf{1}_{\{0\le j<\widetilde{j^\star}\}} & \text{if } \widetilde{j^\star} > 0 \\ 0 & \text{if } \widetilde{j^\star} = 0 \\ -\mathbf{1}_{\{\widetilde{j^\star}\le j<0\}} & \text{if } \widetilde{j^\star} < 0 \end{cases}$$

where

$$\widetilde{j^\star} = \widetilde{j^\star}(n,t) = \frac{1}{2}\, 2^{n/2} Y(T_{2\lfloor 2^{n-1}t\rfloor,n}).$$

Consequently, by combining Lemma 4.1 with (35), we deduce:

$$S_n^{(\kappa)}(f,t) = \begin{cases} \sum_{j=0}^{\widetilde{j^\star}-1} f(X^+_{(2j+1)2^{-n/2}})\Big[\Big(X^+_{(2j+2)2^{-n/2}} - X^+_{(2j+1)2^{-n/2}}\Big)^{\kappa} \\ \qquad\qquad +(-1)^{\kappa+1}\big(X^+_{(2j+1)2^{-n/2}} - X^+_{(2j)2^{-n/2}}\big)^{\kappa}\Big] & \text{if } \widetilde{j^\star} > 0 \\ 0 & \text{if } \widetilde{j^\star} = 0 \\ \sum_{j=0}^{|\widetilde{j^\star}|-1} f(X^-_{(2j+1)2^{-n/2}})\Big[\Big(X^-_{(2j+2)2^{-n/2}} - X^-_{(2j+1)2^{-n/2}}\Big)^{\kappa} \\ \qquad\qquad +(-1)^{\kappa+1}\big(X^-_{(2j+1)2^{-n/2}} - X^-_{(2j)2^{-n/2}}\big)^{\kappa}\Big] & \text{if } \widetilde{j^\star} < 0 \end{cases}.$$

Here, as in the proof of (16), X^+ (resp. X^-) represents X restricted to $[0,\infty)$ (resp. $(-\infty,0]$). For $t \geqslant 0$, set

$$\begin{aligned} \widetilde{J}_n^{\pm}(f,t) &= 2^{(\kappa-1)\frac{n}{4}} \sum_{j=0}^{\left\lfloor \frac{1}{2}2^{\frac{n}{2}}t\right\rfloor} f(X^{\pm}_{(2j+1)2^{-\frac{n}{2}}})\Big[\Big(X^{\pm}_{(2j+2)2^{-\frac{n}{2}}} - X^{\pm}_{(2j+1)2^{-\frac{n}{2}}}\Big)^{\kappa} \\ &\qquad +(-1)^{\kappa+1}\big(X^{\pm}_{(2j+1)2^{-\frac{n}{2}}} - X^{\pm}_{(2j)2^{-\frac{n}{2}}}\big)^{\kappa}\Big] \end{aligned}$$

and, for $u \in \mathbb{R}$:

$$\widetilde{J}_n(f,u) = \begin{cases} \widetilde{J}_n^+(f,u), & \text{if } u \geqslant 0, \\ \widetilde{J}_n^-(f,-u), & \text{if } u \geqslant 0. \end{cases}$$

Also, let

$$\widetilde{Y}_n(t) = Y(T_{2\lfloor 2^{n-1}t\rfloor,n}).$$

Observe that

$$2^{(\kappa-1)\frac{n}{4}}\, S_n^{(\kappa)}(f,t) = \widetilde{J}_n\big(f, \widetilde{Y}_n(t)\big). \tag{36}$$

Finally, using Corollary 2.8 (for κ even) and Corollary 2.10 (for κ odd), and arguing exactly as in the proof of (16), we obtain that the statement of Theorem 1.4 holds.

□

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