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Exchangeable Fragmentation-Coalescence Processes and their Equilibrium Measures

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Abstract

We define and study a family of Markov processes with state space the compact set of all partitions of $\mathbb{N}$ that we call exchangeable fragmentation-coalescence processes. They can be viewed as a combination of homogeneous fragmentation as defined by Bertoin and of homogenous coalescence as defined by Pitman and Schweinsberg or Möhle and Sagitov. We show that they admit a unique invariant probability measure and we study some properties of their paths and of their equilibrium measure.

Key words and phrases: Fragmentation, coalescence, invariant distribution.

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1 Introduction

Coalescence phenomena (coagulation, gelation, aggregation,...) and their dual fragmentation phenomena (splitting, erosion, break-ups,...), are present in a wide variety of contexts.

References as to the fields of application of coalescence and fragmentation models (physical chemistry, astronomy, biology, computer sciences...) may be found in Aldous [2] -mainly for coalescence- and in the proceedings [9] for fragmentation (some further references can be found in the introduction of [4]). Clearly, many fragmentation or coalescence phenomena are not "pure" in the sense that both are present at the same time. For instance, in the case of polymer formation there is a regime near the critical temperature where molecules break up and recombine simultaneously. Another example is given by Aldous [2], when, in his *one specific application* section, he discusses how certain liquids (e.g., olive oil and alcohol) mix at high temperature but separate below some critical level. When one lowers very slowly the temperature through this threshold, droplets of one liquid begin to form, merge and dissolve back very quickly.

It appears that coalescence-fragmentation processes are somewhat less tractable mathematically than pure fragmentation or pure coalescence. One of the reasons is that by combining these processes we lose some of the nice properties they exhibit when they stand alone, as for instance their genealogic or branching structure. Nevertheless, it is natural to investigate such processes, and particularly to look for their equilibrium measures.

In this direction Diaconis, Mayer-Wolf, Zeitouni and Zerner [11] considered a coagulation-fragmentation transformation of partitions of the interval $(0,1)$ in which the merging procedure corresponds to the multiplicative coalescent while the splittings are driven by a quadratic fragmentation. By relating it to the random transposition random walk on the group of permutations, they were able to prove a conjecture of Vershik stating that the unique invariant measure of this Markov process is the Poisson-Dirichlet law. We would also like to mention the work of Pitman [21] on a closely related split and merge transformation of partitions of $(0,1)$ as well as Durrett and Limic [12] on another fragmentation-coalescence process of $(0,1)$ and its equilibrium behavior. However, a common characteristic of all these models is that they only allow for binary splittings (a fragment that splits creates exactly two new fragments) and pairwise coalescences. Furthermore the rate at which a fragment splits or merges depends on its size and on the size of the other fragments.

Here, we will focus on a rather different class of coagulation-fragmentations that can be deemed *exchangeable* or *homogeneous*. More precisely, this paper deals with processes which describe the evolution of a countable collection of masses which results from the splitting of an initial object of unit mass. Each fragment can split into a countable, possibly finite, collection of sub-fragments and each collection of fragments can merge. One can have simultaneously infinitely many clusters that merge, each of them containing infinitely many masses.

We will require some homogeneity property in the sense that the rate at which fragments split or clusters merge does not depend on the fragment sizes or any other characteristic and is not time dependent.

Loosely speaking, such processes are obtained by combining the semigroups of a homogenous fragmentation and of an exchangeable coalescent. Exchangeable coalescents, or rather Ξ-coalescents, were introduced independently by Schweinsberg [23] [2] and by Möhle and Sagitov [19] who obtained them by taking the limits of scaled ancestral processes in a population model with exchangeable family sizes. Homogeneous fragmentations were introduced and studied by Bertoin [6, 7, 8].

The paper is organized as follows. Precise definitions and first properties are given in Section 3. Next, we prove that there is always a unique stationary probability measure for these processes and we study some of their properties. Section 5 is dedicated to the study of the paths of exchangeable fragmentation-coalescence processes.

The formalism used here and part of the following material owe much to a work in preparation by Bertoin based on a series of lectures given at the IHP in 2003, [5].

2 Preliminaries

Although the most natural state space for processes such as fragmentation or coalescence might be the space of all possible ordered sequence of masses of fragments

$$\mathcal{S}^{\downarrow} = \{1 \geq x_1, \geq x_2 \geq ... \geq 0, \sum_i x_i \leq 1\},$$

[2] Schweinsberg was extending the work of Pitman [20] who treated a particular case, the so-called Λ-coalescent in which when a coalescence occurs, the involved fragments always merge into a single cluster.

as in the case of pure fragmentation or pure coalescence, we prefer to work with the space $\mathcal{P}$ of partitions of $\mathbb{N}$. An element π of $\mathcal{P}$ can be identified with an infinite collection of blocks (where a block is just a subset of $\mathbb{N}$ and can be the empty set) $\pi = (B_1, B_2, ...)$ where $\cup_i B_i = \mathbb{N}$, $B_i \cap B_j = \emptyset$ when $i \neq j$ and the labelling corresponds to the order of the least element, i.e., if w_i is the least element of B_i (with the convention $\min \emptyset = \infty$) then $i \leq j \Rightarrow w_i \leq w_j$. The reason for such a choice is that we can discretize the processes by looking at their restrictions to $[n] := \{1, ..., n\}$.

As usual, an element $\pi \in \mathcal{P}$ can be identified with an equivalence relation by setting

$$i \overset{\pi}{\sim} j \Leftrightarrow i \text{ and } j \text{ are in the same block of } \pi.$$

Let $B \subseteq B' \subseteq \mathbb{N}$ be two subsets of $\mathbb{N}$, then a partition π' of B' naturally defines a partition $\pi = \pi'_{|B}$ on B by taking $\forall i, j \in B, i \overset{\pi}{\sim} j \Leftrightarrow i \overset{\pi'}{\sim} j$, or otherwise said, if $\pi' = (B'_1, B'_2, ...)$ then $\pi = (B'_1 \cap B, B'_2 \cap B, ...)$ and the blocks are relabelled.

Let $\mathcal{P}_n$ be the set of partitions of $[n]$. For an element π of $\mathcal{P}$ the restriction of π to $[n]$ is $\pi_{|[n]}$ and we identify each $\pi \in \mathcal{P}$ with the sequence $(\pi_{|[1]}, \pi_{|[2]}, ...) \in \mathcal{P}_1 \times \mathcal{P}_2 \times ....$ We endow $\mathcal{P}$ with the distance

$$d(\pi^1, \pi^2) = 1/\max\{n \in \mathbb{N} : \pi^1_{|[n]} = \pi^2_{|[n]}\}.$$

The space $(\mathcal{P}, d)$ is then compact. In this setting it is clear that if a family $(\Pi^{(n)})_{n \in \mathbb{N}}$ of $\mathcal{P}_n$-valued random variable is *compatible*, i.e., if for each n

$$\Pi^{(n+1)}_{|[n]} = \Pi^{(n)} \text{ a.s.},$$

then, almost surely, the family $(\Pi^{(n)})_{n \in \mathbb{N}}$ uniquely determines a $\mathcal{P}$-valued variable Π such that for each n one has

$$\Pi_{|[n]} = \Pi^{(n)}.$$

Thus we may define the exchangeable fragmentation-coalescence processes by their restrictions to $[n]$.

Let us now define deterministic notions which will play a crucial role in the forthcoming constructions. We define two operators on $\mathcal{P}$, a coagulation operator, $\pi, \pi' \in \mathcal{P} \mapsto \mathit{Coag}(\pi, \pi')$ (the coagulation of π by π') and a fragmentation operator $\pi, \pi' \in \mathcal{P}, k \in \mathbb{N} \mapsto \mathit{Frag}(\pi, \pi', k)$ (the fragmentation of the k-th block of π by π').

- Take $\pi = (B_1, B_2, ...)$ and $\pi' = (B'_1, B'_2, ...)$. Then $\mathit{Coag}(\pi, \pi') = (B''_1, B''_2, ...)$, where $B''_1 = \cup_{i \in B'_1} B_i, B''_2 = \cup_{i \in B'_2} B_i, ...$ Observe that the labelling is consistent with our convention.

- Take $\pi = (B_1, B_2, ...)$ and $\pi' = (B'_1, B'_2, ...)$. Then, for $k \leq \#\pi$, where $\#\pi$ is the number of non-empty blocks of π, the partition $Frag(\pi, \pi', k)$ is the relabelled collection of blocks formed by all the B_i for $i \neq k$, plus the sub-blocks of B_k given by $\pi'_{|B_k}$.

Similarly, when $\pi \in \mathcal{P}_n$ and $\pi' \in \mathcal{P}$ or $\pi' \in \mathcal{P}_k$ for $k \geq \#\pi$ one can define $Coag(\pi, \pi')$ as above and when $\pi' \in \mathcal{P}$ or $\pi' \in \mathcal{P}_m$ for $m \geq \mathrm{Card}(B_k)$ (and $k \leq \#\pi$) one can define $Frag(\pi, \pi', k)$ as above.

Define $\mathbf{0} := (\{1\}, \{2\}, ...)$ the partition of $\mathbb{N}$ into singletons, $\mathbf{0}_n := \mathbf{0}_{|[n]}$, and $\mathbf{1} := (\{1, 2, ...\})$ the trivial partition of $\mathbb{N}$ in a single block, $\mathbf{1}_n := \mathbf{1}_{|[n]}$. Then $\mathbf{0}$ is the neutral element for *Coag*, i.e., for each $\pi \in \mathcal{P}$

$$Coag(\pi, \mathbf{0}) = Coag(\mathbf{0}, \pi) = \pi,$$

(for $\pi \in \cup_{n \geq 2} \mathcal{P}_n$, as $Coag(\mathbf{0}, \pi)$ is not defined, one only has $Coag(\pi, \mathbf{0}) = \pi$) and $\mathbf{1}$ is the neutral element for *Frag*, i.e., for each $\pi \in \mathcal{P}$ one has

$$Frag(\mathbf{1}, \pi, 1) = Frag(\pi, \mathbf{1}, k) = \pi.$$

Similarly, when $\pi \in \cup_{n \geq 2} \mathcal{P}_n$, for each $k \leq \#\pi$ one only has

$$Frag(\pi, \mathbf{1}, k) = \pi.$$

Note also that the coagulation and fragmentation operators are not really reciprocal because *Frag* can only split one block at a time.

Much of the power of working in $\mathcal{P}$ instead of $\mathcal{S}^{\downarrow}$ comes from Kingman's theory of exchangeable partitions. For the time being, let us just recall the basic definition. Define the action of a permutation $\sigma : \mathbb{N} \mapsto \mathbb{N}$ on $\mathcal{P}$ by

$$i \overset{\sigma(\pi)}{\sim} j \Leftrightarrow \sigma(i) \overset{\pi}{\sim} \sigma(j).$$

A random element Π of $\mathcal{P}$ or a $\mathcal{P}$ valued process $\Pi(\cdot)$ is said to be exchangeable if for any permutation σ such that $\sigma(n) = n$ for all large enough n one has $\sigma(\Pi) \overset{d}{=} \Pi$ or $\Pi(\cdot) \overset{d}{=} \sigma(\Pi(\cdot))$.

3 Definition, characterization and construction of EFC processes

3.1 Definition and characterization

We can now define precisely the exchangeable fragmentation-coalescence processes and state some of their properties. Most of the following material is very close to the analogous definitions and arguments for pure fragmentations (see [6]) and coalescences (see [20, 23]).

Definition 1. *A $\mathcal{P}$-valued Markov process $(\Pi(t), t \geq 0)$, is an exchangeable fragmentation-coalescent process ("EFC process" thereafter) if it has the following properties:*

- *It is exchangeable.*
- *Its restrictions $\Pi_{|[n]}$ are càdlàg finite state Markov chains which can only evolve by fragmentation of one block or by coagulation.*

More precisely, the transition rate of $\Pi_{|[n]}(\cdot)$ from π to π', say $q_n(\pi, \pi')$, is non-zero only if $\exists \pi''$ such that $\pi' = Coag(\pi, \pi'')$ or $\exists \pi'', k \geq 1$ such that $\pi' = Frag(\pi, \pi'', k)$.

Observe that this definition implies that $\Pi(0)$ should be exchangeable. Hence the only possible deterministic starting points are $\mathbf{1}$ and $\mathbf{0}$ because the measures $\delta_{\mathbf{1}}(\cdot)$ and $\delta_{\mathbf{0}}(\cdot)$ (where $\delta_{\bullet}(\cdot)$ is the Dirac mass in $\bullet$) are the only exchangeable measures of the form $\delta_\pi(\cdot)$. If $\Pi(0) = \mathbf{0}$ we say that the process is started from dust, and if $\Pi(0) = \mathbf{1}$ we say it is started from unit mass.

Note that the condition that the restrictions $\Pi_{|[n]}$ are càdlàg implies that Π itself is also càdlàg.

Fix n and $\pi \in \mathcal{P}_n$. For convenience we will also use the following notations for the transition rates: For $\pi' \in \mathcal{P}_m \backslash \{\mathbf{0}_m\}$ where $m = \#\pi$ the number of non-empty blocks of π, call

$$C_n(\pi, \pi') := q_n(\pi, Coag(\pi, \pi'))$$

the rate of coagulation by π'. For $k \leq \#\pi$ and $\pi' \in \mathcal{P}_{|B_k|} \backslash \{\mathbf{1}_{|B_k|}\}$ where $|B_k|$ is the cardinality of the k-th block, call

$$F_n(\pi, \pi', k) := q_n(\pi, Frag(\pi, \pi', k))$$

the rate of fragmentation of the kth block by π'.

We will say that an EFC process is non-degenerated if it has both a fragmentation and coalescence component, i.e., for each n there are some $\pi'_1 \neq \mathbf{1}_n$ and $\pi'_2 \neq \mathbf{0}_n$ such that $F_n(\mathbf{1}_n, \pi'_1, 1) > 0$ and $C_n(\mathbf{0}_n, \pi'_2) > 0$.

Of course the compatibility of the $\Pi_{|[m]}$ and the exchangeability requirement entail that not every family of transition rates is admissible. In fact, it is enough to know how $\Pi_{|[m]}$ leaves $\mathbf{1}_m$ and $\mathbf{0}_m$ for every $m \leq n$ to know all the rates $q_n(\pi, \pi')$.

Proposition 2. *There exist two families $((C_n(\pi))_{\pi \in \mathcal{P}_n \backslash \{\mathbf{0}_n\}})_{n \in \mathbb{N}}$ and $((F_n(\pi))_{\pi \in \mathcal{P}_n \backslash \{\mathbf{1}_n\}})_{n \in \mathbb{N}}$ such that for every $m \leq n$ and for every $\pi \in \mathcal{P}_n$ with m blocks ($\#\pi = m$) one has*

1. *For each* $\pi' \in \mathcal{P}_m \backslash \{\mathbf{0}_m\}$
$$q_n(\pi, Coag(\pi, \pi')) = C_n(\pi, \pi') = C_m(\pi').$$
2. *For each* $k \leq m$ *and for each* $\pi' \in \mathcal{P}_{|B_k|} \backslash \{\mathbf{1}_{|B_k|}\}$,
$$q_n(\pi, Frag(\pi, \pi', k)) = F_n(\pi, \pi', k) = F_{|B_k|}(\pi').$$
3. *All other transition rates are zero.*

Furthermore, these rates are exchangeable, i.e., for any permutation σ *of* $[n]$*, for all* $\pi \in \mathcal{P}_n$ *one has* $C_n(\pi) = C_n(\sigma(\pi))$ *and* $F_n(\pi) = F_n(\sigma(\pi))$.

As the proof of this result is close to the arguments used for pure fragmentation or pure coalescence and is rather technical, we postpone it until section 6.

Observe that, for n fixed, the finite families $(C_n(\pi))_{\pi \in \mathcal{P}_n \backslash \{\mathbf{0}_n\}}$ and $(F_n(\pi))_{\pi \in \mathcal{P}_n \backslash \{\mathbf{1}_n\}}$ may be seen as measures on $\mathcal{P}_n$. The compatibility of the $\Pi_{|[n]}(\cdot)$ implies the same property for the (C_n, F_n), i.e., as measures, the image of C_{n+1} (resp. F_{n+1}) by the projection $\mathcal{P}_{n+1} \mapsto \mathcal{P}_n$ is C_n (resp. F_n), see Lemma 1 in [6] for a precise demonstration in the case where there is only fragmentation ($C \equiv 0$), the general case being a simple extension. Hence, by Kolmogorov's extension Theorem, there exists a unique measure C and a unique measure F on $\mathcal{P}$ such that for each n and for each $\pi \in \mathcal{P}_n$ such that $\pi \neq \mathbf{1}_n$ (resp. $\pi \neq \mathbf{0}_n$)
$$C_n(\pi) = C(\{\pi' \in \mathcal{P} : \pi'_{|[n]} = \pi\}) \text{ resp. } F_n(\pi) = F(\{\pi' \in \mathcal{P} : \pi'_{|[n]} = \pi\}).$$

Furthermore, as we have observed, the measures C_n and F_n are exchangeable. Hence, C and F are exchangeable measures. They must also satisfy some integrability conditions because the $\Pi_{|[n]}(\cdot)$ are Markov chains and have thus a finite jump rate at any state. For $\pi \in \mathcal{P}$ define $Q(\pi, n) := \{\pi' \in \mathcal{P} : \pi'_{|[n]} = \pi_{|[n]}\}$. Then for each $n \in \mathbb{N}$ we must have
$$C(\mathcal{P} \backslash Q(\mathbf{0}, n)) < \infty \tag{1}$$
and
$$F(\mathcal{P} \backslash Q(\mathbf{1}, n)) < \infty. \tag{2}$$
It is clear that we can suppose without loss of generality that C and F assign no mass to the respective neutral elements for *Coag* and *Frag*, i.e., $C(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 0$.

Here are three simple examples of exchangeable measures.

1. Let ϵ_n the partition that has only two non empty blocks: $\mathbb{N}\backslash\{n\}$ and $\{n\}$. Then the (infinite) measure $\mathbf{e}(\cdot) = \sum_{n\in\mathbb{N}} \delta_{\epsilon_n}(\cdot)$ (where δ is the Dirac mass) is exchangeable. We call it the erosion measure .

2. For each $i \neq j \in \mathbb{N}$, call $\epsilon_{i,j}$ the partition that has only one block which is not a singleton: $\{i, j\}$. Then the (infinite) measure $\kappa(\cdot) = \sum_{i<j\in\mathbb{N}} \delta_{\epsilon_{i,j}}(\cdot)$ is exchangeable. We call it the Kingman measure.

3. Take $x \in \mathcal{S}^{\downarrow} := \{x_1 \geq x_2 \geq ... \geq 0; \sum_i x_i \leq 1\}$. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent variables with respective law given by $P(X_i = k) = x_k$ for all $k \geq 1$ and $P(X_i = -i) = 1 - \sum_j x_j$. Define a random variable π with value in $\mathcal{P}$ by letting $i \overset{\pi}{\sim} j \Leftrightarrow X_i = X_j$. Following Kingman, we call π the x-paintbox process and denote by μ_x its distribution. Let ν be a measure on $\mathcal{S}^{\downarrow}$, then the mixture μ_ν of paintbox processes directed by ν, i.e., for $A \subseteq \mathcal{P}$
$$\mu_\nu(A) = \int_{\mathcal{S}^{\downarrow}} \mu_x(A)\nu(dx),$$
is an exchangeable measure. We call it the ν-paintbox measure.

Extending seminal results of Kingman [17], Bertoin has shown in [6] and in [5] that any exchangeable measure on $\mathcal{P}$ that satisfies (1) (respectively (2)) is a combination of κ and a ν-paintbox measure (resp. $\mathbf{e}$ and a ν-paintbox process). Hence the following proposition merely restates these results.

Proposition 3. *For each exchangeable measure C on $\mathcal{P}$ such that $C(\{\mathbf{0}\}) = 0$, and $C(\mathcal{P}\backslash Q(\mathbf{0}, n)) < \infty, \forall n \in \mathbb{N}$ there exists a unique $c_k \geq 0$ and a unique measure ν_{Coag} on $\mathcal{S}^{\downarrow}$ such that*
$$\begin{gathered}\nu_{Coag}(\{(0,0,...)\}) = 0,\\ \int_{\mathcal{S}^{\downarrow}} \left(\sum_{i=1}^{\infty} x_i^2\right) \nu_{Coag}(dx) < \infty, \\ \text{and } C = c_k\kappa + \mu_{\nu_{Coag}}.\end{gathered} \tag{3}$$

For each exchangeable measure F on $\mathcal{P}$ such that $F(\{\mathbf{1}\}) = 0$ and $F(\mathcal{P}\backslash Q(\mathbf{1}, n)) < \infty, \forall n \in \mathbb{N}$ there exists a unique $c_e \geq 0$ and a unique measure ν_{Disl} on $\mathcal{S}^{\downarrow}$ such that
$$\begin{gathered}\nu_{Disl}(\{(1,0,...)\}) = 0,\\ \int_{\mathcal{S}^{\downarrow}} \left(1 - \sum_{i=1}^{\infty} x_i^2\right) \nu_{Disl}(dx) < \infty, \\ \text{and } F = c_e\mathbf{e} + \mu_{\nu_{Disl}}.\end{gathered} \tag{4}$$

The two integrability conditions on ν_{Disl} and ν_{Coag} (4) and (3) ensure that $C(\mathcal{P}\backslash Q(\mathbf{0},n)) < \infty$ and $F(\mathcal{P}\backslash Q(\mathbf{1},n)) < \infty$. See [6] for the demonstration concerning F. The part that concerns C can be shown by the same arguments.

The condition on ν_{Disl} (4) may seem at first sight different from the condition that Bertoin imposes in [6] and which reads

$$\int_{\mathcal{S}^{\downarrow}} (1-x_1)\nu_{Disl}(dx) < \infty$$

but they are in fact equivalent because

$$1 - \sum_i x_i^2 \leq 1 - x_1^2 \leq 2(1-x_1)$$

and on the other hand

$$1 - \sum_i x_i^2 \geq 1 - x_1 \sum_i x_i \geq 1 - x_1.$$

Thus the above proposition implies that for each EFC process Π there is a unique (in law) exchangeable fragmentation $\Pi^{(F)}(t)$ -the fragmentation whose law is characterized by the measure F- and a unique (in law) exchangeable coalescence $\Pi^{(C)}(t)$ -the coalescent whose law is characterized by the measure C- such that Π is a combination of $\Pi^{(F)}$ and $\Pi^{(C)}$ in the sense that its transition rates respectively in the coalescence and the fragmentation sense are the same as those of $\Pi^{(F)}$ and $\Pi^{(C)}$. This was not obvious a priori because some kind of compensation phenomenon could have allowed weaker integrability conditions.

One can sum up the preceding analysis in the following characterization of exchangeable fragmentation-coalescence processes.

Proposition 4. *The distribution of an EFC process $\Pi(\cdot)$ is completely characterized by the initial condition (i.e., the law of $\Pi(0)$), the measures ν_{Disl} and ν_{Coag} as above and the parameters $c_e, c_k \in \mathbb{R}+$.*

Remark : The above results are well known for pure fragmentation or pure coalescence. If, for instance, we impose $F(\mathcal{P}) = 0$ (i.e., there is only coalescence and no fragmentation, the EFC process is degenerated), the above proposition shows that our definition agrees with Definition 3 in Schweinsberg [23]. On the other hand if there is only fragmentation and no coalescence, our definition is equivalent to that given by Bertoin in [6], which relies on some fundamental properties of the semi-group. There, the Markov chain property of the restrictions is deduced from the definition as well as the characterization of the distribution by c and ν_{Disl}.

Nevertheless, the formulation of Definition 1 is new. More precisely, for pure fragmentations, Definition 1 only requires that the process Π and its restrictions should be Markov and exchangeable and furthermore that only one block can fragmentate at a time. Point 2 of Proposition 2 then implies that Π has the fragmentation and homogeneity properties. We say that Π has the fragmentation property if each fragment evolves independently of the past and of the other fragments. This is obvious from the observation that the splitting rates in Proposition 2 of the blocks of $\Pi_{|[n]}$ only depend on their size. The fact that all transition rates can be expressed in terms of $F_n(\pi)$ -the rates at which $\mathbf{1}_n$ splits- implies the homogeneity property, i.e., each fragment splits according to a fragmentation which has the same law as the original one, up to the scale-factor. In [6] homogeneous fragmentations are rather *defined* as exchangeable Markov processes whose semi-group have the fragmentation and homogeneity properties.

3.2 Poissonian construction

As for exchangeable fragmentation or coalescence, one can construct EFC processes by using Poisson point processes (PPP in the following). More precisely let $P_C = ((t, \pi^{(C)}(t)), t \geq 0)$ and $P_F = ((t, \pi^{(F)}(t), k(t)), t \geq 0)$ be two independent PPP in the same filtration. The atoms of the PPP P_C are points in $\mathbb{R}^+ \times \mathcal{P}$ and its intensity measure is given by $dt \otimes (\mu_{\nu_{Coag}} + c_k \kappa)$. The atoms of P_F are points in $\mathbb{R}^+ \times \mathcal{P} \times \mathbb{N}$ and its intensity measure is $dt \otimes (c_e \mathbf{e} + \mu_{\nu_{Disl}}) \otimes \#$ where $\#$ is the counting measure on $\mathbb{N}$ and dt is the Lebesgue measure.

Take $\pi \in \mathcal{P}$ an exchangeable random variable and define a family of $\mathcal{P}_n$-valued processes $\Pi^n(\cdot)$ as follows: for each n fix $\Pi^n(0) = \pi_{|[n]}$ and

- if t is not an atom time neither for P_C or P_F then $\Pi^n(t) = \Pi^n(t-)$,
- if t is an atom time for P_C such that $(\pi^{(C)}(t))_{|[n]} \neq \mathbf{0}_n$ then

$$\Pi^n(t) = Coag(\Pi^n(t-), \pi^{(C)}(t)),$$

- if t is an atom time for P_F such that $k(t) < n$ and $(\pi^{(F)}(t))_{|[n]} \neq \mathbf{1}_n$ then

$$\Pi^n(t) = Frag(\Pi^n(t-), \pi^{(F)}(t), k(t)).$$

Note that the Π^n are well defined because on any finite time interval, for each n, one only needs to consider a finite number of atoms. Furthermore P_C and P_F being independent in the same filtration, almost surely there is no t which is an atom time for both PPP's. This family is constructed to be compatible and thus defines uniquely a process Π such that $\Pi_{|[n]} = \Pi^n$ for each n. By analogy with homogeneous fragmentations ([6]) and exchangeable coalescence ([20, 23]) the following should be clear.

Proposition 5. *The process* Π *constructed above is an EFC process with characteristics* c_k, ν_{Coag}, c_e *and* ν_{Disl}.

Proof. It is straightforward to check that the restrictions $\Pi_{|[n]}(t)$ are Markov chains whose only jumps are either coagulations or fragmentations. The transition rates are constructed to correspond to the characteristics c_k, ν_{Coag}, c_e and ν_{Disl}. The only thing left to check is thus exchangeability. Fix $n \in \mathbb{N}$ and σ a permutation of $[n]$, then $(\sigma(\Pi^n(t)))_{t\geq 0}$ is a jump-hold Markov process. Its transition rates are given by $q_n^{(\sigma)}(\pi, \pi') = q_n(\sigma^{-1}(\pi), \sigma^{-1}(\pi'))$.

Suppose first that $\pi' = Frag(\pi, \pi'', k)$ for some π''. Note that there exists a unique $l \leq \#\pi$ and a permutation σ' of $[m]$ (where $m = |B_k|$ is the cardinality of the k-th block of π we want to split) such that

$$\sigma^{-1}(\pi') = Frag(\sigma^{-1}(\pi), \sigma'(\pi''), l).$$

Using Proposition 2 we then obtain that

$$\begin{aligned} q_n^{(\sigma)}(\pi, \pi') &= q_n(\sigma^{-1}(\pi), \sigma^{-1}(\pi')) \\ &= q_n(\sigma^{-1}(\pi), Frag(\sigma^{-1}(\pi), \sigma'(\pi''), l)) \\ &= F_m(\sigma'(\pi'')) \\ &= F_m(\pi'') \\ &= q_n(\pi, \pi') \end{aligned}$$

The same type of arguments show that when $\pi' = Coag(\pi, \pi'')$ for some π'' we also have

$$q_n^{(\sigma)}(\pi, \pi') = q_n(\pi, \pi').$$

Thus, Π^n and $\sigma(\Pi^n)$ have the same transition rates and hence the same law.

As this is true for all n, it entails that Π and $\sigma(\Pi)$ also have the same law.

$\square$

Let $\Pi(\cdot)$ be an EFC process and define P_t as its semi-group, i.e., for a continuous function $\phi : \mathcal{P} \mapsto \mathbb{R}$

$$P_t\phi(\pi) := \mathbf{E}_\pi(\phi(\Pi(t)))$$

the expectation of $\phi(\Pi(t))$ conditionally on $\Pi(0) = \pi$.

Corollary 6. *An EFC process $\Pi(\cdot)$ has the Feller property, i.e., for each continuous function $\phi : \mathcal{P} \mapsto \mathbb{R}$,*

- *for each $\pi \in \mathcal{P}$ one has*

$$\lim_{t\to 0+} P_t\phi(\pi) = \phi(\pi),$$

- *for all $t > 0$ the function $\pi \mapsto P_t\phi(\pi)$ is continuous.*

Proof. Call C_f the set of functions

$$C_f = \{f : \mathcal{P} \mapsto \mathbb{R} : \exists n \in \mathbb{N} \text{ s.t. } \pi_{|[n]} = \pi'_{|[n]} \Rightarrow f(\pi) = f(\pi')\}$$

which is dense in the space of continuous functions of $\mathcal{P} \mapsto \mathbb{R}$. The first point is clear for a function $\Phi \in C_f$ (because the first jump-time of $\Phi(\Pi(\cdot))$ is distributed as an exponential variable with finite mean). We conclude by density. For the second point, consider $\pi, \pi' \in \mathcal{P}$ such that $d(\pi, \pi') < 1/n$ (i.e., $\pi_{|[n]} = \pi'_{|[n]}$) then use the same PPP P_C and P_F to construct two EFC processes, $\Pi(\cdot)$ and $\Pi'(\cdot)$, with respective starting points $\Pi(0) = \pi$ and $\Pi'(0) = \pi'$. By construction $\Pi_{|[n]}(\cdot) = \Pi'_{|[n]}(\cdot)$ in the sense of the identity of the paths. Hence

$$\forall t \geq 0, d(\Pi(t), \Pi'(t)) < 1/n.$$

□

Hence, when considering an EFC process, one can always suppose that one works in the usual augmentation of the natural filtration $\mathcal{F}_t$ which is then right continuous.

As a direct consequence, one also has the following characterization of EFC's in terms of the infinitesimal generator : Let $(\Pi(t), t \geq 0)$ be an EFC process, then the infinitesimal generator of Π, denoted by $\mathcal{A}$, acts on the functions $f \in C_f$ as follows:

$$\begin{aligned}\forall \pi \in \mathcal{P}, \mathcal{A}(f)(\pi) = \quad & \int_{\mathcal{P}} C(d\pi')(f(Coag(\pi, \pi')) - f(\pi)) \\ & + \sum_{k\in\mathbb{N}} \int_{\mathcal{P}} F(d\pi')(f(Frag(\pi, \pi', k)) - f(\pi)),\end{aligned}$$

where $F = c_e \mathbf{e} + \mu_{\nu_{Disl}}$ and $C = c_k \kappa + \mu_{\nu_{Coag}}$. Indeed, take $f \in C_f$ and n such that $\pi_{|[n]} = \pi'_{|[n]} \Rightarrow f(\pi) = f(\pi')$, then as $\Pi_{|[n]}(\cdot)$ is a Markov chain the above formula is just the usual generator for Markov chains. Transition rates have thus the required properties and hence this property characterizes EFC processes.

3.3 Asymptotic frequencies

When A is a subset of $\mathbb{N}$ we will write

$$\bar{\lambda}_A = \limsup_{n \to \infty} \frac{\#\{k \leq n : k \in A\}}{n}$$

and

$$\underline{\lambda}_A = \liminf_{n \to \infty} \frac{\#\{k \leq n : k \in A\}}{n}.$$

When the equality $\bar{\lambda}_A = \underline{\lambda}_A$ holds we call $\|A\|$, the asymptotic frequency of A, the common value which is also the limit

$$\|A\| = \lim_{n \to \infty} \frac{\#\{k \leq n : k \in A\}}{n}.$$

If all the blocks of $\pi = (B_1, B_2, ...) \in \mathcal{P}$ have an asymptotic frequency we define

$$\Lambda(\pi) = (\|B_1\|, \|B_2\|, ...)^{\downarrow}$$

the decreasing rearrangement of the $\|B_i\|$'s.

Theorem 7. *Let $\Pi(t)$ be an EFC process. Then*

$$X(t) = \Lambda(\Pi(t))$$

exists almost surely simultaneously for all $t \geq 0$, and $(X(t), t \geq 0)$ is a Feller process.

The proof (see section 6), which is rather technical, uses the regularity properties of EFC processes and the existence of asymptotic frequencies simultaneously for all rational times $t \in \mathbb{Q}$. We call the process $X(t)$ the associated ranked-mass EFC process.

Remark The state space of a ranked mass EFC process X is $\mathcal{S}^{\downarrow}$. Thus, our construction of EFC processes Π in $\mathcal{P}$ started from $\mathbf{0}$ gives us an entrance law $Q_{(0,0,...)}$ for X. More precisely, call $Q_{(0,0,...)}(t)$ the law of $X(t)$ conditionally on $X(0) = (0, 0, ...)$. Then, for all $t \geq 0$, there is the identity

$Q_{(0,0,\ldots)}(t) = \Lambda(P_{\mathbf{0}}(t))$ where $\Lambda(P_{\mathbf{0}}(t))$ is the image of $P_{\mathbf{0}}(t)$, the distribution of $\Pi(t)$ conditionally on $\Pi(0) = \mathbf{0}$, by the map Λ. The ranked frequencies of an EFC process started from $\mathbf{0}$ defines a process with this entrance law that comes from dust at time 0+, i.e., the largest mass vanishes almost surely as $t \searrow 0$. The construction of this entrance law is well known for pure coalescence processes, see Pitman [20] for a general treatment, but also Kingman [16] and Bolthausen-Sznitman [10, Corollary 2.3] for particular cases.

4 Equilibrium measures

Consider an EFC process Π which is not trivial, i.e., $\nu_{Coag}, \nu_{Disl}, c_e$ and c_k are not zero simultaneously.

Theorem 8. *There exists a unique (exchangeable) stationary probability measure ρ on $\mathcal{P}$ and one has*

$$\rho = \delta_{\mathbf{0}} \Leftrightarrow c_k = 0 \text{ and } \nu_{Coag} \equiv 0$$

and

$$\rho = \delta_{\mathbf{1}} \Leftrightarrow c_e = 0 \text{ and } \nu_{Disl} \equiv 0$$

where δ_π is the Dirac mass at π.

Furthermore, $\Pi(\cdot)$ converges in distribution to ρ as $t \to \infty$.

Proof. If the process Π is a pure coalescence process (i.e., $c_e = 0$ and $\nu_{Disl}(\cdot) \equiv 0$) it is clear that $\mathbf{1}$ is an absorbing state towards which the process converges almost surely. In the pure fragmentation case it is $\mathbf{0}$ that is absorbing and attracting.

In the non-degenerated case, for each $n \in \mathbb{N}$, the process $\Pi_{|[n]}(\cdot)$ is a finite state Markov chain. Let us now check the irreducibility in the non-degenerated case. Suppose first that $\nu_{Disl}(\mathcal{S}^\downarrow) > 0$. For every state $\pi \in \mathcal{P}_n$, if $\Pi_{|[n]}(t) = \pi$ there is a positive probability that the next jump of $\Pi_{|[n]}(t)$ is a coalescence. Hence, for every starting point $\Pi_{|[n]}(0) = \pi \in \mathcal{P}_n$ there is a positive probability that $\Pi_{|[n]}(\cdot)$ reaches $\mathbf{1}_n$ in finite time T before any fragmentation has occurred. Now take $x \in \mathcal{S}^\downarrow$ such that $x_2 > 0$ and recall that μ_x is the x-paintbox distribution. Then for every $\pi \in \mathcal{P}_n$ with $\#\pi = 2$ (recall that $\#\pi$ is the number of non-empty blocks of π) one has

$$\mu_x(Q(\pi, n)) > 0.$$

That is the n-restriction of the x-paintbox partition can be any partition of $[n]$ into two blocks with positive probability. More precisely if $\pi \in \mathcal{P}_n$ is such that $\pi = (B_1, B_2, \emptyset, \emptyset ...)$ with $|B_1| = k$ and $|B_2| = n-k$ then

$$\mu_x(Q(\pi, n)) \geq x_1^k x_2^{n-k} + x_2^k x_1^{n-k}.$$

Hence, for any $\pi \in \mathcal{P}$ with $\#\pi = 2$, the first transition after T is $\mathbf{1}_n \to \pi$ with positive probability. As any $\pi \in \mathcal{P}_n$ can be obtained from $\mathbf{1}_n$ by a finite series of binary fragmentations we can iterate the above idea to see that with positive probability the jumps that follow T are exactly the sequence of binary splitting needed to get to π and the chain is hence irreducible.

Suppose now that $\nu_{Disl} \equiv 0$, there is only erosion $c_e > 0$, and that at least one of the following two conditions holds

- for every $k \in \mathbb{N}$ one has $\nu_{Coag}(\{x \in \mathcal{S}^{\downarrow} : \sum_{i=1}^{i=k} x_i < 1\}) > 0$,
- there is a Kingman component, $c_k > 0$,

then almost the same proof applies. We first show that the state $\mathbf{0}_n$ can be reached from any starting point by a series of splittings corresponding to erosion, and that from there any $\pi \in \mathcal{P}_n$ is reachable through binary coagulations.

In the remaining case (i.e., $c_k = 0, \nu_{Disl} \equiv 0$ and there exists $k > 0$ such that $\nu_{Coag}(\{x \in \mathcal{S}^{\downarrow} : \sum_{i=1}^{i=k} x_i < 1\}) = 0$) the situation is slightly different in that $\mathcal{P}_n$ is not the irreducible class. It is easily seen that the only partitions reachable from $\mathbf{0}_n$ are those with at most k non-singletons blocks. But for every starting point π one reaches this class in finite time almost surely. Hence there is no issues with the existence of an invariant measure for this type of $\Pi_{|[n]}$, it just does not charge partitions outside this class.

Thus there exists a unique stationary probability measure $\rho^{(n)}$ on $\mathcal{P}_n$ for the process $\Pi_{|[n]}$. Clearly by compatibility of the $\Pi_{|[n]}(\cdot)$ one must have

$$\text{Proj}_{\mathcal{P}_n}(\rho^{(n+1)})(\cdot) = \rho^{(n)}(\cdot)$$

where $\text{Proj}_{\mathcal{P}_n}(\rho^{(n+1)})$ is the image of $\rho^{(n+1)}$ by the projection on $\mathcal{P}_n$. This implies that there exists a unique probability measure ρ on $\mathcal{P}$ such that for each n one has $\rho^{(n)}(\cdot) = \text{Proj}_{\mathcal{P}_n}(\rho)(\cdot)$. The exchangeability of ρ is a simple consequence of the exchangeability of Π. Finally, the chain $\Pi_{|[2]}(\cdot)$ is specified by two transition rates $\{1\}\{2\} \to \{1,2\}$ and $\{1,2\} \to \{1\}\{2\}$, which are both non-zero as soon as the EFC is non-degenerated. Hence,

$$\text{Proj}_{\mathcal{P}_2}(\rho)(\cdot) \notin \{\delta_{\mathbf{1}_2}(\cdot), \delta_{\mathbf{0}_2}(\cdot)\}.$$

Hence, when we have both coalescence and fragmentation $\rho \notin \{\delta_{\mathbf{1}}, \delta_{\mathbf{0}}\}$.

The $\Pi_{|[n]}(\cdot)$ being finite state Markov chains, it is well known that they converge in distribution to $\rho^{(n)}$, independently of the initial state. By definition of the distribution of Π this implies that $\Pi(\cdot)$ converges in distribution to ρ.

$\square$

Although we cannot give an explicit expression for ρ in terms of c_k, ν_{Coag}, c_e and ν_{Disl}, we now relate certain properties of ρ to these parameters. In particular we will ask ourselves the following two natural questions:

- under what conditions does ρ charge only partitions with an infinite number of blocks, resp. a finite number of blocks, resp. both ?
- under what conditions does ρ charge partitions with dust (i.e., partitions such that $\sum_i \|B_i\| \leq 1$ where $\|B_i\|$ is the asymptotic frequency of block B_i) ?

The proofs of the results in the remainder of this section are placed in section 6.

4.1 Number of blocks

We will say that an EFC process *fragmentates quickly* if $c_e > 0$ or $\nu_{Disl}(\mathcal{S}^{\downarrow}) = \infty$. If this is not the case (i.e., $c_e = 0$ and $\nu_{Disl}(\mathcal{S}^{\downarrow}) < \infty$) we say that it *fragmentates slowly.*

We first examine whether of not ρ charges partitions with a finite number of blocks.

Theorem 9. *1. Let $\Pi(\cdot)$ be an EFC process that fragmentates quickly. Then*

$$\rho(\{\pi \in \mathcal{P} : \#\pi < \infty\}) = 0.$$

2. Let $\Pi(\cdot)$ be an EFC process that fragmentates slowly and such that

$$\nu_{Disl}(\{x \in \mathcal{S}^{\downarrow} : \forall k\ x_k > 0\}) = 0$$

and

$$\sum_k \log(k)\nu_{Disl}(\{x \in \mathcal{S}^{\downarrow} : \sum_{i=1}^{k+1} x_i = 1\}) < \infty.$$

Assume furthermore that $\nu_{Coag}(\mathcal{S}^{\downarrow}) = 0$ and $c_k > 0$, then

$$\rho(\{\pi \in \mathcal{P} : \#\pi < \infty\}) = 1.$$

Remarks :

1. The proof of the second point uses the connection with a work of Lambert [18] on which some details are given in section 5. In the case where we drop the hypothesis $\nu_{Coag}(\mathcal{S}^{\downarrow}) = 0$, although adding coalescence should reinforce the conclusion, we are only able to prove that it holds under the stronger condition
$$\nu_{Disl}(\{x \in \mathcal{S}^{\downarrow} : x_1 + x_2 < 1\}) = 0.$$
2. This implies that for an EFC process with a binary fragmentation component, a Kingman coalescence component and no erosion (i.e., $c_k > 0, c_e = 0$ and $\nu_{Disl}(\{x \in \mathcal{S}^{\downarrow} : x_1 + x_2 < 1\}) = 0$) we have the equivalence
$$\rho(\{\pi \in \mathcal{P} : \#\pi = \infty\}) = 1 \Leftrightarrow \nu_{Disl}(\mathcal{S}^{\downarrow}) = \infty$$
and when $\nu_{Disl}(\mathcal{S}^{\downarrow}) < \infty$ then $\rho(\{\pi \in \mathcal{P} : \#\pi = \infty\}) = 0$.
3. Finally, an interesting question is the case of an EFC process for which the measure ν_{Coag} satisfies the condition given in [23] for a coalescent to come down from infinity. It is not self-evident that the condition of slow fragmentation is enough to ensure that ρ only charges partitions with finitely many blocks. The reason is that even though each fragment then splits at a finite rate, as we start with an infinite number of fragments the fragmentation could fix the process at an infinite level.

4.2 Dust

For any fixed time t the partition $\Pi(t)$ is exchangeable. Hence, by Kingman's theory of exchangeable partition (see [1] for a simple proof of this result), its law is a mixture of paintbox processes. A direct consequence is that every block $B_i(t)$ of $\Pi(t)$ is either a singleton or an infinite block with strictly positive asymptotic frequency. Recall that the asymptotic frequency of a block $B_i(t)$ is given by
$$\|B_i(t)\| = \lim_{n\to\infty} \frac{1}{n}\#\{k \leq n : k \in B_i(t)\}$$
so part of Kingman's result is that this limit exists almost surely for all i simultaneously. The asymptotic frequency of a block corresponds to its

mass, thus singletons have zero mass, they form what we call dust. More precisely, for $\pi \in \mathcal{P}$ define the set

$$\text{dust}(\pi) \quad := \bigcup_{j:\|B_j\|=0} B_j.$$

When π is exchangeable we have almost surely

$$\text{dust}(\pi) = \{i \in \mathbb{N} : \exists j \text{ s.t. } \{i\} = B_j\}$$

and

$$\sum_i \|B_i\| + \|\text{dust}(\pi)\| = 1.$$

For fragmentation or EFC processes, dust can be created *via* two mechanisms: either from erosion (that's the atoms that correspond to the erosion measure $c_e\mathbf{e}$ when $c_e > 0$), or from sudden splitting which corresponds to atoms associated to the measure $\mu_{\nu'_{Disl}}$ where ν'_{Disl} is simply ν_{Disl} restricted to $\{x \in \mathcal{S}^\downarrow : \sum_i x_i < 1\}$. Conversely, in the coalescence context mass can condensate out of dust, thus giving an entrance law in $\mathcal{S}^\downarrow$, see [20].

The following theorem states that when the coalescence is strong enough in an EFC process, the equilibrium measure does not charge partitions with dust. We say that an EFC process *coalesces quickly* (resp. *slowly*) if $\int_{\mathcal{S}^\downarrow} (\sum_i x_i)\, \nu_{Coag}(dx) = \infty$ or $c_k > 0$ (resp. $\int_{\mathcal{S}^\downarrow} (\sum_i x_i)\, \nu_{Coag}(dx) < \infty$ and $c_k = 0$).

Theorem 10. *Let $(\Pi(t), t \geq 0)$ be an EFC process that coalesces quickly and ρ its invariant probability measure. Then*

$$\rho(\{\pi \in \mathcal{P} : dust(\pi) \neq \emptyset\}) = 0.$$

In case of no fragmentation, this follows from Proposition 30 in [23].

4.3 Equilibrium measure for the ranked mass EFC process

For ρ the equilibrium measure of some EFC process with characteristics $\nu_{Disl}, \nu_{Coag}, c_e$ and c_k, let θ be the image of ρ by the map $\mathcal{P} \mapsto \mathcal{S}^\downarrow : \pi \mapsto \Lambda(\pi)$.

Proposition 11. *Let X be a ranked-mass EFC process with characteristics $\nu_{Disl}, \nu_{Coag}, c_e$ and c_k. Then θ is its unique invariant probability measure.*

Proof. As for each fixed t one has

$$P_\rho(\Lambda(\Pi(t)) \in A) = \rho(\{\pi : \Lambda(\pi) \in A\}) = \theta(A)$$

it is clear that θ is an invariant probability measure.

Suppose that θ is an invariant measure for X and fix $t \geq 0$. Hence if $X(0)$ has distribution θ so does $X(t) = \Lambda(\Pi(t))$. As $\Pi(t)$ is exchangeable it is known by Kingman's theory of exchangeable partitions (see [1]) that $\Pi(t)$ has law $\mu_\theta(\cdot)$ the mixture of paintbox processes directed by θ. This implies that $\mu_\theta(\cdot)$ is invariant for Π and hence $\mu_\theta(\cdot) = \rho(\cdot)$ and thus θ is the unique invariant measure for X. □

5 Path properties

5.1 Number of blocks along the path.

One of the problems tackled by Pitman [20] and Schweinsberg [24, 23] about coalescent processes is whether or not they come down from infinity. Let us first recall some of their results. By definition if $\Pi^C(\cdot)$ is a standard coalescent $\Pi^C(0) = \mathbf{0}$ and thus $\#\Pi^C(0) = \infty$. We say that Π^C comes down from infinity if $\#\Pi^C(t) < \infty$ a.s. for all $t > 0$. We say it stays infinite if $\#\Pi^C(t) = \infty$ a.s. for all $t > 0$.

Define $\Delta_f := \{x \in \mathcal{S}^{\downarrow} : \exists i \in \mathbb{N} \text{ s.t. } \sum_{j=1}^{i} x_j = 1\}$. We know by Lemma 31 in [24], which is a generalization of Proposition 23 in [20], that if $\nu_{Coag}(\Delta_f) = 0$ the coalescent either stays infinite or comes down from infinity.

For $b \geq 2$ let λ_b denote the total rate of all collisions when the coalescent has b blocks

$$\lambda_b = \mu_{\nu_{Coag}}(\mathcal{P} \backslash Q(\mathbf{0}, b)) + c_k \frac{b(b-1)}{2}.$$

Let γ_b be the total rate at which the number of blocks is decreasing when the coalescent has b blocks,

$$\gamma_b = c_k \frac{b(b-1)}{2} + \sum_{k=1}^{b-1} (b-k) \mu_{\nu_{Coag}}(\{\pi : \#\pi_{|[b]} = k\}).$$

If $\nu_{Coag}(\Delta_f) = \infty$ or $\sum_{b=2}^{\infty} \gamma_b^{-1} < \infty$, then the coalescent comes down from infinity. The converse is not always true but holds for instance for the important case of the Λ-coalescents (i.e., those for which many fragments can merge into a single block, but only one such merger can occur simultaneously).

These kinds of properties concerns the paths of the processes, and it seems that they bear no simple relation with properties of the equilibrium measure. For instance the equilibrium measure of a coalescent that stays infinite is $\delta_{\mathbf{1}}(\cdot)$ and it therefore only charges partitions with one block, but its path lies entirely in the subspace of $\mathcal{P}$ of partitions with an infinite number of blocks.

Let $\Pi(\cdot)$ be an EFC process. Define the sets

$$G := \{t \geq 0 : \#\Pi(t) = \infty\}$$

and

$$\forall k \in \mathbb{N}, G_k := \{t \geq 0 : \#\Pi(t) > k\}.$$

Clearly every arrival time t of an atom of P_C such that $\pi^{(C)}(t) \in \Delta_f$ is in G^c the complementary of G. In the same way an arrival time t of an atom of P_F such that $\pi^{(F)}(t) \in \mathcal{S}^{\downarrow}\backslash\Delta_f$ and $B_{k(t)}(t-)$ (the fragmented block) is infinite immediately before the fragmentation, must be in G. Hence, if $\nu_{Disl}(\mathcal{S}^{\downarrow}\backslash\Delta_f) = \infty$ and $\nu_{Coag}(\Delta_f) = \infty$, then both G and G^c are everywhere dense, and this independently of the starting point which may be $\mathbf{1}$ or $\mathbf{0}$.

The following proposition shows that when the fragmentation rate is infinite, G is everywhere dense. Recall the notation $\Pi(t) = (B_1(t), B_2,(t), ...)$ and define $\Delta_f(k) := \{x \in \Delta_f : \sum_{i=1}^{k} x_i = 1\}$.

Theorem 12. *Let Π be an EFC process that fragmentates quickly. Then, a.s. G is everywhere dense. More precisely*

$$G^c = \{t : \pi^{(C)}(t) \in \Delta_f\} \tag{5}$$

and for all $n \geq 2$

$$G_n^c = \{t : \pi^{(C)}(t) \in \Delta_f(n)\}. \tag{6}$$

We begin with the proof that G is a.s. everywhere dense.

As $G = \cap G_k$ we only need to show that a.s. for each $k \in \mathbb{N}$ the set G_k is everywhere dense and open to conclude with Baire's Theorem. The proof relies on two lemmas.

Lemma 13. *Let Π be an EFC process that fragmentates quickly started from* $\mathbf{1}$. *Then, a.s. for all $k \in \mathbb{N}$*

$$\inf\{t \geq 0 : \#\Pi(t) > k\} = 0.$$

Proof. Fix $k \in \mathbb{N}$ and $\epsilon > 0$, we are going to show that there exists $t \in [0, \epsilon[$ such that

$$\exists n \in \mathbb{N} : \#\Pi_{|[n]}(t) \geq k.$$

Let $B(i,t)$ be the block of $\Pi(t)$ that contains i. As $\nu_{Disl}(\mathcal{S}^{\downarrow}) = \infty$ (or $c_e > 0$) it is clear that almost surely $\exists n_1 \in \mathbb{N} : \exists t_1 \in [0, \epsilon[$ such that $\Pi_{|[n_1]}(t_1-) = \mathbf{1}_{|[n_1]}$ and t_1 is a fragmentation time such that $\Pi_{|[n_1]}(t_1)$ contains at least two distinct blocks, say $B(i_1, t_1) \cap [n_1]$ and $B(i_2, t_1) \cap [n_1]$, of which at least one is not a singleton and is thus in fact infinite when seen in $\mathbb{N}$. The time of coalescence of i_1 and i_2 -the first time at which they are in the same block again- is exponentially distributed with parameter

$$\int_{\mathcal{S}^{\downarrow}} (\sum_i x_i^2) \nu_{Coag}(dx) + c_k < \infty.$$

Hence if we define

$$\tau_{i_1,i_2}(t_1) := \inf\{t \geq t_1 : i_1 \overset{\Pi(t)}{\sim} i_2\}$$

then almost surely we can find $n_2 > n_1$ large enough such that the first time t_2 of fragmentation of $B(i_1, t_1) \cap [n_2]$ or $B(i_2, t_1) \cap [n_2]$ is smaller than $\tau_{i_1,i_2}(t_1)$ (i.e., i_1 and i_2 have not coalesced yet), $t_2 < \epsilon$ and t_2 is a fragmentation time at which $B(i_1, t_2-) \cap [n_2]$ or $B(i_2, t_2-) \cap [n_2]$ is split into (at least) two blocks. Furthermore we can always choose n_2 large enough so that one of them is not a singleton. Hence at t_2 there are at least 3 non-empty blocks in $\Pi_{|[n_2]}(t_2)$, and at least one of them is not a singleton. By iteration, almost surely, $\exists n_k : \exists t_k \in [0, \epsilon[$ such that t_k is a fragmentation time and

$$\#\Pi_{|[n_k]}(t_k) \geq k.$$

$\square$

Lemma 14. *Let Π be an EFC process that fragmentates quickly. Then, a.s. G_k is everywhere dense and open for each $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$, call $\Gamma_k = \{t_1^{(k)} < t_2^{(k)} < ...\}$ the collection of atom times of P_C such that a coalescence occurs on the $k+1$ first blocks if there are more than $k+1$ blocks, i.e.,

$$\pi^{(C)}(t) \notin Q(\mathbf{0}, k+1)$$

(recall that $Q(\mathbf{0}, k+1) = \{\pi \in \mathcal{P} : \pi_{|[k+1]} = \mathbf{0}_{k+1}\}$). Suppose $t \in G_k$, then by construction $\inf\{s > t : s \in G_k^c\} \in \Gamma_k$ (because one must at least coalesce the first $k+1$ distinct blocks present at time t before having less than k blocks). As the $t_i^{(k)}$ are stopping times, the strong Markov property and the first lemma imply that $G_k^c \subseteq \Gamma_k$. Hence G_k is a dense open subset of $\mathbb{R}+$. $\square$

We can apply Baire's theorem to conclude that $\cap_k G_k = G$ is almost surely everywhere dense in $\mathbb{R}+$.

We now turn to the proof of (5) and (6). As $G^c = \cup G_n^c$, it suffices to show (6) for some $n \in \mathbb{N}$.

Recall from the proof of Lemma 14 that $G_k^c \subseteq \Gamma_k = \{t_1^{(k)}, t_2^{(k)}, ..\}$ the set of coalescence times at which $\pi^{(C)}(t) \notin Q(\mathbf{0}, k+1)$.

Now fix $n \in \mathbb{N}$ and consider simultaneously the sequence $(t_i^{(k)})_{i\in\mathbb{N}}$ and $(t_i^{(k+n)})_{i\in\mathbb{N}}$. It is clear that for each $i \in \mathbb{N}, \exists j \in \mathbb{N}$ such that

$$t_i^{(k)} = t_j^{(k+n)}$$

because $\pi^{(c)}(t) \notin Q(\mathbf{0}, k+1) \Rightarrow \pi^{(c)}(t) \notin Q(\mathbf{0}, k+n+1)$. Furthermore the $t_i^{(k+n)}$ have no other accumulation points than ∞, thus, by Theorem 12, there exists $r_1^{(k+n)} < t_1^{(k)}$ and $n_1 < \infty$ such that for all $s \in]r_1^{(k+n)}, t_1^{(k)}[$: $\#\Pi_{[n_1]}(s) > k+n$. Hence, a necessary condition to have $\#\Pi_{|[n_1]}(t_1^{(k)}) \leq k$ is that $t_1^{(k)}$ is a multiple collision time, and more precisely $t_1^{(k)}$ must be a collision time such that $\#\pi^{(c)}_{|[k+n]}(t_1^{(k)}) \leq k$. Hence

$$G_k^c \subseteq \{t_i^{(k)} \text{ s.t. } \#\pi^{(c)}_{|[k+n]}(t_i^{(k)}) \leq k\}.$$

As this is true for each n almost surely, the conclusion follows.

This finishes the proof of Theorem 12.

As recently noted by Lambert [18], there is an interpretation of some EFC processes in terms of population dynamics. More precisely if we consider an EFC process $(\Pi(t), t \geq 0)$ such that $\nu_{Disl}(\mathcal{S}^{\downarrow}) < \infty$ and

$$(H) \qquad \begin{cases} \nu_{Disl}(\mathcal{S}^{\downarrow} \setminus \Delta_f) = 0 \\ c_e = 0 \\ \nu_{Coag}(\mathcal{S}^{\downarrow}) = 0 \\ c_k > 0 \end{cases}$$

then, if at all time all the blocks of $\Pi(t)$ are infinite we can see the number of blocks $(Z(t) = \#\Pi(t), t \geq 0)$ as the size of a population where each individual gives rise (without dying) to a progeny of size i with rate $\nu_{Disl}(\Delta_f(i+1))$

and there is a negative density-dependence due to competition pressure. This is reflected by the Kingman coalescence phenomenon which results in a quadratic death rate term. The natural death rate is set to 0, i.e., there is no linear component in the death rate. In this context, an EFC process that comes down from infinity corresponds to a population started with a very large size. Lambert has shown that a sufficient condition to be able to define what he terms a logistic branching process started from infinity is

$$(L) \qquad \sum_k p_k \log k < \infty$$

where $p_k = \nu_{Disl}(\Delta_f(k+1))$.

More precisely, this means that if P_n is the law of the $\mathbb{N}$-valued Markov chain $(Y(t), t \geq 0)$ started from $Y(0) = n$ with transition rates

$$\forall i \in \mathbb{N} \begin{cases} i \to i+j \text{ with rate } ip_j \text{ for all } j \in \mathbb{N} \\ i \to i-1 \text{ with rate } c_k i(i-1)/2 \text{ when } i > 1. \end{cases},$$

then P_n converge weakly to a law P_∞ which is the law of a $\mathbb{N} \cup \infty$-valued Markov process $(Z(t), t \geq 0)$ started from ∞, with the same transition semi-group on $\mathbb{N}$ as Y and whose entrance law can be characterized. Moreover, if we call $\tau = \inf\{t \geq 0 : Z(t) = 1\}$ we have that $\mathbb{E}(\tau) < \infty$.

As $\#\Pi(\cdot)$ has the same transition rates as $Y(\cdot)$ and the entrance law from ∞ is unique, these processes have the same law. Hence the following is a simple corollary of Lambert's result.

Proposition 15. *Let Π be an EFC process started from dust (i.e., $\Pi(0) = \mathbf{0}$) and satisfying the conditions (H) and (L). Then one has*

$$\forall t > 0, \ \#\Pi(t) < \infty \ a.s.$$

Proof. If $T = \inf\{t : \#\Pi(t) < \infty\}$, Lambert's result implies that $\mathbb{E}(T) < \infty$ and hence T is almost surely finite. A simple application of Proposition 23 in [20] and Lemma 31 in [23] shows that if there exists $t < \infty$ such that $\#\Pi(t) < \infty$ then $\inf\{t : \#\Pi(t) < \infty\} = 0$. To conclude, we can use Theorem 3.1 in [18] to see that as $\#\Pi(t)$ is positive recurrent in $\mathbb{N}$, if $\Pi(0) = \mathbf{1}$ (or any partition with a finite number of blocks) then $\inf\{t \geq 0 : \#\Pi(t) = \infty\} = \infty$. This entails that when an EFC process satisfying (H) and (L) reaches a finite level it cannot go back to infinity. As $\inf\{t : \#\Pi(t) < \infty\} = 0$, this means that

$$\forall t > 0, \#\Pi(t) < \infty.$$

□

Remark : Let $\Pi(\cdot) = (B_1(\cdot), B_2(\cdot), ...)$ be a "(H)-(L)" EFC process started from dust, $\Pi(0) = \mathbf{0}$. Then for all $t > 0$ one has a.s. $\sum_i \|B_i(t)\| = 1$. This is clear because at all time $t > 0$ there are only a finite number of blocks.

If we drop the hypothesis $\nu_{Disl}(\mathcal{S}^{\downarrow}) < \infty$ (i.e., we drop (L) and we suppose $\nu_{Disl}(\mathcal{S}^{\downarrow}) = \infty$), the process Π stays infinite (Theorem 12). We now show that nevertheless, for a fixed t, almost surely $\|B_1(t)\| > 0$. We define by induction a sequence of integers $(n_i)_{i\in\mathbb{N}}$ as follows: we fix $n_1 = 1, t_1 = 0$ and for each $i > 1$ we chose n_i such that there exists a time $t_i < t$ such that t_i is a coalescence time at which the block 1 coalesces with the block n_i and such that $n_i > w_{n_{i-1}}(t_{i-1})$ where $w_k(t)$ is the least element of the kth block at time t. This last condition ensures that $(w_{n_i}(t_i))$ is a strictly increasing sequence because one always has $w_n(t) \geq n$. The existence of such a construction is assured by the condition $c_k > 0$. Hence at time t one knows that for each i there has been a coalescence between 1 and $w_{n_i}(t_i)$. Consider $(\Pi_t^{(F)}(s), s \in [0, t[)$ a coupled fragmentation process defined as follows: $\Pi_t^{(F)}(0)$ has only one block which is not a singleton which is

$$B_1^{(F)}(0) = \{1, w_{n_2}(t_2), w_{n_3}(t_3), ...\}.$$

The fragmentations are given by the same PPP P_F used to construct Π (and hence the processes are coupled). It should be clear that if $w_{n_i}(t_i)$ is in the same block with 1 for $\Pi^{(F)}(t)$ the same is true for $\Pi(t)$ because it means that no dislocation separates 1 from $w_{n_i}(t_i)$ during $[0, t]$ for $\Pi^{(F)}$ and hence also for Π, thus

$$\forall i \in \mathbb{N}, 1 \overset{\Pi(t)}{\sim} w_{n_i}(t_i).$$

As by construction $B_1^{(F)}(t) \subseteq \{1, w_{n_2}(t_2), w_{n_3}(t_3), ...\}$ one has

$$\|B_1(t)\| \geq \|B_1^{(F)}(t)\| > 0.$$

Hence for all $t > 0$ one has $P(\{1\} \subset \mathrm{dust}(\Pi(t))) = 0$ and thus

$$P(\mathrm{dust}(\Pi(t)) \neq \emptyset) = 0.$$

Otherwise said, when $\nu_{Disl}(\mathcal{S}^{\downarrow}) = \infty$ the fragmentation part does not let a "(H)" EFC process come down from infinity, but it let the dust condensates into mass. Note that "binary-binary"[3] EFC processes are a particular case. The question of the case $\nu_{Disl}(\mathcal{S}^{\downarrow}) < \infty$ but (L) is not true remains open.

[3] i.e., $c_e = 0, \nu_{Disl}(\{x : x_1 + x_2 < 1\} = 0, \nu_{Coag} \equiv 0$ and $c_k > 0$.

5.2 Missing mass trajectory

This last remark prompts us to study in more generality the behavior of the process of the missing mass

$$D(t) = \|\mathrm{dust}(t)\| = 1 - \sum_i \|B_i(t)\|.$$

In [20] it was shown (Proposition 26) that for a pure Λ-coalescence started from $\mathbf{0}$ (i.e., such that $\nu_{Coag}(\{x \in \mathcal{S}^{\downarrow} : x_2 > 0\}) = 0$)

$$\xi(t) := -\log(D(t))$$

has the following behavior:

- either the coalescence is quick ($c_k > 0$ or $\int_{\mathcal{S}^{\downarrow}} (\sum_i x_i)\nu_{Coag}(dx) = \infty$) and then $D(t)$ almost surely jumps from 1 to 0 immediately (i.e., $D(t) = 0$ for all $t > 0$,)
- or the coalescence is slow ($c_k = 0$ and $\int_{\mathcal{S}^{\downarrow}} (\sum_i x_i)\nu_{Coag}(dx) < \infty$) and one has that $\xi(t)$ is a drift-free subordinator whose Lévy measure is the image of $\nu_{Coag}(dx)$ via the map $x \mapsto -\log(1 - x_1)$.

In the following we make the following hypothesis about the EFC process we consider

$$\text{(H')} \qquad \begin{cases} c_k = 0 \\ \int_{\mathcal{S}^{\downarrow}} (\sum_i x_i)\nu_{Coag}(dx) < \infty \\ \nu_{Disl}(\{x \in \mathcal{S}^{\downarrow} : \sum_i x_i < 1\} = 0 \end{cases}$$

The last assumption means that sudden dislocations do not create dust.

Before going any further we should also note that without loss of generality we can slightly modify the PPP construction given in Proposition 5 : We now suppose that P_F is the sum of two point processes $P_F = P_{Disl} + P_e$ where P_{Disl} has measure intensity $\mu_{\nu_{Disl}} \otimes \#$ and P_e has measure intensity $c_e \mathbf{e}$. If t is an atom time for P_{Disl} one obtains $\Pi(t)$ from $\Pi(t-)$ as before, if P_e has an atom at time t, say $(t, \epsilon_{k(t)})$, then $\Pi(t-)$ is left unchanged except for $k(t)$ which becomes a singleton if this was not already the case. Furthermore, if t is an atom time for P_C we will coalesce $B_i(t-)$ and $B_j(t-)$ at time t if and only if $w_i(t-)$ and $w_j(t-)$ (i.e., the least elements of $B_i(t-)$ and $B_j(t-)$ respectively) are in the same block of $\pi^{(C)}(t)$. This is equivalent to saying that from the point of view of coalescence the labelling of the block is the following: if i is not the least element of its block B_i is empty, and if it is the least element of its block then B_i is this block. To check this, one can for instance satisfy that the transition rates of the restrictions $\Pi_{|[n]}(\cdot)$ are left unchanged.

Proposition 16. *Let* Π *be an EFC process satisfying (H'). Then* ξ *is a solution of the SDE*

$$d\xi(t) := d\sigma(t) - c_e(e^{\xi(t)} - 1)dt$$

where σ *is a drift-free subordinator whose Lévy measure is the image of* $\nu_{Coag}(dx)$ *via the map* $x \mapsto -\log(1 - \sum_i x_i)$.

The case when $c_e = 0$ is essentially a simple extension of Proposition 26 in [20] which can be shown with the same arguments. More precisely, we use a coupling argument. If we call $(\Pi^{(C)}(t), t \geq 0)$ the coalescence process started from $\Pi(0)$ and constructed with the PPP P_C, we claim that for all t

$$\mathrm{dust}(\Pi(t)) = \mathrm{dust}(\Pi^{(C)}(t)).$$

This is clear by observing that for a given i if we define

$$T_i^{(C)} = \inf\{t > 0 : i \notin \mathrm{dust}(\Pi^{(C)}(t))\}$$

we have that $T_i^{(C)}$ is necessarily a collision time which involves $\{i\}$ and the new labelling convention implies that

$$T_i^{(C)} = \inf\{t > 0 : i \notin \mathrm{dust}(\Pi(t))\}.$$

Furthermore, given a time t, if $i \notin \mathrm{dust}(\Pi(t))$ then $\forall s \geq 0 : i \notin \mathrm{dust}(\Pi(t + s))$. Hence for all $t \geq 0$ one has $\mathrm{dust}(\Pi(t)) = \mathrm{dust}(\Pi^{(C)}(t))$ and thus Proposition 26 of [20] applies.

We now concentrate on the case $c_e > 0$. Define

$$D_n(t) := \frac{1}{n}\#\{\mathrm{dust}(\Pi(t)) \cap [n]\}.$$

Note that $\mathrm{dust}(\Pi(t)) \cap [n]$ can be strictly included in the set of the singletons of the partition $\Pi_{|[n]}(t)$. Observe that the process D_n is a Markov chain with state-space $\{0, 1/n, ..., (n-1)/n, 1\}$. We already know that D is a càdlàg process and that almost surely, for all $t \geq 0$ one has $D_n(t) \to D(t)$.

First we show that

Lemma 17. *With the above notations* $D_n \Rightarrow D$, *i.e., the process* D_n *converges weakly to* D *in the Skorokhod space of càdlàg paths* $[0, \infty) \mapsto \mathbb{R}$.

Proof. One only has to show that the sequence D_n is tight because we have convergence of the finite dimensional marginal laws (see for instance [15, VI.3.20]).

The idea is to use Aldous' tightness criterion ([15, VI.4.5]). The processes D_n are bounded by 0 and 1 and hence one only has to check that $\forall \epsilon > 0$

$$\lim_{\theta \searrow 0} \limsup_{n} \sup_{S,T \in \mathcal{T}_N^n; S \leq T \leq S+\theta} P(|D_n(T) - D_n(S)| \geq \epsilon) = 0$$

where $\mathcal{T}_N^n$ is the set of all stopping times in the natural filtration of D_n bounded by N.

First note that

$$\begin{aligned} \sup_{S,T \in \mathcal{T}_N^n; S \leq T \leq S+\theta} & P(|D_n(T) - D_n(S)| \geq \epsilon) \\ & \leq \sup_{S \in \mathcal{T}_N^n} P(\sup_{t \leq \theta} |D_n(S+t) - D_n(S)| \geq \epsilon) \end{aligned}$$

hence we will work on the right hand term.

Fix $S \in \mathcal{T}_N^n$. First we wish to control $P(\sup_{t \leq \theta}(D_n(S+t) - D_n(S)) \geq \epsilon)$. Observe that the times t at which $\Delta(D_n(t)) = D_n(t) - D_n(t-) > 0$ all are atom times of P_F such that $\pi^{(F)}(t) = \epsilon_i$ for some $i \leq n$ (recall that ϵ_i is the partition of $\mathbb{N}$ that consists of two blocks: $\{i\}$ and $\mathbb{N}\backslash\{i\}$) because under (H'), the only way in which dust can be created is erosion. Hence, clearly,

$$P(\sup_{t \leq \theta}(D_n(S+t) - D_n(S)) \geq \epsilon) \quad \leq \quad P(\frac{1}{n} \sum_{s \in [S,S+\theta]} {}_{\{\pi^{(F)}(s) = \epsilon_i, i=1,\ldots,n\}} \geq \epsilon).$$

The process

$$\Big(\sum_{i=1}^{n} \sum_{s \in [S,S+\theta]} {}_{\{\pi^{(F)}(s) = \epsilon_i\}} \Big)_{\theta \geq 0}$$

is a sum of n independent standard Poisson processes with intensity c_e, hence for each $\eta > 0$ and $\epsilon > 0$ there exists $\theta_0 \leq \epsilon / c_e$ and n_0 such that for each $\theta \leq \theta_0$ and $n \geq n_0$ one has

$$P(\frac{1}{n} \sum_{i=1}^{n} \sum_{s \in [S,S+\theta]} {}_{\{\pi^{(F)}(s) = \epsilon_i\}} > \epsilon) = P(\frac{1}{n} \sum_{i=1}^{n} \sum_{s \in [0,0+\theta]} {}_{\{\pi^{(F)}(s) = \epsilon_i\}} > \epsilon) < \eta$$

where the first equality is just the strong Markov property in S and the last inequality is a consequence of the law of large numbers (observe that

$c_e\theta = \mathbb{E}(\sum_{s\in[S,S+\theta]} \ _{\{\pi^{(F)}(s)=\epsilon_i\}}))$. Hence, the bound is uniform in S and one has that for each $\theta \le \theta_0$ and $n \ge n_0$

$$\sup_{S\in\mathcal{T}_N^n} P(\sup_{t\le\theta}(D_n(S+t) - D_n(S)) \ge \epsilon) < \eta.$$

Let us now take care of $P(\sup_{t\le\theta}(D_n(S) - D_n(S+t)) \ge \epsilon)$. We begin by defining a coupled coalescence process as follows: we let $\Pi_S^{(C)}(0) = \mathbf{0}$, and the path of $\Pi_S^{(C)}(\cdot)$ corresponds to P_C. More precisely, if P_C has an atom at time $S+t$, say $\pi^{(C)}(S+t)$, we coalesce $\Pi_S^{(C)}(t-)$ by $\pi^{(C)}(S+t)$ (using our new labelling convention). For each n we define

$$\mathrm{dust}_n^{\mathrm{coag}}(S,\cdot) := \mathrm{dust}(\Pi_S^{(C)}(\cdot)) \cap [n],$$

$$D_n^{\mathrm{coag}}(S,\cdot) := \frac{1}{n}\#\mathrm{dust}_n^{\mathrm{coag}}(S,\cdot,)$$

and

$$\mathrm{dust}_n(\cdot) := \mathrm{dust}(\Pi(\cdot)) \cap [n].$$

We claim that for each $t \ge 0$

$$\{i \le n : \forall s \in [S,S+t]\ i \in \mathrm{dust}_n(s)\} \subseteq \mathrm{dust}_n^{\mathrm{coag}}(S,t).$$

Indeed suppose $j \in \{i \le n : \forall s \in [S,S+t]\ i \in \mathrm{dust}_n(s)\}$, then for $r \le t$ one has $j \in \mathrm{dust}(\pi^{(C)}(S+r))$ (recall that the $\pi^{(C)}(\cdot)$ are the atoms of P_C) and hence j has not yet coalesced for the process $\Pi_S^{(C)}(\cdot)$. On the other hand, if there exists a coalescence time $S+r$ such that $j \in \mathrm{dust}_n(S+r-)$ and $j \notin \mathrm{dust}_n(S+r)$ then it is clear that j also coalesces at time r for $\Pi_S^{(C)}(.)$ and hence $j \notin \mathrm{dust}_n^{\mathrm{coag}}(S,r)$. Thus we have that

$$D_n(S) - \frac{1}{n}\{i \le n : \forall s \in [S,S+t]\ i \in \mathrm{dust}_n(s)\} \le 1 - D_n^{\mathrm{coag}}(S,t).$$

Now note that

$$\{i \le n : \forall s \in [S,S+t]\ i \in \mathrm{dust}_n(s)\} \subseteq \mathrm{dust}_n(S+t)$$

and thus

$$\begin{aligned} D_n(S) - D_n(S+t) &\le D_n(S) - \frac{1}{n}\{i \le n : \forall s \in [S,S+t]\ i \in \mathrm{dust}_n(s)\} \\ &\le D_n(S) - \frac{1}{n}\{i \le n : \forall s \in [S,S+\theta]\ i \in \mathrm{dust}_n(s)\} \\ &\le 1 - D_n^{\mathrm{coag}}(S,\theta) \end{aligned}$$

(for the second inequality observe that $t \mapsto \{i \leq n : \forall s \in [S, S+t]\ i \in \mathrm{dust}_n(s)\}$ is decreasing). We can now apply the strong Markov property for the PPP P_C at time S and we see that

$$\begin{aligned} P(1 - D_n^{\text{coag}}(S,\theta) > \epsilon) &= P(1 - D_n^{\text{coag}}(0,\theta) > \epsilon) \\ &= P\left(-\log\left(D_n^{\text{coag}}(0,\theta)\right) > -\log(1-\epsilon)\right). \end{aligned}$$

Define

$$\xi_n(t) := -\log(D_n^{\text{coag}}(0,t)).$$

We know that almost surely, for all $t \geq 0$ one has $\xi_n(t) \to \xi(t)$ where $\xi(t)$ is a subordinator whose Lévy measure is given by the image of ν_{Coag} by the map $x \mapsto -\log(1 - \sum_i x_i)$. Hence, $P(\xi_n(\theta) > -\log(1-\epsilon)) \to P(\xi(\theta) > -\log(1-\epsilon))$ when $n \to \infty$. Thus, for any $\eta > 0$ there exists a θ_1 such that if $\theta < \theta_1$ one has $\limsup_n P(\xi_n(\theta) > -\log(1-\epsilon)) < \eta$. This bound being uniform in S, the conditions for applying Aldous' criterion are fulfilled. $\square$

It is not hard to see that $D_n(\cdot)$, which takes its values in $\{0, 1/n, ..., n/n\}$, is a Markov chain with the following transition rates:

- if $k < n$ it jumps from k/n to $(k+1)/n$ with rate $c_e n(1-k/n)$,
- if $k > 0$ it jumps from k/n to r/n for any r in $0, ..., k$ with rate $\binom{k}{r}\int_0^1 x^r(1-x)^{k-r}\tilde{\nu}(dx)$ where $\tilde{\nu}$ is the image of ν_{Coag} by the map $\mathcal{S}^{\downarrow} \mapsto [0,1] : x \mapsto (1 - \sum_i x_i)$.

Hence, if A_n is the generator of the semi-group of D_n one necessarily has for any non-negative continuous f

$$\begin{aligned} A_n f(k/n) &= \frac{f((k+1)/n) - f(k/n)}{1/n} c_e(1-k/n) \\ &\quad + \sum_{r=1}^{k} (f((k-r)/n) - f(k/n))\binom{k}{r} \int_0^1 x^r(1-x)^{k-r}\tilde{\nu}(dx). \end{aligned} \tag{7}$$

We wish to define the A_n so they will have a common domain, hence we will let A_n be the set of pairs of functions f, g such that $f : [0,1] \mapsto \mathbb{R}$ is continuously differentiable on $[0,1]$ and $f(D_n(t)) - \int_0^t g(D_n(s))ds$ is a martingale. Note that continuously differentiable functions on $[0,1]$ are dense in $C([0,1])$ the space of continuous functions on $[0,1]$ for the L_∞ norm.

Hence A_n is multivalued because for each function f, any function g such that $g(k/n)$ is given by (7)will work. In the following we focus on the only such g_n which is linear on each $[k/n, (k+1)/n]$.

We know that $D_n \Rightarrow D$ in the space of càdlàg functions and that D_n is solution of the martingale problem associated to A_n. Define

$$Af(x) := f'(x)(1-x)c_e + \int_0^1 (f(\theta x) - f(x))\tilde{\nu}(d\theta).$$

In the following $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

Lemma 18. *One has*

$$\lim_{n\to\infty} \|g_n - Af\| = 0.$$

Proof. We decompose g_n into $g_n = g_n^{(1)} + g_n^{(2)}$ where both $g_n^{(1)}$ and $g_n^{(2)}$ are linear on each $[k/n, (k+1)/n]$ and

$$g_n^{(1)}(k/n) = \frac{f((k+1)/n) - f(k/n)}{1/n} c_e(1-k/n)$$

while

$$g_n^{(2)}(k/n) = \sum_{r=1}^{k} (f((k-r)/n) - f(k/n))\binom{k}{r} \int_0^1 \theta^r (1-\theta)^{k-r} \tilde{\nu}(d\theta).$$

One has that $\frac{f((k+1)/n)-f(k/n)}{1/n} \to f'(x)$ when $n \to \infty$ and $k/n \to x$. Hence, as f' is continuous on $[0,1]$, one has that

$$\|g_n^{(1)}(x) - f'(x)c_e(1-x)\| \to 0.$$

Let us now turn to the convergence of $g_n(2)$. For a fixed x and a fixed θ one has that

$$\sum_{r=1}^{[nx]} (f(r/n) - f([nx]/n))\binom{[nx]}{r}\theta^r(1-\theta)^{[nx]-r} \to f(\theta x) - f(x)$$

when $n \to \infty$ because $\binom{[nx]}{r}\theta^r(1-\theta)^{[nx]-r} = P(B_{[nx],\theta} = r)$ where $B_{[nx],\theta}$ is a $[nx], \theta$-binomial variable. More precisely,

$$\sum_{r=1}^{[nx]} f([nx]/n)\binom{[nx]}{r}\theta^r(1-\theta)^{[nx]-r} = f([nx]/n) \to f(x)$$

and

$$\sum_{r=1}^{[nx]} f(r/n)\binom{[nx]}{r}\theta^r(1-\theta)^{[nx]-r} = \mathbb{E}(f(B_{[nx],\theta}/n)) \to f(\theta x)$$

when $n \to \infty$. We need this convergence to be uniform in x. We proceed in two steps: first it is clear that

$$\lim_n \sup_x (f(x) - f([nx]/n)) = 0.$$

For the second part fix $\epsilon > 0$. There exists $\eta > 0$ such that $\forall x, y \in [0,1]$ one has $|x-y| \le \eta \Rightarrow |f(x) - f(y)| < \epsilon$.

Next it is clear that for each $\eta > 0$ there is a $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$ and $\forall x \in [\eta, 1]$ one has

$$\begin{array}{ll} & P(B_{[nx],\theta} \in [[nx](\theta-\eta), [nx](\theta+\eta)]) \\ \ge & P(B_{[n\eta],\theta} \in [[n\eta](\theta-\eta), [n\eta](\theta+\eta)]) \\ > & 1-\epsilon. \end{array}$$

Hence, for $n \ge n_0$ and $x > \theta$

$$\begin{aligned} \sum_{r=1}^{[nx]} \quad & f(r/n)\tbinom{[nx]}{r}\theta^r(1-\theta)^{[nx]-r} \\ & = \mathbb{E}(f(B_{[nx],\theta}/n)) \\ & \ge (1-\epsilon) \inf_{r \in [[nx](\theta-\eta), [nx](\theta+\eta)]} f(r/n) \\ & \ge (1-\epsilon) \inf_{\theta' \in [\theta-\eta, \theta+\eta]} f(\frac{[nx]}{n}\theta') \end{aligned}$$

and

$$\begin{aligned} \sum_{r=1}^{[nx]} \quad & f(r/n)\tbinom{[nx]}{r}\theta^r(1-\theta)^{[nx]-r} \\ & = \mathbb{E}(f(B_{[nx],\theta}/n)) \\ & \le \sup_{r \in [[nx](\theta-\eta), [nx](\theta+\eta)]} f(r/n) + \epsilon\|f\| \\ & \le \sup_{\theta'' \in [\theta-\eta, \theta+\eta]} f(\frac{[nx]}{n}\theta'') + \epsilon\|f\|. \end{aligned}$$

Hence, for any $\epsilon' > 0$, by choosing ϵ and η small enough, one can ensure that there exists a n_1 such that for all $n \geq n_1$ one has

$$\begin{aligned}
\sup_{x\in[\eta,1]} & \left| f(\theta x) - \sum_{r=1}^{[nx]} f(r/n) \tbinom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r} \right| \\
\leq & \max\left\{ \left| f(\theta x) - (1-\epsilon) \inf_{\theta' \in [\theta-\eta,\theta+\eta]} f(\frac{[nx]}{n}\theta') \right| ; \right. \\
& \left. \left| f(\theta x) - \sup_{\theta'' \in [\theta-\eta,\theta+\eta]} f(\frac{[nx]}{n}\theta'') + \epsilon \|f\| \right| \right\} \\
\leq & \epsilon'
\end{aligned}$$

where we have used $\sup_{|x-y|<\varepsilon} |f(x) - f(y)| \to 0$ when $\varepsilon \to 0$ for the last inequality.

For $x < \eta$ note that

$$\begin{aligned}
& \left| f(\theta x) - \sum_{r=1}^{[nx]} f(r/n) \tbinom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r} \right| \\
\leq & |f(\theta x) - f(0)| + \left| f(0) - \sum_{r=1}^{[nx]} f(r/n) \tbinom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r} \right| .
\end{aligned}$$

We can bound $\sum_{r=1}^{[nx]} f(r/n) \binom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r}$ as follows:

$$\begin{aligned}
& P(B_{[nx],\theta} < [n\eta]+1) \inf_{s\leq\eta} f(s) \\
\leq & \sum_{r=1}^{[nx]} f(r/n) \tbinom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r} \\
\leq & P(B_{[nx],\theta} < [n\eta]+1) \sup_{s\leq\eta} f(s) + \|f\| P(B_{[nx],\theta} \geq [n\eta]+1).
\end{aligned}$$

Hence one has that

$$\lim_n \sup_x \left(\sum_{r=1}^{[nx]} (f(r/n)) \tbinom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r} - f(\theta x) \right) = 0.$$

Finally we conclude that

$$\sup_x \left| \left[\sum_{r=1}^{[nx]} (f(r/n) - f([nx]/n)) \tbinom{[nx]}{r} \theta^r (1-\theta)^{[nx]-r} \right] - \left[f(\theta x) - f(x) \right] \right| \to 0$$

We can then apply the dominated convergence theorem and we get

$$\sup_x \left| g_n^{(2)}(x) - \int_0^1 [f(\theta x) - f(x)]\, \tilde{\nu}(d\theta) \right| \to 0.$$

Hence, one has $\|g_n^{(2)} - g^{(2)}\| \to 0$ where $g^{(2)}(x) = \int_0^1 f(\theta x) - f(x) \tilde{\nu}(d\theta)$.

□

One can now use Lemma IV.5.1 (p. 196) in [14] to see that D must be solution of the Martingale Problem associated to A. Hence one can use Theorem III.2.26 in [15] to see that D is solution of

$$dD(t) = c_e(1 - D(t))dt - D(t-)dN(t)$$

where $N(t)$ is a PPP with values in $[0, 1]$ whose measure intensity is the image of ν_{Coag} by the map $x \mapsto \sum_i x_i$. Call $(\theta_s, s \geq 0)$ the atoms of N_t. Recall that $\xi(t) = -\log(D(t))$ and observe that D is a bounded variation process. Some straightforward calculus then shows

$$\begin{aligned} \xi(t) &= \xi(0) + \int_0^t c_e(e^{\xi(s)} - 1)ds \\ &\quad + \sum_{s \leq t} (-\log((1 - \theta_s)D(s-)) + \log(D(s-))). \end{aligned}$$

Hence, we conclude that

$$d\xi(t) = d\sigma(t) - c_e(e^{\xi(t)} - 1)dt$$

where σ is a drift-free subordinator whose Lévy measure is the image of ν_{Coag} by $x \mapsto -\log(1 - \sum_i x_i)$.

6 Proofs

6.1 Proof of Proposition 2

The compatibility of the chains $\Pi_{|[n]}$ can be expressed in terms of transition rates as follows: For $m < n \in \mathbb{N}$ and $\pi, \pi' \in \mathcal{P}_n$ one has

$$q_m(\pi_{|[m]}, \pi'_{|[m]}) = \sum_{\pi'' \in \mathcal{P}_n : \pi''_{|[m]} = \pi'_{|[m]}} q_n(\pi, \pi'').$$

Consider $\pi \in \mathcal{P}_n$ such that $\pi = (B_1, B_2, ..., B_m, \emptyset, ...)$ has $m \leq n$ non-empty blocks. Call $w_i = \inf\{k \in B_i\}$ the least element of B_i and σ a permutation of $[n]$ that maps every $i \le m$ on w_i. Let π' be an element of $\mathcal{P}_m$, then the restriction of the partition $\sigma(Coag(\pi, \pi'))$ to $[m]$ is given by: for $i, j \le m$

$$\begin{aligned} i \overset{\sigma(Coag(\pi,\pi'))}{\sim} j &\Leftrightarrow \sigma(i) \overset{Coag(\pi,\pi')}{\sim} \sigma(j) \\ &\Leftrightarrow \exists k, l : \sigma(i) \in B_k, \sigma(j) \in B_l, k \overset{\pi'}{\sim} l \\ &\Leftrightarrow i \overset{\pi'}{\sim} j \end{aligned}$$

and hence

$$\sigma(Coag(\pi, \pi'))_{|[m]} = \pi'. \tag{8}$$

By definition $C_n(\pi, \pi')$ is the rate at which the process $\sigma(\Pi_{|[n]}(\cdot))$ jumps from $\sigma(\pi)$ to $\sigma(Coag(\pi, \pi'))$. Hence, by exchangeability

$$C_n(\pi, \pi') = q_n(\sigma(\pi), \sigma(Coag(\pi, \pi'))).$$

Observe that $\sigma(\pi)_{|[m]} = \mathbf{0}_m$. Hence if π'' is a coalescence of $\sigma(\pi)$ it is completely determined by $\pi''_{|[m]}$. Thus, for all $\pi'' \in \mathcal{P}_n$ such that $\pi''_{|[m]} = \sigma(Coag(\pi, \pi'))_{|[m]}$ and $\pi'' \ne \sigma(Coag(\pi, \pi'))$ one has

$$q_n(\sigma(\pi), \pi'') = 0. \tag{9}$$

For each $\pi \in \mathcal{P}_n$ define

$$Q_n(\pi, m) := \{\pi' \in \mathcal{P}_n : \pi_{|[m]} = \pi'_{|[m]}\}$$

(for $\pi \in \mathcal{P}$ we will also need $Q(\pi, m) := \{\pi' \in \mathcal{P} : \pi_{|[m]} = \pi'_{|[m]}\}$). Clearly, (9) yields

$$q_n(\sigma(\pi), \sigma(Coag(\pi, \pi')) = \sum_{\pi'' \in Q_n(\sigma(Coag(\pi,\pi')),m)} q_n(\sigma(\pi), \pi'')$$

because there is only one non-zero term in the right hand-side sum. Finally recall (8) and use the compatibility relation to have

$$\begin{aligned} C_n(\pi, \pi') &= q_n(\sigma(\pi), \sigma(Coag(\pi, \pi')) \\ &= \sum_{\pi'' \in Q_n(\sigma(Coag(\pi,\pi')),m)} q_n(\sigma(\pi), \pi'') \\ &= q_m(\sigma(\pi)_{|[m]}, \sigma(Coag(\pi, \pi'))_{|[m]}) \\ &= q_m(\mathbf{0}_m, \pi') \\ &= C_m(\mathbf{0}_m, \pi') \\ &:= C_m(\pi'). \end{aligned}$$

Let us now take care of the fragmentation rates. The argument is essentially the same as above. Suppose $B_k = \{n_1, ..., n_{|B_k|}\}$. Let σ be a permutation of $[n]$ such that of all $j \leq |B_k|$ one has $\sigma(j) = n_j$. Hence, in $\sigma(\pi)$ the first block is $[|B_k|]$. The process $\sigma(\Pi_{|[n]}(\cdot))$ jumps from $\sigma(\pi)$ to the state $\sigma(\mathit{Frag}(\pi, \pi', k))$ with rate $F_n(\pi, \pi', k)$. Note that for $i, j \leq |B_k|$

$$\begin{aligned} i \overset{\sigma(\mathit{Frag}(\pi,\pi',k))}{\sim} j &\Leftrightarrow \sigma(i) \overset{\mathit{Frag}(\pi,\pi',k)}{\sim} \sigma(j) \\ &\Leftrightarrow n_i \overset{\mathit{Frag}(\pi,\pi',k)}{\sim} n_j \\ &\Leftrightarrow i \overset{\sigma(\pi')}{\sim} j \end{aligned}$$

and hence

$$\sigma(\mathit{Frag}(\pi, \pi', k)) = \mathit{Frag}(\sigma(\pi), \sigma(\pi'), 1). \tag{10}$$

Thus by exchangeability $F_n(\pi, \pi', k) = F_n(\sigma(\pi), \sigma(\pi'), 1)$, and it is straightforward to see that by compatibility

$$F_n(\sigma(\pi), \sigma(\pi'), 1) = F_{|B_k|}(\mathbf{1}_{|B_k|}, \sigma(\pi'), 1) = F_{|B_k|}(\sigma(\pi')).$$

The invariance of the rates $C_n(\mathbf{0}_n, \pi')$ and $F_n(\mathbf{1}_n, \pi', 1)$ by permutations of π' is also a direct consequence of exchangeability. In particular $F_{|B_k|}(\sigma(\pi')) = F_{|B_k|}(\pi')$ and thus we conclude that $F_n(\pi, \pi', k) = F_{|B_k|}(\pi')$.

6.2 Proof of Theorem 7

We first have to introduce a bit of notation: let $B(i, t)$ denote the block that contains i at time t and define

- $\bar{\lambda}_i(t) = \bar{\lambda}_{B_i(t)}$ and $\underline{\lambda}_i(t) = \underline{\lambda}_{B_i(t)}$,
- $\bar{\lambda}(i, t) = \bar{\lambda}_{B(i,t)}$ and $\underline{\lambda}(i, t) = \underline{\lambda}_{B(i,t)}$.

In the following we will use repeatedly a coupling technique that can be described as follows: Suppose Π is an EFC process constructed with the PPP P_F and P_C, we choose T a stopping time for Π, at time T we create a fragmentation process $(\Pi^{(F)}(T+s), s \geq 0)$ started from $\Pi^{(F)}(T) = \Pi(T)$ and constructed with the PPP $(P_F(T+s), s \geq 0)$. We call $(B_1^{(F)}(T+s), B_2^{(F)}(T+s), ...)$ the blocks of $\Pi^{(F)}(T+s)$ and $\bar{\lambda}_i^{(F)}(T+s), \underline{\lambda}_i^{(F)}(T+s)$ the corresponding limsup and liminf for the frequencies. The processes $\Pi(T+s)$ and $\Pi^{(F)}(T+s)$ are coupled. More precisely, observe that for instance

$$B_1^{(F)}(T+s) \subseteq B_1(T+s), \forall s \geq 0$$

because if $i \in B_1^{(F)}(T+s)$ it means that there is no $r \in [T, T+s]$ such that $k(r) = 1$ and $1 \overset{\pi^{(F)}(r)}{\not\sim} i$ and hence $i \in B_1(T+s)$.

Consider an exchangeable variable $\Pi = (B_0, B_1, B_2, ...)$ where as before B_0 is used to contain all the singletons, and a fixed subset $A = (a_1, a_2, ...) \subset \mathbb{N}$. Call σ_A the only increasing bijection from A to $\mathbb{N}$. By exchangeability $\Pi \overset{d}{=} \sigma_A^{-1}(\Pi)$ and almost surely $\|\sigma_A(B_i \cap A)\| = \|B_i\|$ for each $i \geq 0$. Thus almost surely,

$$\begin{aligned}\forall i \geq 0,\ \lim_{n\to\infty} \frac{\#\{k \leq n, k \in B_i \cap A\}}{\#\{k \leq n, k \in A\}} &= \lim_{n\to\infty} \frac{\#\{k \leq n, k \in \sigma_A(B_i \cap A)\}}{n} \\ &= \|B_i\|.\end{aligned}$$

This implies that almost surely

$$\forall i \geq 0,\ \limsup_{n\to\infty} \frac{\#\{k \leq n : k \in B_i \cap A\}}{n} = \bar{\lambda}_A \|B_i\|$$

and

$$\forall i \geq 0,\ \liminf_{n\to\infty} \frac{\#\{k \leq n : k \in B_i \cap A\}}{n} = \underline{\lambda}_A \|B_i\|.$$

This result can easily be extended to the case of a random set A which is independent of Π by conditioning on A.

Hence, if we start a homogeneous fragmentation $(\Pi^{(F)}(T+s), s \geq 0)$ from a partition that does not necessarily admit asymptotic frequencies, say $\Pi^{(F)}(T) = (..., A, ...)$ (i.e., A is one of the block in $\Pi^{(F)}(0)$), we still have that if a designates the least element of A then almost surely

$$\bar{\lambda}^{(F)}(a, T+s) \to \bar{\lambda}_A \tag{11}$$

and

$$\underline{\lambda}^{(F)}(a, T+s) \to \underline{\lambda}_A$$

when $s \searrow 0$.

To prove Theorem 7, it suffices to prove the existence of the asymptotic frequency of $B_1(t)$ simultaneously for all t, the same proof then applies to the $B(i,t)$ for each i. As $\Pi(t)$ is an exchangeable process we already know that $\|B_1(q)\|$ exists simultaneously for all $q \in \mathbb{Q}$. For such q we thus have that $\bar{\lambda}_1(q) = \underline{\lambda}_1(q)$. Hence, it suffices to show that $\bar{\lambda}_1(t)$ and $\underline{\lambda}_1(t)$ are both càdlàg processes. In the following we write $q \searrow\searrow t$ or $q \nearrow\nearrow t$ to mean q converges to t in $\mathbb{Q}$ from below (resp. from above).

The first step is to show that:

Lemma 19. *Almost surely, the process* $(L(t), t \geq 0)$ *defined by*

$$\forall t \geq 0 : L(t) := \lim_{q \searrow\searrow t} \bar{\lambda}_1(q) = \lim_{q \searrow\searrow t} \underline{\lambda}_1(q)$$

exists and is càdlàg.

Proof. Using standard results (see for instance [22, Theorem 62.13]), and recalling that $\bar{\lambda}_1$ and $\underline{\lambda}_1$ coincide on $\mathbb{Q}$, one only need to show that $q \mapsto \bar{\lambda}_1(q) = \underline{\lambda}_1(q)$ is a regularisable process, that is

$$\lim_{q \searrow\searrow t} \bar{\lambda}_1(q) = \lim_{q \searrow\searrow t} \underline{\lambda}_1(q) \text{ exist for every real } t \geq 0,$$
$$\lim_{q \nearrow\nearrow t} \bar{\lambda}_1(q) = \lim_{q \nearrow\nearrow t} \underline{\lambda}_1(q) \text{ exist for every real } t \geq 0.$$

Using [22, Theorem 62.7], one only has to check that whenever $N \in \mathbb{N}$ and $a, b \in \mathbb{Q}$ with $a < b$, almost surely we have

$$\sup\{\bar{\lambda}_1(q) : q \in \mathbb{Q}^+ \cap [0, N]\} = \sup\{\underline{\lambda}_1(q) : q \in \mathbb{Q}^+ \cap [0, N]\} < \infty$$

and

$$U_N(\bar{\lambda}_1; [a, b]) = U_N(\underline{\lambda}_1; [a, b]) < \infty$$

where $U_N(\bar{\lambda}_1; [a, b])$ is the number of upcrossings of $\bar{\lambda}_1$ from a to b during $[0, N]$. By definition $\sup\{\bar{\lambda}_1(q) : q \in \mathbb{Q}^+ \cap [0, N]\} \leq 1$ and $\sup\{\underline{\lambda}_1(q) : q \in \mathbb{Q}^+ \cap [0, N]\} \leq 1$. Suppose that $q \in \mathbb{Q}$ is such that $\bar{\lambda}_1(q) > b$. Then if we define $s = \inf\{r \geq 0 : \bar{\lambda}_1(q + r) \leq a\}$ one can use the Markov property and the coupling with a fragmentation $(\Pi^{(F)}(q + r), r \geq 0)$ started from $\Pi(q)$, constructed with the PPP $(P_F(q + r), r \geq 0)$ to see that $s \geq \theta$ where θ is given by $\theta := \inf\{r \geq 0 : \bar{\lambda}_1^{(F)}(t + r) \leq a\}$. If one has a sequence $L_1 < R_1 < L_2 < R_2, \ldots.$ in $\mathbb{Q}$ such that $\bar{\lambda}_1(L_i) < a < b < \bar{\lambda}_1(R_i)$, then one has that for each i, $R_i - L_i > \theta_i$ where $(\theta_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence with same distribution as θ. Hence $P(U_N(\bar{\lambda}_1; [a, b]) = \infty) = 0$. □

The next step is the following:

Lemma 20. *Let* T *be a stopping time for* Π. *Then one has*

$$\sum_{i \in \mathbb{N}} \bar{\lambda}_i(T) \leq 1$$

and $\bar{\lambda}_1$ *and* $\underline{\lambda}_1$ *are right continuous at* T.

Proof. For the first point, suppose that $\sum_i \bar{\lambda}_i(T) = 1+\gamma > 1$. Then there exists $N \in \mathbb{N}$ such that $\sum_{i\le N} \bar{\lambda}_i(T) > 1+\gamma/2$. Call $w_i(t)$ the least element of $B_i(t)$. Let S be the random stopping time defined as the first time after T such that S is a coalescence involving at least two of the $w_N(T)$ first blocks

$$S = \inf\{s \ge T : \pi^{(C)}(s) \notin Q(\mathbf{0}, w_N(T))\}.$$

Hence, between T and S, for each $i \le N$ one has that $w_i(T)$ is the least element of its block. Applying the Markov property in T we have that $S-T$ is exponential with a finite parameter and is thus almost surely positive.

Define $(\Pi^{(F)}(T+s), s \ge 0)$ as the fragmentation process started from $\Pi(T)$ and constructed from the PPP $(P_F(T+s), s \ge 0)$. On the time interval $[T, S]$ one has that for each i, the block of $\Pi^{(F)}$ that contains w_i is included in the block of Π that contains w_i (because the last might have coalesced with blocks whose least element is larger than $w_N(T)$).

Fix $\epsilon > 0$, using (11) and the above remark, one has that for each $i \le N$ there exists a.s. a $\theta_i > 0$ such that for all $t \in [T, T+\theta_i]$ one has

$$\bar{\lambda}(w_i(T), t) > (1-\epsilon)\bar{\lambda}(w_i(T), T).$$

Thus, if $\theta = \min_{i\le N} \theta_i > 0$ one has that

$$\min_{s\in[T,T+\theta]} \sum_i \bar{\lambda}_i(s) > (1+\gamma/2)(1-\epsilon).$$

As the random set $\mathbb{Q}\cap[T, T+\theta]$ is almost surely not empty, choosing ϵ small enough yields a contradiction with the fact that almost surely for all $t \in \mathbb{Q}$ one has $\sum \bar{\lambda}_i(t) \le 1$.

Fix $\epsilon > 0$, the first part of the lemma implies that there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{i\ge N_\epsilon} \bar{\lambda}_i(T) \le \epsilon.$$

Let $(\Pi^{(F)}(T+s), s \ge 0)$ be as above a fragmentation started from $\Pi(T)$ and constructed with the PPP $(P_F(T+s), s \ge 0)$. As we have noted

$$\bar{\lambda}_1(T+s) \ge \bar{\lambda}_1^{(F)}(T+s) \to \bar{\lambda}_1^{(F)}(T) \tag{12}$$

$$\underline{\lambda}_1(T+s) \ge \underline{\lambda}_1^{(F)}(T+s) \to \underline{\lambda}_1^{(F)}(T) \tag{13}$$

when $s \searrow 0$.

Now consider

$$S = \inf\{s \ge T : \pi^{(C)}_{|[N_\epsilon]}(s) \ne \mathbf{0}_{N_\epsilon}\}$$

the first coalescence time after T such that $\pi^{(C)}_{|[N_\epsilon]}(s) \neq \mathbf{0}_{N_\epsilon}$. One has $\forall s \in [T,S]$

$$\begin{aligned} \bar{\lambda}_1(T+s) &\leq \bar{\lambda}_1(T) + \sum_{i \geq N_\epsilon} \bar{\lambda}_i(T) \leq \bar{\lambda}_1(T) + \epsilon \\ \underline{\lambda}_1(T+s) &\leq \underline{\lambda}_1(T) + \sum_{i \geq N_\epsilon} \bar{\lambda}_i(T) \leq \underline{\lambda}_1(T) + \epsilon. \end{aligned}$$

Thus $\bar{\lambda}_1(T+s) \to \bar{\lambda}_1(T)$ and $\underline{\lambda}_1(T+s) \to \underline{\lambda}_1(T)$ when $s \searrow 0$. $\square$

To conclude the proof of the first point of Theorem 7, observe that as the map $\Pi(t) \mapsto \bar{\lambda}_1(t)$ is measurable in $\mathcal{F}_t$, the right-continuous usual augmentation of the filtration, one has that for any $\epsilon > 0$

$$\inf\{t : |\limsup_{s \searrow 0} \bar{\lambda}_1(t+s) - \bar{\lambda}_1(t)| > \epsilon\}$$

or

$$\inf\{t : |\liminf_{s \searrow 0} \bar{\lambda}_1(t+s) - \bar{\lambda}_1(t)| > \epsilon\}$$

are stopping times for Π in $\mathcal{F}$. The above lemma applies and hence this stopping times are almost surely infinite. The same argument works for $\underline{\lambda}_1$. This shows that $\bar{\lambda}_1$ and $\underline{\lambda}_1$ are almost surely right-continuous processes. As they coincide almost surely with L on the set of rationals, they coincide everywhere and hence their paths are almost surely càdlàg.

Before we can prove rigourously that $X(t)$ is a Feller process, as stated in Theorem 7, we have to pause for a moment to define a few notions related to the laws of EFC processes conditioned on their starting point. By our definition, an EFC process Π is exchangeable. Nevertheless, if P is the law of Π and P_π is the law of Π conditionally on $\Pi(0) = \pi$, one has that as soon as $\pi \neq \mathbf{0}$ or $\mathbf{1}$, the process Π is not exchangeable under P_π (because for instance $\Pi(0)$ is not exchangeable). The process Π conditioned by $\Pi(0) = \pi$ (i.e., under the law P_π) is called an EFC evolution. Clearly one can construct every EFC evolution exactly as the EFC processes, or more precisely, given the PPP's P_F and P_C one can then choose any initial state π and construct the EFC evolution $\Pi, \Pi(0) = \pi$ with P_F and P_C as usually. Let us first check quickly that under P_π we still have the existence of $X(t)$ simultaneously for all t.

In the following we will say that a partition $\pi \in \mathcal{P}$ is *good* if $\Lambda(\pi)$ exists, there are no finite blocks of cardinality greater than 1 and either $\mathrm{dust}(\pi) = \emptyset$ or $\|\mathrm{dust}(\pi)\| > 0$.

Lemma 21. *For each $\pi \in \mathcal{P}$ such that π is good, then P_π-a.s. the process $X(t) = \Lambda(\Pi(t))$ exists for all t simultaneously and we call Q_π its law.*

Proof. Consider $\pi = (B_1, B_2, ...)$ a good partition. For each $i \in \mathbb{N}$ such that $\# B_i = \infty$, let $f_i : \mathbb{N} \mapsto \mathbb{N}$ be the only increasing map that send B_i on $\mathbb{N}$. Let $B_0 = \cup_{i:\# B_i < \infty} B_i$ and if B_0 is infinite(which is the case whenever it is not empty) set $g : \mathbb{N} \mapsto \mathbb{N}$ the unique increasing map that send B_0 onto $\mathbb{N}$.

Using the exchangeability properties attached to the PPP's P_F and P_C one can easily see that for each $i \in \mathbb{N}$ such that $\# B_i = \infty$,

$$f_i(\Pi(t)_{|B_i})$$

and

$$g(\Pi(t)_{|B_0})$$

are EFC processes with initial state $\mathbf{1}$ for the first ones and $\mathbf{0}$ for the later. Hence for each i one has that $f_i(\Pi(t)_{|B_i})$ has asymptotic frequencies $X^{(i)}(t) := \Lambda(f_i(\Pi(t)_{|B_i}))$ simultaneously for all t. Thus it is not hard to see from this that $\Pi(t)_{|B_i}$ has asymptotic frequencies simultaneously for all t, namely $\|B_i\| X^{(i)}(t)$.

Fix $\epsilon > 0$, there exists N_ϵ such that

$$\|B_0\| + \sum_{i \leq \mathbb{N}_\epsilon} \|B_i\| \geq 1 - \epsilon.$$

If we call $\Pi(t) = (B_1(t), B_2(t), ...)$ the blocks of $\Pi(t)$, we thus have that for $j \in \mathbb{N}$ fixed

$$\bar{\lambda}_j(t) \leq \sum_{i \leq N_\epsilon} \|B_j(t) \cap B_i\| + \epsilon$$

and

$$\underline{\lambda}_j(t) \geq \sum_{i \leq N_\epsilon} \|B_j(t) \cap B_i\|.$$

Hence

$$\sup_{t \geq 0} \sup_{i \in \mathbb{N}} (\bar{\lambda}_i(t) - \underline{\lambda}_i(t)) \leq \epsilon.$$

As ϵ is arbitrary this shows that almost surely $\sup_{t\geq 0} \sup_{i \in \mathbb{N}} (\bar{\lambda}_i(t) - \underline{\lambda}_i(t)) = 0$. We call Q_π the law of $X(t)$ under P_π. □

Although EFC evolutions are not exchangeable, they do have a very similar property:

Lemma 22. *Let* $(\Pi_1(t), t \geq 0)$ *be an EFC evolution with law* P_{π_1} *(i.e.,* $P(\Pi_1(0) = \pi_1) = 1$*) and with characteristics* $\nu_{Disl}, \nu_{Coag}, c_k$ *and* c_e*. Then for any bijective map* $\sigma : \mathbb{N} \mapsto \mathbb{N}$ *the process* $\Pi_2(t) := (\sigma^{-1}(\Pi_1(t)), t \geq 0)$ *is an EFC evolution with law* $P_{\sigma^{-1}(\pi_1)}$ *and same characteristics.*

Proof. Consider $\Pi_1(t) = (B_1^{(1)}(t), B_2^{(1)}(t), ...)$ an EFC evolution with law P_{π_1} (i.e., started from π_1) and constructed with the PPP's P_F and P_C. Let $\pi_2 = \sigma^{-1}(\pi_1)$ and $(\Pi_2(t), t \geq 0) = (\sigma^{-1}(\Pi_1(t)), t \geq 0)$. For each $t \geq 0$ and $k \in \mathbb{N}$ call $\phi(t, k)$ the label of the block $\sigma(B_k^{(1)}(t-))$ in $\Pi_2(t-)$. By construction, $\Pi_2(t)$ is a $\mathcal{P}$-valued process started from π_2. When P_F has an atom, say $(k(t), \pi^{(F)}(t))$ the block of $\Pi_2(t-)$ which fragments has the label $\phi(t, k(t))$ and the fragmentation is done by taking the intersection with $\sigma^{-1}(\pi^{(F)}(t))$. Call $\tilde{P}_F$ the point process of the images of the atoms of P_F by the transformation

$$(t, k(t), \pi^{(F)}(t)) \mapsto (t, \phi(t, k(t)), \sigma^{-1}(\pi^{(F)}(t))).$$

If t is an atom time for P_C, say $\pi^{(C)}(t)$, then Π_2 also coalesces at t, and if the blocks i and j merge at t in Π_1 then the blocks $\phi(t, i)$ and $\phi(t, j)$ merge at t for Π_2, hence the coalescence is made with the usual rule by the partition $\phi^{-1}(t, \pi^{(C)}(t))$. Call $\tilde{P}_C$ the point process image of P_C by the transformation

$$(t, \pi^{(C)}(t)) \mapsto (t, \phi^{-1}(t, \pi^{(C)}(t))).$$

We now show that $\tilde{P}_C$ and $\tilde{P}_F$ are PPP with the same measure intensity as P_C and P_F respectively. The idea is very close to the proof of Lemma 3.4 in [3]. Let us begin with $\tilde{P}_F$. Let $A \subset \mathcal{P}$ such that $(\mu_{\nu_{Disl}} + c_e \mathbf{e})(A) < \infty$ and define

$$N_A^{(i)}(t) := \#\{u \leq t : \sigma(\pi^{(F)}(u)) \in A, k(u) = i\}.$$

Then set

$$N_A(t) := \#\{u \leq t : \sigma(\pi^{(F)}(u)) \in A, \phi(u, k(u)) = 1\}.$$

By definition

$$dN_A(t) = \sum_{i=1}^{\infty} {}_{\{\phi(t,i)=1\}} dN_A^{(i)}(t).$$

The process N_A is increasing, càdlàg and has jumps of size 1 because by construction the $N_A^{(i)}$ do not jump at the same time almost surely. Define the counting processes $\bar{N}_A^{(i)}(t)$ by the following differential equation

$$d\bar{N}_A^{(i)}(t) = {}_{\{\phi(t,i)=1\}} dN_A^{(i)}(t).$$

It is clear that ${}_{\{\phi(t,i)=1\}}$ is adapted and left-continuous in $(\mathcal{F}_t)$ the natural filtration of Π_1 and hence predictable. The $N_A^{(i)}(\cdot)$ are i.i.d. Poisson process with intensity $(\mu_{\nu_{Disl}} + c_e\mathbf{e})(A) = (\mu_{\nu_{Disl}} + c_e\mathbf{e})(\sigma^{-1}(A))$ in $(\mathcal{F}_t)$. Thus for each i the process

$$\begin{aligned} M_A^{(i)}(t) &= \bar{N}_A^{(i)}(t) - (\mu_{\nu_{Disl}} + c_e\mathbf{e})(A) \int_0^t {}_{\{\phi(u,i)=1\}} du \\ &= \int_0^t {}_{\{\phi(u,i)=1\}} d\big(N_A^{(i)}(u) - (\mu_{\nu_{Disl}} + c_e\mathbf{e})(A)u\big) \end{aligned}$$

is a square-integrable martingale.

Define

$$M_A(t) := \sum_{i=1}^{\infty} \int_0^t {}_{\{\phi(u,i)=1\}} d(N_A^{(i)}(u) - (\mu_{\nu_{Disl}} + c_e\mathbf{e})(A)u).$$

For all $i \neq j$, for all $t \geq 0$ one has ${}_{\{\phi(t,i)=1\}}\ {}_{\{\phi(t,j)=1\}} = 0$ and for all $t \geq 0$ one has $\sum_{i=1}^{\infty}\ {}_{\{\phi(u,i)=1\}} = 1$, the $M_A^{(i)}$ are orthogonal (because they do not share any jump-time) and hence the oblique bracket of M_A is given by

$$\begin{aligned} < M_A > (t) &= \sum_{i=1}^{\infty} \left\langle \int_0^t {}_{\{\phi(u,i)=1\}} d(N_A^{(i)}(u) - (\mu_{\nu_{Disl}} + c_e\mathbf{e})(A)u) \right\rangle \\ &= (\mu_{\nu_{Disl}} + c_e\mathbf{e})(A)t. \end{aligned}$$

Hence M_A is a L_2 martingale. This shows that $N_A(t)$ is increasing càdlàg with jumps of size 1 and has $(\mu_{\nu_{Disl}} + c_e\mathbf{e})(A)t$ as compensator. We conclude that $N_A(t)$ is a Poisson process of intensity $(\mu_{\nu_{Disl}}+c_e\mathbf{e})(A)$. Now take $B \subset \mathcal{P}$ such that $A \cap B = \emptyset$ and consider $N_A(t)$ and $N_B(t)$, clearly they do not share any jump time because the $N_A^{(i)}(t)$ and $N_B^{(i)}(t)$ don't. Hence

$$P_F^{(1)}(t) = \{\sigma(\pi^{(F)}(u)) : u \leq t, \phi(u, k(t)) = 1\}$$

is a PPP with measure-intensity $(\mu_{\nu_{Disl}} + c_e\mathbf{e})$. Now, by the same arguments

$$P_F^{(2)}(t) = \{\sigma(\pi^{(F)}(u)) : u \leq t, \phi(u, k(t)) = 2\}$$

is also a PPP with measure-intensity $(\mu_{\nu_{Disl}} + c_e\mathbf{e})$ independent of $P_F^{(1)}$. By iteration we see that $\tilde{P}_F$ is a PPP with measure intensity $(\mu_{\nu_{Disl}} + c_e\mathbf{e}) \otimes \#$.

Let us now treat the case of $\tilde{P}_C$. The main idea is very similar since the first step is to show that for $n \in \mathbb{N}$ fixed and $\pi \in \mathcal{P}$ such that $\pi_{|[n]} \neq \mathbf{0}_n$ one has that the counting process

$$N_{\pi,n}(t) = \#\{u \leq t : \phi^{-1}(u, \pi^{(C)}(u))_{|[n]} = \pi_{|[n]}\}$$

is a Poisson process with intensity $(\mu_{\nu_{Coag}} + c_k\kappa)(Q(\pi, n))$.

For each unordered collection of n distinct elements in $\mathbb{N}$, say $\mathbf{a} = a_1, a_2, ..., a_n$, let $\sigma_{\mathbf{a}}$ be a permutation such that for each $i \leq n$, $\sigma_{\mathbf{a}}(i) = a_i$.

For each $\mathbf{a}$ define

$$N_{\mathbf{a},\pi}(t) = \#\{u \leq t : (\sigma_{\mathbf{a}}(\pi^{(C)}(u)))_{|[n]} = \pi_{|[n]}\}.$$

By exchangeability $N_{\mathbf{a},\pi}(t)$ is a Poisson process with measure intensity $(\mu_{\nu_{Coag}} + c_k\kappa)(Q(\pi, n))$.

By construction

$$dN_{\pi,n}(t) = \sum_{\mathbf{a}} \prod_{i=1}^{n} \quad {}_{\{\phi(t,a_i)=i\}} dN_{\mathbf{a},\pi}(t).$$

We see that we are in a very similar situation as before: the $N_{\mathbf{a},\pi}(t)$ are not independent but at all time t there is exactly one $\mathbf{a}$ such that

$$\prod_{i=1}^{n} \quad {}_{\{\phi(t,a_i)=i\}} = 1$$

and hence one can define orthogonal martingales $M_{\mathbf{a}}(t)$ as we did for the $M_A^{(i)}(t)$ above and conclude in the same way that $N_{\pi,n}(t)$ is a Poisson process with measure intensity $(\mu_{\nu_{Coag}} + c_k\kappa)(Q(\pi, n))$. If we now take $\pi' \in \mathcal{P}$ such that $\pi'_{|[n]} \neq \pi_{|[n]}$ we have that $N_{\pi,n}(t)$ and $N_{\pi',n}(t)$ are independent because for each fixed $\mathbf{a}$ the processes given by the equation

$$dM_{\mathbf{a},\pi}(t) = \prod_{i=1}^{n} \quad {}_{\{\phi(t,a_i)=i\}} dN_{\mathbf{a},\pi}(t)$$

and

$$dM_{\mathbf{a},\pi'}(t) = \prod_{i=1}^{n} \quad {}_{\{\phi(t,a_i)=i\}} dN_{\mathbf{a},\pi'}(t)$$

respectively do not have any common jumps. Hence $N_{\pi,n}(t)$ and $N_{\pi',n}(t)$ are independent and thus we conclude that $\tilde{P}_C$ is a PPP with measure intensity $\mu_{\nu_{Coag}} + c_k\kappa$.

Putting the pieces back together we see that Π_2 is an EFC evolution with law P_{π_2} and same characteristics as Π_1.

$\square$

For each $\pi \in \mathcal{P}$ such that $\Lambda(\pi) = x$ exists, and for each $k \in \mathbb{N}$ we define $n_\pi(k)$ the label of the block of π which corresponds to x_k, i.e., $\|B_{n_\pi(k)}\| = x_k$. In the case where two B_k's have the same asymptotic frequency we use the order of the least element, i.e., if there is an i such that $x_i = x_{i+1}$ one has $n_\pi(i) < n_\pi(i+1)$. The map $i \mapsto n_\pi(i)$ being bijective, call m_π its inverse. Furthermore we define $B_0 = \cup_{i:\|B_i\|=0} B_i$ and $x_0 = \|B_0\| = 1 - \sum_{i\in\mathbb{N}} x_i$. In the following we will sometimes write $\pi = (B_0, B_1, ...)$ and $x = (x_0, x_1, ...)$.

Let $\pi = (B_1, B_2, ...), \pi' = (B'_1, B'_2, ...) \in \mathcal{P}$ be two good partitions. We write $\Lambda(\pi) = x = (x_1, x_2, ...)$ and $\Lambda(\pi') = x' = (x'_1, x'_2, ...)$. Suppose furthermore that either $x_0 = 0$ and $x'_0 = 0$ or they are both strictly positive and that

$$\inf\{k \in \mathbb{N} : x_k = 0\} = \inf\{k \in \mathbb{N} : x'_k = 0\}.$$

Define $\sigma_{\pi,\pi'}$ as the unique bijection $\mathbb{N} \mapsto \mathbb{N}$ that maps every $B_{n_\pi(i)}$ onto $B'_{n_{\pi'}(i)}$ such that if $j, k \in B_{n_\pi(i)}$ with $j < k$ then $\sigma_{\pi,\pi'}(j) < \sigma_{\pi,\pi'}(k)$. Note that this definition implies that $\sigma_{\pi,\pi'}(B_0) = B'_0$. Furthermore we have $\pi' = \sigma^{-1}_{\pi,\pi'}(\pi)$. We will also need the convention $\sigma_{\pi,\pi'}(0) = 0$.

We will use the following technical lemma:

Lemma 23. *For π, π' fixed in $\mathcal{P}$ satisfying the above set of hypotheses, let $\Pi(t) = (B_1(t), B_2(t), ...)$ be an EFC evolution started from π with law P_π, then*

- *$\Lambda(\pi \cap \Pi(t))$ exists almost surely for all $t \geq 0$ simultaneously where $\pi \cap \Pi(t)$ is defined by $i \overset{\pi\cap\Pi(t)}{\sim} j$ if and only if we have both $i \overset{\Pi(t)}{\sim} j$ and $i \overset{\pi}{\sim} j$.*

- *$\Lambda(\sigma^{-1}_{\pi,\pi'}(\Pi(t) \cap \pi))$ also exists a.s. for all $t \geq 0$ and for each $j, k \in \mathbb{N}$ one has*
$$\|\sigma_{\pi,\pi'}(B_j(t) \cap B_k)\| = \frac{x'_{m_\pi(k)}}{x_{m_\pi(k)}} \|B_j(t) \cap B_k\|.$$

- *$\Lambda(\sigma^{-1}_{\pi,\pi'}(\Pi(t)))$ exists a.s. $\forall t \geq 0$ and for each $j \in \mathbb{N}$*
$$\|\sigma_{\pi,\pi'}(B_j(t))\| = \sum_{k\geq 0} \frac{x'_{m_\pi(k)}}{x_{m_\pi(k)}} \|B_j(t) \cap B_k\|.$$

Proof. For an infinite subset $B \subset \mathbb{N}$ call F_B the increasing map that sends $\mathbb{N}$ onto B. Then by construction for each $k \in \mathbb{N}$ one has that $F_{B_k}(\Pi(t)_{|B_k})$ is a $\mathcal{P}$ valued EFC process started from $\mathbf{1}$ and $F_{B_0}(\Pi(t)_{|B_0})$ is a $\mathcal{P}$ valued EFC process started from $\mathbf{0}$. Hence, $\Lambda(\pi \cap \Pi(t))$ exists (as well as the asymptotic frequencies of the blocks of the form $B_k(t) \cap B_0$).

Now for the second point, for each $j, k \in \mathbb{N}$ define

$$s_{j,k}(n) := \max\{m \leq n : k \in \sigma_{\pi,\pi'}(B_j(t) \cap B_k)\}$$

and for each $k \in \mathbb{N}$

$$s_k(n) := \max\{m \leq n : k \in \sigma_{\pi,\pi'}(B_k)\}$$

with the convention that $\max \varnothing = 0$ and $0/0 = 0$.

First we observe that if A and B are two subsets of $\mathbb{N}$ such that A and $A \cap B$ both have asymptotic frequencies then

$$\frac{\max\{i \leq n, i \in (A \cap B)\}}{\max\{i \leq n, i \in B\}} \to 1$$

as $n \to \infty$. Hence, using that if $B_k \neq \varnothing$ then $\sigma^{-1}_{\pi,\pi'}(s_k(n)) \to \infty$ when $n \nearrow \infty$ one has

$$\frac{\sigma^{-1}_{\pi,\pi'}(s_{j,k}(n))}{\sigma^{-1}_{\pi,\pi'}(s_k(n))} = \frac{\max\{i \leq \sigma^{-1}_{\pi,\pi'}(s_k(n)), i \in (B_j(t) \cap B_k)\}}{\max\{i \leq \sigma^{-1}_{\pi,\pi'}(s_k(n)), i \in B_k\}} \to 1. \tag{14}$$

Thus

$$\frac{\#\{k \leq \sigma^{-1}_{\pi,\pi'}(s_{j,k}(n)) : k \in B_k\}}{\#\{k \leq \sigma^{-1}_{\pi,\pi'}(s_k(n)) : k \in B_k\}} \to 1,\ n \to \infty. \tag{15}$$

Note that as either $s_{j,k}(n) \nearrow \infty$ or $s_{j,k}(n) \equiv 0$ we can use (15) to get

$$\frac{\#\{k \leq \sigma^{-1}_{\pi,\pi'}(s_{j,k}(n)) : k \in B_j(t) \cap B_k\}}{\#\{k \leq \sigma^{-1}_{\pi,\pi'}(s_k(n)) : k \in B_k\}} \to \frac{\|B_j(t) \cap B_k\|}{x_{m_\pi(k)}} \quad \text{as } n \to \infty.$$

Furthermore, by definition

$$\#\{m \leq n : m \in \sigma_{\pi,\pi'}(B_k)\} = \#\{m \leq \sigma^{-1}_{\pi,\pi'}(s_k(n)) : m \in B_k\}.$$

Hence the following limit exists and

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n} \#\{m \leq n : m \in \sigma_{\pi,\pi'}(B_j(t) \cap B_k)\} \\
= \quad & \lim_{n\to\infty} \frac{1}{n} \Bigg(\frac{\#\{m \leq \sigma^{-1}_{\pi,\pi'}(s_{j,k}(n)) : m \in B_j(t) \cap B_k\}}{\#\{m \leq \sigma^{-1}_{\pi,\pi'}(s_k(n)) : m \in B_k\}} \\
& \qquad\qquad \#\{m \leq n : m \in \sigma_{\pi,\pi'}(B_k)\} \Bigg) \\
= \quad & x'_{m_{\pi'}(k)} \frac{\|B_j(t) \cap B_k\|}{x_{m_\pi(k)}}.
\end{aligned}$$

The same argument works when $k = 0$. For the last point it is enough to note that for each k

$$\begin{aligned}\|\sigma_{\pi,\pi'}(B_k(t))\| &= \| \cup_{i=0}^{\infty} \sigma_{\pi,\pi'}(B_k(t) \cap B_i)\| \\ &= \sum_{i=0}^{\infty} \|\sigma_{\pi,\pi'}(B_k(t) \cap B_i)\|.\end{aligned}$$

□

The key lemma to prove the proposition is the following:

Lemma 24. *Consider $\pi_1, \pi_2 \in \mathcal{P}$ with the same hypothesis as in the above lemma. Suppose furthermore that $\Lambda(\pi_1) = \Lambda(\pi_2)$. Then*

$$Q_{\pi_1} = Q_{\pi_2}.$$

Proof. We have $\pi_1 = (B_1^{(1)}, B_2^{(1)}, ...)$ and $\pi_2 = (B_1^{(2)}, B_2^{(2)}, ...)$. To ease the notations, call $\sigma = \sigma_{\pi_1,\pi_2}$. Note that we have $\pi_2 = \sigma^{-1}(\pi_1)$.

Lemma 22 implies that the law of $\sigma^{-1}(\Pi_1(t))$ is P_{π_2}. Lemma 23 yields that for each k one has

$$\forall t \geq 0 : \|\sigma(B_k^{(1)}(t))\| = \|B_k^{(1)}(t)\|$$

and hence $\Lambda(\sigma^{-1}(\Pi_1(\cdot)) = \Lambda(\Pi_1(\cdot))$. As the distributions of $\Lambda(\Pi_1(\cdot))$ and $\Lambda(\sigma^{-1}(\Pi_1(\cdot))$ are respectively Q_{π_1} and Q_{π_2} one has

$$Q_{\pi_1} = Q_{\pi_2}.$$

□

A simple application of Dynkin's criteria for functionals of Markov processes to be Markov (see Theorem 10.13 (page 325) in [13]) concludes the proof of the "Markov" part of Proposition 7. For the "Fellerian" part, for $x \in \mathcal{S}^{\downarrow}$, call $(Q_x(t), t \geq 0)$ the law of $X(t)$ conditionally on $X(0) = x$. As X is right-continuous we must only show that for t fixed $x \mapsto Q_x(t)$ is continuous.

Let $x^{(n)} \to x$ when $n \to \infty$. The idea is to construct a sequence of random variables $X^{(n)}(t)$ each one with law $Q_{x_n}(t)$ and such that $X^{(n)}(t) \to X(t)$ almost surely and where $X(t)$ has law $Q_x(t)$.

Take $\pi = (B_0, B_1, B_2, ...) \in \mathcal{P}$ such that $\Lambda(\pi) = x$. For each n let π_n be a partition such that $\Lambda(\pi_n) = x^{(n)}$ and call $\sigma_n = \sigma_{\pi,\pi_n}$. [4] Furthermore

[4]To be rigorous one should extend the definition of σ_{π,π_n} to allow for the cases where π and π_n do not have the same number of blocks. To do so, the extra blocks should then be assimilated to dust.

it should be clear that we can choose π_n such that for each $k \leq n$ one has $m_{\pi_n}(k) = m_\pi(k)$. Hence, one has that for each $j \geq 0$: $x^{(n)}_{m_\pi(j)} \to x_{m_\pi(j)}$ when $n \to \infty$ because $x^{(n)} \to x$.

As we have observed, for each n the process $X^{(n)}(t) = \Lambda((\sigma_n)^{-1}(\Pi(t)))$ where $\Pi(\cdot) = (B_1(\cdot), B_2(\cdot), ...)$ has law P_π exists and has law $Q_{x^{(n)}}(t)$.

Using the Lemma 23 one has that

$$\|\sigma_n(B_j(t))\| = \sum_{k\geq 0} \frac{x^{(n)}_{m_{\pi_n}(k)}}{x_{m_\pi(k)}} \|B_j(t) \cap B_k\|.$$

This entails that for each j one has a.s.

$$\|\sigma_n(B_j(t))\| \to \|B_j(t)\|, \text{ when } n \to \infty.$$

Hence, almost surely, $X^{(n)}(t) \to X(t)$ as $n \to \infty$ and thus $Q_{x^{(n)}}(t) \to Q_x(t)$ in the sense of convergence of finite dimensional marginals.

6.3 Proof of Theorem 9, part 1

Proof. We will prove that for each $K \in \mathbb{N}$ one has $\rho(\{\pi : \#\pi = K\}) = 0$.

Let us write the equilibrium equations for $\rho^{(n)}(\cdot)$, the invariant measure of the Markov chain $\Pi_{|[n]}$. For each $\pi \in \mathcal{P}_n$

$$\rho^{(n)}(\pi) \sum_{\pi' \in \mathcal{P}_n \setminus \{\pi\}} q_n(\pi, \pi') = \sum_{\pi'' \in \mathcal{P}_n \setminus \{\pi\}} \rho^{(n)}(\pi'') q_n(\pi'', \pi)$$

where $q_n(\pi, \pi')$ is the rate at which $\Pi_{|[n]}$ jumps from π to π'. Fix $K \in \mathbb{N}$ and for each $n \geq K$, call $A_{n,K} := \{\pi \in \mathcal{P}_n : \#\pi \leq K\}$ and $D_{n,K} := \mathcal{P}_n \setminus A_{n,K}$ where $\#\pi$ is the number of non-empty blocks of π.

Summing over $A_{n,K}$ yields

$$\begin{aligned} & \sum_{\pi \in A_{n,K}} \rho^{(n)}(\pi) \left[\sum_{\pi' \in A_{n,K} \setminus \{\pi\}} q_n(\pi, \pi') + \sum_{\pi' \in D_{n,K}} q_n(\pi, \pi') \right] \\ = \quad & \sum_{\pi \in A_{n,K}} \left[\sum_{\pi'' \in A_{n,K} \setminus \{\pi\}} \rho^{(n)}(\pi'') q_n(\pi'', \pi) + \sum_{\pi'' \in D_{n,K}} \rho^{(n)}(\pi'') q_n(\pi'', \pi) \right] \end{aligned}$$

but as

$$\sum_{\pi\in A_{n,K}} \rho^{(n)}(\pi) \left[\sum_{\pi'\in A_{n,K}\setminus\{\pi\}} q_n(\pi,\pi') \right]$$
$$= \sum_{\pi\in A_{n,K}} \left[\sum_{\pi''\in A_{n,K}\setminus\{\pi\}} \rho^{(n)}(\pi'') q_n(\pi'',\pi) \right]$$

one has

$$\sum_{\pi\in A_{n,K}} \rho^{(n)}(\pi) \left[\sum_{\pi'\in D_{n,K}} q_n(\pi,\pi') \right] = \sum_{\pi\in A_{n,K}} \left[\sum_{\pi''\in D_{n,K}} \rho^{(n)}(\pi'') q_n(\pi'',\pi) \right].$$

That is, if we define $q_n(\pi, C) = \sum_{\pi'\in C} q_n(\pi,\pi')$ for each $C \subseteq \mathcal{P}_n$,

$$\sum_{\pi\in A_{n,K}} \rho^{(n)}(\pi) q_n(\pi, D_{n,K}) = \sum_{\pi''\in D_{n,K}} \rho^{(n)}(\pi'') q_n(\pi'', A_{n,K}). \quad (16)$$

Therefore

$$\sum_{\pi\in A_{n,K}\setminus A_{n,K-1}} \rho^{(n)}(\pi) q_n(\pi, D_{n,K}) \leq \sum_{\pi''\in D_{n,K}} \rho^{(n)}(\pi'') q_n(\pi'', A_{n,K}). \quad (17)$$

Hence, all we need to prove that $\rho(\{\pi \in \mathcal{P} : \#\pi_{|[n]} = K\}) \to 0$ when $n \to \infty$ is to give an upper bound for the right hand-side of (17) which is uniform in n and to show that

$$\min_{\pi\in A_{n,K}\setminus A_{n,K-1}} q_n(\pi, D_{n,K}) \underset{n\to\infty}{\to} \infty. \quad (18)$$

Let us begin with (18). Define

$$\Phi(q) := c_e(q+1) + \int_{\mathcal{S}^\downarrow} (1 - \sum_i x_i^{q+1}) \nu_{Disl}(dx).$$

This function was introduced by Bertoin in [6], where it plays a crucial role as the Laplace exponent of a subordinator; in particular, Φ is a concave increasing function. When k is an integer greater or equal than 2, $\Phi(k-1)$ is the rate at which $\{[k]\}$ splits, i.e., it is the arrival rate of atoms $(\pi^{(F)}(t), k(t), t)$ of P_F such that $\pi^{(F)}_{|[k]}(t) \neq \mathbf{1}_k$ and $k(t) = 1$. More precisely $c_e k$ is the rate of arrival of atoms that correspond to erosion and $\int_{\mathcal{S}^\downarrow} (1 - \sum_i x_i^k) \nu_{Disl}(dx)$

is the rate of arrival of dislocations. Hence, for $\pi \in \mathcal{P}_n$ such that $\#\pi = K$, say $\pi = (B_1, B_2, ..., B_K, \emptyset, \emptyset, ...)$, one has

$$q_n(\pi, D_{n,K}) = \sum_{i:|B_i|>1} \Phi(|B_i| - 1)$$

because it only takes a fragmentation that creates at least one new block to enter $D_{n,K}$.

First, observe that

$$\sum_{i:|B_i|>1} c_e|B_i| \geq c_e(n - K + 1),$$

next, note that

$$\tilde{\Phi} : q \mapsto \tilde{\Phi}(q) = \int_{\mathcal{S}^\downarrow} (1 - \sum_i x_i^{q+1}) \nu_{Disl}(dx)$$

is also concave and increasing for the same reason that Φ is and furthermore

$$\tilde{\Phi}(0) \geq 0.$$

This ensure that for any $n_1, n_2, ..., n_K \in (\mathbb{N}^*)^K$ such that $\sum_{i=1}^K n_i = n$ one has

$$\sum_{i=1}^K \tilde{\Phi}(n_i - 1) \geq \tilde{\Phi}(n - K) + (K - 1)\tilde{\Phi}(0).$$

Hence, for every $(B_1, ..., B_K) \in \mathcal{P}_n$ one has the lower bound

$$\begin{aligned} q_n(\pi, D_{n,K}) &= \sum_{i:|B_i|>1} \Phi(|B_i| - 1) \\ &\geq \int_{\mathcal{S}^\downarrow} (1 - \sum_i x_i^{(n-K)+1}) \nu_{Disl}(dx) + c_e(n - K + 1). \end{aligned}$$

As $\Phi(x) \underset{x\to\infty}{\to} \infty \Leftrightarrow \nu_{Disl}(\mathcal{S}^\downarrow) = \infty$ or $c_e > 0$ one has

$$c_e > 0 \text{ or } \nu_{Disl}(\mathcal{S}^\downarrow) = \infty \Rightarrow \lim_{n\to\infty} \min_{\pi:\#\pi=K} q_n(\pi, D_{n,K}) = \infty.$$

On the other hand it is clear that for $\pi \in D_{n,K}$ the rate $q_n(\pi, A_{n,K})$ only depends on $\#\pi$ and K (by definition the precise state π and n play no role in this rate). By compatibility it is easy to see that if π, π' are such that $\#\pi' > \#\pi = K+1$ then

$$q_n(\pi, A_{n,K}) \geq q_n(\pi', A_{n,K}).$$

Hence, for all $\pi \in D_{n,K}$ one has

$$q_n(\pi, A_{n,K}) \leq \tau_K$$

where $\tau_K = q_n(\pi', A_{n,K})$ for all n and any $\pi' \in \mathcal{P}_n$ such that $\#\pi' = K+1$, and hence τ_K is a constant that only depends on K.

Therefore

$$\begin{aligned}
\min_{\pi \in \mathcal{P}_n : \#\pi = K} q_n(\pi, D_{n,K}) \sum_{\pi \in \mathcal{P}_n : \#\pi = K} \rho^{(n)}(\pi) &\leq \sum_{\pi \in \mathcal{P}_n : \#\pi = K} \rho^{(n)}(\pi) q_n(\pi, D_{n,K}) \\
&\leq \sum_{\pi'' \in D_{n,K}} \rho^{(n)}(\pi'') q_n(\pi'', A_{n,K}) \\
&\leq \tau_K \sum_{\pi'' \in D_{n,K}} \rho^{(n)}(\pi''),
\end{aligned}$$

where, on the second inequality, we used (17). Thus

$$\rho^{(n)}(\{\pi \in \mathcal{P}_n : \#\pi = K\}) \leq \tau_K / \min_{\pi \in \mathcal{P}_n : \#\pi = K} q_n(\pi, D_{n,K}).$$

This show that for each $K \in \mathbb{N}$, one has $\lim_{n \to \infty} \rho^{(n)}(\{\pi \in \mathcal{P}_n : \#\pi = K\}) = 0$ and thus $\rho(\#\pi < \infty) = 0$. $\square$

6.4 Proof of Theorem 9, part 2

Proof. We use the connection explained before Proposition 15. The set of conditions in Theorem 9, part 2 is just (L) and (H). Hence we can apply Theorem 3.1 in [18] to see that the process $\#\Pi(t)$ started from ∞ is positive recurrent in $\mathbb{N}$ and converges in distribution to some probability distribution on $\mathbb{N}$.

The proof of the first remark after Theorem 9 is the following.

For each $n \in \mathbb{N}$ we define the sequence $(a_i^{(n)})_{i \in \mathbb{N}}$ by

$$a_i^{(n)} := \rho^{(n)}(A_{n,i} \backslash A_{n,i-1}) = \rho^{(n)}(\{\pi \in \mathcal{P}_n : \#\pi = i\}).$$

We also note $p := \nu_{Disl}(\mathcal{S}^{\downarrow})$ the total rate of fragmentation. The equation (16) becomes for each $K \in [n]$

$$\sum_{\pi : \#\pi = K} \rho^{(n)}(\{\pi\}) q_n(\pi, D_{n,K}) = \sum_{\pi'' \in D_{n,K}} \rho^{(n)}(\{\pi''\}) q_n(\pi'', A_{n,K}) \qquad (19)$$

because the fragmentation is binary. When $\#\pi = K$ one has $q_n(\pi, D_{n,K}) \leq Kp$, thus

$$\begin{aligned} a_K^{(n)} Kp &\geq \sum_{\pi'' \in D_{n,K}} \rho^{(n)}(\{\pi''\}) q_n(\pi'', A_{n,K}) \\ &\geq \sum_{\pi'' : \#\pi'' = K+1} \rho^{(n)}(\{\pi''\}) q_n(\pi'', A_{n,K}) \\ &\geq \sum_{\pi'' : \#\pi'' = K+1} \rho^{(n)}(\{\pi''\}) c_k K(K+1)/2 \\ &\geq a_{K+1}^{(n)} c_k K(K+1)/2. \end{aligned} \tag{20}$$

Hence for all $K \in [n-1]$

$$a_K^{(n)} p \geq a_{K+1}^{(n)} c_k (K+1)/2$$

and thus

$$1 = \sum_{i=1}^{n} a_i^{(n)} < a_1^{(n)} (1 + \sum_{i=1}^{n-1} (p/c_k)^i 2^{i-1}/i!).$$

We conclude that $a_1^{(n)}$ is uniformly bounded from below by $(1 + \sum_{i=1}^{\infty} (p/c_k)^i 2^{i-1}/i!)^{-1}$. On the other hand, as $a_1^{(n)} \leq 1$ one has

$$\begin{aligned} \lim_{n\to\infty} \sum_{i>K}^{n} a_i^{(n)} &\leq \lim_{n\to\infty} a_1^{(n)} \sum_{i>K-1}^{n-1} \frac{(2p/c_k)^i}{2i!} \\ &\leq \sum_{i>K-1}^{n-1} \frac{(2p/c_k)^i}{2i!} \\ &\leq \sum_{i>K-1}^{\infty} \frac{(2p/c_k)^i}{2i!} \to 0 \end{aligned}$$

when $K \to \infty$. Hence if we define $a_i := \lim_{n\to\infty} a_i^{(n)} = \rho(\{\pi \in \mathcal{P} : \#\pi = i\})$ we have proved that the series $\sum_i a_i$ is convergent and hence $\lim_{K\to\infty} \sum_{i>K} a_i = 0$. This shows that $\rho(\{\pi \in \mathcal{P} : \#\pi = \infty\}) = 0$.

□

6.5 Proof of Theorem 10

Proof. Define $I_n := \{\pi = (B_1, B_2, ...) \in \mathcal{P} : B_1 \cap [n] = \{1\}\}$ (when no confusion is possible we sometime use $I_n := \{\pi \in \mathcal{P}_n : B_1 = \{1\}\}$) i.e., the partitions of $\mathbb{N}$ such that the only element of their first block in $[n]$ is $\{1\}$. Our proof relies on the fact that

$$\rho(\{\pi \in \mathcal{P} : \mathrm{dust}(\pi) \neq \emptyset\}) > 0 \Rightarrow \rho(\{\pi \in \mathcal{P} : \pi \in \cap_n I_n\}) > 0.$$

As above let us write down the equilibrium equations for $\Pi_{|[n]}(\cdot)$:

$$\sum_{\pi \in \mathcal{P}_n \cap I_n} \rho^{(n)}(\pi) q_n(\pi, I_n^c) = \sum_{\pi' \in I_n^c} \rho^{(n)}(\pi') q_n(\pi', I_n).$$

Recall that $A_{n,b}$ designates the set of partitions $\pi \in \mathcal{P}_n$ such that $\#\pi \leq b$ and $D_{n,b} = \mathcal{P}_n \backslash A_{n,b}$. For each b observe that

$$\min_{\pi \in D_{n,b} \cap I_n} \{q_n(\pi, I_n^c)\} = q_n(\pi', I_n^c)$$

where π' can be any partition in $\mathcal{P}_n$ such that $\pi' \in I_n$ and $\#\pi' = b+1$. We can thus define

$$f(b) := \min_{\pi \in D_{n,b} \cap I_n} \{q_n(\pi, I_n^c)\}.$$

If $c_k > 0$ and $\pi \in D_{n,b} \cap I_n$ one can exit from I_n by a coalescence of the Kingman type. This happens with rate greater than $c_k b$. If $\nu_{Coag}(\mathcal{S}^{\downarrow}) > 0$ one can also exit *via* a coalescence with multiple collision, and this happens with rate greater than

$$\zeta(b) := \int_{\mathcal{S}^{\downarrow}} \left(\sum_i x_i \left(1 - (1 - x_i)^{b-1} \right) \right) \nu_{Coag}(dx).$$

This $\zeta(b)$ is the rate of arrival of atoms $\pi^{(C)}(t)$ of P_C such that $\pi^{(C)}(t) \notin I_b$ and which do not correspond to a Kingman coalescence. Thus $\sup_{b \in \mathbb{N}} \zeta(b)$ is the rate of arrival of "non-Kingman" atoms $\pi^{(C)}(t)$ of P_C such that $\pi^{(C)}(t) \notin I := \cap_n I_n$. This rate being $\int_{\mathcal{S}^{\downarrow}} \left(\sum_i x_i\right) \nu_{Coag}(dx)$ and $\zeta(b)$ being an increasing sequence one has

$$\lim_{b \to \infty} \zeta(b) = \int_{\mathcal{S}^{\downarrow}} \left(\sum_i x_i \right) \nu_{Coag}(dx).$$

Thus it is clear that, under the conditions of the proposition, $f(b) \to \infty$ when $b \to \infty$.

On the other hand, when $\pi \in I_n^c$, the rate $q_n(\pi, I_n)$ is the speed at which 1 is isolated from all the other points, thus by compatibility it is not hard to see that

$$q_2 := \int_{\mathcal{S}^{\downarrow}} (1 - \sum_i x_i^2) \nu_{Disl}(dx) \geq q_n(\pi, I_n)$$

where q_2 is the rate at which 1 is isolated from its first neighbor (the inequality comes from the inclusion of events).

Hence,

$$\begin{aligned}
\sum_{\pi \in I_n \cap D_{n,b}} \rho^{(n)}(\pi) f(b) &\leq \sum_{\pi \in I_n \cap D_{n,b}} \rho^{(n)}(\pi) q_n(\pi, I_n^c) \\
&\leq \sum_{\pi' \in I_n^c} \rho^{(n)}(\pi') q_n(\pi', I_n) \\
&\leq \sum_{\pi' \in I_n^c} \rho^{(n)}(\pi') q_2 \\
&\leq q_2
\end{aligned}$$

which yields

$$\rho^{(n)}(I_n \cap D_{n,b}) \leq q_2/f(b).$$

Now as ρ is exchangeable one has $\rho(I \cap A_b) = 0$ where $I = \cap_n I_n$ and $A_b = \cap_n A_{n,b}$ (exchangeable partitions who have dust have an infinite number of singletons, and thus cannot have a finite number of blocks). Hence $\rho^{(n)}(I_n \cap A_{n,b}) \to 0$.

Fix $\epsilon > 0$ arbitrarily small and choose b such that $q_2/f(b) \leq \epsilon/2$. Then choose n_0 such that for all $n \geq n_0, \rho^{(n)}(I_n \cap A_{n,b}) \leq \epsilon/2$. Hence

$$\forall n \geq n_0 : \rho^{(n)}(I_n) = \rho^{(n)}(I_n \cap A_{n,b}) + \rho^{(n)}(I_n \cap D_{n,b}) \leq \epsilon/2 + \epsilon/2.$$

Thus $\lim_{n\to\infty} \rho^{(n)}(I_n) = 0$ which entails $\rho(B_1 = \{1\}) = 0$. As ρ is an exchangeable probability measure, it is a mixture of paintbox processes (see [1]) and hence

$$\begin{aligned}
\rho(B_1 = \{1\}) &= \mathbb{E}\left[\mathbb{E}\Big(\ {}_{\{B_1=\{1\}\}} \big| (\|B_1\|, \|B_2\|, \ldots)\Big)\right] \\
&= \mathbb{E}\left[(1 - \sum_i \|B_i\|)\right] \\
&= \int_{\mathcal{P}} (1 - \sum_i \|B_i\|) \rho(d\pi)
\end{aligned}$$

to see that $\rho(\text{dust}(\pi) \neq \emptyset) = 0$. $\square$

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