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Curvilinear Integrals Along Enriched Paths

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Abstract

[1] Inspired by the fundamental work of T.J. Lyons (loc.cit), we develop a theory of curvilinear integrals along a new kind of enriched paths in $\mathbb{R}^d$. We apply these methods to the fractional Brownian Motion, and prove a support theorem for SDE driven by the Skorohod fBM of Hurst parameter $H > 1/4$.

Key words: Curvilinear integrals, Hölder continuity, rough paths, stochastic integrals, stochastic differential equations, fractional Brownian motion

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[1]The first version of this paper has been substantially improved in the first part. The part relative to the fBM follows the same line and results.

I. Introduction

In a remarkable work, T. Lyons [20,21,24] defined a differential calculus along continuous rough paths $X(t)$. He used the notion of multiplicative functional which are bounded in p-variation. He then developped a calculus for geometric rough path for any p. In the case $p < 3$, the calculus also holds for the non-geometric case.

Here we extend a calculus even for the non-geometric case $p < 4$, with application to the Skorohod-fractional Brownian motion. We are led to a new notion of *enriched* paths, and the calculus works for Lyons enhanced path, if it is also enriched.

In fact we only use the α-Hölder continuity, this is not a restriction (*cf.* below).

Section I is the Introduction. In Section II, we prove a fundamental lemma (independant of the theory of rough paths, the so-called sewing lemma), which allows us to prove the convergence of very general Riemann-type sums. The proof uses a Hölder control, but it also holds with any control function as explained in corollary 2.3. Examples are given: existence of the Young integral, stochastic integral (Ito or Stratonovich), fractional Brownian motion, and also a very simple proof of the theorem of Lyons concerning the almost rough paths.

In Section III, we introduce the notion of an enriched path, which allows us to define curvilinear integrals ($\alpha \in]0, 1]$ arbitrary). Notice that for $\alpha > 1/3$, this notion is equivalent to the notion of p-rough path ($p < 3$). The equivalence does not hold for $\alpha \leq 1/3$. An application is given to enrich the Peano curve.

In Section IV, we deal with the case $\alpha > 1/3$ for to solve the differential equation naturally associated to the theory of enriched paths. Here, an important tool is the Schauder fixed point theorem which applies thanks to the linearity of the enrichement ($\alpha > 1/3$). For the unicity, we use the Banach fixed point theorem (of course with stronger hypotheses).

In this Section is also included a precise comparison with the theory of Lyons.

In Section V, in view of application to the fBM, we deeply study the case $\alpha > 1/4$. It turns out that for to solve a differential equation by the Picard iteration procedure, the good hypothesis is to have an enriched and enhanced path. Then the procedure works even in the non-geometric case.

In Section VI, we deal with the fBM with Hurst parameter $H > 1/4$. We first approximate the geometric rough path by $\mathcal{C}^\infty$ functions thanks to a martingale procedure. In a second time, this allows us to approximate the Skorohod fBM. Using the results of Section V, we solve SDE driven by the fBM.

In Section VII, we prove a support theorem for general stochastic enriched paths including the geometric or the Skorohod fBM.

Section VIII is an appendix: a technical lemma extending the Kolmogorov lemma, in relation with the sewing lemma.

We also show how to reparametrize a continuous bounded p-variation function $F(s,t)$ to obtain a $1/p$-Hölder control function.

We must point out two interesting alternative approaches on connected subjects ([13],[14]).

II. The sewing lemma: Abstract Riemann sums

2.1 Lemma : *Let μ be a continuous function on $[0,T]^2$ with values in a Banach space B, and $\varepsilon > 0$. Suppose that μ satisfies*

$$|\delta\mu(a,b,c)| = |\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq K|b-a|^{1+\varepsilon} \tag{2.1}$$

for every $a,b,c \in [0,T]$ with $c \in [a,b]$. Then there exists a function $\varphi(t)$ unique up to an additive constant such that for $a \leq b$

$$|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon} \tag{2.2}$$

Moreover the least constant is at most $K\theta(\varepsilon)$ with $\theta(\varepsilon) = (1-2^{-\varepsilon})^{-1}$.

Proof : Put $\mu_n(a,b) = \sum_{i=0}^{2^n-1} \mu(t_i, t_{i+1})$ where $t_i = a + (b-a)i/2^n$. We get

$$|\mu_{n+1}(a,b) - \mu_n(a,b)| \leq K \sum_i |t_{i+1} - t_i|^{1+\varepsilon} = K|b-a|^{1+\varepsilon} 2^{-n\varepsilon}$$

Hence the sequence $\mu_n(a,b)$ has a uniform limit $u(a,b)$ which satisfies

$$|u(a,b) - \mu(a,b)| \leq K \sum_{n\geq 0} |b-a|^{1+\varepsilon} 2^{-n\varepsilon} = K\theta(\varepsilon)|b-a|^{1+\varepsilon}$$

Moreover u is semi-additive, that is $u(a,b) = u(a,c) + u(c,b)$ everytime c is the midpoint of $[a,b]$, as follows from the equality $\mu_{n+1}(a,b) = \mu_n(a,c) + \mu_n(c,b)$. This is the unique semi-additive function such that $|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon}$, for if v is another such function, the difference $w = v - u$ satisfies

$$|w(a,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon} \quad \Rightarrow \quad |w(a,b)| \leq |w(a,c)| + |w(c,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon} 2^{-\varepsilon}$$

By induction we get $|w(a,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon} 2^{-n\varepsilon}$ for every n, and $w = 0$.

Let k be an integer, and put $v(a,b) = \sum_{i=0}^{k-1} u(a + (b-a)i/k, a + (b-a)(i+1)/k)$. Then v is semi-additive and has $|v(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon}$. Hence $v = u$ by uniqueness. It then follows that $u(a,b) = u(a,c) + u(c,b)$ for every rational barycenter $c \in [a,b]$. The same holds for every $c \in [a,b]$ by continuity of u. Putting $\varphi(t) = u(0,t)$ gives the result. The uniqueness of φ follows in the same way as u.

2.2 Remark: Suppose μ is continuous and satisfies only the inequality $|\delta\mu(a,b,c)| \leq |V(b) - V(a)|^{1+\varepsilon}$ with a non-decreasing continuous function V on $[0,T]$. Then the reparametrization $t' = \lambda t + V(t)$ with $\lambda > 0$ yields a function φ such that $|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)|\lambda(b -$

$a) + V(b) - V(a)|^{1+\varepsilon}$. By unicity φ does not depend on λ so that $|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)|V(b) - V(a)|^{1+\varepsilon}$.

2.3 Corollary: *Suppose μ is continuous and satisfies $|\delta\mu(a,b,c)| \leq \omega(a,b)^{1+\varepsilon}$ where ω is a continuous control function as in* [24]. *Then there exists a unique function φ such that*

$$|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)\omega(a,b)^{1+\varepsilon}$$

Proof : Recall that according to Lyons [24], ω is super-additive. If $\alpha \in [0,T]$ we get a function φ on $[\alpha,T]$ such that $|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)|\omega(\alpha,b) - \omega(\alpha,a)|^{1+\varepsilon}$ for $[a,b] \subset [\alpha,T]$. We have *a fortiori* $|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)|\omega(\alpha,b) - \omega(\alpha,a) + \omega(\beta,b) - \omega(\beta,a)|^{1+\varepsilon}$ for $[a,b] \subset [\beta,T] \subset [\alpha,T]$. By unicity φ does not depend on α, that is $\varphi_\alpha = \varphi_\beta$ up to a constant on $[\beta,T]$. Hence for $[a,b] \subset [0,T]$ we get by taking $a = \alpha$

$$|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)|\omega(a,b) - \omega(a,a)|^{1+\varepsilon} = \theta(\varepsilon)\omega(a,b)^{1+\varepsilon}$$

2.4 Corollary: (Riemann sums) *Let $\sigma = \{t_1,\ldots,t_n\}$ an arbitrary subdivision of $[a,b]$ with mesh $\delta = \operatorname{Sup}_i \omega(t_i,t_{i+1})$. Define the Riemann sum*

$$J_\sigma = \sum_i \mu(t_i,t_{i+1})$$

then J_σ converges to $\varphi(b) - \varphi(a)$ as δ shrinks to 0.

Proof : $|\varphi(b) - \varphi(a) - J_\sigma| \leq \sum_i |\varphi(t_{i+1}) - \varphi(t_i) - \mu(t_i,t_{i+1})| \leq \theta(\varepsilon)\omega(a,b)\delta^\varepsilon$ which converges to 0 as $\delta \to 0$.

Applications

Example 1. The Young integral: Let x and y be α and β-Hölder continuous functions with $\alpha + \beta > 1$, and put $\mu(a,b) = x_a(y_b - y_a)$. Then the sewing lemma yields a function φ, and the difference $\varphi(b) - \varphi(a)$ is called the Young integral

$$\int_a^b x_t\, dy_t$$

Another way to define the Young integral is to take $\mu'(a,b) = \int_{[a,b]} x_t\, dy_t$ where $[a,b]$ is the oriented line segment from the point (x_a,y_a) to the point (x_b,y_b). The resulting function φ is the same as above, as follows from the estimate $|\mu'(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{\alpha+\beta}$.

It should be remarked that this extends to Banach valued functions x and y with a bilinear continuous map $B(x,y)$ by putting $\mu(a,b) = B(x_a, y_b - y_a)$. The Young integral obtained can be denoted

$$\int_a^b B(x_t, dy_t)$$

Example 2. The stochastic integral: Let X_t be the $\mathbb{R}^d$-valued standard Brownian motion. If f is a $\mathcal{C}^2$-vector field, consider the scalar product

$$\mu(a,b) = f(X_a).(X_b - X_a) + \int_a^b \big[\nabla f(X_a)(X_t - X_a)\big].\, dX_t \tag{2.3}$$

with the Ito integral in the right member. The sewing lemma applies thanks to

$$N_p(\delta\mu(a,b,c)) \le \mathrm{Cst}\,|b-a|^{3/2}$$

As easily verified, the obtained difference $\varphi(b)-\varphi(a)$ is the Ito integral of $f(X_t)\,dX_t$. Now, if we had taken

$$\overset{\circ}{\mu}(a,b) = f(X_a).(X_b - X_a) + \int_a^b \left[\nabla f(X_a)(X_t - X_a)\right] \circ dX_t \tag{2.4}$$

we would have obtained a similar estimate, and the obtained $\varphi(b)-\varphi(a)$ would have been the Stratonovich integral of $f(X_t)\circ dX_t$. Moreover, observing that

$$\overset{\circ}{\mu} - \mu = \frac{1}{2}\,\mathrm{div}\, f(X_a)(b-a)$$

we recover the well known difference $\displaystyle\frac{1}{2}\int_a^b \mathrm{div}\, f(X_t)\,dt$ between the two integrals.

Example 3. The fractional Brownian motion: Let X_t^H be the $\mathbb{R}^d$-valued fractional Brownian motion with the Hurst parameter $H>0$ defined by

$$X_t^H = c(H)\int_0^t (t-s)^{H-1/2}\,dX_s$$

with $c(H) = 1/\Gamma(H+1/2)$. In the very regular case $H>3/2$, $t\to X_t^H$ is an L^p-valued $\mathcal{C}^1$ function so that $\displaystyle\int_a^b P(X_t^H)\,dX_t^H$ has an obvious meaning for every polynomial P. We proved in [10] that this integral has an L^p-valued analytic continuation for $H>1/4$. Moreover we have the estimates for $i=0,1,2$

$$N_p\left(\int_a^b (X_t^H - X_a^H)^{\otimes i}\otimes dX_t^H\right) \le \mathrm{Cst}\,|b-a|^{(i+1)H}$$

The same trick as above (2.4) allows us to define the integral

$$\int_a^b f(X_t^H)\,dX_t^H$$

for $H>1/3$. For $H>1/4$ we must put

$$\begin{aligned}\mu(a,b) = f(X_a^H).(X_b^H - X_a^H) &+ \int_a^b \left[\nabla f(X_a^H)(X_t^H - X_a^H)\right] dX_t^H \\ &+ \frac{1}{2}\int_a^b \left[\nabla^2 f(X_a^H).(X_t^H - X_a^H)^{\otimes 2}\right] dX_t^H\end{aligned}$$

For $H=1/2$ we recover the Stratonovich integral. As we shall see below, we can also extend the "Ito integral" with respect to dX_t^H for $H>1/4$.

Exemple 4. The Chen-Lyons multiplicative functionals: An n-truncated multiplicative functional in the sense of Chen-Lyons is a sequence

$$X_{ab} = (1, X_{ab}^1, X_{ab}^2, \ldots, X_{ab}^n)$$

where X_{ab}^k belongs to $T^k(V)$ the k-tensor product of a linear space V, indexed by the pairs $\{a,b\} \subset [0,S] \subset \mathbb{R}$, and satisfying the multiplicative relation

$$X_{ab} = X_{ac}X_{cb}$$

for $a \le c \le b$, that is for every $k \le n$ (for more clarity we dropped the symbol $\otimes$)

$$X_{ab}^k = \sum_{i=0}^{k} X_{ac}^i X_{cb}^{k-i}$$

It is continuous if every function $(a,b) \to X_{ab}^k$ is continuous (we suppose V of finite dimension).

An n-truncated series is an n-almost multiplicative functional if for every $a \le c \le b$, we have only

$$|X_{ab}^k - (X_{ac}X_{cb})^k| \le \omega(a,b)^{1+\varepsilon}$$

with a control ω and an $\varepsilon > 0$. Hence we have the following, substantially due to Lyons [24, theorem 3.2.1]:

2.5 Theorem: *Let X be an n-truncated continuous multiplicative functional, and let $Y_{ab}^{n+1} \in T^{n+1}(V)$ be continuous and such that*

$$Y = (1, X^1, X^2, \ldots, X^n, Y^{n+1})$$

is an $n+1$-almost multiplicative functional. Then there exists a unique X_{ab}^{n+1} such that

$$Z = (1, X^1, X^2, \ldots, X^n, X^{n+1})$$

is an $n+1$-multiplicative functional with the condition

$$|X_{ab}^{n+1} - Y_{ab}^{n+1}| \le \mathrm{Cst}\, \omega(a,b)^{1+\varepsilon}$$

The least constant is at most $\theta(\varepsilon) = (1 - 2^{-\varepsilon})^{-1}$.

Proof : Put

$$\mu(a,b) = Y_{ab}^{n+1} + \sum_{k=1}^{n} X_{0a}^k X_{ab}^{n+1-k}$$

For $a \le c \le b$ we get

$$\delta\mu(a,b,c) = Y_{ab}^{n+1} - Y_{ac}^{n+1} - Y_{cb}^{n+1} - \sum_{k=1}^{n} X_{ac}^k X_{cb}^{n+1-k}$$

hence $|\delta\mu| \le \omega(a,b)^{1+\varepsilon}$. By the sewing lemma, we get a function $\varphi(t) \in T^{n+1}(V)$ such that

$$|\varphi(b) - \varphi(a) - \mu(a,b)| \le \theta(\varepsilon)\omega(a,b)^{1+\varepsilon}$$

Put

$$X^{n+1}_{ab} = \varphi(b) - \varphi(a) - \sum_{k=1}^{n} X^k_{0a} X^{n+1-k}_{ab}$$

we then get

$$X^{n+1}_{ab} - X^{n+1}_{ac} - X^{n+1}_{cb} = \sum_{k=1}^{n} X^k_{ac} X^{n+1-k}_{cb}$$

and

$$|X^{n+1}_{ab} - Y^{n+1}_{ab}| = |\varphi(b) - \varphi(a) - \mu(a,b)| \leq \theta(\varepsilon)\omega(a,b)^{1+\varepsilon}$$

Uniqueness follows by the routine argument.

2.6 Remark: Suppose moreover than we have $|X^k_{ab}| \leq \omega(a,b)^{k\alpha}$ for every $k \leq n$, and that $(n+1)\alpha > 1$ (*cf.* also [24, theorem 3.1.2]). Then $(1, X^1, X^2, \ldots, X^n, 0)$ is an $n+1$-almost multiplicative functional, so that we can take $Y^{n+1}_{ab} = 0$. We then get (with $\varepsilon = (n+1)\alpha - 1$)

$$|X^{n+1}_{ab}| \leq \theta(\varepsilon)\omega(a,b)^{(n+1)\alpha}$$

Repeating this process yields an infinite Chen-Lyons series $X = (1, X^1, X^2, \ldots)$ with

$$|X^{n+p}_{ab}| \leq \left[\prod_{k=1}^{p} \theta[(n+k)\alpha - 1]\right] \omega(a,b)^{(n+p)\alpha}$$

Observe that the infinite product converges, so that we have for every p

$$|X^{n+p}_{ab}| \leq K(\alpha, n)\omega(a,b)^{(n+p)\alpha}$$

for every p, with a universal constant. After computations we get the estimate $K(\alpha, n) \leq \theta[(n+1)\alpha - 1]^{\theta(\alpha)}$.

III. Enriching the paths ($\alpha > 1/(m+2)$)

In this section we consider an $\mathbb{R}^d$ valued-path y defined on $[0,T]$ which is α-Hölder continuous with $\alpha > 1/(m+2)$ where $m \geq 1$ is an integer.

Let $\mathcal{P}_m$ be the space of all real polynomials on $\mathbb{R}^d$ with degree $\leq m$. We assume that we are given a linear map denoted

$$P \to I(P)$$

with values in the space $\mathcal{C}^\alpha([0,T], \mathbb{R}^d)$. We shall also denote

$$I_{ab}(P) = I(P)(b) - I(P)(a)$$

We suppose the following property: for every ξ belonging to the convex hull of the image $y[a,b]$ and every linear function f on $\mathbb{R}^d$ we have for $k \in \{0, \ldots, m\}$

$$\left| I_{ab}[f^k(\cdot - \xi)] \right| \leq \mathrm{Cst}\, \|f\|^k |b-a|^{(k+1)\alpha} \tag{3.1}$$

The least constant in (3.1) is denoted $\|I\|_\alpha$.

If we take $P = 1$, we get a $\mathcal{C}^\alpha$-function x with values in $\mathbb{R}^d$.

We can introduce the notation

$$I_{ab}(P) = \int_a^b P(y_t)\, dx_t$$

By a canonical tensor extension, we define

$$\int_a^b P(y_t) \otimes\, dx_t \in \mathbb{R}^n \otimes \mathbb{R}^d$$

for every $\mathbb{R}^n$-valued polynomial on $\mathbb{R}^d$. We then get

$$\left| \int_a^b (y-\xi)^{\otimes k} \otimes\, dx_t \right| \leq \|I\|_\alpha |b-a|^{(k+1)\alpha} \tag{3.2}$$

3.1 Definition: *The pair (y, I) is called an enriched path, I is the enrichment of the path y.*

3.2 Remark: Even if y or x is $\mathcal{C}^\infty$, the definition does not necessarily agree with the usual path integral.

3.3 Theorem: *Let f be a $\mathcal{C}^{m+1}$ function on $\mathbb{R}^d$. There exists a function $\varphi(t)$ unique up to an additive constant, such that for every $a, b \in [0,1]$*

$$\left| \varphi(b) - \varphi(a) - \int_a^b T(y_a, y_t)\, dx_t \right| \leq \mathrm{Cst}\, |b-a|^{(m+2)\alpha} Q_{ab}(\nabla^{m+1} f) \tag{3.3}$$

where $T(y_a, y)$ is the Taylor polynomial of f at the point y_a of degree m and Q_{ab} the uniform norm on the image $y[a,b]$.

Proof : Put

$$\mu(a,b) = \int_a^b T(y_a, y_t)\, dx_t \tag{3.4}$$

We have for $c \in [a,b]$

$$\delta\mu = \mu(a,b) - \mu(a,c) - \mu(c,b) = \int_c^b [T(y_a, y_t) - T(y_c, y_t)]\, dx_t \tag{3.5}$$

Put $\xi_s = y_a + s(y_c - y_a)$ for $s \in [0,1]$, and $\psi(s) = T(\xi_s, y_t)$. We get

$$\psi'(s) = (m!)^{-1} \nabla^{m+1} f(\xi_s).(y_c - y_a).(y_t - \xi_s)^{\otimes m}$$

where the dots stand for contracted tensor products. Hence

$$\delta\mu = -(m!)^{-1} \int_0^1 ds \int_c^b \left[\, \nabla^{m+1} f(\xi_t).(y_c - y_a).(y_t - \xi_s)^{\otimes m} \right]\, dx_t \tag{3.6}$$

$$|\delta\mu| \leq (m!)^{-1}\|y\|_\alpha \|I\|_\alpha Q_{ab}(\nabla^{m+1} f)|b-a|^{(m+2)\alpha} \tag{3.7}$$

where $\|y\|_\alpha$ is the norm of y in $\mathcal{C}^\alpha$. As $(m+2)\alpha > 1$ the result follows from theorem 2.1. Besides, the least constant in (3.7) is $\leq \theta((m+2)\alpha - 1)M\|x\|_\alpha$.

We shall denote

$$\int_a^b f(y_t)\, dx_t = \varphi(b) - \varphi(a)$$

In particular, if f is a polynomial of degree m, we recover the integral $\displaystyle\int_a^b f(y_t)\, dx_t$, since f coincides with every of its Taylor polynomial of degree m.

IV. The case ($\alpha > 1/3$)

The hypotheses are the same as in section III, with $m = 1$.

4.1 Proposition: *Put*

$$z_t = \int_0^t y_s \otimes\, dx_s \tag{4.1}$$

We then have

$$|z_b - z_a - y_a \otimes (x_b - x_a)| \leq \|I\|_\alpha |b-a|^{2\alpha} \tag{4.2}$$

Conversely, let x and z be $\mathcal{C}^\alpha$-functions with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$ such that (4.2) holds. Then we can define an enrichment of y by putting

$$\int_a^b dx_t = x_b - x_a, \qquad \int_a^b y_t \otimes\, dx_t = z_b - z_a$$

Proof : Obvious.

4.2 Lemma : *Assume that $x \in \mathcal{C}^\alpha$, $\alpha \leq 1$ and $0 < H \leq T$. On the space $\mathcal{C}^\alpha[0,T]$, the following semi-norms are equivalent*

$$\|x\|_{\alpha,H} = \mathrm{Sup}\{|x_t - x_s|/|b-a|^\alpha \ / \ 0 < |b-a| \leq H\}$$

Proof : Obviously $\|x\|_{\alpha,H} \leq \|x\|_{\alpha,K}$ for $H \leq K$, and

$$\|x\|_{\alpha,2H} \leq 2^{1-\alpha}\|x\|_{\alpha,H}$$

4.3 Proposition: *Let $K_H(R)$ be the set of couples $(y,z) \in \mathcal{C}^\alpha[0,T] \times \mathcal{C}^\alpha[0,T]$ such that $y_0 = 0$ and for $|b-a| \leq H$*

$$|y_b - y_a| \leq R|b-a|^\alpha \quad \text{and} \quad |z_b - z_a - y_a \otimes (x_b - x_a)| \leq R|b-a|^{2\alpha}$$

This is a non void convex set, and it is compact in the topology induced by $\mathcal{C}^\beta[0,T] \times \mathcal{C}^\beta[0,T]$ for $\beta < \alpha$.

Proof : Left to the reader.

Iteration

Let x_t and y_t be two enriched paths, such that $I_{ab}(1) = J_{ab}(1) = x_b - x_a$. We shall use the notations

$$X^1_{ab} = x_b - x_a, \quad Y^1_{ab} = y_b - y_a, \quad X^2_{ab} = \int_a^b X^1_{at} \otimes dx_t, \quad Y^2_{ab} = \int_a^b Y^1_{at} \otimes dx_t \tag{4.3}$$

Observe that $Y^2_{ab} = z_b - z_a - y_a \otimes (x_b - x_a)$.

Let σ be a matrix valued $\mathcal{C}^2$ function, put (with the dot for the matrix product)

$$\overline{y}_t = \int_0^t \sigma(y_s).\, dx_s$$

4.4 Proposition: *There exists a $\mathcal{C}^\alpha$ function $\overline{z}_t$ such that*

$$|\overline{Y}^2_{ab}| = |\overline{z}_b - \overline{z}_a - \overline{y}_a \otimes X^1_{ab}| \leq \mathrm{Cst}\, |b-a|^{2\alpha}$$

Proof : Put

$$\mu^2(a,b) = \overline{y}_a \otimes X^1_{ab} + \sigma(y_a).X^2_{ab}$$

where the dot is a natural contracted tensor product. We have

$$\delta\mu^2(a,b,c) = -\left[\int_a^c [\sigma(y_t) - \sigma(y_a)].\, dx_t\right] \otimes X^1_{cb} - [\sigma(y_c) - \sigma(y_a)].X^2_{cb}$$

which implies

$$|\delta\mu^2| \leq \mathrm{Cst}\, |b-a|^{3\alpha}$$

Applying the sewing lemma yields the expected function $\overline{z}_t$.

We shall denote

$$\overline{z}_t = \int_0^t \overline{y}_s \otimes dx_s$$

Denote $\mathcal{C}^2_b$ the space of functions which are bounded with its derivatives up to order 2.

4.5 Theorem: *Suppose that $\sigma \in \mathcal{C}^2_b$. The set $K_H(R)$ is invariant under the map $(y,z) \to (\overline{y}, \overline{z})$ for R large enough and H sufficiantly small.*

Proof : Put $\sigma_a = \sigma(y_a)$, $\quad \sigma'_a = \nabla\sigma(y_a)$ and as above

$$\mu^1(a,b) = \sigma_a.X^1_{ab} + \sigma'_a.Y^2_{ab}, \qquad \mu^2(a,b) = \overline{y}_a \otimes X^1_{ab} + \sigma_a.X^2_{ab}$$

then

$$\delta\mu^1(a,b,c) = (\sigma_a - \sigma_c + \sigma'_a Y^1_{ac}).X^1_{cb} + (\sigma'_a - \sigma'_c).Y^2_{cb}$$

$$\delta\mu^2(a,b,c) = (\overline{y}_a - \overline{y}_c + \sigma_a X^1_{ac}) \otimes X^1_{cb} + (\sigma_a - \sigma_c).X^2_{cb}$$

First we have

$$|\mu^1(a,b)| \leq M|b-a|^\alpha + MR|b-a|^{2\alpha}$$

Assume that $RH^\alpha \leq 1$, then for $|b-a| \leq H$

$$|\mu^1(a,b)| \leq 2M|b-a|^\alpha, \quad \Rightarrow \quad |\delta\mu^1| \leq 6M|b-a|^\alpha$$

$$|\overline{y}_b - \overline{y}_a| \leq 2M(1+3\theta)|b-a|^\alpha \quad \Rightarrow \quad \|\overline{y}\|_{\alpha,H} \leq 2M(1+3\theta)$$

Take $R \geq 2M(1+3\theta)$, then $\|\overline{y}\|_{\alpha,H} \leq R$. Using again $RH^\alpha \leq 1$, we get

$$\left| \int_a^c (\sigma_t - \sigma_a)\, dx_t \right| \leq M[1+6\theta]|b-a|^\alpha$$

$$|\delta\mu^2| \leq M^2(1+6\theta)|b-a|^{2\alpha} + MR|b-a|^{3\alpha} = M'|b-a|^{2\alpha}$$

$$|\overline{Y}^2_{ab}| \leq |\mu^2(a,b) - \overline{y}_a X^1_{ab}| + \theta|\delta\mu^2| \leq (M+\theta M')|b-a|^{2\alpha}$$

Hence we can take finally

$$R \geq 2M(1+3\theta), \quad R \geq (M+\theta M'), \quad H^\alpha \leq 1/R$$

4.6 Corollary: *The map $(y,z) \to (\overline{y},\overline{z})$ has a fixed point in $K_H(R)$.*

Proof : $K_H(R)$ is convex and compact in $\mathcal{C}^\beta \times \mathcal{C}^\beta$ for $\beta < \alpha$ by proposition 4.3, moreover the map is obviously continuous in this topology. So the Schauder theorem applies.

4.7 Corollary: *Assume that σ is $\mathcal{C}^2_b$. Then the "differential equation"*

$$y_t = \int_0^t \sigma(y_s).\, dx_s, \qquad z_t = \int_0^t y_s \otimes dx_s \tag{4.4}$$

with $|z_b - z_a - y_a \otimes (x_b - x_a)| \leq \mathrm{Cst}\,|b-a|^{2\alpha}$ *has a solution.*

Uniqueness

4.8 Proposition: *(Variation of the sewing lemma) Assume that $\nu_t(a,b)$ is continuous with respect to the three variables (t,a,b) with*

$$|\nu_t(a,b) - \nu_t(a,c) - \nu_t(c,b)| \leq M|b-a|^{1+\varepsilon} \quad \text{for } a \leq c \leq b$$

Then $\mu_t = \int_0^t \nu_s\, ds$ has the same property, and the resulting function φ_t is derivable in t.

Proof : Let ψ_t be the unique function such that $\psi_t(0) = 0$ and

$$|\psi_t(b) - \psi_t(a) - \nu_t(a,b)| \leq M\theta(\varepsilon)|b-a|^{1+\varepsilon}$$

It is continuous with respect to (t, a, b). We get the result by putting

$$\varphi_t(x) = \int_0^t \psi_s(x)\, ds$$

4.9 Theorem: *Assume that σ is $\mathcal{C}_b^3$. Let (y, z) be such that the inequality $|z_b - z_a - y_a \otimes X_{ab}^1| \leq R|b-a|^{2\alpha}$ holds. Put as in proposition 4.4*

$$\overline{y}_b - \overline{y}_a = \int_a^b \sigma(y_t).\, dx_t, \qquad \overline{z}_b - \overline{z}_a = \int_a^b \overline{y}_t \otimes\, dx_t$$

with $(y, z) \in K_H(R)$. Suppose that $(y + u, z + v)$ is another couple belonging to $K_H(R)$. Then $(\overline{y}, \overline{z})$ is weakly derivable in the direction of (u, v). Moreover the map $(y, z) \to (\overline{y}, \overline{z})$ is a contraction in the set $K_H(R)$ for R large enough, H and T small enough, in the the topology of $\mathcal{C}^\alpha \times \mathcal{C}^\alpha$.

Proof : Write again

$$\mu^1(a, b) = \sigma(y_a).X_{ab}^1 + \sigma'(y_a).Y_{ab}^2, \qquad \mu^2(a, b) = \overline{y}_a \otimes X_{ab}^1 + \sigma(y_a).X_{ab}^2$$

Put

$$\mu_t^1(a, b) = \sigma(y_a + tu_a).X_{ab}^1 + \sigma'(y_a + tu_a).(Y_{ab}^2 + tU_{ab}^2)$$
$$\mu_t^2(a, b) = (\overline{y}_a + t\overline{u}_a) \otimes X_{ab}^1 + \sigma(y_a + tu_a).X_{ab}^2$$

Denote D the derivative at $t = 0$, and assume that $\|u\|_\alpha \leq \rho$, $|U_{ab}^2| \leq \rho|b-a|^{2\alpha}$ and $RT^\alpha \leq 1$

$$D\mu^1(a, b) = \sigma'(y_a).u_a.X_{ab}^1 + \sigma''(y_a).u_a.Y_{ab}^2 + \sigma'(y_a).U_{ab}^2$$

$$D\mu^2(a, b) = \overline{u}_a \otimes X_{ab}^1 + \sigma'(y_a).u_a.X_{ab}^2$$

Denote M a constant only depending on X and σ. We have

$$|D\mu^1(a, b)| \leq M\rho T^\alpha|b-a|^\alpha + M\rho T^\alpha R|b-a|^{2\alpha} + M\rho|b-a|^{2\alpha} \leq 3M\rho T^\alpha|b-a|^\alpha$$

$$\begin{aligned}\delta D\mu^1(a, b, c) = {} & [\sigma'(y_a) - \sigma'(y_c) + \sigma''(y_a)Y_{ac}^1].u_a.X_{cb}^1 + [\sigma''(y_a) - \sigma''(y_c)].u_a.Y_{cb}^2 \\ & + [\sigma'(y_a) - \sigma'(y_c)].U_{cb}^2 - \sigma''(y_c).U_{ac}^1.Y_{cb}^2\end{aligned}$$

As $RT^\alpha \leq 1$, each term in the right hand side is majorized by $MR\rho|b-a|^{3\alpha}$, so that $|\delta D\mu^1| \leq 4MR\rho|b-a|^{3\alpha}$. By proposition 4.8, the variation $\overline{u} = \Delta\overline{y}$ satisfies

$$\|\overline{u}\|_\alpha \leq (3 + 4\theta)MT^\alpha\rho \leq k\rho$$

with $k = (3 + 4\theta(3\alpha - 1))T^\alpha < 1$ for T small enough.

Now compute the variation of $\overline{Y}^2$, denoted $\overline{U}^2$. Let $\Delta\mu^2$ be the variation of μ^2. We have

$$\overline{U}_{ab}^2 = \Delta\mu^2(a, b) - \overline{u}_a \otimes X_{ab}^1 + \sum_n \delta\Delta\mu_n^2(a, b)$$

where μ_n^2 is defined as in the proof of the sewing lemma. We have

$$|\Delta\mu^2(a,b) - \overline{u}_a \otimes X^1_{ab}| = |[\sigma(y_a+u_a) - \sigma(y_a)].X^2_{ab}| \leq M\rho T^\alpha |b-a|^{2\alpha}$$

Next we have to estimate $\delta\Delta\mu^2$. As above, we estimate $\delta D\mu^2 = D\delta\mu^2$. Recall that

$$\delta\mu^2 = -[\overline{y}_c - \overline{y}_a - \mu^1(a,c)] \otimes X^1_{cb} - \sigma'(y_a).Y^2_{ac} \otimes X^1_{cb} + [\sigma(y_a) - \sigma(y_c)].X^2_{cb}$$

with dots as contracted tensor products. We get

$$\begin{aligned} D\delta\mu^2 = &- [\overline{u}_c - \overline{u}_a - D\mu^1(a,c)] \otimes X^1_{cb} - \sigma''(y_a).u_a.Y^2_{ac} \otimes X^1_{cb} \\ &- \sigma'(y_a).U^2_{ac} \otimes X^1_{cb} + [\sigma'(y_a).u_a - \sigma'(y_c).u_c].X^2_{cb} \end{aligned}$$

As seen above, the first term is majorized by $MR\rho|b-a|^{4\alpha} \leq M\rho|b-a|^{3\alpha}$, the other ones are also majorized by $M\rho|b-a|^{3\alpha}$. We then get

$$\left|\sum_n \delta\Delta\mu_n^2(a,b)\right| \leq 4\theta M|b-a|^{3\alpha} \leq M'\rho T^\alpha |b-a|^{2\alpha}$$

Finally we obtain

$$|\overline{U}^2_{ab}| \leq (M+M')\rho T^\alpha |b-a|^{2\alpha} \leq k\rho|b-a|^{2\alpha}$$

with $k = (M+M')T^\alpha < 1$ for T small enough.

4.10 Corollary: *The differential equation (4.4) has a unique solution.*

Proof : The Picard sequence converges in $K_R(T)$, so that the fixed point is unique.

4.11 Remark: In the frame of enhanced paths, this theorem is due to Lyons [24, corollary 6.2.2].

Comparison with the Lyons rough paths

First observe that $X = (1, X^1, X^2)$ defined in formulae (4.3) is a rough path in the sense of Lyons, which is easy to verify. Conversely, if we are given a rough path $X = (1, X^1, X^2)$, we can put

$$v_t = x_0 \otimes X^1_{0t} + X^2_{0t}, \quad \text{or equivalently} \quad \int_a^b (x_t - x_a) \otimes dx_t = X^2_{ab}$$

We then get an enriched path (x, v).

If we are given an enriched path y with $X^1_{ab} = \displaystyle\int_a^b dx_t$ then (Y^1, Y^2) defined in proposition 4.4 satisfies

$$Y^2_{ab} = Y^2_{ac} + Y^1_{ac} \otimes X^1_{cb} + Y^2_{cb}$$

Conversely if we are given the couple (Y^1, Y^2), satisfying the preceding relation, putting

$$y_t = Y^1_{0t}, \quad \text{and} \quad z_t = y_0 \otimes X^1_{0t} + Y^2_{0t}$$

is an enriched path (y, z) in the sense of proposition 4.4.

Miscellaneous

4.12 Proposition: *Let (x_t, z_t) be an enriched path, then*

$$\eta_t = x \otimes x\Big|_0^t - \int_0^t (x_s \otimes \, dx_s + dx_s \otimes x_s)$$

is a symmetric tensor of class $\mathcal{C}^{2\alpha}$. Moreover, if we put

$$\overline{z}_t = \int_0^t x_s \circ dx_s = \int_0^t x_s \otimes \, dx_s + \frac{1}{2}\eta_t$$

we get

$$\int_a^b x_s \circ dx_s + \int_a^b dx_s \circ x_s = x \otimes x\Big|_a^b$$

We then say that $\overline{z}_t$ is a normal enrichment of x_t, it is the normal part of z_t.

Proof : Left to the reader.

4.13 Remark: We also have

$$\overline{z}_t = \frac{1}{2} x \otimes x\Big|_0^t + \frac{1}{2}\int_0^t x_s \otimes \, dx_s - dx_s \otimes x_s = \frac{1}{2} x \otimes x\Big|_0^t + \frac{1}{2}\int_0^t x_s \circ dx_s - dx_s \circ x_s$$

If we write $d\eta_t = dx_t \otimes dx_t$, we can write

$$d(x_t \otimes x_t) \;=\; x_t \circ dx_t + \, dx_t \circ x_t \;=\; x_t \otimes \, dx_t + \, dx_t \otimes x_t + dx_t \otimes dx_t$$

Observe that $dx_t \circ dx_t = 0$.

4.14 Theorem: *Let f be of class $\mathcal{C}^3$. We then have an "Ito formula"*

$$f(x_b) - f(x_a) = \int_a^b \nabla f(x_t).dx_t + \frac{1}{2}\int_a^b \langle \nabla^2 f(x_t), dx_t.dx_t \rangle$$

where the last integral is Young, or in differential form

$$d[f(x_t)] = \nabla f(x_t).dx_t + \frac{1}{2}\langle \nabla^2 f(x_t), dx_t \otimes dx_t \rangle$$

Proof : Obvious.

4.15 Remark: Taking the normal part $\overline{z}_t$ of z_t we get

$$d[f(x_t)] = \nabla f(x_t) \circ dx_t$$

4.16 Corollary: *Put* $m_t = \exp[\langle k, x_t\rangle - \frac{1}{2}\langle k \otimes k, \eta_t\rangle]$. *We then have*

$$m_b - m_a = \int_a^b m_t \langle k, dx_t\rangle$$

An example: The enriched Peano's curve

There is a description of the curve $\gamma(t) = z_t = (x_t, y_t)$ in Favard [7]. Recall that we have a sequence of piecewise linear curves $t \to z_t^n = (x_t^n, y_t^n) \in \mathbb{R}^2$ with vertices for $t = k.9^{-n}$, $0 \leq k < 9^n$. The sequence uniformly converges to a continuous curve z_t, which is the Peano curve.

Let Δ_n be the set of numbers $k.9^{-n}$, $\Delta = \bigcup_{n \geq 0} \Delta_n$. The sequence z^n is stationnary at each point of Δ. One then has $|z_{t'} - z_t| \leq \sqrt{2}\ |t' - t|^{1/2}$ for every $t \in \Delta_n$, and $t' = t + 9^{-n}$. Hence, there exists by the the combinational lemma of the appendix, a constant M_1 such that $|z_b - z_a| \leq M_1 |b - a|^{1/2}$ for every a, b, so that z_t is 1/2-Hölder continuous.

Put $A_{ab}^n = \displaystyle\int_a^b (z_t^n - z_a^n) \otimes dz_t^n$. We first have $A_{01}^k = \frac{1}{2} z_1 \otimes z_1$ for every k (easy computation), so that by the similarities of the Peano curve

$$A_{tt'}^{n+k} = \frac{1}{2} z_{tt'} \otimes z_{tt'}$$

for $t \in \Delta_n$, $t' = t + 9^{-n}$ and $k \geq 0$. Next, if $t'' = t' + 9^{-n}$ we have

$$A_{tt''}^{n+k} = A_{tt'}^{n+k} + A_{t't''}^{n+k} + z_{tt'} \otimes z_{t't''}$$

Hence for every $a, b \in \Delta$, the sequence A_{ab}^n converges stationarily to a limit A_{ab}.

Now, for $t \in \Delta_n$ and $t' = t + 9^{-n}$, we have $|A_{tt'}| \leq |t' - t|$, and for $a \leq c \leq b \in \Delta$

$$|A_{ab} - A_{ac} - A_{cb}| = |z_{ac} \otimes z_{cb}| \leq M_1^2 |c - a|^{1/2} |b - c|^{1/2}$$

From the combinational lemma we first deduce an M_2 such that $|A_{ab}| \leq M_2 |b - a|$ for every $n \geq 0$, $a, b \in \Delta$, next that A extends continuously on $[0, 1]^2$. Finally $(1, z, A)$ is a rough path, and we have an α-enriched curve for $\alpha = 1/2 > 1/3$.

V. The case $\alpha > 1/4$

Suppose that x and y are two enriched paths in the sense of section III, with $I(1)_{ab} = J(1)_{ab} = x_b - x_a$. Put

$$Y_{ab}^1 = y_b - y_a, \quad Y_{ab}^2 = \int_a^b Y_{at}^1 \otimes\ dx_t, \quad Z_{ab} = \int_a^b (Y_{at}^1)^{\otimes 2} \otimes\ dx_t$$

5.1 Proposition: *For $a \leq c \leq b$ we have*

$$Y^2_{ab} - Y^2_{ab} - Y^2_{ab} = Y^1_{ac} \otimes X^1_{cb} \tag{5.1}$$

$$Z_{ab} - Z_{ac} - Z_{cb} = (Y^1_{ac})^{\otimes 2} \otimes X^1_{cb} + Y^1_{ac} \otimes Y^2_{cb} + S(Y^1_{ac} \otimes Y^2_{cb}) \tag{5.2}$$

where S is the symmetry of the tensor product defined by $S(u \otimes v \otimes w) = v \otimes u \otimes w$.

Proof : Straightforward.

5.2 Remark: These two relations (5.1) and (5.2) are the only ones we need to compute integrals as in section III, for $m = 2$. Indeed, we have (formula (3.3))

$$\left| \varphi(b) - \varphi(a) - \int_a^b [f(y_a) + \nabla f(y_a).Y^1_{at} + \frac{1}{2}\nabla^2 f(y_a).Y^{1\otimes 2}_{at}].\, dx_t \right| \leq \text{Cst}\, |b-a|^{4\alpha}$$

that is

$$\left| \varphi(b) - \varphi(a) - f(y_a).X^1_{ab} - \nabla f(y_a).Y^2_{ab} - \frac{1}{2}\nabla^2 f(y_a).Z_{ab} \right| \leq \text{Cst}\, |b-a|^{4\alpha}$$

Let σ be a $\mathcal{C}^3$ matrix function on $\mathbb{R}^d$. We want to define a triple $(\overline{Y}^1, \overline{Y}^2, \overline{Z})$ with the same properties as the triple (Y^1, Y^2, Z). First define

$$\overline{y}_t = \int_0^t \sigma(y_s) \otimes\, dx_s, \qquad \overline{Y}^1_{ab} = \overline{y}_b - \overline{y}_a = \int_a^b \sigma(y_s) \otimes\, dx_s$$

Hypothesis: there exists X^3 such that $(1, X^1, X^2, X^3)$ is a rough path, and Y^3 such that for $a \leq c \leq b$

$$Y^3_{ab} = Y^3_{ac} + Y^1_{ac} \otimes X^2_{cb} + Y^2_{ac} \otimes X^1_{cb} + Y^3_{cb} \tag{5.3}$$

and

$$|Y^3_{ab}| \leq \text{Cst}\, |b-a|^{3\alpha} \tag{5.4}$$

Iteration

5.3 Proposition: *Put*

$$\mu^2(a,b) = \overline{y}_a \otimes X^1_{ab} + \sigma(y_a).X^2_{ab} + \nabla\sigma(y_a).Y^3_{ab}$$

where the dots are obvious contracted tensor products. There exists a function U such that $|U_b - U_a - \mu^2(a,b)| \leq \text{Cst}\, |b-a|^{4\alpha}$. Put $\overline{Y}^2_{ab} = U_b - U_a - \overline{y}_a \otimes X^1_{ab}$, we then have for $a \leq c \leq b$

$$\overline{Y}^2_{ab} = \overline{Y}^2_{ac} + \overline{Y}^1_{ac} \otimes X^1_{cb} + \overline{Y}^2_{cb}$$

and

$$|\overline{Y}^2_{ab}| \leq \text{Cst}\, |b-a|^{2\alpha}$$

Proof : Put $\sigma_a = \sigma(y_a)$, $\quad \sigma'_a = \nabla\sigma(y_a)$. We get for $a \leq c \leq b$

$$\delta\mu^2(a,b,c) = [\overline{y}_a - \overline{y}_c + \sigma_a.X^1_{ac} + \sigma'_a.Y^2_{ac}] \otimes X^1_{cb} + [\sigma_a - \sigma_c + \sigma'_a.Y^1_{ac}] \otimes X^2_{cb} + [\sigma'_a - \sigma'_c].Y^3_{cb}$$

Every term is majorized by Cst $|b-a|^{4\alpha}$, so that the sewing lemma applies, and the function U exists. The last claims are straightforward.

5.4 Proposition: *Put*

$$\mu^3(a,b) = \overline{y}_a \otimes X^1_{ab} \otimes x_b + \overline{Y}^2_{ab} \otimes x_b - \overline{y}_a \otimes X^2_{ab} - \sigma_a.X^3_{ab}$$

There exists a function V such that $|V_b - V_a - \mu^3(a,b)| \leq \text{Cst}\,|b-a|^{4\alpha}$. Put

$$\overline{Y}^3_{ab} = \overline{y}_a \otimes X^1_{ab} \otimes x_b + \overline{Y}^2_{ab} \otimes x_b - \overline{y}_a \otimes X^2_{ab} - (V_b - V_a)$$

We then have for $a \leq c \leq b$

$$\overline{Y}^3_{ab} = \overline{Y}^3_{ac} + \overline{Y}^1_{ac} \otimes X^2_{cb} + \overline{Y}^2_{ac} \otimes X^1_{cb} + \overline{Y}^3_{cb}$$

and

$$|\overline{Y}^3_{ab}| \leq \text{Cst}\,|b-a|^{3\alpha}$$

Proof : Write $\mu^3 = A + B - C - D$, so that $\delta\mu^3(a,b,c) = \delta A + \delta B - \delta C - \delta D$. We get

$$\delta A = \overline{y}_a \otimes X^1_{ac} \otimes X^1_{cb} - \overline{Y}^1_{ac} \otimes X^1_{cb} \otimes x_b$$

$$\delta B = \overline{Y}^2_{ac} \otimes X^1_{cb} + \overline{Y}^1_{ac} \otimes X^1_{cb} \otimes x_b$$

$$\delta C = \overline{y}_a \otimes X^1_{ac} \otimes X^1_{cb} - \overline{Y}^1_{ac} \otimes X^2_{cb}$$

$$\delta D = \sigma_a.X^2_{ac} \otimes X^1_{cb} + \sigma_a.X^1_{ac} \otimes X^2_{cb} + (\sigma_a - \sigma_c).X^3_{cb}$$

Hence

$$\delta\mu^3 = [\overline{Y}^2_{ac} - \sigma_a.X^2_{ac}] \otimes X^1_{cb} + [\overline{Y}^1_{ac} - \sigma_a.X^1_{ac}] \otimes X^2_{cb} - (\sigma_a - \sigma_c).X^3_{cb}$$

Every term is majorized by Cst $|b-a|^{4\alpha}$, so that the sewing lemma applies, and the function V exists. The last claims are straightforward.

5.5 Proposition: *Put*

$$\nu(a,b) = \overline{y}_a^{\otimes 2} \otimes X^1_{ab} + \overline{y}_a \otimes \overline{Y}^2_{ab} + S(\overline{y}_a \otimes \overline{Y}^2_{ab}) + \int_a^b [\sigma(y_a).X^1_{at}]^{\otimes 2} \otimes dx_t$$

Then ν satisfies the conditions of the sewing lemma. Let W be a function such that $|W_b - W_a - \nu(a,b)| \leq \text{Cst}\,|b-a|^{4\alpha}$. Then

$$\overline{Z}_{ab} = W_b - W_a - [\overline{y}_a^{\otimes 2} \otimes X^1_{ab} + \overline{y}_a \otimes \overline{Y}^2_{ab} + S(\overline{y}_a \otimes \overline{Y}^2_{ab})]$$

satisfies the following relation

$$\overline{Z}_{ab} - \overline{Z}_{ac} - \overline{Z}_{cb} = \overline{Y}^{1\ \otimes 2}_{ac} \otimes X^1_{cb} + \overline{Y}^1_{ac} \otimes \overline{Y}^2_{cb} + S(\overline{Y}^1_{ac} \otimes \overline{Y}^2_{cb})$$

$$|\overline{Z}_{ab}| \leq \text{Cst}\,|b-a|^{3\alpha}$$

Proof : If a function $u(a,b)$ is such that $u(a,b)/|b-a|^{4\alpha}$ is bounded, then we write $u \underset{4\alpha}{\approx} 0$. In the same way we write $u \underset{4\alpha}{\approx} v$ for $u - v \underset{4\alpha}{\approx} 0$.

We have

$$\nu(a,b) = A + B + S(B) + C$$

For $a \leq c \leq b$, we have

$$\delta\nu(a,b,c) = \delta A + \delta B + S(\delta B) + \delta C$$

$$\delta A = (\overline{y}_a^{\otimes 2} - \overline{y}_c^{\otimes 2}) \otimes X^1_{cb} = [(\overline{Y}^1_{ac})^{\otimes 2} - \overline{y}_c \otimes \overline{Y}^1_{ac} - \overline{Y}^1_{ac} \otimes \overline{y}_c] \otimes X^1_{cb}$$

$$\delta B = \overline{y}_a \otimes \overline{Y}^1_{ac} \otimes X^1_{cb} - \overline{Y}^1_{ac} \otimes \overline{Y}^2_{cb}$$

$$\delta C \underset{4\alpha}{\approx} \int_c^b [(\sigma_c.X^1_{at})^{\otimes 2} - (\sigma_c.X^1_{ct})^{\otimes 2}] \otimes dx_t$$

$$\delta C \underset{4\alpha}{\approx} (\sigma_c.X^1_{ac})^{\otimes 2} X^1_{cb} + \sigma_c.X^1_{ac} \otimes \sigma_c.X^2_{cb} + S(\sigma_c.X^1_{ac} \otimes \sigma_c.X^2_{cb})$$

Hence

$$\delta\nu \underset{4\alpha}{\approx} [(\sigma_c.X^1_{ac})^{\otimes 2} - (\overline{Y}^1_{ac})^{\otimes 2}] \otimes X^1_{cb} + D + S(D) \tag{5.5}$$

with

$$D = \sigma_c.X^1_{ac} \otimes \sigma_c X^2_{cb} - \overline{Y}^1_{ac} \otimes \overline{Y}^2_{cb}$$

As we have

$$\overline{Y}^1_{ac} = \sigma_c.X^1_{ac} + \int_a^c (\sigma_t - \sigma_c).\, dx_t$$

we infer that the first term in (5.5) is $\underset{4\alpha}{\approx} 0$.

As we have

$$\overline{Y}^2_{cb} \underset{4\alpha}{\approx} \sigma_c.X^2_{cb} + \int_c^b \sigma'_c.Y^1_{ct}.dx_t \otimes X^1_{tb}$$

we infer that $D \underset{4\alpha}{\approx} 0$. Finally $\delta\nu \underset{4\alpha}{\approx} 0$.

It remains to prove the last claims, this is straightforward.

Let $(1, X^1, X^2, X^3)$ be a rough path. Let $Y = (Y^1, Y^2, Y^3, Z)$ as above. As we have seen, there exists $\overline{Y} = (\overline{Y}^1, \overline{Y}^2, \overline{Y}^3, \overline{Z})$ with the same properties as Y. By induction we get a sequence $Y_n = (Y^1_n, Y^2_n, Y^3_n, Z_n)$ of enriched paths with respect to X. The problem is to know if this sequence converges as for $\alpha > 1/3$. For $\sigma \in \mathcal{C}^3_b$, a proof analogous to the case $\alpha > 1/3$ could be possible, by the use of many tedious computations, which are left to an upcoming paper.

In the particular case of geometric rough paths, this problem was solved by Lyons [24, theorem 6.3.1].

VI. The Fractional Brownian Motion

Many authors have studied the fBM with Hurst index $H > 1/4$, recent publications are [3,4,5,6,8,9,10,12,13].

Let X_t be the $\mathbb{R}^d$-valued linear standard Brownian motion. For $H > 0$, put

$$X_t^H = c(H) \int_0^t (t-r)^{H-1/2} \, dX_r, \qquad \text{with} \qquad c(H) = \frac{1}{\Gamma(H+1/2)}$$

We get a fractional Brownian motion with Hurst parameter H. Recall that we have for $s, t \geq 0$

$$N_2(X_t^H - X_s^H) \leq K_H \, |t-s|^H$$

where K_H is a continuous function of $H > 0$. Besides, for every $H' < H$, there exists a function F which belongs to every $L^p(\mu)$ and such that

$$|X_t^H - X_s^H| \leq |t-s|^{H'} F$$

so that almost every path belongs to $\mathcal{C}^{H'}$.

Let c_n be a Hilbert basis of $L^2([0,1])$, such that each c_n is $\mathcal{C}^\infty$ with compact support in $]0,1[$. If x_t is the standard linear Brownian motion, put

$$U_n = \int_0^1 c_n(t) \, dx_t$$

We get a Hilbert basis of the fist Wiener chaos. One has

$$x_t = \sum_{n \geq 1} U_n s_n \qquad \text{with} \qquad s_n(t) = \int_0^t c_n(u) \, du$$

Hence the linear fractional Brownian motion writes

$$x_t^H = \sum_{n \geq 1} U_n s_n^H(t)$$

with

$$s_n(t) = c(H) \int_0^t (t-r)^{H-1/2} c_n(r) \, dr$$

If n is an integer, put

$$x_t^{H,n} = \sum_{i \leq n} U_i s_i^H(t)$$

Then the process $x^{H,n}$ has $\mathcal{C}^\infty$ paths. Moreover $x^{H,n}$ is a L^p- martingale with values in the separable Banach space $\mathcal{C}_0^{H'}$ which is the closure of $\mathcal{C}^\infty$ in $\mathcal{C}^{H'}$. (Recall that $H' < H$). Hence $x^{H,n}$ converges to x^H in the space $L^p(\mathcal{C}_0^{H'})$ for every $p \geq 1$, and almost everywhere.

6.1 Proposition: *Let $y \in \mathcal{C}^H(L^2)$ be a process which is independant of x. Put*

$$J_{ab}^n = \int_a^b [y_t - y_a] \, dx_t^{H,n} = \sum_{i \leq n} j^i U_i$$

Then J_{ab}^n converges as $n \to \infty$ to a process

$$J_{ab} = \int_a^b [y_t - y_a]\, dx_t^H$$

and

$$N_2(J\,) \leq \mathrm{Cst}\, |b-a|^{2H} \tag{6.1}$$

Proof : Put

$$j^n = c(H-1) \int_a^b [y_t - y_a]\, dt \int_0^t (t-r)^{H-3/2} c_n(r)\, dr = \int_0^b Z_r c_n(r)\, dr$$

where

$$W_r = c(H-1) \int_{a \vee r}^b [y_t - y_a](t-r)^{H-3/2}\, dt$$

Then we have for $r < a$

$$N_2(W_r) \leq |c(H-1)| K_H |b-a|^H \int_a^b (t-r)^{H-3/2} c_n\, dt \leq c(H) K_H |b-a|^H \chi^H(r)$$

with

$$\chi^H(r) = (b-r)^{H-1/2} - (a-r)^{H-1/2}$$

For $r > a$, one has $W_r = U_r + V_r$ with

$$U_r = c(H-1)[y_r - y_a] \int_r^b (t-r)^{H-3/2}\, dt = c(H)[y_r - y_a](b-r)^{H-1/2}$$

$$V_r = c(H-1) \int_r^b [y_t - y_r](t-r)^{H-3/2}\, dt$$

$$N_2(V_r) \leq |c(H-1)| k \int_r^b (t-r)^{2H-3/2}\, dt \leq k(b-a)^{2H-1/2}$$

We get

$$\mathbb{E}([J^n]^2) = \sum_{i \leq n} \mathbb{E}([j^n]^2) \leq \mathbb{E} \int_0^b W_r^2\, dr \leq \mathrm{Cst}\, |b-a|^{4H}$$

so that the sequence J_{ab}^n is bounded. The convergence and (6.1) follow from the Pythagoras theorem.

6.2 Definition: *For $H > 1/4$, we put*

$$\int_a^b (y_t - y_a)\, dx_t^H = \operatorname*{Lim}_{n \to \infty} J_{ab}^n$$

6.3 Proposition: *Assume that y belongs to $\mathcal{C}^H(L^4)$ and is independant from x. Put*

$$K^n = \int_a^b [y_t - y_a]^2\, dx_t^{H,n}$$

Then K^n converges as $n \to \infty$ to a limit K and

$$N_2(K) \leq \mathrm{Cst}\,|b-a|^{3H} \tag{6.2}$$

Proof : Observe that y^2 belongs to L^2 so that

$$K = \int_a^b [y_t - y_a]^2\, dx_t^H$$

is defined. Introduce as above

$$W_r = c(H-1)\int_{a\vee r}^b [y_t - y_a]^2 (t-r)^{H-3/2}\, dt$$

For $r < a$

$$N_2(W_r) \leq |c(H-1)| K_H |b-a|^{2H} \int_a^b (t-r)^{H-3/2} c_n\, dt \leq c(H) K_H |b-a|^{2H} \chi^H(r)$$

For $r > a$, one has $W_r = U_r + V_r$ with

$$U_r = c(H-1)[y_r - y_a]^2 \int_r^b (t-r)^{H-3/2}\, dt = c(H)[y_r - y_a]^2 (b-r)^{H-1/2}$$

$$V_r = c(H-1)\int_r^b [y_t + y_r - 2y_a][y_t - y_r](t-r)^{H-3/2}\, dt$$

$$N_2(V_r) \leq 2|c(H-1)| k |b-a|^H \int_r^b (t-r)^{2H-3/2}\, dt \leq k(b-a)^{3H-1/2}$$

Finally

$$\mathbb{E}([J^n]^2) = \mathbb{E}\int_0^b W_r^2\, dr \leq \mathrm{Cst}\,|b-a|^{6H}$$

Hence the sequence J_{ab}^n is bounded, the convergence and (6.2) follow as above.

6.4 Definition: *For $H > 1/4$, we put*

$$\int_a^b (y_t - y_a)^2\, dx_t^H = \operatorname*{Lim}_{n\to\infty} K_{ab}^n$$

Application: If x and $\overline{x}$ are two independant linear Brownian motions, one can define

$$\int_a^b (\overline{x}_t^H - \overline{x}_a^H)\, dx_t^H, \qquad \int_a^b (\overline{x}_t^H - \overline{x}_a^H)^2\, dx_t^H \qquad \text{by propositions 6.1 and 6.3}$$

$$\int_a^b (x_t^H - x_a^H)\, dx_t^H = \frac{1}{2}(x_b^H - x_a^H)^2, \qquad \int_a^b (x_t^H - x_a^H)^2\, dx_t^H = \frac{1}{3}(x_b^H - x_a^H)^3$$

$$\int_a^b (x_t^H - x_a^H)(\overline{x}_t^H - \overline{x}_a^H)\, dx_t^H = \frac{1}{2}(x_b^H - x_a^H)^2(\overline{x}_b^H - \overline{x}_a^H) - \frac{1}{2}\int_a^b (x_t^H - x_a^H)^2\, d\overline{x}_t^H$$

All of these expressions are analytic functions of $H > 1/4$, and are all majorized by $\mathrm{Cst}\,|b-a|^{2H}$ for the three first, and by $\mathrm{Cst}\,|b-a|^{3H}$ for the last ones.

6.5 Proposition: *Let X_t^H an $\mathbb{R}^d$-valued fBM as defined in the beginning of this section. Denote $X_{ab}^H = X_b^H - X_a^H$ as usual, and put*

$$X_{ab}^{H,2} = \int_a^b X_{at}^H \otimes dX_t^H, \qquad Z_{ab}^H = \int_a^b (X_{at}^H)^{\otimes 2} \otimes dX_t^H$$

$$X_{ab}^{H,3} = \int_a^b X_{at}^H \otimes dX_t^H \otimes X_{at}^H$$

all these expressions are defined, they are analytic functions of $H > 1/4$. For $a \leq c \leq b$, one has

$$X_{ab}^{H,2} - X_{ac}^{H,2} - X_{cb}^{H,2} = X_{ac}^H \otimes X_{cb}^H \tag{6.3}$$

$$Z_{ab}^H - Z_{ac}^H - Z_{cb}^H = (X_{ac}^H)^{\otimes 2} \otimes X_{cb}^H + X_{ac}^H \otimes X_{cb}^{H,2} + S_{12}(X_{ac}^H \otimes X_{cb}^{H,2}) \tag{6.4}$$

$$X_{ab}^{H,3} - X_{ac}^{H,3} - X_{cb}^{H,3} = X_{ac}^{H,2} \otimes X_{cb}^H + X_{ac}^H \otimes X_{cb}^{H,2} \tag{6.5}$$

where S_{12} is the symmetry of $\mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ defined by $(u, v, w) \to (v, u, w)$.

Moreover we have the estimates

$$N_2(X_{ab}^{H,2}) \leq \mathrm{Cst}\,|b-a|^{2H}, \quad N_2(Z_{ab}^H) \leq \mathrm{Cst}\,|b-a|^{3H}, \quad N_2(X_{ab}^{H,3}) \leq \mathrm{Cst}\,|b-a|^{3H} \tag{6.6}$$

Proof : The three expressions are combinations of coordinates as defined above. The three equalities are obvious for $H > 1$, they follow for $H > 1/4$ by analyticity. The three inequalities follow from (6.1) and (6.2), as seen above.

6.6 Remark: The triple $(X^H, X^{H,2}, Z^H)$ is an enriched path in the sense of Section III, the quadruple $(1, X^H, X^{H,2}, X^{H,3})$ is a vector valued geometric rough path in the sense of Lyons [24].

Path regularity

In the sequel, we assume that $1/4 < H' < H$.

6.7 Theorem: *There exists a function F which belongs to every L^p, such that*

$$|X_{ab}^H| \leq |b-a|^{H'} F, \qquad |X_{ab}^{H,2}| \leq |b-a|^{2H'} F$$

$$|Z_{ab}^H| \leq |b-a|^{3H'} F, \qquad |X_{ab}^{H,3}| \leq |b-a|^{3H'} F$$

Proof : The first inequality follows from the classical Kolmogorov lemma. The other three depend on the following lemma.

6.8 Lemma : *Let Δ_n be the subset of numbers $k.2^{-n}$ for $k \in \{0, \ldots, 2^n - 1\}$. If $H' < H$, there exists a function F which belongs to every L^p such that for every $n \geq 0$ and every $t, t' \in \Delta_n$ such that $t' - t = 2^{-n}$, one has*

$$|X_{tt'}^{H,2}| \leq |t' - t|^{2H'} F, \qquad |Z_{tt'}^{H}| \leq |t' - t|^{3H'} F \qquad |X_{tt'}^{H,3}| \leq |t' - t|^{3H'} F \tag{6.7}$$

Proof : For any s, t one has with $\varepsilon = 2H - 2H'$, $p > 1/\varepsilon$

$$\mathbb{E}\left|\frac{X_{st}^{H,2}}{|t-s|^{2H'}}\right|^p \leq c_p |t-s|^{2p(H-H')} = c_p |t-s|^{\varepsilon p}$$

$$\mathbb{E}\left|\frac{X_{tt'}^{H,2}}{|t'-t|^{2H'}}\right|^p \leq c_p \, 2^{-n\varepsilon p} \qquad \Rightarrow \qquad \mathbb{E}\sum_{t \in \Delta_n}\left|\frac{X_{tt'}^{H,2}}{|t'-t|^{2H'}}\right|^p \leq c_p \, 2^{n(1-\varepsilon p)}$$

Put
$$F_p = \left\{\sum_{n \geq 0}\sum_{t \in \Delta_n}\left|\frac{X_{tt'}^{H,2}}{|t'-t|^{2H'}}\right|^p\right\}^{1/p} \qquad \Rightarrow \qquad \mathbb{E}F_p^p \leq \frac{c_p}{1 - 2^{1-\varepsilon p}}$$

The function F_p belongs to L^p. Putting $F = \operatorname*{Inf}_{p > 1/\varepsilon} F_p$ yields the same inequality (the first in (6.7)) and F belongs to every L^p. The same argument holds for the other formulae (6.7).

Proof of theorem 6.7: We first deal with $X_{ab}^{H,2}$. Put

$$\mu(a,b) = X_{ab}^{H,2} = \int_a^b X_{at}^H \otimes dX_t^H$$

For $c \in [a,b]$ we get (formula (6.3))

$$|\delta\mu(a,b,c)| = |X_{ac}^H \otimes X_{cb}^H| \leq |c-a|^{H'} |b-c|^{H'} F$$

where F belongs to every L^p. The first conclusion follows by the combinational lemma of the appendix.

Next take $\mu(a,b) = Z_{ab}^H$. We get by formula (6.4)

$$\delta\mu(a,b,c) = (X_{ac}^H)^{\otimes 2} \otimes X_{cb}^H + X_{ac}^H \otimes X_{cb}^{H,2} + S(X_{ac}^H \otimes X_{cb}^{H,2})$$

This yields the estimate

$$|\delta\mu(a,b,c)| \leq |c-a|^{H'} |b-c|^{H'} |b-a|^{H'} F$$

with a function F in every L^p. Applying the combinational lemma gives the result.

In the same way, put $\mu(a,b) = X_{ab}^{H,3}$, we have

$$\delta\mu(a,b,c) = X_{ac}^{H,2} \otimes X_{cb}^H + X_{ac}^H \otimes X_{cb}^{H,2}$$

$$|\delta\mu(a,b,c)| \leq |c-a|^{H'} |b-c|^{H'} |b-a|^{H'} F$$

and the result follows as above.

Hence we can claim

6.9 Theorem: *For $H > H' > 1/3$, almost every $(X_t^H, X_{st}^{H,2})$ is a H'-enriched path of degree 1.*

For $H > H' > 1/4$, almost every $(X_t^H, X_{st}^{H,2}, Z_{st}^H)$ is a H'-enriched path of degree 2, and $(1, X_t^H, X_{st}^{H,2}, X_{st}^{H,3})$ is a H'-rough path.

6.10 Theorem: *Let $f(x)$ be a $\mathcal{C}_b^3$ matrix-valued function on $\mathbb{R}^d$. Let X_t^H the d-dimensional fractional Brownian motion. For $H > H' > 1/4$, the following integral*

$$Z_t = \int_0^t f(X_s^H).\, dX_s^H$$

along almost every H'-enriched path, makes sense. Moreover we have

$$|Z_b - Z_a| \leq |b-a|^{H'} F$$

where F belongs to every L^p.

Besides for $H = 1/2$ (standard Brownian motion), Z_t coincides with the Stratonovich integral (it suffices to take $f \in \mathcal{C}_b^2$).

Proof : The only point is to verify that Z_t is the the Stratonovich integral for $H = 1/2$. This holds by definition if f is an affine polynomial. In general, for to define the integral of f, we put pathwise

$$\mu(a,b) = f(X_a).(X_b - X_a) + \nabla f(X_a).X_{ab}^{H,2}$$

We have (with dots as contracted tensor products)

$$\delta\mu = [f(X_a) - f(X_c) + \nabla f(X_a).(X_c - X_a)].(X_b - X_c) + [\nabla f(X_a) - \nabla f(X_c)].X_{ab}^{H,2}$$

There exists $F \in \bigcap_p L^p$ such that $|\delta\mu| \leq |b-a|^{3H'} F$, so that we get $N_2(\delta\mu) \leq \mathrm{Cst}\, |b-a|^{3H'}$. As we have seen in Section II, the Stratonovich integral satisfies

$$N_2\left(\int_a^b f(X_t) \circ dX_t - \mu(a,b)\right) \leq \mathrm{Cst}\, |b-a|^{3H'}$$

so that it coincides with the pathwise integral.

6.11 Remark: Many authors use a fractional Brownian motion which is a centered Gaussian process G_t^H with the covariance function $|t-s|^{2H}$. Thanks to [5], it can be represented in terms of the standard Brownian motion X_t by the formula

$$G_t^H = \int_0^t K^H(t,s)\, dX_s$$

where the kernel K^H is analytic with respect to H. Then for $H > 1/4$, the analytic continuation arguments of this section extends to this case.

Ito's formulae

Here we shall modify the enrichment of X^H in order to get Ito's type integral for $H > 1/3$ and for $H > 1/4$.

First we deal with $H > 1/3$.

Put

$$\widetilde{X}^{H,2}_{ab} = \int_a^b X^H_{at} \odot dX^H_t$$

where $\odot$ is the tensorization of the Skorohod integral, as defined in [10]. Recall that we have

$$\widetilde{X}^{H,2}_{ab} - X^{H,2}_{ab} = -\frac{c(H)^2}{4H}[b^{2H} - a^{2H}]\, I$$

where I is the Kronecker tensor. For $H = 1/2$, the Skorohod integral coincides with the Ito integral, as well known. It is straightforward to check that $(1, X^H, X^{H,2})$ is a rough path and also an enrichment in the sense of section III. Notice that this is a pathwise analytic function of $H > 1/3$, so that the Ito-Skorohod formula which was proved in [10] holds

$$F(X^H_b) - F(X^H_a) = \int_a^b \nabla F(X^H_t) \odot dX^H_t + \frac{1}{2}c(H)^2 \int_a^b \Delta F(X^H_t)\, t^{2H-1}\, dt$$

for every real polynomial F (here we have again a contracted tensor product).

For $H > 1/4$, we must define an enrichment $(X^H, \widetilde{X}^{H,2}, \widetilde{Z}^H)$. We shall also define $\widetilde{X}^{H,3}$ such that $(1, X^H, \widetilde{X}^{H,2}, \widetilde{X}^{H,3})$ be a rough path.

Put

$$\eta^H_{ab} = \frac{c(H)^2}{4H}(b^{2H} - a^{2H})\, I$$

$$\widetilde{Z}^H_{ab} = Z^H_{ab} - \left[\int_a^b X^H_{at} \otimes d\eta^H_t + S_{12}\left(\int_a^b X^H_{at} \otimes d\eta^H_t\right)\right]$$

For $b > a > 0$, it is easy to check that there exists $F \in \bigcap_p L^p$ such that

$$|\widetilde{Z}^H_{ab}| \leq |b-a|^{2H'} F$$

$$\widetilde{Z}^H_{ab} - \widetilde{Z}^H_{ac} - \widetilde{Z}^H_{cb} = (X^H_{ac})^{\otimes 2} \otimes X^H_{cb} + X^H_{ac} \otimes \widetilde{X}^{H,2}_{ac} + S_{12}(X^H_{ac} \otimes \widetilde{X}^{H,2}_{ac})$$

where S_{12} is the symmetry of $\mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$ defined by $(u \otimes v \otimes w) \to (v \otimes u \otimes w)$.

In the same way, put for $b > a > 0$

$$\widetilde{X}^{H,3}_{ab} = X^{H,3}_{ab} + \int_a^b X^H_{at} \otimes d\eta^H_t + \int_a^b (\eta^H_t - \eta^H_a) \otimes dX^H_t$$

As above, we get F such that

$$|\widetilde{X}^{H,3}_{ab}| \leq |b-a|^{3H'} F, \qquad \widetilde{X}^{H,3}_{ab} - \widetilde{X}^{H,3}_{ac} - \widetilde{X}^{H,3}_{cb} = X^H_{ac} \otimes \widetilde{X}^{H,2}_{cb} + \widetilde{X}^{H,2}_{ac} \otimes X^H_{cb}$$

The three functions $\widetilde{X}^{H,2}$, $\widetilde{Z}^H$ and $\widetilde{X}^{H,3}$ are pathwise analytic for $H > 1/4$, so that the Ito-Skorohod formula holds for $H > 1/4$.

More generally we obtain for $H > 1/4$ the formula

$$\int_a^b f(X_t^H).\, dX_t^H = \int_a^b f(X_t)^H \odot dX_t^H + \frac{c(H)^2}{2} \int_a^b (\operatorname{div} f)(X_t) t^{2H-1}\, dt$$

for a vector field f of class $\mathcal{C}^3$ on $\mathbb{R}^d$, where div is the Euclidean divergence.

6.12 Remarks: a) For $H > 1/4$, t^{2H} is not $\mathcal{C}^{3H'}$ until the origin. Nevertheless, as $a \to 0$, $\widetilde{Z}_{ab}^H$ and $\widetilde{X}_{ab}^{H,3}$ converge to limits which satisfy inequalities (6.7) on all of $[0,1]$.

b) In fact, we can replace $I\,\eta_t$ by any other tensor of class $\mathcal{C}^{3\alpha}$ with $\alpha > 3/4$.

VII. Support theorem for the fBM

This theorem was established for the ordinary Brownian motion in [18,25,26] and for the geometric fBM in [11,12].

Generalities

Let α be real, $1/4 < \alpha < 1$. Consider $\mathcal{C}_0^\alpha[0,1]$ which is the closure of $\mathcal{C}_0^\infty[0,1]$ in $\mathcal{C}^\alpha[0,1]$. This is a separable Banach space under the Hölder norm. Denote $\mathcal{E}^\alpha[0,1]$ the space of elements $\mathcal{X} = (X, X^2, X^3, Z)$ such that $(1, X, X^2, X^3)$ is a rough path, and (X, X^2, Z) is an enrichment. Put

$$d(\mathcal{X}, \mathcal{X}') = \|X - X'\|_\alpha + \operatorname*{Sup}_{t \neq s} \left\{ \frac{|X_{st}^2 - X_{st}^2{}'|}{|t-s|^{2\alpha}} + \frac{|X_{st}^3 - X_{st}^3{}'|}{|t-s|^{3\alpha}} + \frac{|Z_{st} - Z_{st}{}'|}{|t-s|^{3\alpha}} \right\}$$

This a Polish space (a separable complete metric space).

Let $h \in \mathcal{C}_0^\infty[0,1]$, put

$$T^h(\mathcal{X}) = \mathcal{X}^h = (X + h, X^{h,2}, X^{h,3}, Z^h)$$

with

$$X_{ab}^{h,2} = \int_a^b [X_{at} + h_{at}] \otimes d(X_t + h_t), \qquad Z_{ab}^h = \int_a^b (X_{at} + h_{at})^{\otimes 2} \otimes d(X_t + h_t)$$

where the integrals are taken in the sense of $\mathcal{X}$ and in the sense of Young. It remains to define $X^{h,3}$.

7.1 Lemma : *$(1, X^h, X^{h,2}, X^3)$ is an almost rough path.*

Proof : Straightforward computation.

7.2 Corollary: *There exists a unique $X^{h,3}$ such that $(1, X^h, X^{h,2}, X^{h,3})$ is a rough path.*

We then have a canonical system $\mathcal{X}^h$.

7.3 Proposition: *The map T^h is continuous (even locally Lipschitz) on $\mathcal{E}^\alpha$, and we have the group property $T^h \circ T^k = T^{h+k}$, so that $(T^h)^{-1} = T^{-h}$.*

Proof : Left to the reader.

7.4 Definition: *The skeleton $S(\mathcal{X})$ of $\mathcal{E}^\alpha$ is the subset constituted with the $\mathcal{C}_0^\infty$ elements of $\mathcal{E}^\alpha$ (where the integrals are taken in the ordinary sense).*

Observe that the group T^h transitively operates on the skeleton.

We denote J^α the closure of the skeleton, we have $T^h(J^\alpha) = J^\alpha$.

7.5 Remark: For every $\mathcal{X} \in J^\alpha$, $(1, X, X^2, X^3)$ is a geometric rough path.

7.6 Proposition: *Let θ be a measure on J^α which is quasi-invariant under T^h, that is $T^h(\theta)$ is absolutely continuous with respect to θ, for every $h \in \mathcal{C}_0^\infty$. If $(0,0)$ belongs to the support of θ, then J^α is the support of θ.*

Proof : Let $\mathcal{X}$ which belongs to J^α. Let V be a neighbourhood of $\mathcal{X}$ in J^α. There exists h such that $T^h(V)$ is a neighbourhood of $(0,0)$, so that V is of positive θ-measure.

Application to the fBM

Let μ^H be the law of the fBM X^H as defined in section VI. This is a Gaussian measure, so that it is quasi-invariant by the Cameron-Martin translations. Observe that $\mathcal{C}_0^\infty$ is included in the Cameron-Martin space. Hence

7.7 Proposition: *Let θ^H be the law of the enriched fBM defined in section VI. Then θ^H is supported by $J^{H'}$. Moreover θ^H is quasi-invariant under the T^h.*

Proof : Consider the approximation $\mathcal{X}^n$ of $\mathcal{X}$ defined with the help of the martingale $X_t^{H,n}$ in section VI. As $\mathcal{X}^n$ belongs to $J^{H'}$ and converges in $L^p(\mathcal{E}^{H'})$ and almost everywhere as proved in section VI, then $\mathcal{X}$ belongs to $J^{H'}$, so that θ^H is supported by $J^{H'}$. The quasi-invariance follows from the quasi-invariance of μ^H.

7.8 Lemma : *For $\theta^{H'}$-almost every $\mathcal{X}$, the following property holds*

$$T_{-X^{H,n}}(\mathcal{X}) \to 0$$

in the space $\mathcal{E}^\alpha$.

Proof : For $H > 1/3$, the only point is to verify that if x_t and $\overline{x}_t$ are two independant linear Brownian motions, then with the notations of section VI

$$\int_a^b [\overline{x}_t^{H,n} - \overline{x}_a^{H,n}]\, dx_t^{H,n} \to \int_a^b [\overline{x}_t^H - \overline{x}_a^H]\, dx_t^H$$

For every a, b the convergence holds almost surely. Moreover, we have the domination with $F \in \bigcap_p L^p$

$$\left| \int_a^b [\overline{x}_t^{H,n} - \overline{x}_a^{H,n}] \, dx_t^{H,n} \right| \leq |b-a|^{2H'} F$$

then for every $H'' < H'$, the convergence takes place in $\mathcal{E}^{H''}$. Finally, the convergence holds in $\mathcal{E}^{H'}$ for every $H' < H$.

For $H > 1/4$, a similar proof yields the result.

7.9 Theorem: *(Support theorem) The support of θ^H in $\mathcal{E}^{H'}$ is exactly the closure of the skeleton.*

Proof : By proposition 7.7, we only have to prove that every neighbourhood of $(0,0)$ has a positive $\theta^{H'}$-measure. Let V be such a neighbourhood, and let $\mathcal{X}$ a point in the support of $\theta^{H'}$ for which the property of the lemma holds. Let n be such $T_{-X^{H,n}}(\mathcal{X}) \in V$. We have $\theta^H[T_{X^{H,n}}(V)] > 0$, so that $\theta^H(V) >$ by the quasi-invariance.

7.10 Corollary: *Let $\sigma \in \mathcal{C}_b^3$ be matrix valued. Put*

$$y_t = \int_0^t \sigma(X_t^H).\, dX_t^H$$

Then the support of the law of y is the closure in $\mathcal{C}^\alpha$ of the $\displaystyle\int_0^t \sigma(h_t).\, h'(t)\, dt$ for $h \in \mathcal{C}_0^\infty$. Moreover y^n converges to y with

$$y_t^n = \int_0^t \sigma(X^{H,n}).\, dX_t^{H,n}$$

7.11 Corollary: *Same claiming for the solution of*

$$y_t = \int_0^t \sigma(y_t).\, dX_t^H$$

in the sense of sections IV and V.

More generally, let η_t be a tensor of class $\mathcal{C}_0^\infty$. Consider the map $\mathcal{X} \to \widetilde{\mathcal{X}}$ as defined in section VI

$$\widetilde{\mathcal{X}} = (x, \widetilde{x}^2, \widetilde{x}^3, z)$$

$$\widetilde{x}^2 = x^2 - \eta_t$$

$$\widetilde{z} = z - \int_a^b x_{at} \otimes d\eta_t - S_{12}\left(\int_a^b x_{at} \otimes d\eta_t \right)$$

$$\widetilde{x}^3 = x^3 + \int_a^b x_{at} \otimes d\eta_t + \int_a^b (\eta_t - \eta_a) \otimes dx_t$$

This map is injective continuous from $J^{H'}$ onto a closed set $\widetilde{J}^{H'}$ in $\mathcal{E}^{H'}$. Let $\widetilde{\theta}^{H'}$ be the image of $\theta^{H'}$. We then have

7.12 Proposition: *The support of the law of the η-peturbated fBM is exactly $\widetilde{J}^{H'}$, that is the closure of the η-skeleton (the elements $\widetilde{\mathcal{X}}$ which are $\mathcal{C}_0^\infty$). The only point is that integration along the $\mathcal{C}^\infty$-skeleton is not taken in the usual sense.*

7.13 Remark: In the case of the Skorohod fBM, we have $\eta_t = c(H)^2 I.t^{2H}/4H$ which is not $\mathcal{C}^\infty$ until the origin. Nevertheless, analogous results hold (corollaries and proposition) with less regular skeleton, thanks to the remark 6.12 a).

VIII. Appendix

The combinational lemma

Let Δ_n the set of dyadic numbers $k.2^{-n}$ for $0 \leq k < 2^n$.

8.1 Lemma : *Let $\mu(a,b)$ be a real or Banach valued function defined on $\Delta \times \Delta$, such that for $n \geq 0$, $t \in \Delta_n$, $t' = t + 2^{-n}$ one has*

$$|\mu(t,t')| \leq k|t'-t|^{2\alpha}$$

and for every $a \leq b$ and $c \in [a,b]$

$$|\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq k|c-a|^\alpha |b-c|^\alpha$$

Then one has for every $a \leq b$

$$|\mu(a,b)| \leq 4k(1-2^{-\alpha})^{-1}|b-a|^{2\alpha}$$

and μ extends continuously to $[0,1]^2$.

Proof : Take $a < b$, and $n_0 = \mathrm{Min}\{n \;/\;]a,b[\cap \Delta_n \neq \emptyset\}$. Let c be the common point. One has $b - c < 2^{-n_0}$. Write

$$b = c + \sum 2^{-n_i}$$

where $n_i > n_0$ is an increasing sequence of integers (finite since $b \in \Delta$). Put

$$c_i = c + 2^{-n_1} + \cdots + 2^{-n_i} \qquad \Rightarrow \qquad c_{i+1} - c_i = 2^{-n_{i+1}} \leq 2^{-i}(b-c)$$

By induction we have $c_i, c_{i+1} \in \Delta_{n_{i+1}}$, then

$$|\mu(c, c_{i+1}) - \mu(c, c_i)| \leq 2k(b-c)^{2\alpha} 2^{-i\alpha} \qquad \Rightarrow \qquad |\mu(c,b)| \leq \frac{2k(b-c)^{2\alpha}}{1-2^{-\alpha}}$$

In the same way $|\mu(a,c)|$ is majorized, and finally

$$|\mu(a,b)| \leq k'(b-a)^{2\alpha}$$

with $k' = 4k(1-2^{-\alpha})^{-1}$. It remains to prove that μ is uniformly continuous on $\Delta \times \Delta$. This follows from the easy inequality

$$|\mu(a,b) - \mu(a',b')| \leq (k+k')\Big[|a-a'|^{\alpha} + |b-b'|^{\alpha}\Big]$$

8.2 Remark: In this lemma we can replace the basis 2 by another one (for example the basis 9 which is useful for the Peano curve).

Hölder norms, and p-variation

Let $F(s,t)$ be a continuous function defined on $[a,b]^2$. Define a α-Hölder functional for $0 < \alpha \leq 1$ by

$$\|F\|_{\alpha} = \operatorname*{Sup}_{s \neq t} \frac{|F(s,t)|}{|t-s|^{\alpha}}$$

Note that if $F(s,t) = f(t) - f(s)$, we recover the classical α-Hölder norm of f.

The p-variation functional for $p \geq 1$ is defined by

$$V_p(F) = \operatorname*{Sup}_{\sigma} \left[\sum_i |F(t_i, t_{i+1})|^p\right]^{1/p}$$

where the supremum is taken over all the finite subdivisions $\sigma = \{t_1, t_2, \ldots, t_n\}$ of $[a,b]$.

If $p = 1/\alpha$, we have the obvious inequality

$$V_p(F) \leq (b-a)^{\alpha} \|F\|_{\alpha}$$

There is a kind of converse property. Suppose that $V_p(F)$ is finite. Define $V(s,t)$ as the p-variation of F restricted to the segment $[s,t]$, and put $w(s,t) = V(s,t)^p$. For $s < t < u$ we have $w(s,u) \geq w(s,t) + w(t,u)$.

8.3 Lemma : *As $s \to t$, $w(s,t)$ converges to 0.*

Proof : Put $v(t) = w(a,t)$, $u(t) = w(t,b)$. As F is continuous, v is a non- decreasing l.s.c. function, then a left continuous function. In the same way, u is right continuous.

We have $w(s,t) + v(s) \leq v(t)$. As $s \uparrow t$, we get

$$\operatorname*{Lim\,Sup}_{s \uparrow t}\ w(s,t) + v(t) \leq v(t)$$

and $w(s,t)$ converges to 0. By the same argument, we have $w(s,t) + u(t) \leq u(s)$. As $t \downarrow s$, we get

$$\operatorname*{Lim\,Sup}_{t \downarrow s}\ w(s,t) + u(s) \leq u(s)$$

and $w(s,t)$ converges to 0.

8.4 Proposition: *The function $v(t) = w(a,t)$ is continuous.*

Proof : Let $a \leq s \leq t$. One has $v(t) = \mathrm{Sup}\,(A(t), B(t))$, where

$$A(t) = \operatorname*{Sup}_{\sigma \in \Sigma_s^t} \sum_i |F(t_i, t_{i+1})|^p, \qquad B(t) = \operatorname*{Sup}_{\sigma \notin \Sigma_s^t} \sum_i |F(t_i, t_{i+1})|^p$$

where Σ_s^t denotes the set of subdivisions σ of $[a, t]$ such that s is a point of subdivision. First $A(t) = v(s) + w(s, t)$ converges to $v(s)$ as $t \downarrow s$.

Next, suppose that $t > s$ converges to s along a sequence such that $B(t) > M$. For each t there exists $\sigma \notin \Sigma_s^t$ such that $s \in]t_i, t_{i+1}[$ and

$$M < \sum_{j<i} |F(t_j, t_{j+1})|^p + |F(t_i, t_{i+1})|^p + \sum_{j>i} |F(t_j, t_{j+1})|^p$$

$$M + |F(t_i, s)|^p + |F(s, t_{i+1})|^p < |F(t_i, t_{i+1})|^p + A(t)$$

Then $t_{i+1} \in [s, t]$ converges to s, and $|F(t_i, s)|^p - |F(t_i, t_{i+1})|^p$ converges to 0 (uniform continuity), hence $M \leq v(s)$. It follows that

$$\operatorname*{Lim\,Sup}_{t \downarrow s} B(t) \leq v(s)$$

8.5 Corollary: *We have*

$$|F(s, t)| \leq |v(t) - v(s)|^{1/p}$$

There exists a new parametrization $t = v^{-1}(u)$ such that $G = F \circ [v^{-1} \times v^{-1}]$ is α-Hölder for $\alpha = 1/p$.

Proof : Obvious.

8.6 Remarks: a) If v is not strictly increasing, replace $v(t)$ with $kt + v(t)$.

b) Take $F(s, t) = f(t) - f(s)$ where f is a continuous function with finite p-variation, then f is $1/p$-Hölder continuous for a suitable reparametrization.

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