Electronic Journal of Probability

Vol. 3 (1998) Paper no. 2, pages 1–26.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol3/paper2.abs.html

Hypercontractivity and comparison of moments of iterated maxima and minima of independent random variables

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Abstract: We provide necessary and sufficient conditions for hypercontractivity of the minima of nonnegative, i.i.d. random variables and of both the maxima of minima and the minima of maxima for such r.v.'s. It turns out that the idea of hypercontractivity for minima is closely related to small ball probabilities and Gaussian correlation inequalities.

Keywords: hypercontractivity, comparison of moments, iterated maxima and minima, Gaussian correlation inequalities, small ball probabilities.

Submitted to EJP on May 17, 1997. Final version accepted on January 7, 1998.

AMS 1991 Subject Classification: Primary 60B11, 60E07, 60E15. Secondary 52A21, 60G15

* Supported in part by an NSF grant

† Participant, NSF Workshop in Linear Analysis & Probability, Texas A&M University

** Supported in part by Polish KBN grant

⋆ Supported in part by US-Israel Binational Science Foundation

‡ Supported in part by a Texas Advanced Research Program grant

Section 1. Introduction.

Let h be a real valued function defined on $\cup_{n=1}^{\infty}\mathbb{R}^n$ and let $0 < p < q$. We say that a random variable X is $\{p,q\} - h-$*hypercontractive* if, for some positive σ,

$$(Eh^q(\sigma X_1, \sigma X_2, \ldots, \sigma X_n))^{1/q} \leq (Eh^p(X_1, X_2, \ldots, X_n))^{1/p}$$

for all n where $X, X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables.

For linear functions h this notion (without the name we attached to it) appears in many inequalities (see, e.g., Nelson (1966), Bonami (1970) Gross (1975), Beckner, W. (1975), Krakowiak and Szulga (1988) and Borell (1979)). In a recent paper of de la Peña, Montgomery-Smith and Szulga (1994), this notion appears for the function $h(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n) = \max_{i \leq n} x_i$. In this paper we study this notion for this max function as well as for the min function $h(x_1, \ldots, x_n) = m_n(x_1, \ldots, x_n) = \min_{i \leq n} x_i$ and for iterated min's and max's, $M_{n_1} m_{n_2} M_{n_3} \ldots m_{n_k}$ where, for example,

$$M_n m_k(x_1, \ldots, x_k, \ldots, x_{k(n-1)+1}, \ldots, x_{kn}) = \max_{i \leq n}(\min_{k(i-1) < j \leq ki} x_i).$$

We give necessary and sufficient conditions for a positive random variable X to be $\{p,q\} - h-$hypercontractive for each of these functions. One surprising consequence (Theorem 5.5) is that in order that X be $\{p,q\} - M_{n_1} m_{n_2} M_{n_3} \ldots m_{n_k}-$hypercontractive it is sufficient (and, of course, necessary) that it is separately $\{p,q\} -$ min and max hypercontractive. Note that the name hypercontractivity is usually attached to inequalities of the form $(Eh^q(X_1, \sigma X_2))^{1/q} \leq (Eh^p(X_1, X_2))^{1/p}$, which in turn are used to prove inequalities of the form discussed here. The reason we permit ourselves to use this notion with a somewhat different interpretation is that we prove below that, in our context the two notions are equivalent (see, e.g., Theorem 5.1).

The main technical tool of the paper of de la Peña, Montgomery-Smith and Szulga is a comparison result for tail distributions of two positive random variables X and Y of the type: There exists a constant, c, such that $P(X > ct) \leq cP(Y > t)$ for all $t > c$. There are two conditions under which they prove that such comparison holds. The first is a hypercontractivity of the max for one of the two variables (and some $p < q$). The second is an inequality of the type $\|\max_{i \leq n} X_i\|_p \leq C\|\max_{i \leq n} Y_i\|_p$, for some C and every n. In such a case we'll say that X and Y are $p-max-$*comparable*, and if one replaces max with a general function h we'll call them $p - h-$*comparable*. We consider here this notion also for the function min and for iterated min's and max's. Among other things we prove an analogous theorem to that of de la Peña, Montgomery-Smith and Szulga (Theorem 3.3) giving a sufficient condition for $P(X \leq ct) \leq \delta P(Y \leq t)$ for all $t \leq c\|X\|_p$ for some $0 < c, \delta < 1$. We also combine our theorem with a version of that of de la Peña, Montgomery-Smith and Szulga (Theorem 5.4) to give a sufficient condition for the comparison of the tail distributions of X and Y to hold for all $t \in \mathbb{R}^+$.

Another application of the technique we develop here is contained in Corollary 5.2, where we give a sufficient condition for a random variable X to be hypercontractive with respect to another interesting (family of) function(s) - the k-order statistic(s).

Our initial motivation for attacking the problems addressed in this paper was an approach we had to solve (a somewhat weaker version of) the so called Gaussian Correlation Problem. Although we still can not solve this problem, we indicate (in and around Theorem 6.8) the motivation and the partial result we have in this direction. As a byproduct we also obtain (in Theorem 6.4) an inequality, involving the Gaussian measure of symmetric convex sets, stated by Szarek (1991) (who proved a somewhat weaker result) as well as a similar inequality for symmetric stable measures.

The rest of the paper is organized as follows. Section 2 provides some basic lemmas and notations. Hypercontractivity for minima and some equivalent conditions are given in section 3. Section 4 presents hypercontractivity for maxima in a way suitable for our applications. In section 5, we combine the results in section 3 and 4 to obtain hypercontractivity for iterated min's and max's, and comparison results for the small ball probabilities of possibly different random vectors. We also give there the sufficient condition for the comparison of moments of order statistics. In section 6, we apply our results to show that the α symmetric stable random variables with $0 < \alpha \le 2$ are minmax and maxmin hypercontractive, which is strongly connected to the regularity of the α-stable measure of small balls. In this section we also indicate our initial motivation, related to the modified correlation inequality as well as a partial result in this direction. Finally, in the last section, we mention some open problems and final remarks.

Section 2. Notations and Some Basic Lemmas.

For nonnegative i.i.d. r.v.'s $\{Z_j\}$, let $m_n = m_n(Z) = \min_{j\le n} Z_j$ and $M_n = M_n(Z) = \max_{j\le n} Z_j$. The r-norm of the random variable W is

$$\|W\|_r = (E|W|^r)^{1/r} \quad \text{for} \quad r > 0$$

and

$$\|W\|_0 = \lim_{r\to 0_+} \|W\|_r = \exp(E(\ln|W|)).$$

We will denote

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}.$$

If $s < t$ then $s \vee (x \wedge t) = (s \vee x) \wedge t$ and it is denoted by $s \vee x \wedge t$. Here and in the rest of the paper we always assume, unless otherwise specified, that $0 < p < q$.

Lemma 2.1. *Assume $\|W\|_q \le C\|W\|_p$. Then*
(a) for $\alpha = 2^{1/(q-p)}C^{q/(q-p)}$, $EW^p \le 2EW^pI_{\{W\le\alpha\|W\|_p\}}$, and
(b) for $0 \le \lambda \le 1$, $P(W > \lambda\|W\|_p) \ge \big((1-\lambda^p)C^{-p}\big)^{q/(q-p)}$.

Proof. (a). Note that

$$EW^pI_{\{W>\alpha\|W\|_p\}} \le E\frac{W^q}{(\alpha\|W\|_p)^{q-p}} \le \frac{C^q\|W\|_p^p}{\alpha^{q-p}} = \frac{1}{2}EW^p.$$

Thus $EW^pI_{\{W>\alpha\|W\|_p\}} \le EW^pI_{\{W\le\alpha\|W\|_p\}}$ and $EW^p \le 2EW^pI_{\{W\le\alpha\|W\|_p\}}$.

(b). The result follows from the Paley-Zygmund inequality

$$EW^p \leq a^p + (EW^q)^{p/q} P^{1-p/q}(W > a)$$

with $a = \lambda \|W\|_p$. $\blacksquare$

Lemma 2.2. *Let $0 < \beta < 1$, then*

(a) *$x \geq \beta^{p/(q-p)}$ and $y^{1/q} \leq x^{1/p}$ imply $(1-x) \leq \beta^{-1} p q^{-1} (1-y)$, for $0 < x, y < 1$*

(b) *$pq^{-1} x \geq y$ implies $(1-x)^{1/q} \leq (1-y)^{1/p}$, for $0 < x, y < 1$*

(c) *$pq^{-1} x \leq y$ implies $(1+x)^{1/q} \leq (1+y)^{1/p}$, for $x, y > 0$*

Proof. (a). We have

$$1 - y \geq 1 - x^{q/p} = (1-x) q p^{-1} \eta^{(q-p)/p} \geq (1-x) q p^{-1} x^{(q-p)/p} \geq (1-x) q p^{-1} \beta$$

where the equality follows from the mean value theorem with $x \leq \eta \leq 1$.

(b). The conclusion follows from the well known fact $(1-y)^\alpha \geq 1 - \alpha y$ with $\alpha = q/p > 1$.

(c) is proved similarly. $\blacksquare$

Lemma 2.3. *Fix $0 < p \leq q < \infty$. Let μ and ν be positive measures on S and T, respectively. If $h\colon \mathbb{R}^n \to \mathbb{R}_+$ is a measurable function and $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ are two sequences of independent r.v.'s such that for each i and each $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ we have*

$$(Eh^q(x_1, \ldots, x_{i-1}, \eta_i, x_{i+1}, \ldots, x_n))^{1/q} \leq (Eh^p(x_1, \ldots, x_{i-1}, \xi_i, x_{i+1}, \ldots, x_n))^{1/p},$$

then

$$(Eh^q(\eta_1, \eta_2, \ldots, \eta_n))^{1/q} \leq (Eh^p(\xi_1, \xi_2, \ldots, \xi_n))^{1/p}.$$

Proof. This follows easily by induction and Minkowski's inequality

$$\left(\int_S \left(\int_T |f(s,t)|^p \mu(dt) \right)^{q/p} \nu(ds) \right)^{1/q} \leq \left(\int_T \left(\int_S |f(s,t)|^q \nu(ds) \right)^{p/q} \mu(dt) \right)^{1/p} \quad \blacksquare.$$

Section 3. Hypercontractivity for minima.

Definition 3.1. *We say that a nonnegative random variable W is $\{p,q\}$-min-hypercontractive (with constant C), if there exists C such that for all n,*

$$\|m_n(W)\|_q \le C\|m_n(W)\|_p.$$

In this case we write $W \in \min\mathcal{H}_{p,q}(C)$.

Lemma 3.2. *If $W \in \min\mathcal{H}_{p,q}(C)$ then for each n*

$$\|m_n(W)\|_p \le K\|m_{2n}(W)\|_p \quad \textit{with} \quad K = 2^{(2q-p)/p(q-p)}C^{q/(q-p)}.$$

Proof. Let $H(t) = H_{p,n}(t) = E(m_n(W) \wedge t)^p$ and note that $H(t)/t^p$ is non-increasing. Taking α as in the Lemma 2.1,

$$\|m_{2n}\|_p^p \ge Em_{2n}^p I_{m_n \le \alpha\|m_n\|_p} = EH(m_n) I_{m_n \le \alpha\|m_n\|_p}.$$

Since $H(t)/t^p$ is non-increasing,

$$EH(m_n)I_{m_n\le\alpha\|m_n\|_p} = E\frac{H(m_n)}{m_n^p} m_n^p I_{m_m \le \alpha\|m_n\|_p} \ge \frac{H(\alpha\|m_n\|_p)}{(\alpha\|m_n\|_p)^p} Em_n^p I_{m_n\le\alpha\|m_n\|_p}.$$

Thus by Lemma 2.1,

$$\|m_{2n}\|_p^p \ge \frac{1}{2}\frac{H(\alpha\|m_n\|_p)}{(\alpha\|m_n\|_p)^p}\|m_n\|_p^p.$$

Furthermore,

$$H(\alpha\|m_n\|_p) \ge Em_n^p I_{m_n\le\alpha\|m_n\|_p} \ge 2^{-1}\|m_n\|_p^p,$$

which gives the conclusion. ∎

The following theorem is a min-analog of a result of de la Peña, Montgomery-Smith and Szulga (1994), proved for maxima, (cf. Theorem 4.4 below).

Theorem 3.3. *Fix $\rho > 0$. Let $0 \le p < q$, and let X, Y be r.v.'s such that $X \in \min\mathcal{H}_{p,q}(C)$ and there exists a constant B such that $\|m_n(Y)\|_q \le B\|m_n(X)\|_q$ for all n. Then $P(X \le \tau t) \le \delta P(Y \le t)$ for all $t \le t_0 = \rho\|X\|_q$ for some constants $0 < \delta < 1$ and $\tau > 0$ depending on p, q, ρ, B and C only.*

Proof. We first prove the assertion of the theorem for $p > 0$ and some $\rho > 0$ depending on p, q, B and C only. Then we'll show how to use this to obtain the result for general ρ and for $p = 0$. By Markov's inequality

$$tP^{1/q}(m_n(Y) > t) \le \|m_n(Y)\|_q \le B\|m_n(X)\|_p.$$

By Lemma 2.1 (b) for each λ, $0 < \lambda < 1$,

$$P^{1/p}(m_n(X) > \lambda\|m_n(X)\|_p) \ge \big((1-\lambda^p)C^{-p}\big)^{q/p(q-p)} = D. \tag{3.1}$$

Hence, taking $t = t_n = BD^{-1}\|m_n(X)\|_p$ we obtain $P^{1/q}(m_n(Y) > t_n) \le P^{1/p}(m_n(X) > \lambda B^{-1}Dt_n)$ which gives $P^{1/q}(Y > t_n) \le P^{1/p}(X > \lambda B^{-1}Dt_n)$ for all n. By Lemmas 2.1 and 3.2 for each $t_{n+1} \le u \le t_n$, this yields

$$P^{1/q}(Y > u) \le P^{1/q}(Y > t_{n+1}) \le P^{1/p}\left(X > \lambda DB^{-1}t_{n+1}\right) \le P^{1/p}\left(X > \lambda D(BK)^{-1}u\right),$$

where K is as in Lemma 3.2. Hence, denoting $\lambda D(BK)^{-1}$ by τ, we have that $P^{1/q}(Y > u) \le P^{1/p}(X > \tau u)$ is satisfied for all u such that $\lim_{n\to\infty} t_n < u \le t_1 = BD^{-1}\|X\|_p$. If $u \le \lim_{n\to\infty} t_n$, then $P(X > \tau u) = 1$ and the above inequality holds true for the obvious reasons. We thus get

$$P^{1/q}(Y > u) \le P^{1/p}(X > \tau u)$$

for all $u \le BD^{-1}\|X\|_p$. Let us observe that by (3.1) for each n

$$P(X > \lambda\|m_{2^n}(X)||_p) \ge D^{p/2^n}$$

and hence by Lemma 3.2

$$P(X > \lambda K^{-n}\|X||_p) \ge D^{p/2^n}.$$

Since $D^{p/2^n} \to 1$, we may choose an n such that $D^{p/2^n} \ge \beta^{p/(q-p)}$, where $\beta > p/q$. Therefore $P(X > \tau s) \ge \beta^{p/(q-p)}$ if $s \le \tau^{-1}K^{-n}\lambda\|X\|_p$ and n is such that $D^{p/2^n} \ge \beta^{p/(q-p)}$.

Also, since $\tau^{-1}\lambda K^{-n} \le BD^{-1}$, we get, by Lemma 2.2 (a), for any $s \le \tau^{-1}\lambda K^{-n}\|X\|_p$,

$$P(X \le \tau s) \le \beta^{-1}pq^{-1}P(Y \le s).$$

Taking δ to be any number from the interval $(pq^{-1}\beta^{-1}, 1)$, $\tau = \lambda D(KB)^{-1}$ and $\rho = \tau^{-1}\lambda K^{-n}C^{-1}$, where n is any positive integer such that $D \ge \beta^{2^n/(q-p)}$, we get

$$P(X \le \tau s) \le \delta P(Y \le s)$$

for all $s \le \rho\|X\|_q \le \tau^{-1}\lambda K^{-n}\|X\|_p$. This proves the result for all $0 < p < q$ and some ρ.

By adjusting τ we can get the inequality for every preassigned ρ. Indeed, given any $\rho_0 > \rho$, $P(X \le \tau\rho\rho_0^{-1}t) \le \delta P(Y \le t)$, as long as $\rho\rho_0^{-1}t \le \rho\|X\|_q$ or, equivalently, as long as $t \le \rho_0\|X\|_q$.

To prove the case $p = 0$, choose $0 < r < q$, say, $r = q/2$. Since we are assuming that $X \in \mathcal{H}_{0,q}(C), X \in \mathcal{H}_{r,q}(C)$. Now apply the theorem for the pair $(p, q) = (q/2, q)$. ∎

To avoid awkward statements in the following theorem, we do not state the exact dependence of each of the constants on the other ones. It is important to note that the constants appearing in conditions $(i) - (iv)$ of that theorem (chosen from $\{\epsilon, \tau, \rho, C, \sigma\}$) depend only on p, q and the constants from the other equivalent conditions. In particular each of these constants depends on the distribution of X only through the constants in the other conditions. This will be useful when considering hypercontractivity of maxima of minima in section 5.

Theorem 3.4. *Fix $\rho > 0$. Let X be a nonnegative r.v. such that $\|X\|_q < \infty$ and let $0 \le p < q$. The following conditions are equivalent*
(i) $X \in \min \mathcal{H}_{p,q}(C)$ for some C,
(ii) there exist $\varepsilon < 1$, $\tau > 0$ such that

$$P(X \le \tau t) \le \varepsilon P(X \le t) \quad \text{for all} \quad t \le t_0 = \rho\|X\|_q,$$

(iii) for each $\varepsilon > 0$, there exists $\tau > 0$ such that

$$P(X \le \tau t) \le \varepsilon P(X \le t) \quad \text{for all} \quad t \le t_0 = \rho\|X\|_q,$$

(iv) there exists $\sigma > 0$ such that

$$(E(t \wedge \sigma X)^q)^{1/q} \le (E(t \wedge X)^p)^{1/p} \quad \text{for all} \quad t \ge 0. \tag{3.2}$$

Proof. (i) $\Rightarrow$ (ii). This implication follows immediately by Theorem 3.3 applied to $Y = X$.
(ii) $\Rightarrow$ (iii). If τ, ε, t_0 are as in (ii) then by induction we obtain for each n,

$$P(X \le \tau^n t) \le \varepsilon^n P(X \le t)$$

for all $t \le t_0$.
(iii) $\Rightarrow$ (iv). For each t, σ and r, $0 < r < 1$ we have

$$\begin{aligned}(E(t \wedge \sigma X)^q)^{1/q} &\le \left(t^q P\left(X > r\sigma^{-1}t\right) + r^q t^q P\left(X \le rt\sigma^{-1}\right)\right)^{1/q} \\ &= t\left(1 - (1 - r^q)P\left(X \le rt\sigma^{-1}\right)\right)^{1/q}.\end{aligned}$$

On the other hand

$$(E(t \wedge X)^p)^{1/p} \ge tP^{1/p}(X > t) = t(1 - P(X \le t))^{1/p}.$$

Therefore, by Lemma 2.2 (b) the inequality (3.2) is satisfied if

$$P(X \le t) \le pq^{-1}(1 - r^q)P\left(X \le rt\sigma^{-1}\right).$$

Thus if $\varepsilon = pq^{-1}(1 - r^q)$, τ, t_0 are as in (iii), then the above inequality and hence (3.2) is fulfilled for $t \le \tau t_0$ and $\sigma \le r\tau$.

If $t > \tau t_0$ then $(E(t \wedge \sigma X)^q)^{1/q} \le \sigma\|X\|_q$ and $(E(t \wedge X)^p)^{1/p} \ge (E(\tau t_0 \wedge X)^p)^{1/p}$. For $p > 0$ we note that

$$\begin{aligned}E(\tau t_0 \wedge X)^p &\ge (\tau t_0)^p P(X > \tau t_0) \\ &= (\tau t_0)^p[1 - P(X \le \tau t_0)] \\ &\ge (\tau t_0)^p[1 - \epsilon P(X \le t_0))] \\ &\ge (\tau t_0)^p(1 - \epsilon).\end{aligned}$$

Therefore it is enough to choose $\sigma = \min\left\{\tau\rho(1 - \epsilon)^{1/p}, \tau r\right\}$ to have the inequality (3.2) be satisfied for all $t \ge 0$.

For $p = 0$ we proceed along similar lines using

$$\begin{aligned}\exp E \ln(\tau t_0 \wedge X) &= \|X\|_q \exp E \ln(\|X\|_q^{-1}(\tau t_0 \wedge X)) \\ &\geq \|X\|_q \exp[\ln(\tau t_0/\|X\|_q) P(X > \tau t_0)] \\ &\geq \|X\|_q \exp[(\ln \tau\rho)(1 - \epsilon P(X \leq t_0))] \\ &\geq \|X\|_q \exp[(\ln \tau\rho)(1-\epsilon)].\end{aligned}$$

(iv) $\Rightarrow$ (i). If $p > 0$, applying Lemma 2.3 to $h(x_1, \cdots, x_n) = x_1 \wedge \cdots \wedge x_n$, $\xi_i = X_i$ and $\eta_i = \sigma X_i$, we get (iv) $\Rightarrow$ (i) with $C = \sigma^{-1}$. The case $p = 0$ follows by a simple limit argument. ∎

Remark 3.5. It follows by Theorem 3.4 that $\{p, q\}$-min-hypercontractivity depends only on the existence of the q-moment and a regularity property of the distribution function at 0, i.e. the following property, which we will call sub-regularity of X (or, more precisely, of the distribution of X) at 0,

$$\lim_{\tau \to 0} \limsup_{t \to 0} \frac{P(X \leq \tau t)}{P(X \leq t)} = 0.$$

Theorem 3.6. *Fix $q > 1$. Let $\{X_i\}_{i \leq n}$ be an i.i.d sequence of nonnegative r.v.'s satisfying condition (ii) of Theorem 3.4 and such that $EX_1^q < \infty$. Then there exists a constant σ such that for each n, and each function $h\colon \mathbb{R}_+^n \to \mathbb{R}_+$ which is concave in each variable separately, we have*

$$(Eh^q(\sigma X_1, \sigma X_2, \ldots, \sigma X_n))^{1/q} \leq Eh(X_1, X_2, \ldots, X_n).$$

Moreover, σ depends only on the constants appearing in the statement of Theorem 3.4 (ii).

Proof. We first simplify by noting that condition (ii) of Theorem 3.4 is satisfied (uniformly in M) by $X \wedge M$ for every M. By Lemma 2.3 it is enough to prove that there exists $\sigma > 0$, such that for each concave g: $\mathbb{R}_+ \to \mathbb{R}_+$, $(Eg^q(\sigma X))^{1/q} \leq Eg(X)$, where we may assume that g is constant on (M, ∞). To prove this inequality for such a g we first note that by Theorem 3.4, X is $\{q, 1\}$-min-hypercontractive, therefore there exists $\sigma > 0$ such that $(Eh_t^q(\sigma X))^{1/q} \leq Eh_t(X)$ for each $t \geq 0$, where h_t is given by $h_t(x) = x \wedge t$.

Since for each bounded concave g: $\mathbb{R}_+ \to \mathbb{R}_+$ there exists a measure μ on $\mathbb{R}_+$ (the measure μ is given by the condition $\mu((x, y]) = g'_+(x) - g'_+(y)$ where $g'_+(x)$ is the right derivative of g at x) such that $g = \int_{\mathbb{R}_+} h_t \mu(dt) + g(0)$ the theorem follows by Minkowski's inequality. ∎

Corollary 3.7. *If $\{X_i\}$, h, q are as in Theorem 3.6 and, additionally, h is α-homogeneous for some $\alpha > 0$ (i.e. $h(tx) = t^\alpha h(x)$), then the random variable, $W = h(X_1, X_2, \ldots, X_n)$, is sub-regular at 0.*

Proof. Theorem 3.6 implies that W is $\{q, 1\}$-min-hypercontractive and the result follows by Theorem 3.4. ∎

Section 4. Hypercontractivity of maxima

In this section we treat the case of maxima in a way similar to that of minima in Section 3. However there are some essential differences which do not allow us to treat these two cases together.

Definition 4.1. *We say that a nonnegative r.v. W is $\{p,q\}$-max-hypercontractive if there exists a constant C such that for all n*

$$\|M_n(W)\|_q \le C\|M_n(W)\|_p.$$

We will write, $W \in \max\mathcal{H}_{p,q}(C)$ in this case.

Lemma 4.2. *Let $\{X_i\}$ be i.i.d. nonnegative r.v.'s. Then*

$$\frac{nP(X>t)}{1+nP(X>t)} \le P(M_n > t) \le nP(X>t).$$

Proof. The right side is obvious and the left follows by taking complements and using the inequality, $nu/(1+nu) \le 1-(1-u)^n$. ∎

Proposition 4.3. *Let $\{X_i\}$ be i.i.d nonnegative r.v.'s. Then for $a>0$ and n a positive integer,*

$$(a)\qquad nP(M_n \le a)EX^r I_{X>a} \le \left(\frac{1-P^n(X\le a)}{P(X>a)}\right) EX^r I_{X>a} \le EM_n^r.$$

If b_n satisfies $P(X>b_n) \le n^{-1} \le P(X \ge b_n)$, then

$$(b)\qquad \begin{aligned} \frac{1}{2}\left(b_n^r + n\int_{b_n}^\infty ru^{r-1}P(X>u)\,du\right) &\le EM_n^r \\ &\le b_n^r + n\int_{b_n}^\infty ru^{r-1}P(X>u)\,du \end{aligned}$$

Proof. (a) Let $\tau = \inf\{j \le n : X_j > a\}$. Then

$$\begin{aligned} E\max_{j\le n} X_j^r \ge E(X_\tau I_{\tau\le n}) &= \sum_{j=1}^n E(X_j^r I_{X_j>a} I_{\max_{i<j} X_i \le a}) \\ &= \sum_{j=1}^n P^{j-1}(X\le a)EX^r I_{X>a} = (\frac{1-P^n(X\le a)}{P(X>a)})EX^r I_{X>a} \\ &\ge nP(M_{n-1} \le a)EX^r I_{X>a} \end{aligned}$$

(b) To see the right hand inequality, just note that, for every $a>0$,

$$EM_n^r = (\int_0^a + \int_a^\infty) ru^{r-1}P(M_n \ge u)\,du \le a^r + \int_a^\infty ru^{r-1}P(M_n\ge u)\,du.$$

For the left hand inequality, we again break up the integral as above and using the defining properties of b_n as well as the monotonicity of $x/(1+x)$ in Lemma 4.2:

$$EM_n^r = (\int_0^{b_n} + \int_{b_n}^\infty) ru^{r-1} P(M_n \geq u)\, du$$
$$\geq \frac{1}{2}\int_0^{b_n} ru^{r-1}\, dr + \frac{n}{2}\int_{b_n}^\infty ru^{r-1} P(X \geq u)\, du. \quad \blacksquare$$

The next Theorem is an extension of Theorem 3.5 of de la Peña, Montgomery-Smith and Szulga (cf. Asmar, Montgomery-Smith (1993)).

Theorem 4.4. *Let $0 \leq p < q$, $\rho > 0$ and let $X \in \max \mathcal{H}_{p,q}(C)$. Let Y be a nonnegative r.v. If there exists a constant D such that if $\|M_n(Y)\|_q \leq D\|M_n(X)\|_q$ for all n, then there are constants A, B such that*

$$EY^q I_{Y>At} \leq B^q t^q P(X > t) \quad \text{for all} \quad t \geq t_0 = \rho\|X\|_p.$$

For $p > 0$, the constants A, B can be chosen in the following way: put $A = 2^{1/q+1/p}CD\lambda^{-1}$, $B = A\big(C^p(1-\lambda^p)^{-1}\big)^{1/(q-p)}$, where $\lambda = (1/2) \wedge \rho$.

Proof. First we note that it is enough to prove the Theorem for $p > 0$. The case $p = 0$ follows easily from this case for the couple $(q/2, q)$ and ρ replaced with $C\rho$. By the Paley-Zygmund inequality (Lemma 2.1 (b)),

$$((1-\lambda^p)C^{-p})^{q/(q-p)} \leq P(M_n(X) > \lambda\|M_n(X)\|_p) \leq nP(X > \lambda\|M_n(X)\|_p). \tag{4.1}$$

We next note that by Markov's inequality and the assumptions: for $\tau = 2^{1/q}CD$,

$$P(M_n(Y) \leq \tau\|M_n(X)\|_p) \geq P(M_n(Y) \leq \tau(CD)^{-1}\|M_n(Y)\|_q) \geq 1/2.$$

Now, by Proposition 4.3, (a), the assumptions above and (4.1),

$$EY^q I_{Y>\tau\|M_n(X)\|_p} \leq 2(CD)^q\|M_n(X)\|_p^q \left(C^p(1-\lambda^p)^{-1}\right)^{q/(q-p)} P(X > \lambda\|M_n(X)\|_p).$$

Since $\|M_{2n}(X)\|_p \leq 2^{1/p}\|M_n(X)\|_p$, we get by interpolation that

$$EY^q I_{Y>2^{1/p}\tau t/\lambda} \leq B^q t^q P(X > t)$$

for

$$B^q = 2^{q/p+1}\big(CD\lambda^{-1}\big)^q \left(C^p(1-\lambda^p)^{-1}\right)^{q/(q-p)}$$

as long as $\lambda\|X\|_p \leq t < \lambda \lim_{n\to\infty}\|M_n(X)\|_p = \lambda\|X\|_\infty$. If $t \geq \lambda\|X\|_\infty$, then since

$$\|Y\|_\infty = \lim_{n\to\infty}\|M_n(Y)\|_q \leq D\liminf_{n\to\infty}\|M_n(X)\|_q = D\|X\|_\infty,$$

$\|Y\|_\infty \leq tD/\lambda$ and, since $2^{1/p}\tau > D$, $EY^q I_{Y>2^{1/p}\tau t/\lambda} = 0$. The conclusion follows trivially. $\blacksquare$

Remark 4.5. Since $A > 1$, Theorem 4.4 yields immediately that

$$P(Y > At) \leq B^q P(X > t) \quad \text{for} \quad t \geq t_0.$$

In the next theorem we make the same convention concerning the constants as we made before the statement of Theorem 3.4.

Theorem 4.6. *Let X be a nonnegative r.v., $0 \le p < q, \rho > 0$. The following conditions are equivalent*

(i) $X \in \max \mathcal{H}_{p,q}(C)$ *for some* $C > 0$;

(ii) there exists a constant B such that

$$EX^q I_{X>t} \le B^q t^q P(X > t) \quad \text{for all} \quad t \ge t_0 = \rho \|X\|_p;$$

(iii) for $\varepsilon > 0$ there exists a constant $D > 1$ such that

$$D^q P(X > Dt) \le \varepsilon P(X > t) \quad \text{for all} \quad t \ge t_0 = \rho \|X\|_p;$$

(iv) there exists a constant $\sigma > 0$ such that

$$E(t \vee \sigma X)^q)^{1/q} \le (E(t \vee X)^p)^{1/p} \quad \text{for all} \quad t \ge 0.$$

Proof. (i) $\Rightarrow$ (ii). By Theorem 4.4 applied to $Y = X$ we derive an existence of constants $A, \overline{B}$ such that

$$EX^q I_{X>At} \le \overline{B}^q t^q P(X > t) \quad \text{for all} \quad t \ge t_0 = \rho \|X\|_p.$$

Hence for any $t \ge t_0$,

$$\begin{aligned} EX^q I_{X>t} &\le EX^q I_{X>At} + EX^q I_{t<X\le At} \\ &\le \overline{B}^q t^q P(X > t) + A^q t^q P(X > t) \\ &\le (\overline{B}^q + A^q) t^q P(X > t). \end{aligned}$$

(ii) $\Rightarrow$ (iii). If t_0, B are as in (ii) then for $t \ge t_0$

$$\begin{aligned} EX^q \ln^+(X/t) &= \int_t^\infty E\frac{X^q I_{X>s}}{s} ds \le B^q \int_t^\infty s^{q-1} P(X > s) ds \le q^{-1} B^{2q} EX^q I_{X>t} \\ &\le q^{-1} B^q t^q P(X > t). \end{aligned}$$

Hence, for any $D > 1$, we have

$$(\ln D) D^q t^q P(X > Dt) \le EX^q \ln^+ X/t \le q^{-1} B^{2q} t^q P(X > t)$$

and it is enough to choose $D > 1$ such that $B^{2q}/(q \ln D) < \varepsilon$.
(iii) $\Rightarrow$ (ii). If (iii) holds with $0 < \varepsilon < 1$, $D > 1$, then by induction $P(X > D^n t) \le \varepsilon^n D^{-nq} P(X > t)$ for $t \ge t_0$. Hence

$$\begin{aligned} EX^q I_{X>t} &= \sum_{k=0}^\infty EX^q I_{D^k t < X \le D^{k+1} t} \le \sum_{k=0}^\infty D^{(k+1)q} t^q P(X > D^k t) \\ &\le \sum_{k=0}^\infty D^{(k+1)q} t^q \cdot \varepsilon^k D^{-kq} P(X > t) = D^q (1-\varepsilon)^{-1} t^q P(X > t). \end{aligned}$$

(ii) and (iii) $\Rightarrow$ (iv). Assume that (ii) and (iii) are fulfilled with constants B, D, ε.

By (ii) we obtain for $t \geq t_0\sigma$, where for the moment σ is any number < 1,

$$\begin{aligned}(E(t \vee \sigma X)^q)^{1/q} &\leq t\Big(P\left(X \leq t\sigma^{-1}\right) + \sigma^q t^{-q} E\left(X^q I\left(X > t\sigma^{-1}\right)\right)\Big)^{1/q} \\ &\leq t\left(1 + (B^q - 1)P\left(X > t\sigma^{-1}\right)\right)^{1/q}\end{aligned}.$$

On the other hand for any $R > 1$

$$(E(t \vee X)^p)^{1/p} \geq t(1 + (R^p - 1)P(X > Rt))^{1/p}.$$

Hence, by Lemma 2.2 (c), the inequality in (iv) holds if

$$pq^{-1}(B^q - 1)(R^p - 1)^{-1} P\left(X > t\sigma^{-1}\right) \leq P(X > Rt).$$

Therefore if we choose R so that

$$pq^{-1}(B^q - 1)(R^p - 1)^{-1} < D^q/\varepsilon \quad \text{and} \quad R \geq \rho 2^{1+1/q}$$

and further choose σ so that $\sigma < (RD)^{-1}$, then the inequality in (iv) is satisfied for all $t \geq \|X\|_p/2^{1+1/q}$.

If $t < \|X\|_p/2^{1+1/q}$, then $(E(t \vee \sigma X)^q)^{1/q} \leq 2^{1/q}(t + \sigma\|X\|_q)$ and $(E(t \vee X)^p)^{1/p} \geq \|X\|_p$ and therefore, using (ii) with $t = t_0$, if additionally $\sigma < (2^{1+1/q}\rho(1 + B^q)^{1/q})^{-1}$ $(< \|X\|_p(2^{1+1/q}\|X\|_q)^{-1})$, then the inequality in (iv) is satisfied for all $t \geq 0$.

(iv) $\Rightarrow$ (i). This implication is proved in the same way as the one in Theorem 3.4. It is enough to replace $\wedge$ by $\vee$ everywhere. ∎

Remark 4.7.

(i) Theorems 4.4 and 4.5 have appeared in a similar form in unpublished notes from a seminar held by the second author and prepared by Rychlik (1992).

(ii) The equivalence of (ii) and (iii) in Theorem 4.6 can be deduced from more general results (cf. Bingham, Goldie and Teugels, page 100).

(iii) It follows from Theorem 4.4 that if X is $\{p, q\}$ max-hypercontractive then for some $\varepsilon > 0$ and all $r < q + \varepsilon$, X is also $\{r, q + \varepsilon\}$-max-hypercontractive.

(iv) The property of $\{p, q\}$-max-hypercontractivity is equivalent to

$$\limsup_{D\to\infty} \limsup_{t\to\infty} \frac{D^q P(X > Dt)}{P(X > t)} = 0$$

which we will call q-sub-regularity at $+\infty$.

Theorem 4.8. *Fix $q > 1$. If $\{X_i\}_{i\leq n}$ is i.i.d sequence of nonnegative r.v.'s satisfying $EX_1 < \infty$ and condition (ii) of Theorem 4.6, then there exists a constant σ such that for each n and each X_i, $i = 1, \ldots, n$ independent copies of X,*

$$(Eh^q(\sigma X_1, \sigma X_2, \ldots, \sigma X_n))^{1/q} \leq Eh(X_1, X_2, \ldots, X_n)$$

for each function $h\colon \mathbb{R}_+^n \to \mathbb{R}_+$ *which is in each variable separately nondecreasing and convex, and* $\lim\limits_{x_i\to+\infty} \left(x_i \frac{\partial h}{\partial x_i}(x) - h(x)\right) \le 0$. *Moreover,* σ *depends only on the constants appearing in the statement of Theorem 4.6 (ii).*

Proof. The proof is the same as in the case of Theorem 3.6, except that we have to replace everywhere $\wedge$ by $\vee$ and that the measure μ is given by $\mu((x,y]) = g'_+(y) - g'_+(x)$ and then

$$g(x) = \int_{\mathbb{R}_+} h_t(x)\mu(dt) + \lim_{t\to\infty} (g(t) - tg'(t)). \quad \blacksquare$$

In analogy to Corollary 3.7 we obtain

Corollary 4.9. *If* $\{X_i\}$, h *are as in Theorem 4.8,* $q > 1$ *and in an addition* h *is* α-*homogeneous for some* $\alpha > 0$, *then the random variable* $W = h(X_1, X_2, \ldots, X_n)$ *is* q-*sub-regular at* $+\infty$.

Section 5. Hypercontractivity of minmax and maxmin.

In this section we will impose on X both the condition of sub-regularity at 0 and that of q-sub-regularity at $+\infty$.

Theorem 5.1. *If* $0 \le p < q$ *and* X *is a nonnegative random variable in* $\min \mathcal{H}_{p,q}(C_1) \cap \max \mathcal{H}_{p,q}(C_2)$, *then there exists a constant* $\sigma > 0$ *such that for each* $0 < s < t < \infty$

$$(E(s \vee \sigma X \wedge t)^q)^{1/q} \le (E(s \vee X \wedge t)^p)^{1/p}. \tag{5.1}$$

Furthermore, σ *depends only on* C_1, C_2, p *and* q.

Proof. Let $R > 1$ be any fixed number, and let $r = R^{-1}$. Let ρ be any positive number, and let τ be such that the inequality in Theorem 3.4 (iii) holds for $\varepsilon = pq^{-1}(1 - r^q)$ for all $t \le t_0 = \rho\|X\|_q$. Then, let $\alpha = 2^{-1/q-1}(1-\epsilon)^{1/p}$. The constant B is such that the inequality in Theorem 4.6 (ii) is true for all $t \ge t_0$ and let D be such that the inequality in Theorem 4.6 (iii) is satisfied for

$$\varepsilon = qp^{-1}(R^p - 1)\left((B^q - 1)^{-1} \wedge (R^q - 1)^{-1}\right) \quad \text{for} \quad t \ge t_0.$$

We will show that for $\sigma = \min\left\{\alpha\rho, \alpha/D, r/D, r\tau\right\}$ the inequality (5.1) holds true for each $0 < s < t < \infty$. Consider the following five cases.

Case 1. $s \le \alpha t_0, t \ge \tau t_0$. We have

$$(E(s \vee \sigma X \wedge t)^q)^{1/q} \le 2^{1/q}(\alpha t_0 + \sigma\|X\|_q)$$

and

$$(E(s \vee X \wedge t)^p)^{1/p} \ge (E(\tau t_0 \wedge X)^p)^{1/p} \ge \tau t_0 P^{1/p}(X > \tau t_0) \ge \tau t_0 (1-\epsilon)^{1/p}.$$

Since $\sigma < \alpha\rho$ the inequality holds by the choice of α.

Case 2. $t \le \tau t_0$, $rt > s$. We have

$$\begin{aligned}(E(s \vee \sigma X \wedge t)^q)^{1/q} &\le \left(t^q P\left(X > r\sigma^{-1}t\right) + r^q t^q P\left(X \le rt\sigma^{-1}\right)\right)^{1/q}\\ &\le t\left(1 + (r^q - 1)P\left(X \le rt\sigma^{-1}\right)\right)^{1/q}\end{aligned}$$

and

$$(E(s \vee X \wedge t)^p)^{1/p} \ge t(1 - P(X \le t))^{1/p}.$$

Therefore by Lemma 2.2 (b) the inequality (5.1) holds if

$$pq^{-1}(1 - r^q)P\left(X \le rt\sigma^{-1}\right) \ge P(X \le t)$$

which is true by the choice of τ since $\sigma < r\tau$, $t \le \tau t_0$.

Case 3. $t \le \tau t_0$, $rt \le s$. We have

$$\begin{aligned}(E(s \vee \sigma X \wedge t)^q)^{1/q} &\le \left(s^q P\left(X \le s\sigma^{-1}\right) + t^q P((X > s\sigma^{-1})\right)^{1/q}\\ &= t\left(1 + ((s/t)^q - 1)\, P\left(X \le s\sigma^{-1}\right)\right)^{1/q}\end{aligned}$$

and

$$\begin{aligned}(E(s \vee X \wedge t)^p)^{1/p} &\ge (s^p P(X \le t) + t^p P(X > t))^{1/p}\\ &= t\,(1 + ((s/t)^p - 1)\, P(X \le t))^{1/p}.\end{aligned}$$

Therefore by Lemma 2.2 (c) to have (5.1) it is enough to show

$$pq^{-1}(1 - (s/t)^q)(1 - (s/t)^p)^{-1} P\left(X \le s\sigma^{-1}\right) \ge P(X \le t).$$

Since the function $(1 - x^q)/(1 - x^p)$ is increasing on $\mathbb{R}_+$ and $s/t \ge r$ it is enough to prove that $pq^{-1}(1 - r^q)(1 - r^p)^{-1}P\big(X \le rt\sigma^{-1}\big) \ge P(X \le t)$ which was proved in the preceding case, because $1 - r^p < 1$.

Case 4. $s > \alpha t_0$, $t > Rs$. We have

$$\begin{aligned}(E(s \vee \sigma X \wedge t)^q)^{1/q} &\le \left(s^q P\left(X \le s\sigma^{-1}\right) + \sigma^q E X^q I\left(X > s\sigma^{-1}\right)\right)^{1/q}\\ &\le s\left(1 + (B^q - 1)P\left(X > s\sigma^{-1}\right)\right)^{1/q},\end{aligned}$$

which follows by the choice of B, since $s\sigma^{-1} > \alpha t_0 \sigma^{-1} \ge t_0$, and

$$\begin{aligned}(E(s \vee X \wedge t)^p)^{1/p} &\ge (s^p P(X \le Rs) + (Rs)^p P(X > Rs))^{1/p}\\ &= s(1 + (R^p - 1)P(X > Rs))^{1/p}.\end{aligned}$$

By Lemma 2.2 (c) it is enough that

$$pq^{-1}(B^q - 1)(R^p - 1)^{-1} P\left(X > s\sigma^{-1}\right) \le P(X > Rs).$$

Since $\sigma \le (\alpha \wedge r)/D$ it is enough to show

$$\frac{P(X > D(\alpha^{-1} \vee R)s)}{P(X > Rs)} \le qp^{-1}(R^p - 1)(B^q - 1)^{-1} = \varepsilon.$$

But, then by the choice of D we have

$$\frac{P(X > D(\alpha^{-1} \vee R)s)}{P(X > Rs)} \le \frac{P(X > D(\alpha^{-1} \vee R)s)}{P(X > (\alpha^{-1} \vee R)s)} \frac{P(X > (\alpha^{-1} \vee R)s)}{P(X > Rs)} \le \frac{\varepsilon}{D^q} < \varepsilon,$$

because $(\alpha^{-1} \vee R)s \ge t_0$.

Case 5. $s > \alpha t_0, t \le Rs$. We have

$$\begin{aligned}(E(s \vee \sigma X \wedge t)^q)^{1/q} &\le \left(s^q P\left(X \le s\sigma^{-1}\right) + t^q P\left(X > s\sigma^{-1}\right)\right)^{1/q} \\ &= s\left(1 + ((t/s)^q - 1)\, P\left(X > s\sigma^{-1}\right)\right)^{1/q}\end{aligned}$$

and

$$\begin{aligned}(E(s \vee X \wedge t)^p)^{1/p} &\ge (s^p P(X \le t) + t^p P(X > t))^{1/p} \\ &= s\,(1 + ((t/s)^p - 1)\, P(X > t))^{1/p}.\end{aligned}$$

By Lemma 2.2 (c) it is enough to prove

$$pq^{-1}((t/s)^q - 1)((t/s)^p - 1)^{-1} P\left(X > s\sigma^{-1}\right) \le P(X \ge Rs).$$

Since $t/s \le R$ it suffices to show that

$$pq^{-1}(R^q - 1)(R^p - 1)^{-1} P\left(X > s\sigma^{-1}\right) \le P(X \ge Rs)$$

which is shown in the same way as in the preceding. ∎

Taking into account the remarks before Theorems 3.4 and 4.6 we check easily that given p, q the constant σ depends only on the min and max hypercontractivity constants of X.

Corollary 5.2. *If X, p, q are as in Theorem 5.1, then there exists a constant C such that if (X_i), $i = 1, \ldots, n$ is a sequence of independent copies of X and $X^{k,n}$ denotes the k-th order statistics of the sequence (X_i), $i = 1, \ldots, n$ then $\|X^{k,n}\|_q \le C\|X^{k,n}\|_p$ and $X^{k,n}$ is q-sub-regular at $+\infty$ and sub-regular at 0.*

Proof. The statistic $X^{k,n}$ can be written as $h(X_1, X_2, \ldots, X_n)$ where for each i and each fixed $x_i, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ the function $f(x_i) = h(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = (s \vee x_i \wedge t)$ for some $0 < s < t$, and all $x_i \in \mathbb{R}_+$. And therefore the first part of the corollary follows by the observation. The second part is obtained easily because we have that $X^{k,n}$ is $\{q, p\}$-max and min hypercontractive. ∎

The preceding corollary can be considerably generalized. At first let us define a class $\mathcal{F}$ of functions $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ which can by written as $g(x) = \int_\Delta h_{s,t}(x)\mu(ds,dt)$ for some positive measure μ on $\Delta = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+\colon\ s \le t\}$ and where $h_{s,t}$ are functions defined by $h_{s,t}(x) = s \vee x \wedge t$. It is possible to give an intrinsic description of functions in $\mathcal{F}$. Instead let us observe that if f is twice continuously differentiable on $\mathbb{R}_+$, then $f \in \mathcal{F}$ if and only if for each $x \in \mathbb{R}_+$, $0 \le xf'(x) \le f(x)$ and $f(0) \ge \int_{\mathbb{R}_+} x(f''(x) \vee 0)dx$. In this case the measure μ is given by the following condition: for measurable $h\colon\ \Delta \to \mathbb{R}_+$

$$\int_\Delta h(s,t)\mu(ds,dt) = \int_{\mathbb{R}_+} \left(\sum_{(s,t)\in I(y)} h(s,t) \right) dy$$

where $I(y)$ is the countable family of open, disjoint intervals with the union equal $\{s \in \mathbb{R}_+\colon\ f'(s) > y\}$. It is not difficult to prove that we have the representation $f(x) = \int_\Delta h_{s,t}(x)\mu(ds,dt) + c$ where $c = f(0) - \int_{\mathbb{R}_+} x(f''(x) \vee 0)dx$.

Theorem 5.3. *Let X be in $\min \mathcal{H}_{p,q}(C_1) \cap \max \mathcal{H}_{p,q}(C_2)$. Then there exists a constant $\sigma > 0$ such that for each n and each $h\colon \mathbb{R}_+^n \to \mathbb{R}_+$, which in each variable separately is in class $\mathcal{F}$, it follows that*

$$(Eh^q(\sigma X_1, \sigma X_2, \ldots, \sigma X_n))^{1/q} \le Eh(X_1, X_2, \ldots, X_n).$$

Moreover if h is α-homogeneous for some $\alpha > 0$ then $h(X_1, \ldots, X_n)$ is q-sub-regular at $+\infty$ and sub-regular at 0.

Proof. The proof follows the same pattern as proofs of Theorems 3.6, 4.8 and Corollaries 3.7, 4.9, and is based on Theorem 5.1 ■

Applying comparison results of Theorems 3.3 and 4.4 we obtain easily

Theorem 5.4. *Let X, Y be nonnegative r.v.'s such that $X \in \min \mathcal{H}_{p,q}(C_1) \cap \max \mathcal{H}_{p,q}(C_2)$ and there exist constants B_1 and B_2 such that $\|m_n(Y)\|_q \le B_1 \|m_n(X)\|_q$ and $\|M_n(Y)\|_q \le B_2 \|M_n(X)\|_q$ for all n, then there exists a constant D, depending only on p, q, C_1, C_2, B_1 and B_2, such that $P(Y \le t) \ge P(DX \le t)$ for all $t \in \mathbb{R}_+$.*

Finally we have

Theorem 5.5. *If $X \in \min \mathcal{H}_{p,q}(C_1) \cap \max \mathcal{H}_{p,q}(C_2)$, then there exists a constant D, depending only on p, q, C_1 and C_2, such that for all l and all $n_1, k_1, n_2, k_2, \ldots n_l, k_l$,*

$$\|M_{n_1} m_{k_1} M_{n_2} m_{k_2} \ldots M_{n_l} m_{k_l}(X)\|_q \le D \|M_{n_1} m_{k_1} M_{n_2} m_{k_2} \ldots M_{n_l} m_{k_l}(X)\|_p.$$

Proof. If X is both min-hypercontractive and max-hypercontractive, by Theorem 5.1 we have a $0 < \sigma < \infty$ for which inequality (5.1) is satisfied. One now applies Lemma 2.3 applied to the functional, $h\colon \mathbb{R}^{n_1 k_1 \cdot \ldots \cdot n_l k_l} \to \mathbb{R}_+$, which is the composition of the min's and max's in the statement of this Theorem. ■

Section 6. Minmax hypercontractivity of norms of stable random vectors.

In this section we apply the results in earlier sections to certain questions concerning Gaussian and symmetric stable measures. In particular, in the second half of this section, we give our initial motivation for initiating this research as well as some partial result concerning a version of the Gaussian Correlation Conjecture.

The following lemma is a consequence of Kanter's inequality, (cf. Ledoux and Talagrand (1991), p. 153) which can be viewed as a concentration result similar to Levy's inequalities. The formulation of the lemma below for Gaussian measures was suggested by X. Fernique.

Lemma 6.1 (Corollary of Kanter's inequality). *Let ν be a symmetric α stable measure with $0 < \alpha \leq 2$ on a separable Banach space F. Then, for any $\kappa \geq 0$, any symmetric, convex set B and any $y \in F$, we have*

$$\nu(\kappa B + y) \leq \frac{3}{2} \frac{\kappa^{\alpha/2}}{\sqrt{1-\nu(B)}}.$$

Proof. Let $\{X, X_i\}_i$ be i.i.d. symmetric α stable random variables with $0 < \alpha \leq 2$. Take $N = [\kappa^{-\alpha}]$. Then using $N\kappa^\alpha \leq 1$ and $(N+1)\kappa^\alpha > 1$, we have by Kanter's inequality

$$\begin{aligned} P(X - y \in \kappa B) &= P(\sum_{i=1}^{N} X_i - N^{1/\alpha} y \in N^{1/\alpha}\kappa B) \\ &\leq \frac{3}{2}\Big(\frac{1}{1 + NP(X \notin N^{1/\alpha}\kappa B)}\Big)^{1/2} \leq \frac{3}{2} \frac{\kappa^{\alpha/2}}{P(X \notin B)^{1/2}} \end{aligned}$$

since $P(X \notin N^{1/\alpha}\kappa B) \geq P(X \notin B)$ and $(1 + NP(X \notin B))^{-1} \leq \kappa^\alpha P^{-1}(X \notin B)$. This finishes the proof. ∎

Lemma 6.2. *Let ν be a symmetric α stable measure with $0 < \alpha \leq 2$ on a separable, Banach space F. Then for any closed, symmetric, convex set $B \subseteq F$, $y \in F$ and $\kappa \leq 1$,*

$$\nu(\kappa B + y) \leq R\kappa^{\alpha/2}\nu(2B + y),$$

where $R = (3/2)(\nu(B))^{-1}(1-\nu(B))^{-1/2}$.

Proof. First consider $y \in B$. Then $\nu(B) \leq \nu(2B + y)$ since $B \subseteq 2B + y$. Thus, to conclude this case, one applies Lemma 6.1.

If $y \notin B$, then let $r = [\kappa^{-1} - 2^{-1}]$. For $k = 0, 1, \cdots, r$ the balls $\{y_k + \kappa B\}$ are disjoint and contained in $y + 2B$, where $y_k = (1 - 2\kappa\|y\|^{-1}k)y$. By Anderson's Theorem, it follows that

$$\nu(y_k + \kappa B) \geq \nu(y + \kappa B)$$

for $k = 0, \cdots, r$. Therefore, $\nu(\kappa B + y) \leq (r+1)^{-1}\nu(2B + y) \leq \kappa\nu(2B + y)$. This proves the lemma, since $2 \leq R$. ∎

Proposition 6.3. *Under the set up of Lemma 6.2, we have for each $\kappa, t \leq 1$,*

$$\nu(\kappa t B) \leq R'\kappa^{\alpha/2}\nu(tB),$$

where $R' = 3(\nu(B/2))^{-1}(1-\nu(B/2))^{-1/2}$.

Proof. Now for any $0 \le t \le 1$, define the probability measure ν_t by $\nu_t(C) = \nu(tC) = P(X/t \in C)$ where X is the symmetric α stable random variable with law ν. Then

$$\nu * \nu_s(C) = P(X + X'/s \in C) = P((1+s^{-\alpha})^{1/\alpha} X \in C) = \nu_t(C),$$

where $t^{-\alpha} = 1 + s^{-\alpha}$ and X' is an independent copy of X. Hence, by Lemma 6.2

$$\begin{aligned}\nu(\kappa t B) = \nu * \nu_s(\kappa B) &= \int_F \nu(2\kappa B/2 + y)\nu_s(dy) \le (2\kappa)^{\alpha/2} R \int_F \nu(B+y)\nu_s(dy) \\ &\le R'\kappa^{\alpha/2}\nu(tB).\end{aligned}$$

Theorem 6.4. *Under the set up of Lemma 6.2, for each $b < 1$, there exists $R(b)$ such that for all $0 \le t \le 1$,*

$$\nu(tB) \le R(b) t^{\alpha/2} \nu(B), \text{ whenever } \nu(B) \le b. \tag{6.1}$$

Proof. Fix B with $\nu(B) \le b$. Choose $s \ge 1$ so that $\nu(sB) = b$. Now, apply the Proposition 6.3 with $\kappa = t$, to get

$$\nu(tB) = \nu(t \cdot \frac{1}{2s}(2sB)) \le R(b)t^{\alpha/2}\nu(\frac{1}{2s}(2sB)) = R(b)t^{\alpha/2}\nu(B),$$

where $R(b) = 3b^{-1}(1-b)^{-1/2}$. ■

Remark 6.5. In the case of $\alpha = 2$ Theorem 6.4 was formulated in Szarek (1991), Lemma 2.6, where a weaker result, which was sufficient for the main results of the paper, was actually proved. Recently, Latała proved that in the case of $\alpha = 2$, the conclusion of Theorem 6.4 holds whenever the measure ν is log concave.

Related results on α-stable measures can be found in Lewandowski, Ryznar and Żak (1992). The key difference is that we need the right hand side of (6.1) to involve $\mu(B)$ for all B such that $\mu(B) \le b$ and the constant R depending only on the number b.

If X satisfies the conclusion of Theorem 5.5 we write $X \in \min\max \mathcal{H}_{p,q}(D)$

Corollary 6.6. *Let $0 < \alpha \le 2$, $0 \le p < q$. If $\alpha \ne 2$ we assume that $q < \alpha$. If W is a α-stable, symmetric vector in a separable Banach space then $\|W\| \in \min\max \mathcal{H}_{p,q}(C)$ for some constant C which depends only on α, p and q.*

Proof. Fix $0 < \alpha < 2$ and let $\{\xi, \xi_i\}_i$ be iid with characteristic function:

$$Ee^{it\xi} = e^{-|t|^\alpha}.$$

By Szulga (1990), there exists $0 < c(\alpha)$ such that for $p < q$, $p, q \in (\alpha - c(\alpha), \alpha)$ there exists $\sigma = \sigma(p,q,\alpha)$ such that for all Banach spaces B and for all $x, y \in B$

$$\|x + \sigma\xi y\|_q \le \|x + \xi y\|_p.$$

This implies that for every n and $\{y_i\}_{i\leq n}$,

$$\|\sum_{i=1}^{n}\xi_i y_i\|_q \leq \sigma^{-1}\|\sum_{i=1}^{n}\xi_i y_i\|_p.$$

Now, if B is a separable Banach space and W is a symmetric stable random variable of index α with values in B, there exists a probability measure Γ on the sphere, S, of the dual space B^* and a constant, c, such that

$$E\exp(i < x^*, W >) = \exp(-c\int_S |< x^*, \theta >|^\alpha \Gamma(d\theta), \text{ for all } x^* \in B^*$$

Now take measures $\Gamma_n = \sum_{i=1}^{n} c_{n,i}\delta_{\theta_i}$ on S which converge weak* to Γ. Let $W_n = \sum_{i=1}^{n}(cc_{n,i})^{1/\alpha}\xi_i\theta_i$. Then

$$E\exp(i < x^*, W_n >) = \exp(-c\int_S |< x^*, \theta >|^\alpha \Gamma_n(d\theta)).$$

So, W_n converges in distribution to W. Hence, for any countable weak*-dense set $\{v_j^*\}_j$ in the unit ball of B^*, we have (since $p < \alpha$ and m is finite):

$$E\sup_{j\leq m} |< v_j^*, W_n >|^p \to E\sup_{j\leq m} |< v_j^*, W >|^p \leq E\|W\|^p.$$

But, then we have

$$\lim_{n\to\infty}(E\sup_{j\leq m} |< v_j^*, W_n >|^q)^{1/q} \leq \sigma^{-1}\|W\|_p.$$

Hence, $\|W\|_q \leq \sigma^{-1}\|W\|_p$. Note that we can interpolate (by Hölder's inequality) to obtain for every $0 < p < q < 2$ a σ for which the last inequality holds. And again, this σ depends only on p, q and α. If W is Gaussian ($\alpha = 2$), then the comparison of the p and q norms is well known and not restricted to $q < 2$ (see, e.g., Ledoux and Talagrand (1990), p. 60). Now, for any $q < \alpha$ (in the Gaussian case any q) and any $q < r < \alpha$ we have $P(\|W\| \leq \frac{1}{2}\|W\|_q)$ bounded below by a positive constant, say, b, depending only on the $\sigma = \sigma(q, r)$ obtained above. This means that, putting $K = \{x : \|x\| \leq \frac{1}{2}\|W\|_q\}$, we have for any $0 \leq u \leq 1$, $P(W \in uK) \leq b$. Hence, by Theorem 6.4 we have

$$P(\|W\| \leq tu\frac{1}{2}\|W\|_q) \leq R(b)t^{\alpha/2}P(\|W\| \leq u\frac{1}{2}\|W\|_q), \text{ for all } 0 \leq u, t \leq 1.$$

Hence, with $\rho = 1/2, \tau = (\epsilon/R(b))^{2/\alpha}$, condition (ii) of Theorem 3.4 holds. Hence, $\|W\| \in \min\mathcal{H}_{0,q}(C)$ for some C depending only on p, q and α. In particular, (using $n = 1$) we now have that $\|W\| \in \max\mathcal{H}_{0,q}(C)$. So, by Theorem 5.5, $\|W\| \in \min\max\mathcal{H}_{p,q}(D)$, for some D depending only on p, q and α. ■

Corollary 6.7. *Let $0 < \alpha \leq 2$, $0 \leq p < q$. If $\alpha \neq 2$ we assume that $q < \alpha$. Let $X_1, X_2, .., X_n$ be symmetric α-stable, independent random vectors in a separable Banach space. Let $h : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function as in Theorem 5.3 which is λ-homogeneous for some λ. Then*

$$h(\|X_1\|, \|X_2\|, .., \|X_n\|) \in \min\max \mathcal{H}_{p,q}(C)$$

and the constant C depends only on α, p, q.

Proof. By Corollary 6.6 a constant σ can be found, which depends only on α, p, q and such that the conclusion of Theorem 5.1 holds true for $X = \|X_i\|$ for $i = 1, 2, .., n$. Now we can proceed as in the proof of Theorem 5.3. ∎

Before proceeding with the next result we would like to explain its connection with the Gaussian Correlation Conjecture.

The conjecture we refer to says that

$$\mu(A \cap B) \geq \mu(A)\mu(B) \tag{6.2}$$

for any symmetric, convex sets A and B in $\mathbb{R}^n$, where μ is a mean zero Gaussian measure on $\mathbb{R}^n$.

In 1977 L. Pitt (1977) proved that the conjecture holds in $\mathbb{R}^2$. Khatri (1967) and Sĭdák (1967, 1968) proved (6.2) when one of the sets is a symmetric slab (a set of the form $\{x \in \mathbb{R}^n : |(x, u)| \leq 1\}$ for some $u \in \mathbb{R}^n$). For more recent work and references on the correlation conjecture, see Schechtman, Schlumprecht and Zinn (1995), and Szarek and Werner (1995). The Khatri-Sĭdák result as a partial solution to the general correlation conjecture has many applications in probability and statistics, see Tong (1980). In particular, it is one of the most important tools discovered recently for the lower bound estimates of the small ball probabilities, see, for example, Kuelbs, Li and Shao (1995), and Talagrand (1994). On the other hand, the Khatri-Sĭdák result only provides the correct lower bound rate up to a constant at the log level of the small ball probability. If the correlation conjecture (6.2) holds, then the existence of the constant of the small ball probability at the log level for the fractional Brownian motion (cf. Li and Shao (1995)) can be shown. Hence, hypercontractivity for minima, small ball probabilities and correlation inequalities for slabs are all related in the setting of Gaussian vectors.

Let $\mathcal{C}_n$ denote the set of symmetric, convex sets in $\mathbb{R}^n$. Since the correlation conjecture iterates, for each $\alpha \geq 1$,the following is a weaker conjecture.

Conjecture C_α. *For any $l, n \geq 1$, and any $A_1, \cdots, A_l \in \mathcal{C}_n$, if μ is a mean zero, Gaussian measure on $\mathbb{R}^n$, then*

$$\mu\Big(\alpha(\bigcap_{i=1}^{l} A_i)\Big) \geq \prod_{i=1}^{l} \mu(A_i).$$

One can restate this (as well as the original conjecture) using Gaussian vectors in $\mathbb{R}^n$ as follows: for $l, n \geq 1$, and any $A = A_1 \times \cdots \times A_l \subseteq \mathbb{R}^{nl}$ let

$$\|\cdot\|_A = \text{the norm on } \mathbb{R}^{nl} \text{ with the unit ball } A,$$
$$\|\cdot\|_l = \text{the norm on } \mathbb{R}^{n} \text{ with the unit ball } A_l.$$

If $G, G_1, \cdots, G_l$ are i.i.d. mean zero Gaussian random variables in $\mathbb{R}^n$, let

$$\mathcal{G} = (G, \cdots, G) \text{ and } \mathcal{H} = (G_1, \cdots, G_l).$$

Then, C_α can be rewritten as:

Restatement of Conjecture C_α. *For all $l, n \geq 1$, and any $t > 0$,*

$$\begin{aligned}\Pr(\|\mathcal{G}\|_A \leq \alpha t) &= \Pr(\mathcal{G} \in \alpha t(A_1 \times \cdots \times A_l)) \\ &\geq \Pr(\mathcal{H} \in t(A_1 \times \cdots \times A_l)) = \Pr(\|\mathcal{H}\|_A \leq t).\end{aligned}$$

By taking complements, reversing the inequalities and raising both sides of the inequality to a power, say N, we get:

$$\Pr(\min_{j \leq N} \|\mathcal{G}^j\|_A > \alpha t) \leq \Pr(\min_{j \leq N} \|\mathcal{H}^j\|_A > t).$$

Again, reversing the inequalities and raising both sides to the power K,

$$\Pr(\max_{k \leq K} \min_{j \leq N} \|\mathcal{G}^{j,k}\|_A > \alpha t) \leq \Pr(\max_{k \leq K} \min_{j \leq N} \|\mathcal{H}^{j,k}\|_A > t).$$

Using the usual formula for p^{th} moments in terms of tail probabilities we would get:

$$\left\| \max_{k \leq K} \min_{j \leq N} \|\mathcal{G}^{j,k}\|_A \right\|_p \leq \alpha \left\| \max_{k \leq K} \min_{j \leq N} \|\mathcal{H}^{j,k}\|_A \right\|_p . \tag{6.3}$$

Note that if the conjecture (6.2) were true then (6.3) would hold with $\alpha = 1$. Even in the case $K = N = 1$, the best that is known is the above inequality with constant $\sqrt{2}$. (Of course, if $N = 1$, the case $K = 1$ is the same as the case of arbitrary K.) To see this first let $T =: \cup_{l=1}^{L} T_l =: \cup_{l=1}^{L} \{(f, l) : f \in A_l^\circ\}$ where A_l° is the polar of A_l. Now define the Gaussian processes Y_t and X_t for $t \in T_l$ by $Y_{f,l} = f(G)$ and $X_{f,l} = f(G_l)$. Then, $\sup_{t \in T} Y_t = \max_{l \leq L} \|G\|_l$ and $\sup_{t \in T} X_t = \max_{l \leq L} \|G_l\|_l$. We now check the conditions of the Chevet-Fernique-Sudakov/Tsirelson version of Slepian's inequality (see also, Marcus-Shepp (1972)). Let $s = (f, p)$ and $t = (g, q)$. If $p = q$, (Y_s, Y_t) has the same distribution as (X_s, X_t), and hence

$$E|Y_s - Y_t|^2 = E|X_s - X_t|^2.$$

If $p \neq q$, then

$$E|Y_s - Y_t|^2 \leq 2\Big(EY_s^2 + EY_t^2\Big) = 2\Big(EX_s^2 + EX_t^2\Big) = 2E|X_s - X_t|^2$$

Therefore, in either case one can use $\sqrt{2}$. Hence, by the version of the Slepian result mentioned above,

$$E \sup_{t \in T} Y_t \leq \sqrt{2} E \sup_{t \in T} X_t.$$

On the other hand the results of de la Peña, Montgomery-Smith and Szulga (mentioned in the introduction) allow one to go from an L_p inequality to a probability inequality if one has one more ingredient, hypercontractivity. By their results if one can prove that there exists a constant $\gamma < \infty$ such that for all K, N and symmetric, convex sets

$$\textbf{(Comparison)} \qquad \left\| \max_{k \le K} \min_{j \le N} \|\mathcal{G}^{j,k}\|_A \right\|_p \le \gamma \left\| \max_{k \le K} \min_{j \le N} \|\mathcal{H}^{j,k}\|_A \right\|_p .$$

and for some $q > p$ and all K, N and symmetric, convex sets

$$\textbf{(Hyper-contr)} \qquad \left\| \max_{k \le K} \min_{j \le N} \|\mathcal{H}^{j,k}\|_A \right\|_q \le \gamma \left\| \max_{k \le K} \min_{j \le N} \|\mathcal{H}^{j,k}\|_A \right\|_p ,$$

then one would obtain for some α,

$$\Pr(\min_{j \le N} \|\mathcal{G}^j\|_A > \alpha t) \le \alpha \Pr(\min_{j \le N} \|\mathcal{H}^j\|_A > t).$$

By using independence to write each side as a power and then taking N^{th} roots and letting $N \to \infty$ we obtain

$$\Pr(\|\mathcal{G}\|_A > \alpha t) \le \Pr(\|\mathcal{H}\|_A > t).$$

Since the constant outside the probability is now 1 we can take complements and reverse the inequality. Now, unraveling the norm and rewriting in terms of μ we return to the inequality C_α. By Theorems 5.4 and 5.5 the two conditions above translate into four conditions, two for max and two for min. The proof of the next theorem consists of checking three of these conditions. Unfortunately we do not know how to check the forth one and must leave it as an assumption.

Theorem 6.8. *Let $Y = \max_{l \le L} \|G\|_l$ and $X = \max_{l \le L} \|G_l\|_l$, where the norms $\|\cdot\|_l$ were defined above. If*

$$\|m_n(Y)\|_q \le C\|m_n(X)\|_q, \tag{6.4}$$

for some $0 < p < q$ then for all $t \ge 0$

$$P(Y \le ct) \ge P(X \le t)$$

where the constant c depends on p and q only.

Proof. In order to apply Theorem 5.4, we need to show that there exist constants C_1, C_2 and C_3 which depend only on p and q such that

$$\text{max hypercontractivity} \quad \|M_n(X)\|_q \le C_1\|M_n(X)\|_p, \tag{6.5}$$

$$\text{min hypercontractivity} \quad \|m_n(X)\|_q \le C_2\|m_n(X)\|_p, \tag{6.6}$$

and

$$\|M_n(Y)\|_q \le C_3 \|M_n(X)\|_q. \tag{6.7}$$

To prove (6.5), note that $M_n(X)$ is a norm of Gaussian vectors. and (6.5) follows from the hypercontractivity of norms of Gaussian vectors (cf. for example, Ledoux and Talagrand (1991), p. 60).

(6.6) follows from Corollary 6.6. Finally (6.7) follows from Slepian's lemma, see the exposition before the statement of this theorem. Now we can apply Theorem 5.4 with (6.4), (6.5), (6.6) and (6.7) in hand. ■

As a consequence of Theorem 6.8, we have the following modified correlation inequality for centered Gaussian measure.

Corollary 6.9. *(modified correlation inequality) Assume (6.4) holds. Then there exists an absolute constant α such that*

$$\mu(\alpha(\bigcap_{l=1}^{L} A_l)) \ge \prod_{l=1}^{L} \mu(A_l) \tag{6.6}$$

for any centered Gaussian measure μ and any convex, symmetric sets A_l, $1 \le l \le L$.

Remark 6.10. From Theorem 4.4 alone one gets the following: There exists an $\alpha < \infty$ such that if, e.g., one has $\prod_{l=1}^{L} \mu(A_l) \ge 8/9$, then

$$\mu(\alpha \bigcap_{l=1}^{L} A_l) \ge \prod_{l=1}^{L} \mu(A_l).$$

This gives some indication of the necessity of handling the case of "small" sets.

Section 7. Final remarks and some open problems.

In this section we mention a few results and open problems that are closely related to the main results in this paper. At first, we give a very simple proof of the following result.

Proposition 7.1. *For $0 < p < q < \infty$, if there exists a constant C, such that for all n,*

$$\|M_n(X)\|_q \le C \|M_n(X)\|_p \tag{7.1}$$

then the following are equivalent:

(i) There exists a constant C, such that for all n,

$$\|M_n(Y)\|_p \le C \|M_n(X)\|_p; \tag{7.2}$$

(ii) There exists a constant C such that

$$P(Y > t) \le C \cdot P(X > t). \tag{7.3}$$

Proof. It follows from de la Peña, Montgomery-Smith and Szulga (1994) that the hypercontractivity of X, (7.1), and the domination relation (7.2) imply the tail domination (7.3). So we only need to show that (ii) implies (i). Without loss of generality, we assume $C > 1$. Let δ be an independent random variable with

$$P(\delta = 1) = 1/C, \quad P(\delta = 0) = 1 - 1/C.$$

Then for all n and all $t \ge 0$

$$\begin{aligned} P(M_n(\delta Y) < t) =& P^n(\delta Y < t) = (1 - P(\delta Y \ge t))^n = (1 - C^{-1} P(Y \ge t))^n \\ &\ge (1 - P(X \ge t))^n = P(M_n(X) < t) \end{aligned}$$

which implies $\|M_n(\delta Y)\|_p \le \|M_n(X)\|_p$. On the other hand, we have

$$E_Y E_\delta \max_{1 \le i \le n} (\delta_i Y_i^p) \ge E_Y \max_{1 \le i \le n} E_\delta(\delta_i Y_i^p) = C^{-1} E \max_{1 \le i \le n} Y_i^p.$$

which finishes the proof. ■

There are many questions related to this work. Let us only mention a few here.

Question 7.2. *Is the best min-hypercontractive constant in (6.4) with $Y = \|X\|$ for symmetric Gaussian vectors X in any separable Banach space*

$$C = \frac{\Gamma^{1/q}(q)}{\Gamma^{1/p}(p)}?$$

The constant follows from the small ball estimates, $P(|X| < s) \sim K \cdot s$ as $s \to 0$, of one-dimensional Gaussian random variable X. Note that if $\beta > 1$ and $P(|X| < s) \sim K \cdot s^\beta$ as $s \to 0$, then the resulting constant in this case is smaller. Thus the conjecture looks reasonable in view of Proposition 6.3.

A related question is, under a max-hyper condition, what can one say about a non-trivial lower bound for $\|M_{k+1}\|_p / \|M_k\|_p$, particularly, in the Gaussian case. This may be useful in answering the question.

A result of Gordon (1987) compares the expected minima of maxima for, in particular, Gaussian processes. We mention this here because a version of Gordon's results could perhaps be used to prove the next Conjecture. Note that if the conjecture holds, then the modified correlation inequality C_α holds.

Conjecture 7.3. *Let G, G_l and norm $\|\cdot\|_l$ be as in Section 1. If $Y = \max_{l \le L} \|G\|_l$ and $X = \max_{l \le L} \|G_l\|_l$, then*

$$\|m_n(Y)\|_q \le C \|m_n(X)\|_q.$$

Our final conjecture is related to stable measures. It is a stronger statement than our Proposition 6.4 and holds for the symmetric Gaussian measures.

Conjecture 7.4. *Let ν be a symmetric α stable measure with $0 < \alpha \leq 2$ on a separable, Banach space F. Then for any closed, symmetric, convex set $B \subseteq F$ and for each $b < 1$, there exists $R(b)$ such that for all $0 \leq t \leq 1$,*

$$\nu(tB) \leq R(b)t\nu(B), \text{ whenever } \nu(B) \leq b.$$

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