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Exit Time, Green Function and Semilinear Elliptic Equations

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Abstract

Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ and τ_D be the first exit time from D by Brownian motion on $\mathbb{R}^n$. In the first part of this paper, we are concerned with sharp estimates on the expected exit time $\mathbb{E}_x[\tau_D]$. We show that if D satisfies a uniform interior cone condition with angle $\theta \in (\cos^{-1}(1/\sqrt{n}), \pi)$, then $c_1\varphi_1(x) \leq \mathbb{E}_x[\tau_D] \leq c_2\varphi_1(x)$ on D. Here φ_1 is the first positive eigenfunction for the Dirichlet Laplacian on D. The above result is sharp as we show that if D is a truncated circular cone with angle $\theta < \cos^{-1}(1/\sqrt{n})$, then the upper bound for $\mathbb{E}_x[\tau_D]$ fails. These results are then used in the second part of this paper to investigate whether positive solutions of the semilinear equation $\Delta u = u^p$ in D, $p \in \mathbb{R}$, that vanish on an open subset $\Gamma \subset \partial D$ decay at the same rate as φ_1 on Γ.

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1 Introduction

Let $n \geq 2$ and X a Brownian motion on $\mathbb{R}^n$. Suppose D is a bounded Lipschitz domain in $\mathbb{R}^n$. Denote by $\tau_D := \inf\{t : X_t \notin D\}$ the first exit time from D by X, and X^D the subprocess obtained from X by letting it be killed upon exiting D. It is well-known that X^D has a jointly continuous Green function $G_D(x, y)$ on $D \times D$ except along the diagonal:

$$\mathbb{E}_x \left[\int_0^{\tau_D} f(X_s)\, ds \right] = \int_D G_D(x, y) f(y) dy, \qquad x \in D,$$

for every Borel function $f \geq 0$ on D. The right hand side of the above display will be denoted as $G_D f(x)$. The infinitesimal generator of X^D is the Laplacian $\frac{1}{2}\Delta$ on D with zero Dirichlet boundary condition. This Dirichlet Laplacian on D has discrete spectrum $0 > -\lambda_1 > -\lambda_2 \geq -\lambda_3 \geq \cdots$. Let φ_1 be the positive eigenfunction corresponding to the eigenvalue $-\lambda_1$ normalized to have $\int_D \varphi_1(x)^2 dx = 1$. The random exit time τ_D, the Green function G_D and the first eigenfunction φ_1 are fundamental objects in probability theory and analysis. In many occasions, one needs to estimate $\mathbb{E}_x[\tau_D] = G_D 1(x)$. When D is a bounded $C^{1,1}$ domain, using the known two-sided Green function estimate on G_D (see Lemma 2.1 below), one can easily deduce

$$\mathbb{E}_x[\tau_D] \asymp \delta_D(x) \qquad \text{on } D. \tag{1.1}$$

Here, and throughout the paper, $\delta_D(x)$ denotes the Euclidean distance between x and D^c, and for two positive functions f, g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 so that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for f and g. Since $\varphi_1(x) = \lambda_1 G_D \varphi_1$, using the two-sided Green function estimate on G_D again, we have $\varphi_1(x) \asymp \delta_D(x)$ on D. Thus on a bounded $C^{1,1}$ domain D,

$$\mathbb{E}_x[\tau_D] \asymp \varphi_1(x) \qquad \text{on } D. \tag{1.2}$$

While it is clear that in general (1.1) no longer holds on bounded Lipschitz domains, it is reasonable to ask if (1.2) remains true for a bounded Lipschitz domain D. We show in this paper that, in fact, (1.2) holds on any bounded Lipschitz domain D with Lipschitz constant strictly less than $1/\sqrt{n-1}$, and fails on some Lipschitz domain with Lipschitz constant strictly larger than $1/\sqrt{n-1}$. In fact, our result is somewhat stronger than that. To state it, let us recall the following notions. For $\theta \in (0, \pi)$, let $\mathscr{C}(\theta)$ be the truncated circular cone in $\mathbb{R}^n$ with angle θ, defined by

$$\mathscr{C}(\theta) := \left\{ x \in \mathbb{R}^n : |x| < 1 \ \text{ and } \ x \cdot e_1 > |x| \cos\theta \right\}, \tag{1.3}$$

where $e_1 := (1, 0, \cdots, 0) \in \mathbb{R}^n$. We say that a bounded Lipschitz domain D satisfies the interior cone condition with common angle θ, if there is some $a > 0$ such that for every point $x \in \partial D$, there is a cone $\mathscr{C} \subset D$ with vertex at x that is conjugate to $a\mathscr{C}(\theta)$; that is, $\mathscr{C}$ is the cone with vertex at x that is obtained from $\mathscr{C}(\theta)$ through parallel translation and rotation.

The result below states that (1.2) holds for bounded Lipschitz domains in $\mathbb{R}^n$ satisfying the interior cone condition with common angle strictly larger than $\cos^{-1}(1/\sqrt{n})$. This includes as special case bounded Lipschitz domains in $\mathbb{R}^n$ whose Lipschitz constant is strictly less than $1/\sqrt{n-1}$.

Theorem 1.1. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ satisfying the interior cone condition with common angle $\theta \in \left(\cos^{-1}(1/\sqrt{n}), \pi\right)$. Then there is a constant $c \geq 1$ such that*

$$c^{-1}\varphi_1(x) \leq \mathbb{E}_x[\tau_D] \leq c\, \varphi_1(x) \qquad \textit{for every } x \in D. \tag{1.4}$$

Remark 1.2. (i) Theorem 1.1 can also be rephrased with φ_1 being replaced by $\varphi(x) := G_D(x, x_0)\wedge 1$, where $x_0 \in D$ is fixed. This is because by Lemma 3.2, $\varphi \asymp \varphi_1$ on D.

(ii) A variant but equivalent form of Theorem 1.1 has also been obtained independently by M. Bieniek and K. Burdzy [4] using a different method. In [4], it is shown under the same condition of Theorem 1.1 that for a fixed compact set $A \subset D$ with non-empty interior, $u(x) := \mathbb{P}_x(\sigma_A < \tau_D)$ is comparable to $\mathbb{E}_x[\tau_D]$ on D, where $\sigma_A := \inf\{t \geq 0 : X_t \in A\}$ is the first entrance time of A by X. Clearly $u = 1$ on K and u is a harmonic function in $D \setminus A$ that vanishes continuously on ∂D. So by the boundary Harnack inequality on bounded Lipschitz domains (see Theorem 1.5 on next page), u is comparable to the function φ in (i). The proof of [4] involves a crucial use of the boundary Harnack inequality on bounded Lipschitz domains and a construction of a particular harmonic function through Kelvin transform. Our proof of Theorem 1.1 establishes a lower bound on φ_1 in terms of the distance function to the boundary and uses a Green function estimate of $G_D(x, y)$ (see Propositions 2.2 and 3.1 below).

(iii) While the lower bound for $\mathbb{E}_x[\tau_D]$ in (1.4) in fact holds for every bounded Lipschitz domain, we show in Theorem 3.3 that the condition on θ is sharp for the upper bound (hence for (1.2)).

(iv) The above result is in sharp contrast to the situation when X is replaced by a rotationally symmetric α-stable process Y on $\mathbb{R}^n$ with $0 < \alpha < 2$. For any bounded Lipschitz domain $D \subset \mathbb{R}^n$ and for the process Y, [17, Theorem 8] shows that $\mathbb{E}_x[\tau_D] \asymp \varphi_0(x)$ on D, where $\varphi_0(x) = G_D 1_K(x)$ for some compact subset K of D. By a similar argument as that for Lemma 3.2, one can show that $\varphi_1 \asymp \varphi_0$ on D. Thus for any rotationally symmetric α-stable process Y on $\mathbb{R}^n$ with $0 < \alpha < 2$, (1.4) holds on *every* bounded Lipschitz domain D. $\square$

We now consider the positive solutions of the following semilinear equation on a bounded Lipschitz domain $D \subset \mathbb{R}^n$:

$$\begin{cases} \frac{1}{2}\Delta u = u^p & \text{in} \quad D, \\ u = \phi & \text{on} \quad \partial D, \end{cases} \tag{1.5}$$

where $p \in \mathbb{R}$, and ϕ is a non-negative continuous function on ∂D that vanishes on an open subset $\Gamma \subset \partial D$. If $\phi > 0$ then existence of positive solutions is standard and we briefly review the vast literature at the end of this section. If ϕ vanishes on a portion of the boundary we show that there is a $p_0 \in \mathbb{R}$ such that above Dirichlet problem has a positive solution if $p \geq p_0$ and does not if $p < p_0$. We investigate whether positive solutions of (1.5) go to zero at the same rate as φ_1 on Γ.

The primary motivation for such a study comes from the BHP for positive harmonic functions. The following classical BHP is due to A. Ancona, B. Dahlberg and J. M. Wu (see [3, p.176] for a proof).

Theorem 1.3. (Ancona, Dahlberg and Wu) *Suppose that D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$. Then there is a constant $c \geq 1$ such that for every $z \in \partial D$, $r > 0$ and two positive harmonic functions u and v in $B(z, 2r) \cap D$ that vanish continuously on $\partial D \cap B(z, 2r)$, we have*

$$\frac{u(x)}{v(x)} \leq c\, \frac{u(y)}{v(y)} \qquad \text{for every } x, y \in D \cap B(z, r). \tag{1.6}$$

From Remark 1.2(i) and the above result it is clear that all harmonic functions that vanish continuously on a part of the boundary go to zero at the same rate as φ_1 on that part. One quickly observes that positive solutions of (1.5) are subharmonic functions on D, and in general, subharmonic functions need not go to zero on $\Gamma \subset \partial D$ at the same rate as φ_1. To state our results precisely, we need some definitions.

Definition 1.4. *We say that $u \in C(\overline{D})$ is a mild solution to (1.5) if*

$$u(x) = h(x) - \int_D G_D(x,y)u^p(y)dy, \quad x \in D,$$

where $h \in C(\overline{D})$ is a harmonic function in D satisfying $h = \phi$ on ∂D.

Here $C(\overline{D})$ denotes the space of continuous functions on $\overline{D}$. It is easy to see that a function $u \in C(\overline{D})$ is a mild solution of (1.5) if and only if it is a weak solution of (1.5) (cf. [6]). We consider the following classes of functions.

- $\mathscr{H}_+ = \mathscr{H}_+(D,\Gamma)$ denotes the class of functions $h \in C(\overline{D})$ that are positive and harmonic in D and vanish on Γ.
- $\mathscr{S}_+^p = \mathscr{S}_+^p(D,\Gamma)$ denotes the class of positive mild solutions $u \in C(\overline{D})$ to (1.5) for some non-negative continuous function ϕ on ∂D that vanishes on Γ.
- $\mathscr{S}_H^p = \mathscr{S}_H^p(D,\Gamma)$ denotes the class of $u \in \mathscr{S}_+^p$ for which $u \asymp h$ in D for some $h \in \mathscr{H}_+$.

In view of Theorem 1.3 and Remark 1.2(i), we see that when D is a bounded Lipschitz domain in $\mathbb{R}^n$, functions in $\mathscr{S}_H^p$ do go to zero on $\Gamma \subset \partial D$ at the same rate as φ_1. So the purpose of the second part of this paper is to investigate how large the class $\mathscr{S}_H^p$ is and, in particular, when is $\mathscr{S}_H^p = \mathscr{S}_+^p$ and when $\mathscr{S}_H^p = \emptyset$.

Theorem 1.5. *Assume that $n \geq 2$ and D is a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

(i) $\mathscr{S}_+^p = \mathscr{S}_H^p \neq \emptyset$ *for $p \geq 1$.*

Assume, in addition, that D satisfies

$$\mathbb{E}_x[\tau_D] \leq c\,\varphi_1(x) \qquad \text{for every } x \in D. \tag{1.7}$$

Then there exists $p_0 \in (-\infty, 0)$ such that

(ii) $\mathscr{S}_H^p \neq \emptyset$ *for $p_0 < p < 1$, and*

(iii) $\mathscr{S}_H^p = \mathscr{S}_+^p = \emptyset$ *for $p < p_0$.*

We conjecture that for $p \in (p_0, 1)$, $\emptyset \neq \mathscr{S}_H^p \subsetneq \mathscr{S}_+^p$. Note that Theorem 1.1 gives a sufficient condition on a Lipschitz D to satisfy condition (1.7). We can say more when D is a bounded $C^{1,1}$-domain. Recall that a bounded domain $D \subset \mathbb{R}^n$ is said to be $C^{1,1}$-smooth if for every point $z \in \partial D$, there is $r > 0$ such that $D \cap B(z,r)$ is the region in $B(z,r)$, under some z-dependent coordinate system, that lies above the graph of a function whose first derivatives are Lipschitz continuous.

Theorem 1.6. *Assume that $n \geq 2$ and D is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$. Then*

(i) $\mathscr{S}_+^p = \mathscr{S}_H^p \neq \emptyset$ *for $p \geq 1$;*

(ii) $\mathscr{S}_H^p \neq \emptyset$ *for $-1 < p < 1$;*

(iii) $\mathscr{S}_H^p = \mathscr{S}_+^p = \emptyset$ *for $p \leq -1$.*

Remark 1.7. (i) Our proof of Theorem 1.6 is based on a two-sided estimate on $G_D(x,y)$; see Proposition 2.2 below. Theorem 1.6 in fact holds not only for the Laplacian but also for a large class of uniformly elliptic operators in a bounded C^2-domain D. Let $\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$, where a_{ij} has continuous derivatives on $\overline{D}$ (i.e. it is $C^1(\overline{D})$), and $A(x) = (a_{ij}(x))$ is a symmetric matrix-valued function that is uniformly bounded and elliptic. Then by Theorem 3.3 of Grüter and Widman [16], the Green function $G_D^{\mathscr{L}}(x,y)$ of $\mathscr{L}$ in D satisfies the following estimate

$$G_D^{\mathscr{L}}(x,y) \le c\delta_D(x)|x-y|^{1-n}, \quad x,y \in D,$$

where $c > 0$. On the other hand, we know from Lemma 4.6.1 and Theorem 4.6.11 of Davies [8] that $G_D^{\mathscr{L}}(x,y) \ge c\,\delta_D(x)\delta_D(y)$. For $r>0$, define $D_r = \{x \in D : \delta_D(x) < r\}$. Thus for fixed $y_0 \in D$ and $r < \delta_D(y_0)$,

$$G_D^{\mathscr{L}}(x,y_0) \asymp \delta_D(x) \quad \text{for } x \in D_r. \tag{1.8}$$

It is well known that Harnack and boundary Harnack principles hold for $\mathscr{L}$ and the Green function

$$G_D^{\mathscr{L}}(x,y) \asymp \begin{cases} |x-y|^{2-n} & \text{when } n \ge 3, \\ \log(1+|x-y|^{-2}) & \text{when } n = 2. \end{cases}$$

for $x,y \in D \setminus D_r$ with $r>0$. Hence by a similar argument as that in Bogdan [5], we conclude that the estimate (2.1)-(2.2) hold for the Green function $G_D^{\mathscr{L}}$ of $\mathscr{L}$ in D . Finally, by imitating the proof of Theorem 1.6 we can obtain the result for $\mathscr{L}$ as well.

(ii) Now suppose $D = [0,1]^n$ and $-1 < p < 1$. Let $u(x) = c_p(x_1)^{\frac{2}{1-p}}$, where $c_p = \left(\frac{2(1+p)}{(1-p)^2}\right)^{\frac{2}{1-p}}$ and x_1 is the first coordinate of $x = (x_1, \ldots, x_n)$. Now $u \in \mathscr{S}_+^p$ clearly and due to the one dimensional nature of this example one can establish $u \notin \mathscr{S}_H^p$. This suggests that Theorem 1.6(ii) could be replaced by:

$$\emptyset \ne \mathscr{S}_H^p \subsetneq \mathscr{S}_+^p, \text{ for } -1 < p < 1;$$

However, we were not able to generalize the above example to general bounded $C^{1,1}$- domains D.

(iii) We have stated all our results for solutions of the equation (1.5). However if we assume that $f(u) \asymp u^p$ then the proofs of our main results can be suitably modified to yield the same quantitative behavior for solutions of the equation

$$\begin{cases} \frac{1}{2}\Delta u = f(u) & \text{in} \quad D, \\ u = \phi & \text{on} \quad \partial D. \end{cases} \tag{1.9}$$

(iv) When D is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, and $p \ge 1$, the result $\mathscr{S}_+^p \ne \emptyset$ is established in [6] (in fact the result is proved to be true for any bounded regular domain D), while for $-1 < p < 1$, $\mathscr{S}_+^p \ne \emptyset$ is shown in [1]. $\square$

There is a wealth of literature on the semilinear elliptic equations. Under certain regularity conditions on $D \subset \mathbb{R}^n$ and ϕ, where $n \ge 3$, the existence of solutions to (1.9), bounded below by a positive harmonic function, was established in [6] when f satisfies the condition that $-u \le f(u) \le u$ for $|u| < \varepsilon$ for some $\varepsilon > 0$, and in [1] the case when $0 \le f(u) \le u^{-\alpha}$ for some $\alpha \in (0,1)$ was resolved.

The equation $\Delta u = u^p$ in D with $u = \phi$ on ∂D has also been widely studied. For $1 \le p \le 2$, it has been studied probabilistically using the exit measure of super-Brownian motion (a measure valued branching process), by Dynkin, Le Gall, Kuznetsov, and others [11; 12; 18]. Properties of solutions when $f(u) = u^p, p \ge 1$, with both finite and singular boundary conditions have also been studied by a number of authors using analytic techniques [2; 13; 15; 19].

Our proofs of Theorem 1.5 and Theorem 1.6 employ implicit probabilistic representation of solutions of (1.5) and Schauder's fixed point theorem. We emphasize that our main new contributions in these two theorems are for subcases (ii)-(iii), that is for $p < 1$. Some part of the results that address the case $p \ge 1$ (Theorem 1.5(i) and Theorem 1.6(i)) may be known (cf. [11; 12; 18]). However, the proofs we provide for these results appear to be more elementary than those available in the literature.

In the sequel, we use $C_\infty(D)$ to denote the space of continuous functions in D that vanish on ∂D. For two real number a and b, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We will use $B(x, r)$ to denote the open ball in $\mathbb{R}^n$ centered at x with radius r.

The rest of the paper is organized as follows. In the next section we present some estimates on the Green function which are required for the proof of Theorem 1.1, Theorem 1.5 and Theorem 1.6. In Section 3, we prove Theorem 1.1 and show that the condition on the common angle is sharp (Theorem 3.3). Finally in Section 4 we prove Theorem 1.5 and in Section 5 we prove Theorem 1.6.

2 Green function estimates

Recall that a bounded domain $D \subset \mathbb{R}^n$ is said to be Lipschitz if there are positive constants r_0 and r so that for every $z \in \partial D$, there is an orthonormal coordinate system CS_z and a Lipschitz function $F_z : \mathbb{R}^{n-1} \to \mathbb{R}$ with $|F_z(\xi) - F_z(\eta)| \le \lambda |\xi - \eta|$ for $\xi, \eta \in \mathbb{R}^{-1}$ so that

$$D \cap B(z, r) = \{ y = (y_1, \cdots, y_n) \in CS_z : |y| < r \text{ and } y_n > F_z(y_1, \cdots, y_{n-1}) \}.$$

The constants (r_0, λ) are called the Lipschitz characteristics of D. If each F_z is a C^1 function whose first order partial derivatives are Lipschitz continuous, then the Lipschitz domain D is said to be a $C^{1,1}$ domain.

We begin with an estimate for Green function in $C^{1,1}$ domains.

Lemma 2.1. *Suppose that D is a bounded $C^{1,1}$ domain. Then*

$$G_D(x, y) \asymp \min\left\{ \frac{1}{|x-y|^{n-2}}, \frac{\delta_D(x)\delta_D(y)}{|x-y|^n} \right\} \qquad \textit{when } n \ge 3 \tag{2.1}$$

$$G_D(x, y) \asymp \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \right) \qquad \textit{when } n = 2. \tag{2.2}$$

Proof. Estimate (2.1) is due to K.-O. Widman and Z. Zhao (see [21]). Estimate (2.2) is established as Theorem 6.23 in [7] for bounded C^2-smooth domain D. However the proof carries over to bounded $C^{1,1}$-domains. □

For the rest of this subsection we will assume that D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$. Recall that we defined $\varphi(x) := G_D(x, x_0) \wedge 1$, where $x_0 \in D$ is fixed.

Let $r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and (r_0, λ) be the Lipschitz characteristics of D. For $x, y \in D$, we let $A_{x,y} = x_0$ if $r(x,y) \geq r_0/32$ and when $r := r(x,y) < r_0/32$, $A_{x,y}$ is any point in D such that

$$B(A_{x,y}, \kappa r) \subset D \cap B(x, 3r) \cap B(y, 3r),$$

with $\kappa := \frac{1}{2\sqrt{1+\lambda^2}}$.

Proposition 2.2. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $n \geq 2$. Then there is a constant $c > 1$ such that on $D \times D \setminus d$,*

$$G_D(x,y) \asymp \frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x-y|^{n-2}} \qquad \text{when } n \geq 3, \tag{2.3}$$

$$c^{-1} \frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \leq G_D(x,y) \leq c \frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \log\left(1 + \frac{1}{|x-y|^2}\right) \qquad \text{when } n = 2. \tag{2.4}$$

Proof. When $n \geq 3$, (2.3) is proved in [5] as Theorem 2. So it remains to show (2.4) when $n = 2$. Since D is bounded, there is a ball $B \supset D$. It follows from [7, Lemma 6.19] that for $x, y \in D$,

$$G_D(x,y) \leq G_B(x,y) \leq \frac{1}{2\pi} \ln\left(1 + 4\frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right) \leq c \ln\left(1 + |x-y|^{-2}\right).$$

On the other hand, by [7, Lemma 6.7], for every c_1, there is a constant $c_2 > 0$ such that

$$G_D(x,y) \geq c_2 \qquad \text{for } x, y \in D \text{ with } |x-y| \leq c_1 \min\{\delta_D(x), \delta_D(y)\}.$$

From these, inequality (2.4) can be proved in the same way as the proofs for [5, Proposition 6 and Theorem 2]. □

3 Exit time and boundary decay rate

In this section we prove Theorem 1.1, and the sharpness of the requirement on the common angle in its statement. To prove Theorem 1.1 we will need the following result.

Proposition 3.1. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $n \geq 2$ and φ_1 be the first positive eigenfunction for the Dirichlet Laplacian in D normalized to have $\int_D \varphi_1(x)^2 dx = 1$. If D satisfies the interior cone condition with common angle $\theta \in \left(\cos^{-1}(1/\sqrt{n}), \pi\right)$, there are positive constants $\varepsilon > 0$ and $a > 0$ such that*

$$\varphi_1(x) \geq a\, \delta_D(x)^{2-\varepsilon} \qquad \textit{for every } x \in D.$$

Proof. By Theorem 4.6.8 of [8] and its proof, there is some constant $a > 0$ such that

$$\varphi_1(x) \geq a\, \delta_D(x)^{\alpha} \qquad \text{for every } x \in D,$$

where $\alpha > 0$ is the constant determined by

$$\alpha(\alpha + n - 2) = \lambda_1(\theta). \tag{3.1}$$

Here $\lambda_1(\theta)$ is the first eigenvalue for the Dirichlet Beltrami-Laplace operator in the unit spherical cap $\mathscr{C}(\theta) \cap \{x \in \mathbb{R}^n : |x| = 1\}$. The first eigenvalue $\lambda_1(\theta)$ can be determined in terms of the hypergeometric function and so can α. Recall the hypergeometric function

$$F(\alpha,\beta,\gamma,z) := 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots$$

Let $\theta(p,n)$ be the smallest positive zero of $F\left(-p, p+n-2, \frac{n-1}{2}, \frac{1-\cos\theta}{2}\right)$. It is known that $p \mapsto \theta(p,n)$ is continuous and strictly decreasing with $\theta(1,n) = \pi/2$ (cf. p.62 of [10]). Let $\theta \mapsto p(\theta,n)$ be the inverse function of $p \mapsto \theta(p,n)$. We know from [10, p.59 and p.63] that α in (3.1) is equal to

$$\alpha = p(\theta,n). \tag{3.2}$$

Note that

$$F\left(-2, n, \frac{n-1}{2}, z\right) = 1 - \frac{4n}{n-1}z + \frac{4n}{n-1}z^2,$$

which has roots $\frac{n-\sqrt{n}}{2n}$ and $\frac{n+\sqrt{n}}{2n}$. Set $z = \frac{1-\cos\theta}{2}$. The corresponding smallest positive root for θ is $\cos\theta_0 = \frac{1}{\sqrt{n}}$ or $\theta_0 = \cos^{-1}(1/\sqrt{n})$. In other words, we have for $n \geq 2$,

$$\theta(2,n) = \cos^{-1}(1/\sqrt{n}), \qquad \text{or equivalently,} \qquad p(\cos^{-1}(1/\sqrt{n}),\, n) = 2. \tag{3.3}$$

As $\theta \mapsto p(\theta,n)$ is strictly decreasing, we have $p(\theta,n) < 2$ for every $\theta > \cos^{-1}(1/\sqrt{n})$. This proves the proposition. $\square$

Recall that $\varphi(x) := G_D(x,x_0) \wedge 1$. The following lemma is known to the experts, but we provide its proof for completeness.

Lemma 3.2. *Suppose that D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$. There is a constant $c \geq 1$ such that*

$$c^{-1}\varphi_1(x) \leq \varphi(x) \leq c\,\varphi_1(x) \qquad \text{for } x \in D.$$

Proof. It is well-known that D is intrinsic ultracontractive (cf. [8]) and so for every $t > 0$, there is a constant $c_t \geq 1$ such that

$$c_t^{-1}\,\varphi_1(x)\varphi_1(y) \leq p^D(t,x,y) \leq c_t\,\varphi_1(x)\varphi_1(y) \qquad \text{for every } x, y \in D. \tag{3.4}$$

For the definition of intrinsic ultracontractivity and its equivalent characterizations, see Davies and Simon [9]. By (3.4), we have

$$\frac{1}{c}\varphi(x) \geq \int_D p^D(1,x,y)\varphi(y)dy \geq c\left(\int_D \varphi(y)\varphi_1(y)dy\right)\varphi_1(x).$$

Thus there is a constant $c_1 > 0$ such that

$$\varphi(x) \geq c_1\varphi_1(x) \qquad \text{for every } x \in D. \tag{3.5}$$

On the other hand, let $K := \{x \in D : G_D(x,x_0) \geq 1\}$, which is a compact subset of D. Observe that both φ and φ_1 are continuous and strictly positive in D. So $a := \sup_{x\in K}\frac{\varphi(x)}{\varphi_1(x)}$ is a positive and finite number. Since φ is harmonic in $D \setminus K$ and $\Delta\varphi_1(x) = -\lambda_1\varphi_1(x)$ with $\lambda_1 > 0$, we have $\Delta(\varphi - a^{-1}\varphi_1) > 0$ on $D \setminus K$. As both φ and φ_1 vanish continuously on ∂D and $\varphi(x) - a\varphi_1(x) \leq 0$ on K, we have by the maximal principle for harmonic functions that

$$\varphi(x) \leq a\,\varphi_1(x) \qquad \text{for every } x \in D \setminus K. \tag{3.6}$$

This proves the Lemma. □

Proof of Theorem 1.1. Since φ_1 is bounded on D, we have

$$\varphi_1(x) = \lambda_1 G_D\varphi_1(x) \leq \lambda_1 \|\varphi_1\|_\infty \, G_D 1(x) = \lambda_1 \|\varphi_1\|_\infty \, \mathbb{E}_x[\tau_D] \qquad \text{for } x \in D.$$

To establish the upper bound on $\mathbb{E}_x[\tau_D]$, set $K = \{x \in D : G_D(x, x_0) \geq 1\}$, which is a compact subset of D. By taking $r_0 > 0$, we may and do assume that the Euclidean distance between K and D^c is at least r_0. Since φ is a positive harmonic function in $D \setminus K$ that vanishes on ∂D, by Carleson's estimate (see, e.g., Theorem III.1.8 of [3]), there is a universal constant $c_1 = c_1(D, K) > 0$ such that $\varphi(y) \leq c_1\, \varphi(A_{x,y})$ whenever $r(x, y) < r_0/32$. Note also that φ is bounded on D and that, by Proposition 3.1 and (3.5), $\varphi(x) \geq c\,\delta_D(x)^{2-\varepsilon}$.

When $n \geq 3$, we have by (2.3),

$$G_D(x, y) \leq c\,\frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2}\frac{1}{|x-y|^{n-2}}, \qquad x, y \in D,$$

Thus we have

$$\begin{aligned}
\frac{G_D 1(x)}{\varphi(x)} &\leq c \int_D \frac{\varphi(y)}{\varphi(A_{x,y})}\frac{1}{\varphi(A_{x,y})|x-y|^{n-2}}dy \\
&= c \int_{\{y\in D: r(x,y)\geq r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})^2}\frac{1}{|x-y|^{n-2}}dy \\
&\quad + c \int_{\{y\in D: r(x,y)< r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})}\frac{1}{\varphi(A_{x,y})\,|x-y|^{n-2}}dy \\
&\leq c \int_D \frac{1}{|x-y|^{n-2}}dy + c \int_{\{y\in D: r(x,y)< r_0/32\}} \frac{1}{(r(x,y)^{2-\varepsilon}\,|x-y|^{n-2}}dy \\
&\leq c + c \int_{\{y\in D: r(x,y)< r_0/32\}} \frac{1}{|x-y|^{n-\varepsilon}}dy \\
&\leq c < \infty.
\end{aligned}$$

When $n = 2$, we have by (2.4),

$$
\begin{aligned}
\frac{G_D 1(x)}{\varphi(x)} &\le c \int_D \frac{\varphi(y)}{\varphi(A_{x,y})} \frac{1}{\varphi(A_{x,y})} \log(1 + |x-y|^{-2}) dy \\
&= c \int_{\{y \in D: r(x,y) \ge r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x-y|^{\varepsilon/2}} dy \\
&\quad + c \int_{\{y \in D: r(x,y) < r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})} \frac{1}{\varphi(A_{x,y}) |x-y|^{\varepsilon/2}} dy \\
&\le c \int_D \frac{1}{|x-y|^{\varepsilon/2}} dy + c \int_{\{y \in D: r(x,y) < r_0/32\}} \frac{1}{(r(x,y)^{2-\varepsilon} |x-y|^{\varepsilon/2}} dy \\
&\le c + c \int_{\{y \in D: r(x,y) < r_0/32\}} \frac{1}{|x-y|^{2-(\varepsilon/2)}} dy \\
&\le c < \infty.
\end{aligned}
$$

The theorem is now proved in view of Lemma 3.2. □

The following result says that Theorem 1.1 is sharp.

Theorem 3.3. *Let $D = \Gamma(\theta)$ be a truncated circular cone in $\mathbb{R}^n$ with common angle $\theta < \cos^{-1}(1/\sqrt{n})$ and $n \ge 2$, defined by (1.3). Then there are constants $c > 0$ and $\alpha > 2$ such that*

$$
G_D 1(x) \ge c\, \delta_D(x)^{2-\alpha} \varphi_1(x) \qquad \text{for every } x = (x_1, 0, \cdots, 0) \text{ with } 0 < x_1 < 1/2. \tag{3.7}
$$

Proof. It is known (see, e.g., two lines above (4.6.6) on page 129 of [8]) that $\varphi_1(x)$ decays at rate $\delta_D(x)^\alpha$ as $x \to 0$ along the axis of the cone $\mathscr{C}(\theta)$, where α is given by (3.1). We see from (3.2)-(3.3) that $\alpha > 2$ when $\theta < \cos^{-1}(1/\sqrt{n})$. Clearly there is $\varepsilon \in (0, 1/2)$ such that $B(x, \delta_D(x)) \subset D \setminus K$ for every $x = (x_1, 0, \cdots, 0)$ with $0 < x_1 < \varepsilon$. This together with (3.6) implies in particular that there is a constant $c > 0$ such that

$$
\varphi(x) \le a\, \varphi_1(x) \le c\, \delta_D(x)^\alpha \qquad \text{for } x = (x_1, 0, \cdots, 0) \text{ with } 0 < x_1 < \varepsilon.
$$

By Harnack inequality,

$$
\varphi(y) \le c\, \varphi(x) \le c \delta_D(x)^\alpha \le c \delta_D(y)^\alpha \tag{3.8}
$$

for every $y \in B(x, \delta_D(x)/2)$ and every $x = (x_1, 0, \cdots, 0)$ with $0 < x_1 < \varepsilon$. By Proposition 2.2,

$$
G_D(x,y) \ge c\, \frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x-y|^{n-2}}, \qquad x, y \in D,
$$

where $A_{x,y}$ is as given in the proof of Theorem 1.1. Here for the case of $n = 2$, we use the convention that $0^0 = 1$. Let $x = (x_1, 0, \cdots, 0)$ with $0 < x_1 < \varepsilon$. For $y \in B(x, \delta_D(x)/4) \setminus B(x, \delta_D(x)/6)$, we can take $A_{x,y} = y$. Note that in this case, $\delta_D(y) \le 5\delta_D(x)/4 \le 15|x-y|$. We therefore have

$$
\begin{aligned}
G_D 1(x) &\geq \int_{B(x,\delta_D(x)/2)\setminus B(x,\delta_D(x)/3)} G_D(x,y)dy \\
&\geq c\,\varphi(x) \int_{B(x,\delta_D(x)/2)\setminus B(x,\delta_D(x)/3)} \frac{1}{\varphi(y)|x-y|^{n-2}} dy \\
&\geq c\,\varphi(x) \int_{B(x,\delta_D(x)/2)\setminus B(x,\delta_D(x)/3)} \frac{1}{|x-y|^{\alpha}|x-y|^{n-2}} dy \\
&\geq c\,\varphi(x)\delta_D(x)^{2-\alpha}.
\end{aligned}
$$

This together with Lemma 3.2 establishes the theorem. □

Remark 3.4. Note that the circular cone $\mathscr{C}(\theta)$ with angle $\theta = \cos^{-1}(1/\sqrt{n})$ has Lipschitz constant $1/\sqrt{n-1}$ at its vertex. So if D is a bounded Lipschitz domain in $\mathbb{R}^n$ with Lipschitz constant strictly less than $1/\sqrt{n-1}$, then D satisfies interior cone condition with common angle $\theta \in \left(\cos^{-1}(1/\sqrt{n}), \pi\right)$. We point out that this is only a sufficient condition. The aforementioned interior cone condition can be satisfied in some bounded Lipschitz domains with Lipschitz constant larger than $1/\sqrt{n-1}$. A smooth domain with an inward sharp cone is such an example. □

4 Semilinear elliptic equations

We start with some technical lemmas for general bounded domains and then proceed to present the proof of Theorem 1.5.

Let Ω be the set of continuous functions from $[0,\infty)$ to $\mathbb{R}^n$, and let $X_t(\omega) = \omega(t)$, $t \ge 0$. Endow Ω with the Borel sigma-field $\mathscr{B}(\Omega)$. Let $\hat{\mathscr{F}}_t$ denote the canonical sigma-field $\sigma\{\omega(s) : 0 \le s \le t\}$. For $x \in \mathbb{R}^n$ let $\mathbb{P}_x$ denote the probability measure on $(\Omega, \mathscr{B}(\Omega))$ under which X is a Brownian motion starting from x. Let $\{\mathscr{F}_t\}$ denote the usual augmentation of the filtration $\{\hat{\mathscr{F}}_t\}$ with respect to the family of measures $\{\mathbb{P}_x, x \in \mathbb{R}^n\}$ (see p. 45 of [20]). For a positive harmonic function h in D and $x \in D$, we denote by $\mathbb{P}^h_x$ the h-transform of $\mathbb{P}_x$ under h (see [3] or [7]). Let $\mathbb{E}_x$ ($\mathbb{E}^h_x$) denote expectation with respect to $\mathbb{P}_x$ (respectively, $\mathbb{P}^h_x$). For any set $A \subset \mathbb{R}^n$ we denote

$$\tau_A = \inf\{t : X_t \notin A\}.$$

The following is a well known result. We provide a proof here for the reader's convenience. This result in fact holds for more general potentials $q \ge 0$, for example when q is in some Kato class (see [7]).

Lemma 4.1. *Assume that every point of ∂D is regular with respect to D^c. Let $h \in \mathscr{H}_+$ and $q \ge 0$. Then the function v given by*

$$v(x) = \mathbb{E}_x\left[h(X_{\tau_D})e^{-\int_0^{\tau_D} q(X_s)ds}\right], \quad x \in D \tag{4.1}$$

satisfies

$$v(x) = h(x) - \int_D G_D(x,y)q(y)v(y)dy, \qquad x \in D. \tag{4.2}$$

The converse is true if q is bounded.

Proof. The proof is along the lines of [6]. Suppose that v is given by (4.1). Then for $x \in D$, by the Markov property of X,

$$\begin{aligned} v(x) &= h(x) + \mathbb{E}_x\left[h(X_{\tau_D})\left(e^{-\int_0^{\tau_D} q(X_s)ds} - 1\right)\right] \\ &= h(x) - \mathbb{E}_x\left[h(X_{\tau_D})\int_0^{\tau_D} q(X_t)e^{-\int_t^{\tau_D} q(X_s)ds}dt\right] \\ &= h(x) - \mathbb{E}_x\left[\int_0^{\tau_D} q(X_t)v(X_t)dt\right] \\ &= h(x) - \int_D G_D(x,y)q(y)v(y)dy. \end{aligned}$$

For the converse, assume $q \geq 0$ is bounded. Suppose now that v satisfies (4.2). Then v is a weak solution to the following equation (cf. [6])

$$\frac{1}{2}\Delta v - qv = 0 \quad \text{in } D \quad \text{with} \quad v|_{\partial D} = h|_{\partial D}. \tag{4.3}$$

As $q \geq 0$ is bounded, it is well known that solutions to equation (4.3) are continuous on $\overline{D}$ and C^1 in D (see, e.g., [14]). Furthermore, the solution of (4.3) enjoys the maximum principle and therefore is unique. This proves the Lemma. □

Lemma 4.2. *There exists a constant $\gamma = \gamma(D) > 0$ such that for every $h \in \mathscr{H}_+$ and $p > 1 - 2\gamma$, we have*

$$\sup_{x \in D} \mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right] \leq C_1 < \infty,$$

where C_1 depends only on h, p and D.

Proof. We will be mainly using the notation in [3], page 200-201. Let $l_k = \{x : h(x) = 2^k\}$ for any $k \in \mathbb{Z}$. Note that there exists k_0 such that $l_k = \emptyset$ for $k \geq k_0$. Define $S_{-1} = 0$ and let $S_0 = \tau_D \wedge \inf\{t : X_t \in \cup l_k\}$. For $i \geq 1$, let $S_i = \tau_D \wedge \inf\{t > S_{i-1} : X_t \in \cup l_k \setminus l_{W_{i-1}}\}$, where W_{i-1} is the number k for which $X_{S_{i-1}} \in l_k$. Let $v_k = \sup_{x \in l_k} \mathbb{E}_x^h[S_1]$. From [3], one has that:
(a) there exists a constant $\gamma(D) > 0$ such that for any k, $v_k \leq c_0 2^{2k\gamma(D)}$

(b) there exists a constant c_1 such that $\sum_{i=0}^{\infty} \mathbb{P}_x^h(W_i = k) \le c_1$ for all $x \in D$. Hence

$$\begin{aligned}
\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s) ds \right] &= \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[\int_{S_{i-1}}^{S_i} h^{p-1}(X_s) ds \right] \\
&\le c_2 \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} (S_i - S_{i-1}) \right] \\
&= c \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} \mathbb{E}_{X_{S_{i-1}}}^h (S_1) \right] \\
&\le c \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} v_{W_{i-1}} \right] \\
&= c \sum_{i=0}^{\infty} \sum_{k=-\infty}^{k_0} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} v_{W_{i-1}} 1(W_{i-1} = k) \right] \\
&= c \sum_{k=-\infty}^{k_0} v_k 2^{k(p-1)} \mathbb{E}_x^h \left[\sum_{i=0}^{\infty} 1(W_{i-1} = k) \right] \\
&\le c \sum_{k=-\infty}^{k_0} 2^{2k\gamma(D)} 2^{k(p-1)} \qquad (4.4)
\end{aligned}$$

Hence if $p > 1 - 2\gamma(D)$ then $\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s) \right] < C_1$ for all $x \in D$. □

Lemma 4.3. *Let $h \in \mathscr{H}_+$. Suppose that p is a real number such that*

$$\sup_{x \in D} \int_D G_D(x, y) h(y)^{p(1+\varepsilon)} dy < \infty$$

for some $\varepsilon > 0$. Then

(i) *The family of functions $\{G_D(x, \cdot) h^p(\cdot) : x \in D\}$ is uniformly integrable over D.*

(ii) *Let $B_{h,p} = \{g : D \to \mathbb{R} : g$ is Borel measurable and $|g(x)| \le h^p(x)$ for all $x \in D\}$. The family of functions $\{\int G_D(\cdot, y) g(y) dy : g \in B_{h,p}\}$ is uniformly bounded and equicontinuous in $C_\infty(D)$, and, consequently, it is relatively compact in $C_\infty(D)$.*

The above assertions hold especially when $p > 1 - 2\gamma(D)$, where $\gamma(D)$ is the constant in Lemma 4.2

Proof. Let $q > 1$ be such that $q^{-1} + (1+\varepsilon)^{-1} = 1$. For any Borel measurable set A, by Hölder inequality,

$$\int_A G_D(x, y) h(y)^p \, dy \le \left(\int_D G_D(x, y) h(y)^{p(1+\varepsilon)} \, dy \right)^{1/(1+\varepsilon)} \left(\int_A G_D(x, y) \, dy \right)^{1/q}.$$

Since D is a bounded, it follows that $\sup_{x \in D} \int_D G_D(x, y) h(y)^p \, dy < \infty$ and

$$\lim_{\delta \to 0} \sup_{A : m(A) < \delta} \sup_{x \in D} \int_A G_D(x, y) h(y)^p \, dy = 0,$$

where m denotes the Lebesgue on $\mathbb{R}^d$. Therefore the family of functions $\{G_D(x,\cdot)h(\cdot)^p, x \in D\}$ is uniformly integrable over D. Since D is Lipschitz domain D, for each $y \in D$, it is known (see [7]) that $x \to G_D(x,y)$ can be extended to be a continuous function on $\overline{D}\backslash\{y\}$ by setting $G_D(x,y) = 0$ for $x \in \partial D$. So the above particularly implies that the function $x \to \int_D G_D(x,y)h(y)^p\,dy$ is continuous on $\overline{D}$ and vanishes on ∂D. On the other hand, by using the triangle inequality, the family of functions $\left\{|G_D(x,\cdot) - G_D(y,\cdot)|\,h(\cdot)^p : x,y \in D\right\}$ is uniformly integrable on D. Therefore the function $(x,y) \to \int_D |G_D(x,z) - G_D(y,z)|\,h(z)^p\,dz$ is continuous on $\overline{D} \times \overline{D}$.

For each $g \in B_{h,p}$, as $|g| \le h^p$, the functions in $B_{h,p}$ are continuous in D, uniformly bounded, and converge uniformly to zero as $x \to \partial D$. For any x, y in D and $g \in B_{h,p}$,

$$\left|\int_D G_D(x,z)g(z)dy - \int_D G_D(y,z)g(z)dz\right| \le \int_D |\,G_D(x,z) - G_D(y,z)\,|\,h(z)^p dz. \tag{4.5}$$

Therefore the family of functions in the statement of the lemma is equi-continuous in D.

When $p > 1-2\gamma(D)$, one can always find an $\varepsilon > 0$ such that $p(1+\varepsilon) > 1-2\gamma$. Thus by Lemma 4.2

$$\sup_{x\in D}\int_D G_D(x,y)h(y)^{p(1+\varepsilon)}\,dy \le \sup_{x\in D} h(x)\mathbb{E}_x^h\left[\int_0^{\tau_D} h(X_s)^{p(1+\varepsilon)-1}ds\right] < \infty$$

and so the hypothesis of the Lemma is satisfied and the result holds. $\square$

Proof of Theorem 1.5. (i) Fix $p \ge 1$ and $h \in \mathscr{H}_+$. Let $\gamma = \gamma(D)$ and C_1 be the positive constants from Lemma 4.2. Define

$$\Lambda = \left\{u \in C(\overline{D}) : e^{-C_1}h \le u \le h \text{ on } \overline{D}\right\}. \tag{4.6}$$

Clearly, Λ is a closed non-empty convex subset of $C(\overline{D})$. Let $G_D(\cdot,\cdot)$ be the Green function of the domain D. Define $T : \Lambda \to C(\overline{D})$ as

$$T(u)(x) = \mathbb{E}_x\left[h(X_{\tau_D})\exp\left(-\int_0^{\tau_D} u^{p-1}(X_s)ds\right)\right]$$

Clearly $Tu(x) \le \mathbb{E}_x\left[h(X_{\tau_D})\right] = h(x)$. Note that for $u \in \Lambda$, $u^{p-1}T(u) \in B_{h,p}$, where $B_{h,p}$ is the space defined in Lemma 4.3(ii). Thus we conclude from Lemma 4.3 that

$$\left\{\int G_D(\cdot,y)u^{p-1}(y)T(u)(y)dy : u \in \Lambda\right\}$$

is relatively compact in $C_\infty(D)$. Since $h \in \mathscr{H}_+$ and by Lemma 4.1,

$$T(u)(x) = h(x) - \int G_D(x,y)u^{p-1}(y)T(u)(y)dy \qquad \text{for } x \in D,$$

we have

$$T(\Lambda) \text{ is relatively compact in } (C(\overline{D}), \|\cdot\|_\infty). \tag{4.7}$$

On the other hand, for any $u \in \Lambda$ and $x \in D$, since $u^{p-1} \le h^{p-1}$,

$$\begin{aligned}
\frac{T(u)(x)}{h(x)} &= \frac{\mathbb{E}_x\left[h(X_{\tau_D})\exp(-\int_0^{\tau_D} u^{p-1}(X_s)ds)\right]}{h(x)} \\
&\ge \mathbb{E}_x^h\left[\exp\left(-\int_0^{\tau_D} h^{p-1}(X_s)ds\right)\right] \\
&\ge \exp\left(-\mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right]\right) \\
&\ge \exp(-C_1)
\end{aligned}$$

where the last inequality follows from Lemma 4.2. By continuity we have $T(u)(x) \ge \exp(-C_1)h(x)$ on $\overline{D}$. Thus we have shown that

$$T(\Lambda) \subset \Lambda. \tag{4.8}$$

If $u_n \in \Lambda$ is such that $\|u_n - u\|_\infty \to 0$, then $u_n^{p-1}(x) \to u^{p-1}(x)$ for all $x \in D$. Now for $u \in \Lambda$, $u^{p-1}(x) \le h^{p-1}(x)$ for all $x \in D$. An application of the Dominated Convergence Theorem implies that $T(u_n)(x) \to T(u)(x)$ for all $x \in D$ and by (4.7), the convergence holds in the uniform norm. We have shown that

$$T : \Lambda \to \Lambda \quad \text{is continuous.} \tag{4.9}$$

Therefore from (4.7), (4.8), (4.9) and Schauder's fixed point theorem [14, Theorem 11.1], T has a fixed point in Λ. Hence there exists a u such that $u(x) = \mathbb{E}_x\left[h(X_{\tau_D})\exp\left(-\int_0^{\tau_D} u^{p-1}(X_s)ds\right)\right]$. From Lemma 4.1 we conclude that $u \in \mathscr{S}_H^p$. Therefore $\mathscr{S}_H^p \neq \emptyset$ for $p \ge 1$.

By definition, $\mathscr{S}_H^p \subset \mathscr{S}_+^p$. Let $u \in \mathscr{S}_+^p$. Since $p \ge 1$, it follows from Lemma 4.1 that

$$u(x) = \mathbb{E}_x\left[h(X_{\tau_D})\exp\left(-\int_0^{\tau_D} u^{p-1}(X_s)ds\right)\right], \qquad x \in D$$

for some $h \in \mathscr{H}_+$. Clearly, $u(x) \le h(x)$ for $x \in D$. By Jensen's inequality

$$\begin{aligned}
\frac{u(x)}{h(x)} &\ge \frac{\mathbb{E}_x\left[h(X_{\tau_D})\exp(-\int_0^{\tau_D} h^{p-1}(X_s)ds)\right]}{h(x)} \\
&\ge \mathbb{E}_x^h\left[\exp\left(-\int_0^{\tau_D} h^{p-1}(X_s)ds\right)\right] \\
&\ge \exp\left(-\mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right]\right) \\
&\ge \exp(-C_1)
\end{aligned}$$

Hence $u(x) \ge h(x)\exp(-C_1)$ for any $x \in D$. This implies that $u \in \mathscr{S}_H^p$, and therefore $\mathscr{S}_H^p = \mathscr{S}_+^p$.

(ii) For any $h \in \mathscr{H}_+$, by (3.4),

$$h(x) \ge \int_D p^D(t,x,y)h(y)dy \ge c\,\varphi_1(x).$$

Hence for $p \geq 0$, by assumption (1.7),

$$\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s) ds \right] \quad = \quad \frac{G_D h^p(x)}{h(x)} \leq \frac{\|h\|_\infty^p}{c} \frac{G_D 1(x)}{\varphi_1(x)} \leq c_1 < \infty.$$

Define

$$\Lambda = \left\{ u \in C(\overline{D}) : \, e^{-c_1} h \leq u \leq h \text{ on } \overline{D} \right\}.$$

Observe that for any $q \in \mathbb{R}$,

$$\min\{e^{-c_1 q}, 1\} \, h^q \leq u^q \leq \max\{e^{-c_1 q}, 1\} \, h^q \quad \text{ in } D. \tag{4.10}$$

For $h \in \mathscr{H}_+$, define

$$\alpha(h) = \inf \left\{ p : \sup_{x \in D} \frac{G_D h^p(x)}{h(x)} < \infty \right\} \tag{4.11}$$

and define

$$p_0 = \inf_{h \in \mathscr{H}_+} \alpha(h). \tag{4.12}$$

It follows from Lemma 4.2 that $p_0 < 0$. We now show that $p_0 > -\infty$. By (4.11) and (4.12), it suffices to show that

$$\text{there exists } q \in \mathbb{R} \text{ such that } \sup_{x \in D} \frac{G_D h^{-q}(x)}{h(x)} = \infty. \tag{4.13}$$

By [7, Lemma 6.7], for every c_1, there is a constant $c_2 > 0$ such that

$$G_D(x,y) \geq c_2 \qquad \text{for } x, y \in D \text{ with } |x - y| \leq c_1 \min\{\delta_D(x), \delta_D(y)\}.$$

Fix $c_1 > 0$ and a corresponding $c_2 > 0$. Note that for a suitable constant $c_3 > 0$, which depends only on c_1, $|y - x| \leq c_3 \delta_D(x)$ implies $|x - y| \leq c_1 \min\{\delta_D(x), \delta_D(y)\}$. Hence

$$\begin{aligned} \frac{G_D h^{-q}(x)}{h(x)} &= \frac{\int_D G_D(x,y) h^{-q}(y) dy}{h(x)} \\ &\geq \frac{c_2 \int_{\{|x-y| < c \min\{\delta_D(x), \delta_D(y)\}} h^{-q}(y) dy}{h(x)} \\ &\geq \frac{c_2 \int_{B(x, c_3 \delta_D(x))} h^{-q}(y) dy}{h(x)}. \end{aligned}$$

By [3, Lemma 1.9, page 185, equation (1.22)], there exist constants $c_4 > 0$ and $\beta > 0$, such that $h(y) \leq c_4 \delta_D(y)^\beta$ for all $y \in B(x, c_3 \delta_D(x))$. Hence, for constants $c_5, c_6 > 0$,

$$\begin{aligned} \frac{G_D h^{-q}(x)}{h(x)} &\geq c_5 \frac{\int_{B(x, c_3 \delta_D(x))} \delta_D(y)^{-q\beta} dy}{\delta_D(x)^\beta} \\ &\geq c_6 \delta(x)^{-q\beta + d - \beta}. \end{aligned}$$

If q is chosen sufficiently large, the last expression above is unbounded over D. This proves (4.13) and thus $p_0 > -\infty$.

For every $p > p_0$, using (4.10) and a fixed point argument very similar to the one used in (i), we have $\mathscr{S}_H^p \neq \emptyset$ for $p > p_0$. Note that the only modifications in the fixed point argument of (i) are the following.

(a) We have $\min\{1, e^{c_1(p-1)}\}u^{p-1}T(u) \in B_{h,p}$ for $u \in \Lambda$ rather than $u^{p-1}T(u) \in B_{h,p}$ in the case of $p \geq 1$.

(b) $u^{p-1} \leq h^{p-1}$ for the case of $p \geq 1$ is now replaced by (4.10) with $q = p-1$.

(iii) Now we show that for every $p < p_0$, $\mathscr{S}_+^p = \emptyset$. Suppose $\mathscr{S}_+^p \neq \emptyset$. Then

$$u(x) = h(x) - \int_D G_D(x,y)u^p(y)\,dy$$

for some $h \in \mathscr{H}_+$. As $u \leq h$ and $p < p_0 \leq 0$,

$$\mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right] = \frac{G_D h^p(x)}{h(x)} \leq \frac{G_D u^p(x)}{h(x)} = \frac{h(x)-u(x)}{h(x)} = 1 - \frac{u(x)}{h(x)} \leq 1.$$

This contradicts the definition of p_0 in (4.12). Hence $\mathscr{S}_+^p = \emptyset$ for every $p < p_0$. □

5 $C^{1,1}$ domain case

In this section we give a proof of Theorem 1.6.

Proof of Theorem 1.6. (i) As any bounded $C^{1,1}$ domain satisfies the hypothesis of Theorem 1.5, the results follows directly from Theorem 1.5(1).

(ii). It is well known that for a bounded $C^{1,1}$ domain D, the Euclidean boundary ∂D is the same as the minimal Martin boundary for Δ in D. So for any $h \in \mathscr{H}_+$, there is a finite positive measure μ on ∂D such that

$$h(x) = \int_{\partial D} K_D(x,z)\,\mu(dz),$$

where $K_D(x,z)$ is the Martin kernel for Δ in D. It is a direct consequence of (2.1) and (2.4) that

$$K_D(x,z) \asymp \frac{\delta_D(x)}{|x-z|^n} \qquad \text{for } x \in D \text{ and } z \in \partial D. \tag{5.1}$$

Hence

$$h(x) \geq c\delta_D(x) \quad \text{for } x \in D. \tag{5.2}$$

Note that for each fixed $z \in \partial D$, $x \mapsto K_D(x,z)$ is a positive harmonic function in D.

We first assume that $n \geq 3$. It follows from Zhao [21] that

$$G_D(x,y) \leq c\min\left\{\delta_D(x)|x-y|^{1-n},\, \delta_D(x)\delta_D(y)|x-y|^{-n}\right\}. \tag{5.3}$$

If $-1 < p < 0$, then by (5.2) and (5.3)

$$\begin{aligned}
\sup_{x\in D}\mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right] &= \sup_{x\in D}\frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\leq c\sup_{x\in D}\left(\delta_D(x)^{-1}\int_D \frac{\delta_D(x)\delta_D(y)^{-p}}{|x-y|^{n-1-p}}\,\delta_D(y)^p\,dy\right) \\
&= c\sup_{x\in D}\int_D \frac{1}{|x-y|^{n-1-p}} < \infty.
\end{aligned}$$

If $0 \le p < 1$, since h is bounded, by (5.2) and (5.3) we have

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] &= \sup_{x\in D} \frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\le c \sup_{x\in D} \left(\delta_D(x)^{-1} \int_D \frac{\delta_D(x)}{|x-y|^{n-1}}\, dy \right) \\
&= c \sup_{x\in D} \int_D \frac{1}{|x-y|^{n-1}} < \infty.
\end{aligned}$$

Now we can imitate the arguments presented in the proof of Theorem 1.5(ii), to conclude that $\mathscr{S}_H^p \neq \emptyset$ when $-1 < p < 1$ and $n \ge 3$.

We now assume $n = 2$. If $-1 < p < 0$, then by (2.2) and (5.2)

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] &= \sup_{x\in D} \frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\le c \sup_{x\in D} \left(\delta_D(x)^{-1} \int_D \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \delta_D(y)^p\, dy \right).
\end{aligned}$$

Observe that $\log(1+ab) \le ab \le ab^{-p}$ for $a > 0$ and $0 < b \le 1$ and $\log(1+ab) \le (-1/p)b^{-p}b^{-p} \le (-1/p)ab^{-p}$ for $a \ge 1$ and $b > 0$. Thus

$$\log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \le c\, \frac{\delta_D(x)}{|x-y|} \frac{\delta(y)^{-p}}{|x-y|^{-p}} \tag{5.4}$$

when either $\delta_D(x) \ge |x-y|$ or $\delta_D(y) \le |x-y|$. It follows that

$$\begin{aligned}
&\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] \\
\le\ & c \sup_{x\in D} \left(\int_D \frac{1}{|x-y|^{1-p}} dy + \delta_D(x)^{-1} \int_{\{y\in D: \delta_D(x)<|x-y|<\delta_D(y)\}} \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \delta_D(y)^p\, dy \right) \\
\le\ & c \sup_{x\in D} \left(1 + \delta_D(x)^{-1} \int_{\{y\in D: \delta_D(x)<|x-y|<\delta_D(y)\}} \frac{\delta_D(x)^{-p}\delta_D(y)^{-p}}{|x-y|^{-2p}}\, \delta_D(y)^p\, dy \right) \\
\le\ & c \sup_{x\in D} \left(1 + \delta_D(x)^{-1-p} \int_{\{y\in D: |x-y|>\delta_D(x)\}} \frac{1}{|x-y|^{-2p}}\, dy \right) \\
\le\ & c + c \sup_{x\in D} \delta_D(x)^{1+p} < \infty.
\end{aligned}$$

Consider $0 \le p < 1$. Since h is bounded, by (2.2), (5.2) and as any $C^{1,1}$-domain satisfies the hypothesis of Theorem 1.5, we have

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] &= \sup_{x\in D} \frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\le c \sup_{x\in D} \left(\delta_D(x)^{-1} G_D 1(x) \right) \\
&= c \sup_{x\in D} \left(\delta_D(x)^{-1} \varphi(x) \right) < \infty.
\end{aligned}$$

The last inequality is due to the fact that $\varphi(\cdot) \asymp \delta_D(\cdot)$ in D, which is a consequence of (5.1) and the BHP (Theorem 1.3).

The arguments presented in the proof of Theorem 1.5(ii) can now be used to conclude that $\mathscr{S}_H^p \neq \emptyset$ when $-1 < p < 0$ and $n = 2$. We thus obtain (ii).

(iii). Suppose that there exists a mild solution u to (1.5) which is positive in D and vanishing on Γ. Then, by definition, there is a positive harmonic function h that vanishes on Γ such that $u = h - G_D u^p$. Hence

$$u(x) \le h(x) \ \text{ and } \ G_D u^p(x) \le h(x) \quad \text{for every } x \in D. \tag{5.5}$$

On the other hand there is a finite positive measure μ on ∂D such that $\mu(\Gamma) = 0$ and

$$h(x) = \int_{\partial D} K_D(x,z)\mu(dz) = \int_{\partial D \setminus \Gamma} K_D(x,z)\mu(dz), \quad x \in D.$$

Take $z_0 \in \Gamma$ and $r_0 > 0$ such that $B(z_0, 2r_0) \subset D_1$. Then by (5.1),

$$h(x) \asymp \delta_D(x) \quad \text{for } x \in D \cap B(z_0, r_0). \tag{5.6}$$

Since $p \le -1$ and $u(y) \le h(y)$, we have

$$u(y)^p \ge c^p\, \delta_D(y)^p \ge c^p\, \delta_D(y)^{-1} \quad \text{for } y \in D \cap B(z_0, r_0).$$

Now take a sequence of points $\{x_k\}$ in $D \cap B(z_0, r_0)$ that converges to z_0. Then for $n \ge 3$, by Fatou's lemma,

$$\begin{aligned}
\liminf_{k\to\infty} \frac{G_D u^p(x_k)}{h(x_k)} &\ge c \liminf_{k\to\infty} \frac{\int_{D\cap B(z_0,r_0)} G_D(x_k,y)\delta_D(y)^{-1}\,dy}{\delta_D(x_k)} \\
&\asymp c \liminf_{k\to\infty} \int_{D\cap B(z_0,r_0)} \delta_D(x_k)^{-1} \min\left\{ \frac{1}{|x_k - y|^{n-2}}, \frac{\delta_D(x_k)\delta_D(y)}{|x_k - y|^n} \right\} \delta_D(y)^{-1}\,dy \\
&\ge c \int_{D\cap B(z_0,r_0)} \liminf_{k\to\infty} \delta_D(x_k)^{-1} \min\left\{ \frac{1}{|x_k - y|^{n-2}}, \frac{\delta_D(x_k)\delta_D(y)}{|x_k - y|^n} \right\} \delta_D(y)^{-1}\,dy \\
&= c \int_{D\cap B(z_0,r_0)} |z_0 - y|^{-n}\,dy \\
&= \infty.
\end{aligned}$$

This contradicts inequality (5.5). Therefore $\mathscr{S}_+^p = \emptyset$ when $n \ge 3$.

Similarly, when $n = 2$, by Fatou's lemma,

$$\begin{aligned}
\liminf_{k\to\infty} \frac{G_D u^p(x_k)}{h(x_k)} &\ge c \liminf_{k\to\infty} \frac{\int_{D\cap B(z_0,r_0)} G_D(x_k,y)\delta_D(y)^{-1}\,dy}{\delta_D(x_k)} \\
&\asymp c \liminf_{k\to\infty} \int_{D\cap B(z_0,r_0)} \delta_D(x_k)^{-1} \log\left(1 + \frac{\delta_D(x_k)\delta_D(y)}{|x_k - y|^2}\right) \delta_D(y)^{-1}\,dy \\
&\ge c \int_{D\cap B(z_0,r_0)} \liminf_{k\to\infty} \delta_D(x_k)^{-1} \log\left(1 + \frac{\delta_D(x_k)\delta_D(y)}{|x_k - y|^2}\right) \delta_D(y)^{-1}\,dy \\
&= c \int_{D\cap B(z_0,r_0)} |z_0 - y|^{-2}\,dy \\
&= \infty.
\end{aligned}$$

This again contradicts inequality (5.5). Therefore $\mathscr{S}_+^p = \emptyset$ when $n = 2$. □

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