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Random walks on infinite self-similar graphs

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Abstract

We introduce a class of rooted infinite self-similar graphs containing the well known Fibonacci graph and graphs associated with Pisot numbers. We consider directed random walks on these graphs and study their entropy and their limit measures. We prove that every infinite self-similar graph has a random walk of full entropy and that the limit measures of this random walks are absolutely continuous.

Key words: random walk ; graph .

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1 Introduction

The Fibonacci graph was first described by Alexander and Zagier [1]. In this work an explicit formula for the entropy of the equal weighted Bernoulli measures on this graph is given. Sidorov and Vershik [14] used a new ergodic theoretic approach to get a similar result. Moreover Lallys work [10] allows us to calculate the entropy of other Bernoulli measures on the Fibonacci graph. The Fibonacci graph is self-similar in the sense that it is isomorphic to all subgraphs taking an arbitrary knot of the graph as the new root. Inspired by the Fibonacci graph we develop here the general concept of rooted infinite self-similar graphs. Especially we will define the growth rate of such graphs and introduce a ζ-function associated with a graph. Examples of self-similar graphs are the full **A**-ary and the Pascal graph. More exciting infinite-self similar graphs are constructed using Pisot numbers, i.e. algebraic integers with all its conjugates in the unit circle. In this view the Fibonacci graph is given by the golden mean.

In a recent work Krön [8] introduced a different class of self-similar graphs; these graphs are not rooted and do not have levels. The concept of self-similarity is quite different in this setting. For instance in our situation the exponential growth rate of a graph is defined by the number of knots at a given level, see Definition 2.2 and Proposition 3.1 and 3.2. In Krön's work the growth dimension is defined using the structure of the cells of the graph.

Having the general definition of rooted infinite self-similar graphs we describe random walks on these graphs. Borel probability measures on the space of infinite sequences induce directed random walks starting at the root of the graph. These random walks are directed from level to level of the graph. Moreover we are able to define induced limit measures of such random walks by projecting the measure on the graph equidistant to an interval. Recently Krön and Teufl [9] studied random walks on self-similar graphs that are not rooted. Such random do not have a distinguished direction and the analysis is again quit different.

Given a shift-invariant measure on the sequence space we are able to introduce the entropy of the random walk on the graph. The self-similarity of the graph is in fact essential to define this quantity. We prove two upper bounds on the entropy of a random walk on a self-similar graph. One is given by the usual metric entropy of the inducing measure, the other one is given by the growth rate of the graph. Beside this the entropy of a random walk shares some well known properties of metric entropy like affinity and upper-semi-continuity.

Two main results in classical entropy theory are the local entropy theorem of Shannon and the variational principle of metric entropy, see [3], [7] or [15]. We prove that the local entropy theorem remains true in our setting for Bernoulli measures on self-similar graphs (see theorem 7.1). We have no hope to prove this theorem for arbitrary random walks, since we rely on the subadditive ergodic theorem. On the other hand the variational principle for entropy is true for all random walks on self-similar graphs, induced by invariant measures. We construct an ergodic measure that induces a random walk of full entropy (see theorem 8.1). Here again the self-similarity of the graph is essential for our construction.

In the end of this work we are concerned with the limit measures of random walks on self-similar graphs. We will prove that if a limit measure is not absolutely continuous with respect to the Lebesgue measure then the corresponding random walk on the self similar graph does not have full entropy. Hence every infinite-self similar graph has a random walk with absolutely continuous limit measures (see theorem 8.1).

2 Infinite self-similar graphs

In this section we present a general construction of infinite self similar graphs and introduce the exponential growth rate and the ζ-function of such graphs.

Let $\mathbf{A}$ be a finite alphabet. Consider the set of all finite words builded from the alphabet,

$$\mathbf{A}^{(\mathbb{N})} := \bigcup_{n=0}^{\infty} \mathbf{A}^n,$$

where we use the convention that $\mathbf{A}^0$ contains the empty word. For $w = (w_1, \dots w_n) \in \mathbf{A}^{(\mathbb{N})}$ we denote the length n of the word w by $l(w)$.
Now we consider an equivalence relation $\simeq$ on $\mathbf{A}^{(\mathbb{N})}$ relating words of the same length,

$$\forall w, v \in \mathbf{A}^{(\mathbb{N})} \quad : \quad w \simeq v \Rightarrow l(w) = l(v)$$

and preserving the relation if two words are continuation by the same word,

$$\forall u, v, w \in \mathbf{A}^{(\mathbb{N})} \quad : \quad w \simeq v \Rightarrow wu \simeq vu.$$

We denote the equivalence class of a word w by $[w] := \{v | v \simeq w\}$ and set

$$\mathbf{K} := \mathbf{A}^{(\mathbb{N})}/_{\simeq} = \{[w] | w \in \mathbf{A}^{(\mathbb{N})}\}.$$

We refer to the elements of $\mathbf{K}$ as knots. We say that two knots $[w]$ and $[v]$ are connected if and only if

$$l(w) = l(v) + 1 \text{ and } \exists i \in \mathbf{A} \ : \ vi \in [w]$$

or

$$l(v) = l(w) + 1 \text{ and } \exists i \in \mathbf{A} \ : \ wi \in [v].$$

Note that by our assumption on $\simeq$ this definition is independent of the representatives w and v of the knots $[w]$ and $[v]$. Furthermore let $\mathbf{E}$ be the set of all edges,

$$\mathbf{E} := \{([u],[v]) \in \mathbf{K}^2 | [u],[v] \text{ are connected}\}.$$

Definition 2.1. *The rooted simplicial infinite graph* $(\mathbf{K}, \mathbf{E})$ *induced by the relation* $\simeq$ *is called self similar if*

$$\forall p, u, v, w \in \boldsymbol{A}^{(\mathbb{N})} \quad : \quad v \simeq u \wedge p \simeq w \Rightarrow vp \simeq uw$$

and

$$\forall p, u, v, w \in \boldsymbol{A}^{(\mathbb{N})} \quad : \quad v \simeq u \wedge vp \simeq uw \Rightarrow p \simeq w$$

In the following we always denote an infinite graph by $\mathbf{K}$ suppressing the set of edges $\mathbf{E}$ in our notation. For each knot $[v] \in \mathbf{K}$ we consider the subgraph

$$\mathbf{K}[v] = \{[uw] | u \in [v] \wedge w \in \mathbf{A}^{(\mathbb{N})}\} \subseteq \mathbf{K}.$$

We may characterize self-similar graphs using their subgraphs:

Proposition 2.1. *A rooted simplicial infinite graph* $\mathbf{K}$ *is self-similar if and only if* $\mathbf{K}$ *is isomorphic to the subgraph* $\mathbf{K}[v]$ *via* $\phi([w]) = [vw]$ *for all* $[v] \in \mathbf{K}$.

Proof. First note that the definition of the map ϕ is independent of the representatives w, v of the knots $[w], [v]$ if and only if

$$v_1 \simeq v_2 \wedge w_1 \simeq w_2 \Rightarrow v_1 w_1 \simeq v_2 w_2.$$

This is the first condition for self-similarity. Then note that the inverse map $\phi^{-1}([vw]) = [w]$ is well defined if and only if

$$v_1 \simeq v_2 \wedge v_1 w_1 \simeq v_2 w_2 \Rightarrow w_1 \simeq w_2.$$

This is the second condition for self-similarity. Furthermore we get by self similarity

$$w_2 i \in [w_1] \Leftrightarrow v w_2 i \in [v w_1]$$

Thus the maps ϕ and ϕ^{-1} preserves the relation of connectedness between two knots. $\square$

Let

$$\mathbf{K}_n = \{[w] | w \in \mathbf{A}^{(\mathbb{N})},\ l(w) = n\}$$

be the knots at the n-the level of graph.

Lemma 2.1. *For an infinite self-similar graph* $\mathbf{K}$ *we have*

$$Card(\mathbf{K}_{n+m}) \le Card(\mathbf{K}_n) Card(\mathbf{K}_m)$$

Proof. The map $\varphi : \mathbf{K}_n \times \mathbf{K}_m \longmapsto \mathbf{K}_{n+m}$ given by $\varphi([w],[v]) = [wv]$ is by self-similarity well defined and onto. Now our result is obvious. $\square$

By this lemma the sequence $\log Card(\mathbf{K}_n)$ is subadditive and we may define:

Definition 2.2. *The exponential growth rate of an infinite self similar graph* $\mathbf{K}$ *is given by*

$$H(\mathbf{K}) := \lim_{n \longmapsto \infty} \frac{\log Card(\mathbf{K}_n)}{n} = \inf_{n \ge 1} \frac{\log Card(\mathbf{K}_n)}{n}$$

We say that $\mathbf{K}$ *has exact growth rate if there are constants* $C, c > 0$ *such that*

$$c e^{nH(\mathbf{K})} \le Card(\mathbf{K}_n) \le C e^{nH(\mathbf{K})}$$

with $H(\mathbf{K}) \neq 0$.

The self-similarity of the graph is essential for the limit to exist. As well self-similarity is essential to define the entropy of a random walk on the graph, see section 5. So all examples of infinite graphs we will study below are self-similar.
At the end of this section we are now able to define a ζ-function associated with growth of $\mathbf{K}_n$:

Definition 2.3. *For an infinite self-similar graph* $\mathbf{K}$

$$\zeta_{\mathbf{K}}(z) = exp(\sum_{n=1}^{\infty} \frac{Card(\mathbf{K}_n)}{n} z^n)$$

is called the ζ*-function of the graph.*

The ζ-function is analytic in the complex disc $D = \{z \mid |z| < e^{-H(\mathbf{K})}\}$ and has singularities on the boundary. We will see in our examples that the function may have an meromorphic continuation to the whole complex plan. The ζ-function encodes information about the growth of an self similar graph into one nice analytic object.

3 Examples of self-similar graphs

3.1 Simple graphs

The simplest self similar graph is given by the relation $u \simeq w \Leftrightarrow l(u) = l(w)$. It has one knot at each level and we call it the trivial graph. Another simple self-similar graph is given by the relation $u \simeq w \Leftrightarrow u = w$ on $\mathbf{A}^{(\mathbb{N})}$. It may be identified with $\mathbf{A}^{(\mathbb{N})}$ itself. We call this the full $\mathbf{A}$-ary graph. Trivially we have $Card(\mathbf{K}_n) = Card(\mathbf{A})^n$ and thus

$$H(\mathbf{K}) = \log Card(\mathbf{A}) \quad \text{and} \quad \zeta_{\mathbf{K}}(z) = 1/(1 - Card(\mathbf{A})z).$$

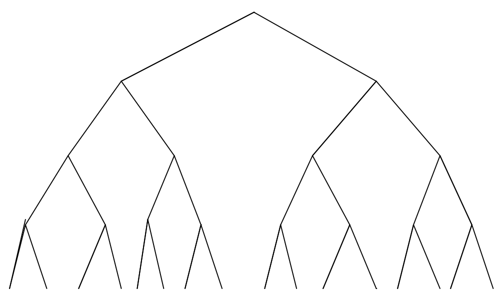

Figure 1: The full binary graph

All other self-similar tress have an strictly smaller exponential growth rate:

Proposition 3.1. *If $\mathbf{K}$ is a self-similar graph with alphabet $\mathbf{A}$, $H(\mathbf{K}) = \log Card(\mathbf{A})$ holds if and only if $\mathbf{K}$ is the full $\mathbf{A}$-ary graph.*

Proof. The if part is obvious. If $\mathbf{K}$ is not the full $\mathbf{A}$-ary graph then $Card(\mathbf{K}_n) < Card(\mathbf{A})^n$ for some n and by self-similarity of the graph

$$H(\mathbf{K}) \leq \frac{\log Card(\mathbf{K}_n)}{n} < \log Card(\mathbf{A}).$$

□

Now consider the self-similar graph given by the relation $v \simeq w \Leftrightarrow l(v) = l(w) = n \wedge \sum_{i=1}^n v_i = \sum_{i=1}^n w_i$ on $\{0,1\}^{(\mathbb{N})}$. In analogy with the Pascal triangle we call corresponding self-similar graph $\mathbf{K}$ the Pascal graph. Obviously $Card(\mathbf{K}_n) = n + 1$ and hence $H(\mathbf{K}) = 0$.

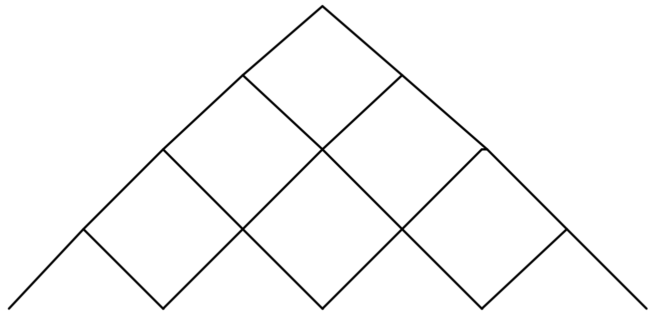

Figure 2: The Pascal graph

In the next subsection we introduce more exciting examples of self-similar graphs.

3.2 Pisot graphs

We describe here the infinite self-similar binary graphs we had in mind inventing the general construction in the last section. These graphs have a strong relationship to algebraic number theory.

Let $\mathbf{A} = \{0,1\}$ and let $\beta \in (0.5,1)$ be the reciprocal of a Pisot number. A Pisot number $\alpha \in (1,2)$ is an algebraic integer with all its Galois conjugates inside the unit circle, see [2]. Examples of $\beta \in (0.5,1)$ such that $\alpha = \beta^{-1}$ is a Pisot number are the solutions of the equations

$$x^n + x^{n-1} + \cdots + x - 1 = 0 \qquad n \geq 2.$$

Clearly, this solutions form a sequence $\beta_n \longmapsto 0.5$. The only example with minimal polynomial of degree 2 is the golden ratio $\beta_2 = (\sqrt{5}-1)/2$. Beside β_3 there are three other examples $\beta_3', \beta_3'', \beta_3'''$ with minimal polynomial of degree 3 given by the solutions of

$$x^3 + x^2 - 1 = 0, \qquad x^3 + x - 1 = 0, \qquad x^3 - x^2 + 2x - 1 = 0.$$

We define an equivalence relation on $\mathbf{A}^n$ by

$$(w_1, \ldots, w_n) \simeq (v_1, \ldots, v_n) :\Leftrightarrow \sum_{k=1}^{n} w_k \beta^k = \sum_{k=1}^{n} v_k \beta^k.$$

This induces an appropriate relation on $\mathbf{A}^{(\mathbb{N})}$ and an infinite graph $\mathbf{K}_\beta$. We call the graph $\mathbf{K}_\beta$ a Pisot graph. If β is the golden ratio the corresponding graph $\mathbf{K}_\beta$ is called the Fibonacci graph, see [14] .

Proposition 3.2. *A Pisot graph $\mathbf{K}_\beta$ is self-similar.*

Proof. We have

$$\sum_{k=1}^{n} w_k \beta^k = \sum_{k=1}^{n} v_k \beta^k \wedge \sum_{k=1}^{m} w_{k+n} \beta^k = \sum_{k=1}^{m} v_{k+n} \beta^k \Rightarrow \sum_{k=1}^{m+n} w_k \beta^k = \sum_{k=1}^{m+n} v_k \beta^k$$

given the first condition of self-similarity and

$$\sum_{k=1}^{n} w_k\beta^k = \sum_{k=1}^{n} v_k\beta^k \wedge \sum_{k=1}^{m+n} w_k\beta^k = \sum_{k=1}^{m+n} v_k\beta^k \Rightarrow \sum_{k=1}^{m} w_{k+n}\beta^k = \sum_{k=1}^{m} v_{k+n}\beta^k$$

given the second one. □

We like to present some figures of Pisot graphs up to the knots of level four.

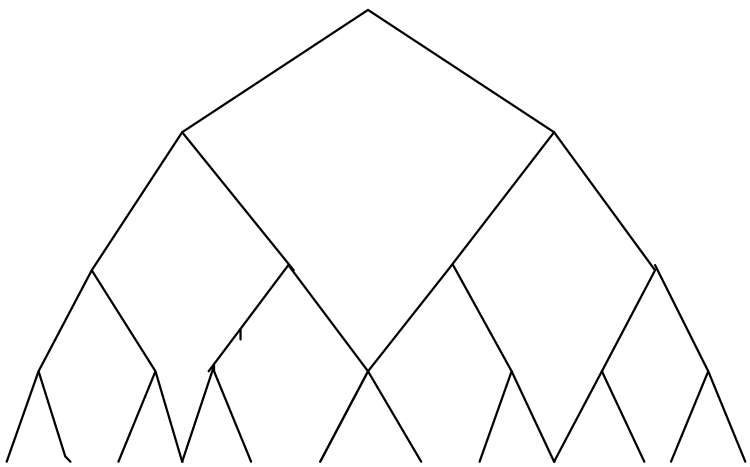

Figure 3: The Fibonacci graph $\mathbf{K}_{\beta_2}$

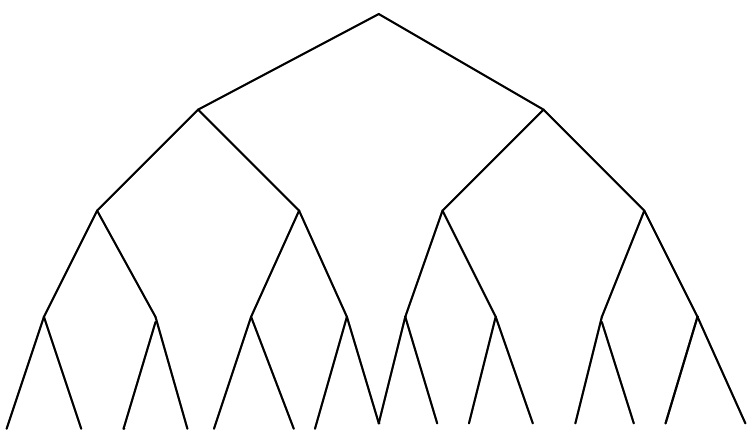

Figure 4: The graph $\mathbf{K}_{\beta_3}$

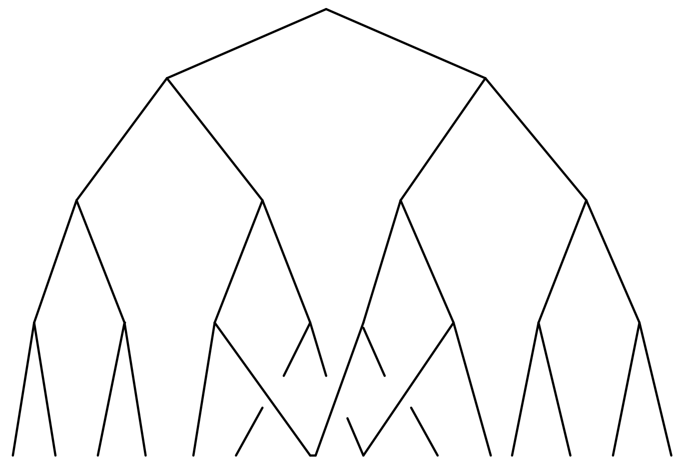

Figure 5: The graph $\mathbf{K}_{\beta_3''}$

We are able to determine the growth rate $H(K_\beta)$ of a Pisot graph using an algebraic result.

Proposition 3.3. *A Pisot graph has exact exponential growth rate with*

$$H(\mathbf{K}_\beta) = -\log\beta.$$

Proof. We know from the famous Garcia lemma [5] that if $\beta^{-1} \in (1,2)$ is a Pisot number, then

$$\sum_{k=1}^{n} w_k\beta^k \neq \sum_{k=1}^{n} v_k\beta^k \Rightarrow c\beta^n \leq |\sum_{k=1}^{n} w_k\beta^k - \sum_{k=1}^{n} v_k\beta^k| \leq C\beta^n$$

holds for constants $c, C > 0$. Hence

$$C^{-1}\frac{\beta-\beta^{n+1}}{1-\beta}\beta^{-n} < Card(\mathbf{K}_{\beta,n}) < c^{-1}\frac{\beta}{1-\beta}\beta^{-n}$$

proving the exact growth rate of $\mathbf{K}_\beta$. Taking logarithm dividing by n and taking the limit leads to the formula for $H(\mathbf{K}_\beta)$. □

An explicit formula for growths of the Fibonacci graph is easy to find. Just notice that $Card(\mathbf{K}_{\beta_2,n}) - Card(\mathbf{K}_{\beta_2,n-1})$ is the Fibonacci sequence. By the formula of Binet we hence have

$$Card(\mathbf{K}_{\beta_2}, n) = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}} - 1.$$

From this we get the following expression for the ζ-function of the Fibonacci graph,

$$\zeta_{\mathbf{K}_{\beta_2}}(z) = \frac{(1-z)(1-z\frac{1-\sqrt{5}}{2})^b}{(1-z\frac{1+\sqrt{5}}{2})^a}$$

with $a = (6+2\sqrt{5})/4\sqrt{5}$ and $b = (6-2\sqrt{5})/4\sqrt{5}$. To find formulas for the growth and the ζ-function of other Pisot graphs is more involved. In special cases you find results on the growth in [6].

4 Random Walks on self-similar graphs

We consider here random walks starting at the root, directed from level to level of the infinite graphs constructed in last sections. We use an ergodic theoretic approach to describe this random walks and their entropy. We first introduced some notations and results from classical ergodic theory, see [15], [3] or [7] for this material.

Consider the sequence space $\Sigma = \mathbf{A}^{\mathbb{N}}$. For sequences $w = (w_i)$ and $v = (v_i)$ in Σ we define a distance d by

$$d(w,v) = \sum_{i=1}^{\infty} \frac{|e(w_i) - e(v_i)|}{Card(\mathbf{A})^i}$$

where $e : \mathbf{A} \longmapsto \{1, \ldots, Card(\mathbf{A})\}$ is an enumeration. (Σ, d) is a perfect, totally disconnected, compact metric space and hence homeomorphic to the cantor set. Given a word $w = (w_1, \ldots, w_n) \in \mathbf{A}^n$ we define the cylinder set of the word by

$$\langle w \rangle := \{(v_i) \in \Sigma | (v_1, \ldots, v_n) = (w_1, \ldots w_n)\}.$$

Given a knot $[w] \in \mathbf{K}$ we define the cylinder set of the knot by

$$\langle [w] \rangle = \bigcup_{v \simeq w} \langle v \rangle = \{(v_i) \in \Sigma | (v_1, \ldots, v_n) \simeq (w_1, \ldots w_n)\}.$$

Both, the cylinder of a word and the cylinder of a knot, as a finite union of cylinder sets of words, are open and closed in Σ. Moreover

$$\mathbf{P}_n = \{\langle w \rangle | w \in \mathbf{A}^n\}$$

and

$$\Pi_n = \{\langle [w] \rangle | [w] \in \mathbf{K}_n\}$$

form partitions of the sequence space Σ, where the partition $\mathbf{P}_n$ is finer than partition Π_n.

Now let $M = M(\Sigma)$ be the space of all Borel probability measures on Σ. With the weak* topology M becomes a compact, convex and metrizable space. A measure $\mu \in M$ induces a directed random walk on the graph $\mathbf{K}$. The directed random walk moves at time n from level $n-1$ to level n. Thus the probability to reach a knot $[w] \in \mathbf{K}$ after $l(w)$ steps is given by

$$\mu([w]) := \mu(\langle [w] \rangle)$$

and the probability to pass a sequence of knots with increasing level $[w_1], \ldots, [w_k] \in \mathbf{K}$ is

$$\mu([w_1], \ldots, [w_k]) = \mu(\langle [w_1] \rangle \cap \cdots \cap \langle [w_k] \rangle).$$

Accordingly the probability to move from $[w_1] \in \mathbf{K}$ to $[w_2] \in \mathbf{K}$ with $l(w_1) = l(w_2) + 1$, is given by

$$\mu([w_2] | [w_1]) = \frac{\mu(\langle [w_1] \rangle \cap \langle [w_2] \rangle)}{\mu(\langle [w_1] \rangle)}$$

By this definitions a measure $\mu \in M$ completely describes a random walk on the class of graphs we are interested in.

Definition 4.1. *Let* $\mathbf{K}$ *be a rooted infinite self similar graph. We call a measure* $\mu \in M$ *together with the definition* $\mu([w]) := \mu(\langle [w] \rangle)$ *for* $[w] \in \mathbf{K}$ *a directed random walk on* $\mathbf{K}$ *starting from the root.*

Now especially consider a Bernoulli measure b on Σ which is characterized by the product property,

$$\forall v, w \in \mathbf{A}^{(\mathbb{N})} \quad b(\langle vw \rangle) = b(\langle v \rangle) b(\langle w \rangle).$$

By the self-similarity of a graph we get:

Proposition 4.1. *If* $b \in M$ *is a Bernoulli measure we have*

$$\forall w, v \in \mathbf{K} \quad b([v])b([w]) \leq b([vw])$$

for the induced random walk on a self-similar graph $\mathbf{K}$*.*

Proof. We have

$$b([v])b([w]) = b(\bigcup_{s \simeq v} \langle s \rangle) b(\bigcup_{t \simeq w} \langle t \rangle) = \sum_{s \simeq v} b(\langle s \rangle) \sum_{t \simeq w} b(\langle t \rangle)$$
$$\leq \sum_{st \simeq vw} b(\langle s \rangle) b(\langle t \rangle) = \sum_{st \simeq vw} b(\langle st \rangle) = b(\bigcup_{st \simeq vw} \langle st \rangle) = b([vw]).$$

□

At the end of this section we introduce limit measures of a random walk on a self similar graph by projecting the measure on $\mathbf{K}$ onto the interval $[0,1]$. To this end first consider the lexicographical order $\prec$ on $\mathbf{A}^{(\mathbb{N})}$. This introduces a linear order on $\mathbf{K}$ by

$$[v] \prec [w] :\Leftrightarrow \min\{\bar{v} | \bar{v} \in [v]\} \prec \min\{\bar{w} | \bar{w} \in [w]\}.$$

It is easy to see that this order is self-similar in the following sense,

$$\forall u \in \mathbf{K} : [v] \prec [w] \Leftrightarrow: [uv] \prec [uw].$$

Now let $\sharp_n$ by the enumeration of $\mathbf{K}_n$ with respect to $\prec$. With the help of the enumeration we define $\pi_n : \mathbf{K}_n \longmapsto [0,1]$ by

$$\pi_n([v]) := \frac{\sharp_n([v])}{Card(\mathbf{K}_n)}.$$

$\mu \in M$ induces a sequence of discrete equidistant probability measures μ_n on the interval $[0,1]$ by

$$\mu_n := \pi_n(\mu) = \mu \circ \pi_n^{-1}.$$

Now we may define:

Definition 4.2. *Let* $\mathbf{K}$ *be a self similar graph. For* $\mu \in M$ *we call a weak** *accumulation point* $\mu_{\mathbf{K}}$ *of the sequence of equidistant probability measures* $\mu_n := \mu \circ \pi_n^{-1}$ *on* $[0,1]$ *a limit measure of the induced random walk on* $\mathbf{K}$*.*

5 The entropy of a random walk on a self-similar graph

We define the shift map $\sigma : \Sigma \longmapsto \Sigma$ by $\sigma((w_i)) = (w_{i+1})$ and consider the set of shift invariant measures $\mathbf{M} = \mathbf{M}(\Sigma) := \{\mu \in M | \sigma(\mu) = \mu\}$. This is a compact, convex and non-empty subspace of M containing the Bernoulli measures. Given a partition $\mathbf{P}$ of Σ and $\mu \in M$ the entropy of the partition is defined as

$$H(\mu, \mathbf{P}) = -\sum_{P \in \mathbf{P}} \mu(P) \log \mu(P).$$

Given a σ-invariant measure the metric entropy of the measure with respect to the map σ is given by

$$h(\mu) = \lim_{n \longmapsto \infty} \frac{1}{n} H(\mu, \mathbf{P}_n)$$

where $\mathbf{P}_n$ is the partition into cylinder sets of words introduced in last section, see again [3], [15] or [7].

We want to define here the entropy of a measure $\mu \in \mathbf{M}$ on an infinite self-similar graph $\mathbf{K}$. In order to do so we need a few notation and one lemma. Let $\mathbf{P}_1, \mathbf{P}_2$ be two partitions. We define their join as

$$\mathbf{P}_1 \vee \mathbf{P}_2 := \{P_1 \cap P_2 | P_1 \in \mathbf{P}_1, P_2 \in \mathbf{P}_2\}.$$

We write $\mathbf{P}_1 \preceq \mathbf{P}_2$ if $\mathbf{P}_2$ is finer then $\mathbf{P}_1$. With this notations we have:

Lemma 5.1.

$$\Pi_{n+m} \preceq \Pi_n \vee \sigma^{-n}(\Pi_m)$$

and

$$H(\mu, \Pi_{n+m}) \leq H(\mu, \Pi_n) + H(\mu, \Pi_m)$$

where Π_n *is the partition given by the knots* $\mathbf{K}$.

Proof. We show that the first inequality follows from the self-similarity of the graph. Let $P \in \Pi_n \vee \sigma^{-n}(\Pi_m)$. Then there exist $[v] \in \mathbf{K}_n$ and $[w] \in \mathbf{K}_m$ such that

$$\begin{aligned} P &= \langle [v] \rangle \cap \sigma^{-n}(\langle [w] \rangle) \\ &= \{(s_k) \in \Sigma | (s_1, \dots, s_n) \simeq (v_1, \dots, v_n) \wedge (s_{n+1}, \dots, s_{n+m}) \simeq (w_1, \dots, w_n)\} \\ &\subseteq \{(s_k) \in \Sigma | (s_1, \dots, s_{n+m}) \simeq (v_1, \dots, v_n, w_1, \dots, w_n)\} \\ &= \langle [vw] \rangle \in \Pi_{n+m}. \end{aligned}$$

From this by well know properties of partition entropy H, see [15], and the invariance of μ we get,

$$\begin{aligned} H(\mu, \Pi_{n+m}) &\leq H(\mu, \Pi_n \vee \sigma^{-n}(\Pi_m)) \leq H(\mu, \Pi_n) + H(\mu, \sigma^{-n}(\Pi_m)) \\ &= H(\mu, \Pi_n) + H(\mu, \Pi_m). \end{aligned}$$

□

From this lemma we know that $H(\mu, \Pi_n)$ is a subadditive sequence and we may define:

Definition 5.1. *The entropy of a random walk on an infinite self-similar graph* $\mathbf{K}$ *induced by an invariant measure* $\mu \in \mathbf{M}$ *is given by*

$$h(\mu, \mathbf{K}) := \lim_{n \longmapsto \infty} \frac{H(\mu, \Pi_n)}{n} = \inf_{n \geq 1} \frac{H(\mu, \Pi_n)}{n}.$$

In the next section we study basic properties of this entropy.

6 Basic properties of the entropy of a random walk on a self-similar graph

There are two obvious upper bounds on the entropy $h(\mu, \mathbf{K})$, one given by the measure and one given by the graph. The first one is:

Proposition 6.1. *For an invariant measure* $\mu \in \mathbf{M}$ *on an infinite self-similar graph* $\mathbf{K}$ *we have*

$$h(\mu, \mathbf{K}) \leq h(\mu).$$

Proof. We have $\Pi_n \preceq \mathbf{P}_n$ implying $H(\mu, \Pi_n) \leq H(\mu, \mathbf{P}_n)$. Dividing by n and taking the limit leads to the inequality. □

The second one is:

Proposition 6.2. *For an invariant measure* $\mu \in \mathbf{M}$ *on an infinite self-similar graph* $\mathbf{K}$ *we have*

$$h(\mu, \mathbf{K}) \leq H(\mathbf{K}).$$

Proof. We have $H(\mu, \Pi_n) \leq \log Card(\Pi_n) = \log Card(\mathbf{K}_n)$. Dividing by n and taking the limit leads to the inequality. □

Of course the interesting question in the entropy theory for self-similar graphs is under what assumption on the graph and on the measure the inequality in the last proposition is an equality. We will discuss the question below. We remark that proposition 6.2 is an analogue of the bound $h(\mu) \leq \log Card(\mathbf{A})$ on the classical metric entropy of a measure $\mu \in \mathbf{M}$. Equality $h(\mu) = \log Card(\mathbf{A})$ holds if and only if μ is the equal weighted Bernoulli measure b, see [3], [7] or [15]. By this fact we conclude:

Proposition 6.3. $h(\mu, \mathbf{K}) = h(\mu)$ *holds for all* $\mu \in \mathbf{M}$ *if and only if* $\mathbf{K}$ *is the full* $\mathbf{A}$*-nary graph.*

Proof. The if part is obvious since the full $\mathbf{A}$-ary graph is described by the space of infinite sequences from $\mathbf{A}$. Now assume $h(b, K) = h(b)$ for the equal weighted Bernoulli measure $b \in \mathbf{M}$. This implies $h(b, K) = \log Card(\mathbf{A})$, thus by Proposition 6.2 $H(K) = \log Card(\mathbf{A})$ and Proposition 3.1 implies that $\mathbf{K}$ is the full $\mathbf{A}$-ary graph. □

Now we study the entropy map $h_{\mathbf{K}} : \mathbf{M} \longmapsto [0, H(K)]$ given by $h_{\mathbf{K}}(\mu) := h(\mu, \mathbf{K})$.

Proposition 6.4. *The entropy map* $h_{\mathbf{K}}$ *associated with a self-similar graph* $\mathbf{K}$ *is affine and upper-semicontinous on* $\mathbf{M}$.

Proof. Let $\mu_1, \mu_2 \in \mathbf{M}$ and set $\mu = p\mu_1 + (1-p)\mu_2$ with $p \in (0,1)$. For all partitions Π_k of Σ the inequality

$$0 \leq -pH_{\mu_1}(\Pi_k) - (1-p)H_{\mu_2}(\Pi_k) + H_\mu(\Pi_k) \leq \log 2$$

holds, see [3]. Thus by the definition of our entropy we have

$$h(\mu, \mathbf{K}) = ph(\mu_1, \mathbf{K}) + (1-p)h(\mu_2, \mathbf{K})$$

meaning affinity of the entropy map.

Now we prove that the entropy map is upper-semi-continuous. Let $\mu, \mu_n \in \mathbf{M}$ with $\mu_n \to \mu$ and fix $\epsilon > 0$. From the definition of the entropy on $\mathbf{K}$ we know that there exists a $k \in \mathbb{N}$ such that,

$$h(\mu, \mathbf{K}) \geq \frac{H(\mu, \Pi_k)}{k} - \frac{\epsilon}{2}.$$

For the elements P of the partition Π_k we have

$$\lim_{n \longrightarrow \infty} \mu_n(P) = \mu(P)$$

since these sets are open and closed. Hence there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{1}{k}|H(\mu, \Pi_k) - H(\mu_n, \Pi_k)| \leq \frac{\epsilon}{2}.$$

Using both inequalities we get,

$$h(\mu, \mathbf{K}) \geq \frac{1}{k} H(\mu_n, \Pi_k) - \epsilon \geq h(\mu_n, \mathbf{K}) - \epsilon.$$

This proves upper-semi-continuity of $h_{\mathbf{K}}$. □

It is well known that the usual metric entropy $h(\mu)$ has the same property on $\mathbf{M}$, see [3], [7] or [15].

7 Local entropy of random walks on self-similar graphs

We now introduce the local entropy of random walks on a self similar graph.

Definition 7.1. *For an infinite self-similar graph* $\mathbf{K}$ *and a measure* $\mu \in \mathbf{M}$ *we call*

$$h(\mu, \mathbf{K})(w) := \limsup_{n \longmapsto \infty} \frac{-\log \mu([w_1, \dots, w_n])}{n}$$

the local entropy of μ *on* $\mathbf{K}$ *in the sequence* $w = (w_1, w_2, \dots) \in \Sigma$.

We have to use the limit superior in this definition since the limit does not have to exist in general. Now we prove a version of the local entropy theorem of Shannon for Bernoulli measures on self-similar graphs.

Theorem 7.1. *If $\mathbf{K}$ is an infinite self-similar graph and $b \in \mathbf{M}$ is a Bernoulli measure, we have*

$$h(b,\mathbf{K})(w) = \lim_{n \longmapsto \infty} \frac{-\log b([w_1,\ldots,w_n])}{n} = h(b,\mathbf{K})$$

for b-almost all $w \in \Sigma$.

Proof. For $w \in \Sigma$ let

$$f_n(w) = -\log b([(w_1,\ldots,w_n)]).$$

By proposition 4.1 we have

$$f_{n+k}(w) \leq f_n(w) + f_k(\sigma^n(w))$$

By the Subadditive Ergodic theorem of Kingman, see [7], the limit

$$\bar{f}(w) = \lim_{n \longmapsto \infty} \frac{1}{n} f_n(w)$$

exists for b-almost all $w \in \Sigma$ and

$$\frac{1}{n}\int f_n db \to \bar{f}.$$

On the other hand by an obvious calculation

$$\int f_n db = -\sum_{[w]\in\mathbf{K}_n} \int \log b([w]) db = \sum_{P\in\Pi_n} b(P)\log b(P) = H(b,\Pi_n)$$

leading to the desired result. □

Proposition 4.1 and therefore the product property of Bernoulli measures is essential for our proof of Shannons theorem in the context of random walks on self-similar graphs.

8 A random walk with full entropy

In this section we are interested in an analogon to the variational principle of metric entropy in classical ergodic theory for random walks on infinite self-similar graphs, see [15], [3] or [7]. Let us first give a definition:

Definition 8.1. *We say that a random walk on an infinite self-similar graphs $\mathbf{K}$ induced by a measure $\mu \in \mathbf{M}$ has full entropy, if*

$$h(\mu,\mathbf{K}) = H(\mathbf{K}).$$

We prove:

Theorem 8.1. *Every infinite self-similar graph has a random walk with full entropy. Moreover the measure inducing this random walk may be chosen ergodic.*

Proof. We first construct a shift invariant measure μ with $h(\mu, \mathbf{K}) = H(\mathbf{K})$ and afterwards prove the existence of an ergodic one.

Let $\sharp(n)$ denote the number of elements of the partition Π_n. Now choose measures $\mu_n \in M$ such that

$$\mu_n(P) = 1/\sharp(n) \qquad \forall P \in \Pi_n$$

and let μ be a weak* accumulation point of the sequence

$$\bar{\mu}_n = \frac{1}{n}\sum_{i=0}^{n-1} \mu_n \circ \sigma^{-i}.$$

By this construction we immediately have that μ is invariant under σ.

Note that $\mathbf{P}_1 \preceq \mathbf{P}_2$ implies $\sigma^{-k}(\mathbf{P}_1) \preceq \sigma^{-k}(\mathbf{P}_2)$ and $\sigma^{-k}(\mathbf{P}_1 \vee \mathbf{P}_2) \preceq \sigma^{-k}(\mathbf{P}_1) \vee \sigma^{-k}(\mathbf{P}_2)$ where $\vee$ denotes the join as usual. From Lemma 5.1 we know $\Pi_{n+m} \preceq \Pi_n \vee \sigma^{-n}(\Pi_m)$. Combining this facts we get by induction

$$\Pi_{aq} \preceq \bigvee_{i=0}^{a-1} \sigma^{-iq}(\Pi_q).$$

Let $\lfloor x \rfloor$ be the integer part of x. Given n and q and k with $0 < q < n$ and $0 \le k < q$ we set $a(k) = \lfloor (n-k)/q \rfloor$ and write $n-k$ in the form $a(k)q + r$ with $0 \le r < q$. With this stipulation we get

$$\begin{aligned}\Pi_n \preceq \sigma^{-k}(\Pi_{n-k}) \vee \Pi_k &\preceq \sigma^{-k}(\Pi_{a(k)q}) \vee \sigma^{-(a(k)q+k)}(\Pi_r) \vee \Pi_k \\ &\preceq \bigvee_{i=0}^{a(k)-1} \sigma^{-iq+r}(\Pi_q) \vee \sigma^{-(a(k)q+k)}(\Pi_r) \vee \Pi_k\end{aligned}$$

and hence

$$\begin{aligned}H(\mu_n, \Pi_n) &\le \sum_{i=0}^{a(k)-1} H(\mu_n, \sigma^{-iq+k}(\Pi_q)) + H(\mu_n, \sigma^{-(a(k)q+k)}(\Pi_r)) + H(\mu_n, \Pi_k) \\ &\le \sum_{i=0}^{a(k)-1} H(\mu_n, \sigma^{-iq+k}(\Pi_q)) + 2q \log 2.\end{aligned}$$

The last inequality follows from the fact, that the partitions Σ_q and Σ_r have less than 2^q elements. Now summing over k gives

$$qH(\mu_n, \Pi_n) \le \sum_{k=0}^{q-1}\sum_{i=0}^{a(k)-1} H(\mu_n, \sigma^{-iq+k}(\Pi_q)) + 2q^2 \log 2 \le nH(\bar{\mu}_n, \Pi_q) + 2q^2 \log 2.$$

This implies

$$\frac{H(\mu_n, \Pi_n)}{n} \le \frac{H(\bar{\mu}_n, \Pi_q)}{q} + \frac{2q}{n}\log 2.$$

By Definition of the measures μ_n we have $H_{\mu_n}(\Pi_n) = \log \sharp(n)$, thus

$$\frac{\log \sharp(n)}{n} \le \frac{H(\bar{\mu}_n, \Pi_q)}{q} + \frac{2q}{n}\log 2.$$

By the Definition of μ as an accumulation point of $\bar{\mu}_n$ and Definition 2.2 we get

$$H(\mathbf{K}) \leq H(\mu, \Pi_q)/q.$$

and by Definition 5.1 and proposition 6.1 we finish with $h(\mu, \mathbf{K}) = H(\mathbf{K})$.

We have shown up to this point that the set

$$\bar{M} := \{\mu \in M | h(\mu, \mathbf{K}) = H(\mathbf{K})\}$$

of invariant Borel measures on Σ is not empty. We know from Proposition 6.4 that $h_{\mathbf{K}}$ is upper semi-continuous and affine, which implies that $\bar{M}$ is compact and convex with respect to the weak* topology. By Krein-Milman theorem, see [3] there exists an extremal point μ of $\bar{M}$. We show that μ is extremal in the space M of all invariant Borel probability measure and hence σ ergodic.
If this is not the case we have $\mu = p\mu_1 + (1-p)\mu_2$ for two distinct σ invariant measures μ_1 and μ_2 and $p \in (0,1)$. Since μ is extremal in $\bar{M}$ we have that μ_1 or μ_2 is not in $\bar{M}$. Hence $h(\mu_1, \mathbf{K}) < H(\mathbf{K})$ or $h(\mu_2, \mathbf{K}) < H(\mathbf{K})$. This implies $h(\mu, \mathbf{K}) < H(\mathbf{K})$ a contradiction to $\mu \in \bar{M}$. □

We remark that in contrast to ergodic theory on the space of infinite sequences the measure of full entropy is in general not Bernoulli. We think that the measures given by our theorem may be an interesting object for further studies. Especially we ask the question if the measure that induces a directed random walk of full entropy on a infinite self-similar graph is unique.

9 Limite measures of random walks on self-similar graphs

We begin here to study limit measures of random walks on self-similar graphs, see Defintion 4.2. We are interested in the relation between the limit measures and the entropy of the random walk on a self-similar graph.

Theorem 9.1. *Let $\mathbf{K}$ be a self similar graph with exact growth rate. If a limit measure of a random walk on $\mathbf{K}$ is not absolutely continuous with respect to the Lebesgue measure then this random walk does not have full entropy.*

Proof. Let $\mu \in \mathbf{M}$ and let $\mu_{\mathbf{K}}$ be a corresponding limit measure of a random walk on a self similar graph $\mathbf{K}$. If $\mu_{\mathbf{K}}$ is not absolutely continues with respect to the Lebesgue measure ℓ, there is a constant $D > 0$ such that for all $\epsilon > 0$ there is a finite union $U \subseteq [0,1]$ of open disjoint intervals with

$$\ell(U) < \epsilon \qquad \text{and} \qquad \mu_{\mathbf{K}}(U) > D.$$

Let

$$\bar{U} = \{\pi_n([u]) | \pi_n([u]) \in U \text{ and } [u] \in \mathbf{K}_n\}$$

For all open intervals and hence for the finite union U of such intervals we have

$$Card(\bar{U})/Card(K_n) \leq \ell(U)$$

which implies

$$Card(\bar{U}) \leq \epsilon Card(K_n)$$

With the help of the sequence of measures $\mu_n = \pi_n(\mu)$ we may estimate

$$H(\mu, \Pi_n) = -\sum_{[u]\in \mathbf{K}_n} \mu([u]) \log \mu([u]) = -\sum_{[u]\in \mathbf{K}_n} \mu_n(\pi_n([u])) \log \mu_n(\pi_n([u]))$$

$$= -\sum_{\pi_n([u])\in \bar{U}} \mu_n(\pi_n([u])) \log \mu_n(\pi_n([u])) - \sum_{\pi_n([u])\notin \bar{U}} \mu_n(\pi_n([u])) \log \mu_n(\pi_n([u]))$$

$$\leq \mu_n(\bar{U}) \log \frac{Card(\bar{U})}{\mu_n(\bar{U})} + (1-\mu_n(\bar{U})) \log \frac{Card(\mathbf{K}_n) - Card(\bar{U})}{1-\mu_n(\bar{U})}$$

$$\leq \mu_n(\bar{U}) \log Card(\bar{U}) + (1-\mu_n)(\bar{U}) \log Card(\mathbf{K}_n) + \log 2$$

$$\leq \mu_n(\bar{U}) \log(\epsilon Card(\mathbf{K}_n)) + (1-\mu_n)(\bar{U}) \log Card(\mathbf{K}_n) + \log 2$$

$$= \log Card(\mathbf{K}_n) + \mu_n(\bar{U}) \log(\epsilon) + \log 2$$

Since $\mu_{\mathbf{K}}$ is an weak* accumulation point of the sequence μ_n there is a subsequence $n_k \longmapsto \infty$ such that for all $k > k_0(\epsilon)$ we have $\mu_{n_k}(\bar{U}) > D$. From our estimate from above we get

$$H(\mu, \Pi_{n_k}) \leq \log Card(\mathbf{K}_{n_k}) + D \log(\epsilon) + \log 2.$$

Using the exact exponential growth rate of $\mathbf{K}$ this implies

$$H(\mu, \Pi_{n_k}) \leq n_k H(\mathbf{K}) + \log(C) + D \log(\epsilon) + \log 2.$$

If $\epsilon > 0$ is small enough and $k > k_0(\epsilon)$ we thus get

$$\frac{H(\mu, \Pi_{n_k})}{n_k} < H(\mathbf{K})$$

and hence $h(\mu, \mathbf{K}) < H(\mathbf{K})$ since $H(\mu, \Pi_{n_k})$ is a subadditive sequence. $\square$

Theorem 8.1 together with Theorem 9.1 have the following striking corollary:

Corollary 9.1. *Every self-similar graph with exact growth rate has a random walk with absolutely continuous limit measures.*

We close our work with a question. What could be said about the density of an absolutely continuous limit measures of random walk on a self-similar graph? Is the density in L^2, could we give an explicit description of the density, lets say for the Fibonacci graph?

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