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A conservative evolution of the Brownian excursion

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Abstract

We consider the problem of conditioning the Brownian excursion to have a fixed time average over the interval $[0,1]$ and we study an associated stochastic partial differential equation with reflection at 0 and with the constraint of conservation of the space average. The equation is driven by the derivative in space of a space-time white noise and contains a double Laplacian in the drift. Due to the lack of the maximum principle for the double Laplacian, the standard techniques based on the penalization method do not yield existence of a solution .

Key words: Brownian meander; Brownian excursion; singular conditioning; Stochastic partial differential equations with reflection.

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1 Introduction

The aim of this paper is to construct a stochastic evolution, whose invariant measure is the law of the Brownian excursion $(e_\theta, \theta \in [0,1])$ conditioned to have a fixed average $\int_0^1 e_\theta \, d\theta = c > 0$ over the interval $[0,1]$.

Since the distribution of the random variable $\int_0^1 e_\theta \, d\theta$ is non-atomic, and the Brownian excursion is not a Gaussian process, it is already not obvious that such conditioning is well defined. The first part of the paper will be dedicated to this problem: we shall write down the density of the random variable $\int_0^1 e_\theta \, d\theta$ and a regular conditional distribution of the law of $(e_\theta, \theta \in [0,1])$, given $\int_0^1 e_\theta \, d\theta$. The same will be done for the Brownian meander $(m_\theta, \theta \in [0,1])$.

After this is done, we shall turn to the problem of finding a natural stochastic dynamics associated with the conditioned laws thus obtained. We recall that a stochastic dynamics whose invariant measure is the law of the Brownian excursion has been studied in [8] and [10], where a stochastic partial differential equation with reflection and driven by space-time white noise is proven to be well posed and associated with a Dirichlet form with reference measure given by the law of the Brownian excursion.

In the present case, we shall see that a natural dynamics with the desired properties solves a fourth order stochastic partial differential equation with reflection and driven by the derivative in space of a space-time white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial^2}{\partial \theta^2}\left(\dfrac{\partial^2 u}{\partial \theta^2} + \eta\right) + \sqrt{2}\,\dfrac{\partial}{\partial \theta}\dot{W}, \\[2ex] u(t,0) = u(t,1) = \dfrac{\partial^3 u}{\partial \theta^3}(t,0) = \dfrac{\partial^3 u}{\partial \theta^3}(t,1) = 0 \\[2ex] u(0,\theta) = x(\theta) \end{cases} \tag{1.1}$$

where $\dot{W}$ is a space-time white noise on $[0,+\infty) \times [0,1]$, $x : [0,1] \mapsto \mathbb{R}_+$ is continuous and $c := \int_0^1 x(\theta)\, d\theta > 0$, u is a continuous function of $(t,\theta) \in [0,+\infty) \times [0,1]$, η is a locally finite positive measure on $(0,+\infty) \times [0,1]$, subject to the constraint:

$$u \geq 0, \qquad \int_{(0,+\infty)\times[0,1]} u \, d\eta \, = \, 0. \tag{1.2}$$

This kind of equations arises as scaling limit of fluctuations of conservative interface models on a wall, as shown in [12], where however different boundary conditions are considered.

Indeed, notice that the boundary conditions in (1.1) are mixed, i.e. Dirichlet for u and Neumann for $\frac{\partial^2 u}{\partial \theta^2}$. In [6] and [12] a similar equation, with Neumann boundary conditions for u and $\frac{\partial^2 u}{\partial \theta^2}$, has been studied, together with the scaling limit of interface models mentioned above. In that case it was possible to prove pathwise uniqueness and existence of strong solutions for the SPDE. In the case of (1.1) we can only prove existence of weak solutions, since we have failed to obtain a uniqueness result: see subsection 2.6 below.

The dynamics is anyway uniquely determined by a natural infinite-dimensional Dirichlet form on the path space of the Brownian excursion, to which it is associated. See Theorem 2.3 below.

We also notice that the Brownian meander $(m_\theta, \theta \in [0,1])$ conditioned to have a fixed average and the density of $\int_0^1 m_\theta \, d\theta$ appear in an infinite-dimensional integration by parts formula in [6, Corollary 6.2].

2 The main results

In this section we want to present the setting and the main results of this paper. We denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product in $L^2(0,1)$:

$$\langle h, k \rangle \;:=\; \int_0^1 h_\theta \, k_\theta \, d\theta$$

and by $\|\cdot\|$ the associated norm. In particular we often use the notation $\langle h, 1\rangle = \int_0^1 h_\theta \, d\theta$.

2.1 Conditioning the Brownian excursion to have a fixed time average

Let $(e_t, t \in [0,1])$ be the normalized Brownian excursion, see [9], and $(\beta_t, t \in [0,1])$ a Brownian bridge between 0 and 0. Let $\{m, \hat{m}, b\}$ be a triple of processes such that:

1. m and $\hat{m}$ are independent copies of a Brownian meander on $[0,1]$
2. conditionally on $\{m, \hat{m}\}$, b is a Brownian bridge on $[1/3, 2/3]$ from $\frac{1}{\sqrt{3}} m_1$ to $\frac{1}{\sqrt{3}} \hat{m}_1$

We introduce the continuous processes:

$$v_t \;:=\; \begin{cases} \dfrac{1}{\sqrt{3}} m_{3t}, & t \in [0, 1/3] \\[2ex] b_t, & t \in [1/3, 2/3], \\[2ex] \dfrac{1}{\sqrt{3}} \hat{m}_{1-3t}, & t \in [2/3, 1], \end{cases} \tag{2.1}$$

$$V_t^c \;:=\; \begin{cases} v_t, & t \in [0,1/3] \cup [2/3,1] \\[2ex] v_t + 18\,(9\,t\,(1-t) - 2)\left(c - \displaystyle\int_0^1 v\right), & t \in [1/3, 2/3]. \end{cases} \tag{2.2}$$

Notice that $\int_0^1 V_t^c \, dt = c$. We set for all $\omega \in C([0,1])$:

$$\rho^c(\omega) \;:=\; \exp\left\{ -162 \left(\int_0^{\frac{1}{3}} (\omega_r + \omega_{1-r}) \, dr + \frac{\omega_{\frac{1}{3}} + \omega_{\frac{2}{3}}}{6} - c \right)^2 - \frac{3}{2} (\omega_{\frac{2}{3}} - \omega_{\frac{1}{3}})^2 \right\}.$$

We recall that $\langle h, 1 \rangle = \int_0^1 h_\theta \, d\theta$ is the average of $h \in L^2(0,1)$. Then the first result of this paper is the following

Theorem 2.1. *Setting for all $c \geq 0$*

$$p_{\langle e,1\rangle}(c) \ =: \ 27\sqrt{\frac{6}{\pi^3}}\,\mathbb{E}\left[\rho^c(V^c)\,1_{\{V^c_t\geq 0,\ \forall t\in[0,1]\}}\right],$$

and for all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$ and $c > 0$

$$\mathbb{E}\left[\Phi(e)\,|\,\langle e,1\rangle = c\right] \ := \ \frac{1}{\mathcal{Z}_c}\,\mathbb{E}\left[\Phi\left(V^c\right)\,\rho^c(V^c)\,1_{\{V^c_t\geq 0,\ \forall t\in[0,1]\}}\right],$$

where $\mathcal{Z}_c > 0$ is a normalization factor, we have

1. *$p_{\langle e,1\rangle}$ is the density of $\langle e,1\rangle$ on $[0,\infty)$, i.e.*

$$\mathbb{P}(\langle e,1\rangle \in dc) \ = \ p_{\langle e,1\rangle}(c)\,1_{\{c\geq 0\}}\,dc.$$

Moreover $p_{\langle e,1\rangle}$ is continuous on $[0,\infty)$, $p_{\langle e,1\rangle}(c) > 0$ for all $c \in (0,\infty)$ and $p_{\langle e,1\rangle}(0) = 0$.

2. *$(\mathbb{P}\left[e \in \cdot\,|\,\langle e,1\rangle = c\right], c > 0)$ is a regular conditional distribution of e given $\langle e,1\rangle$, i.e.*

$$\mathbb{P}(e \in \cdot\,, \langle e,1\rangle \in dc) \ = \ \mathbb{P}\left[e \in \cdot\,|\,\langle e,1\rangle = c\right]\ p_{\langle e,1\rangle}(c)\,1_{\{c>0\}}\,dc.$$

In section 9 below we state and prove analogous results for the Brownian meander.

2.2 Two Hilbert spaces

For the study of the stochastic partial differential equation (1.1) we need to introduce some notation. We denote by A the realization in $L^2(0,1)$ of ∂^2_θ with Neumann boundary condition at 0 and 1, i.e.:

$$D(A) \ := \ \{h \in H^2(0,1) : h'(0) = h'(1) = 0\}, \qquad A \ := \ \frac{\partial^2}{\partial\theta^2}. \tag{2.3}$$

Notice that A is self-adjoint in $L^2(0,1)$. We also introduce another notation for the *average* of $h \in L^2(0,1)$:

$$\overline{h} \ := \ \int_0^1 h \ = \ \langle h,1\rangle.$$

Then we also set for all $c \in \mathbb{R}$:

$$L^2_c \ := \ \left\{h \in L^2(0,1) : \ \overline{h} = c\right\}. \tag{2.4}$$

Now we define the operator $Q : L^2(0,1) \mapsto L^2(0,1)$:

$$Qh(\theta) \ := \ \int_0^1 q(\theta,\sigma)\,h_\sigma\,d\sigma, \qquad \text{where :}$$
$$q(\theta,\sigma) \ := \ \theta\wedge\sigma + \frac{\theta^2+\sigma^2}{2} - \theta - \sigma + \frac{4}{3}, \qquad \theta,\sigma \in [0,1].$$

Then a direct computation shows that for all $h \in L^2(0,1)$:

$$\langle Qh, 1\rangle \;=\; \langle h, 1\rangle, \qquad -AQh \;=\; h - \overline{h},$$

i.e. Q is the inverse of $-A$ on L^2_0 and conserves the average. Then we define H as the completion of $L^2(0,1)$ with respect to the scalar product:

$$(h,k)_H \;:=\; \langle Qh, k\rangle, \qquad \|h\|^2_H := (h,h)_H.$$

For all $c \in \mathbb{R}$ we also set:

$$H_c \;:=\; \{h \in H : \; (h,1)_H = c\}.$$

We remark that H is naturally interpreted as a space of distributions, in particular as the dual space of $H^1(0,1)$.

We also need a notation for the realization A_D in $L^2(0,1)$ of ∂^2_θ with Dirichlet boundary condition at 0 and 1, i.e.:

$$D(A_D) \;:=\; \{h \in H^2(0,1) : \; h(0) = h(1) = 0\}, \qquad A_D \;:=\; \frac{\partial^2}{\partial\theta^2}. \tag{2.5}$$

Notice that A_D is self-adjoint and invertible in $L^2(0,1)$, with inverse:

$$Q_D h(\theta) \;=\; (-A_D)^{-1} h(\theta) \;:=\; \int_0^1 (\theta \wedge \sigma - \theta\sigma)\; h_\sigma \, d\sigma, \qquad \theta \in [0,1]. \tag{2.6}$$

2.3 Weak solutions of (1.1)

We state now the precise meaning of a solution to (1.1).

Definition 2.2. *Let $u_0 \in C([0,1])$, $u_0 \geq 0$, $\int_0^1 u_0 > 0$, $u_0(0) = u_0(1) = 0$. We say that (u, η, W), defined on a filtered complete probability space $(\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t)$, is a* weak *solution to (1.1) on $[0,T]$ if*

1. *a.s. $u \in C((0,T] \times [0,1])$, $u \geq 0$ and $u \in C([0,T]; H)$*
2. *a.s. $u_t(0) = u_t(1) = 0$ for all $t \geq 0$*
3. *a.s. η is a positive measure on $(0,T] \times (0,1)$, such that $\eta([\delta,T] \times [\delta, 1-\delta]) < \infty$ for all $\delta > 0$*
4. *$(W(t,\theta))$ is a Brownian sheet, i.e. a centered Gaussian process such that*

$$\mathbb{E}\left[W(t,\theta)\, W(t',\theta')\right] = t \wedge t' \cdot \theta \wedge \theta', \qquad t, t' \geq 0, \;\; \theta, \theta' \in [0,1]$$

5. *u_0 and W are independent and the process $t \mapsto (u_t(\theta), W(t,\theta))$ is $(\mathcal{F}_t)$-adapted for all $\theta \in [0,1]$ and I interval in $[0,1]$*

6. for all $h \in C^4([0,1])$ *such that* $h'(0) = h'(1) = h''(0) = h''(1) = 0$ *and for all* $0 < \delta \le t \le T$*:*

$$\begin{aligned} \langle u_t, h\rangle &= \langle u_\delta, h\rangle - \int_\delta^t \langle u_s, A_D A h\rangle\, ds \\ &\quad - \int_\delta^t \int_0^1 A h_\theta\, \eta(ds, d\theta) - \sqrt{2} \int_\delta^t \int_0^1 h'_\theta\, W(ds, d\theta) \end{aligned} \tag{2.7}$$

7. a.s. the contact property *holds:* $\mathrm{supp}(\eta) \subset \{(t,\theta) : u_t(\theta) = 0\}$*, i.e.*

$$\int_{(0,T]\times[0,1]} u\, d\eta = 0.$$

2.4 Function spaces

Notice that for all $c \in \mathbb{R}$, $H_c = c1 + H_0$ is a closed affine subspace of H isomorphic to the Hilbert space H_0. If J is a closed affine subspace of H, we denote by $C_b(J)$, respectively $C_b^1(J)$, the space of all bounded continuous functions on J, resp. bounded and continuous together with the first Fréchet derivative (with respect to the Hilbert structure inherited from H). We also denote by $\mathrm{Lip}(J)$ the set of all $\varphi \in C_b(J)$ such that:

$$[\varphi]_{\mathrm{Lip}(J)} := \sup_{h \ne k} \frac{|\varphi(h) - \varphi(k)|}{\|h - k\|_H} < \infty.$$

Finally, we define $\mathrm{Exp}(H) \subset C_b(H)$ as the linear span of $\{\cos((h,\cdot)_H), \sin((h,\cdot)_H) : h \in D(A_D\, A)\}$.

To $\varphi \in C_b^1(H_c)$ we associate a gradient $\nabla_{H_0}\varphi : H_c \mapsto H_0$, defined by:

$$\frac{d}{d\varepsilon}\, \varphi(k + \varepsilon\, h)\bigg|_{\varepsilon=0} = (\nabla_{H_0}\varphi(k), h)_H, \qquad \forall\, k \in A,\ h \in H_0. \tag{2.8}$$

The important point here is that we only allow derivatives along vectors in H_0 and the gradient is correspondingly in H_0. In particular, by the definition of the scalar product in H, each $\varphi \in \mathrm{Exp}(H)$ is also Fréchet differentiable in the norm of $L^2(0,1)$; then, denoting by $\nabla\varphi$ the gradient in the Hilbert structure of $L^2(0,1)$, we have

$$\nabla_{H_0}\varphi = (-A)\nabla\varphi, \qquad \forall\, \varphi \in \mathrm{Exp}(H). \tag{2.9}$$

2.5 The stochastic dynamics

We are going to state the result concerning equation (1.1). We denote by $X_t : H^{[0,\infty[} \mapsto H$ the coordinate process and we define

$$\nu_c := \mathbb{P}\left[e \in \cdot \,|\, \langle e, 1\rangle = c\right] = \text{law of } e \text{ conditioned to have average } c,$$

which is well defined by Theorem 2.1. We notice that the support of ν_c in H is

$$K_c := \text{closure in } H \text{ of } \left\{h \in L^2(0,1) : h \ge 0,\ \langle h, 1\rangle = 1\right\},$$

and the closed affine hull in H of K_c is H_c.

Then the second result of this paper is

Theorem 2.3. *Let $c > 0$.*

(a) The bilinear form $\mathcal{E} = \mathcal{E}_{\nu_c, \|\cdot\|_{H_0}}$ given by

$$\mathcal{E}(u,v) := \int_{K_c} (\nabla_{H_0} u, \nabla_{H_0} v)_H \, d\nu_c, \qquad u, v \in C_b^1(H_c),$$

is closable in $L^2(\nu_c)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet Form. Furthermore, the associated semigroup $(P_t)_{t\geq 0}$ in $L^2(\nu_c)$ maps $L^\infty(\nu_c)$ in $C_b(K_c)$.

(b) For any $u_0 = x \in K_c \cap C([0,1])$ there exists a weak solution (u, η, W) of (1.1) such that the law of u is $\mathbb{P}_x$.

(c) ν_c is invariant for (P_t), i.e. $\nu_c(P_t f) = \nu_c(f)$ for all $f \in C_b(K_c)$ and $t \geq 0$.

By Theorem 2.3, we have a Markov process which solves (1.1) weakly and whose invariant measure is the law of e conditioned to have average equal to c.

2.6 Remarks on uniqueness of solutions to (1.1)

We expect equation (1.1) to have pathwise-unique solutions, since this is typically the case for monotone gradient systems: this is always true in finite dimensions, see [3], and has been proven in several interesting infinite-dimensional situations, see [8] and [6]. In the present situation, the difficulty we encountered in the proof of uniqueness of (1.1) is the following: because of the boundary condition $u(t,0) = u(t,1) = 0$ and of the reflection at 0, it is expected that the reflecting measure η has infinite mass on $]0,T] \times [0,1]$; this is indeed true for second order SPDEs with reflection: see [11]. If this is the case, then it becomes necessary to localize in $]0,1[$ in order to prove a priori estimates; however, in doing so one loses the crucial property that the average is constant. In short, we were not able to overcome these two problems.

3 Conditioning e on its average

3.1 An absolute continuity formula

Let $(X_t)_{t\in[0,1]}$ be a continuous centered Gaussian process with covariance function $q_{t,s} := \mathbb{E}[X_t \, X_s]$. We have in mind the case of X being a Brownian motion or a Brownian bridge. In this section we consider two processes Y and Z, both defined by linear transformations of X, and we write an absolute continuity formula between the laws of Y and Z.

For all h in the space $M([0,1])$ of all signed measures with finite total variation on $[0,1]$ we set:

$$Q : M([0,1]) \mapsto C([0,1]), \qquad Q\lambda(t) := \int_0^1 q_{t,s} \, \lambda(ds), \quad t \in [0,1].$$

We denote by $\langle \cdot, \cdot \rangle : C([0,1]) \times M([0,1]) \mapsto \mathbb{R}$ the canonical pairing,

$$\langle h, \mu \rangle := \int_0^1 h_t \, \mu(dt).$$

where a continuous function $k \in C([0,1])$ is identified with $k_t\, dt \in M([0,1])$. We consider $\lambda, \mu \in M([0,1])$ such that:

$$\langle Q\lambda, \mu\rangle \;=\; 0, \qquad \langle Q\lambda, \lambda\rangle + \langle Q\mu, \mu\rangle \;=\; 1. \tag{3.1}$$

We set for all $\omega \in C([0,1])$:

$$\gamma(\omega) \;:=\; \int_0^1 \omega_s\, \lambda(ds), \qquad \Lambda_t \;:=\; Q\lambda(t),\ t \in [0,1], \qquad I \;:=\; \langle Q\lambda, \lambda\rangle,$$

$$a(\omega) \;:=\; \int_0^1 \omega_s\, \mu(ds), \qquad M_t \;:=\; Q\mu(t),\ t \in [0,1], \qquad 1 - I \;=\; \langle Q\mu, \mu\rangle,$$

and we notice that $\gamma(X) \sim N(0, I)$, $a(X) \sim N(0, 1-I)$ and $\{\gamma(X), a(X)\}$ are independent by (3.1). We fix a constant $\kappa \in \mathbb{R}$ and if $I < 1$ we define the continuous processes

$$Y_t \;:=\; X_t + (\Lambda_t + M_t)\,(\kappa - a(X) - \gamma(X))\,, \quad t \in [0,1],$$

$$Z_t \;:=\; X_t + \frac{1}{1-I}\, M_t\,(\kappa - a(X) - \gamma(X))\,, \quad t \in [0,1].$$

Lemma 3.1. *Suppose that $I < 1$. Then for all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$:*

$$\mathbb{E}\left[\Phi(Y)\right] \;=\; \mathbb{E}\left[\Phi(Z)\, \rho(Z)\right], \tag{3.2}$$

where for all $\omega \in C([0,1])$:

$$\rho(\omega) \;:=\; \frac{1}{\sqrt{1-I}}\, \exp\left(-\frac{1}{2}\, \frac{1}{1-I}\, \left(\gamma(\omega) - \kappa\right)^2 + \frac{1}{2}\, \kappa^2 \right).$$

We postpone the proof of Lemma 3.1 to section 10.

3.2 Proof of Theorem 2.1

If (X, Y) is a centered Gaussian vector and $Y \mapsto \mathbb{R}$ is not a.s. constant, then it is well known that a regular conditional distribution of X given $Y = y \in \mathbb{R}$ is given by the law of

$$X - \frac{\sigma_{XY}}{\sigma_{YY}}(Y - y), \qquad \text{where} \quad \sigma_{XY} = \mathbb{E}(XY), \quad \sigma_{YY} = \mathbb{E}(Y^2).$$

We apply this property to $X = (\beta_t, t \in [0,1])$ and to $Y = \int_0^1 \beta$. Notice that for all $t \in [0,1]$:

$$\mathbb{E}\left[\beta_t \int_0^1 \beta_r\, dr\right] = \frac{t(1-t)}{2}, \qquad \mathbb{E}\left[\left(\int_0^1 \beta_r\, dr\right)^2\right] = \frac{1}{12}.$$

Therefore, for all $c \in \mathbb{R}$, a regular conditional distribution of the law of β conditioned on $\left\{\int_0^1 \beta = c\right\}$ is given by the law of the process:

$$\beta_t^c \;:=\; \beta_t \;+\; 6\, t\, (1-t) \left(c - \int_0^1 \beta \right), \qquad t \in [0,1]. \tag{3.3}$$

Lemma 3.2. *Let $c \in \mathbb{R}$. For all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$:*

$$\mathbb{E}\left[\Phi(\beta) \,\middle|\, \int_0^1 \beta = c\right] \;=\; \mathbb{E}\left[\Phi(\beta^c)\right] \;=\; \mathbb{E}\left[\Phi\left(\Gamma^\beta\right)\, \rho_1\left(\Gamma^\beta\right)\right]$$

where for all $\omega \in C([0,1])$

$$\Gamma^\omega_t \;=\; \begin{cases} \omega_t, & t \in [0,1/3] \cup [2/3,1] \\ \\ \omega_t + 18\,(9\,t\,(1-t) - 2)\left(c - \int_0^1 \omega\right), & t \in [1/3,2/3] \end{cases} \tag{3.4}$$

$$\rho_1(\omega) \;:=\; \sqrt{27}\,\exp\left(-162\left(\int_0^{\frac{1}{3}} (\omega_r + \omega_{1-r})\,dr + \frac{\omega_{\frac{1}{3}} + \omega_{\frac{2}{3}}}{6} - c\right)^2 + 6\,c^2\right).$$

Proof. We shall show that we are in the situation of Lemma 3.1 with $X = \beta$, $Y = \beta^c$ and $Z = \Gamma^\beta$. In the notation of Lemma 3.1, we consider

$$\lambda(dt) \;:=\; \sqrt{12}\left(1_{[0,\frac{1}{3}]\cup[\frac{2}{3},1]}(t)\,dt + \frac{\delta_{\frac{1}{3}}(dt) + \delta_{\frac{2}{3}}(dt)}{6}\right),$$

$$\mu(dt) \;:=\; \sqrt{12}\left(1_{[\frac{1}{3},\frac{2}{3}]}(t)\,dt - \frac{\delta_{\frac{1}{3}}(dt) + \delta_{\frac{2}{3}}(dt)}{6}\right),$$

and $\kappa := \sqrt{12}\,c$. Then:

$$\gamma(\beta) \;=\; \sqrt{12}\int_0^{\frac{1}{3}}\left(\beta_r + \frac{1}{2}\,\beta_{\frac{1}{3}}\right)dr + \sqrt{12}\int_{\frac{2}{3}}^1\left(\beta_r + \frac{1}{2}\,\beta_{\frac{2}{3}}\right)dr, \quad I \;=\; \frac{26}{27}$$

$$a(\beta) \;=\; \sqrt{12}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\beta_r - \frac{1}{2}\,\beta_{\frac{1}{3}} - \frac{1}{2}\,\beta_{\frac{2}{3}}\right)dr,$$

$$\Lambda_t \;=\; 1_{[0,\frac{1}{3}]\cup[\frac{2}{3},1]}(t)\;\sqrt{3}\,t(1-t)\; + 1_{(\frac{1}{3},\frac{2}{3})}(t)\;\frac{2\sqrt{3}}{9}, \quad M_t \;=\; 1_{[\frac{1}{3},\frac{2}{3}]}(t)\;\sqrt{3}\,t(1-t).$$

The desired result follows by tedious direct computations and from Lemma 3.1. □

Lemma 3.3. *For all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$ and $f : \mathbb{R} \mapsto \mathbb{R}$*

$$\mathbb{E}\left[\Phi(e)\,f(\langle e, 1\rangle)\right] \;=\; \int_0^\infty 27\sqrt{\frac{6}{\pi^3}}\,\mathbb{E}\left[\Phi\left(V^c\right)\,\rho^c(V^c)\,1_{K_0}(V^c)\right]\,f(c)\,dc \tag{3.5}$$

Proof. Define $\{B, b, \hat{B}\}$, processes such that:

1. B and $\hat{B}$ are independent copies of a standard Brownian motion over $[0,1/3]$
2. conditionally on $\{B, \hat{B}\}$, b is a Brownian bridge over $[1/3,2/3]$ from $B_{1/3}$ to $\hat{B}_{1/3}$.

We set:

$$r_t := \begin{cases} B_t & t \in [0,1/3] \\ \\ b_t & t \in [1/3,2/3] \\ \\ \hat{B}_{1-t} & t \in [2/3,1]. \end{cases}$$

Moreover we set, denoting the density of $N(0,t)(dy)$ by $p_t(y)$:

$$\rho_2(\omega) := \frac{p_{\frac{1}{3}}(\omega_{\frac{2}{3}} - \omega_{\frac{1}{3}})}{p_1(0)} = \sqrt{3}\, \exp\left(-\frac{3}{2}\,(\omega_{\frac{2}{3}} - \omega_{\frac{1}{3}})^2\right), \qquad \omega \in C([0,1]).$$

By the Markov property of β:

$$\mathbb{E}\left[\Phi(r)\,\rho_2(r)\right] = \mathbb{E}[\Phi(\beta)].$$

Then, recalling the definition of ρ^c above, by Lemma 3.1 and Lemma 3.2:

$$\mathbb{E}\left[\Phi(\beta^c)\right] = \mathbb{E}\left[\Phi\left(\Gamma^\beta\right)\,\rho_1\left(\Gamma^\beta\right)\right] = \mathbb{E}[\Phi\left(\Gamma^r\right)\,\rho_1\left(\Gamma^r\right)\,\rho_2(\Gamma^r)] = 9\,\mathbb{E}[\Phi(\Gamma^r)\,\rho^c(\Gamma^r)]\,e^{6c^2}.$$

We recall now that $\mathbb{P}(\beta \in K_\varepsilon) = 1 - \exp(-2\,\varepsilon^2) \sim 2\,\varepsilon^2$ as $\varepsilon \to 0$, where $K_\varepsilon = \{\omega \in C([0,1]) : \omega \geq -\varepsilon\}$. We want to compute the limit of $\frac{1}{2\,\varepsilon^2}\,\mathbb{E}\left[\Phi(\beta^c)\,1_{K_\varepsilon}(\beta^c)\right]$ as $\varepsilon \to 0$. On the other hand $\mathbb{P}(B_t \geq -\varepsilon, \forall t \in [0,1/3]) \sim \sqrt{\frac{6}{\pi}}\,\varepsilon$ by (9.2). Then by (9.1) and (9.2)

$$\frac{1}{2\,\varepsilon^2}\,\mathbb{E}\left[\Phi(\beta^c)\,1_{K_\varepsilon}(\beta^c)\right] \to \frac{27}{\pi}\,\mathbb{E}\left[\Phi\left(V^c\right)\,\rho^c(V^c)\,1_{K_0}(V^c)\right]e^{6c^2}. \tag{3.6}$$

On the other hand, β conditioned on K_ε tends in law to the normalized Brownian excursion $(e_t, t \in [0,1])$, as proven in [7]. Then we have for all bounded $f \in C(\mathbb{R})$:

$$\frac{1}{2\,\varepsilon^2}\,\mathbb{E}\left[\Phi(\beta)\,1_{K_\varepsilon}(\beta)\,f(\langle\beta,1\rangle)\right] \to \mathbb{E}\left[\Phi(e)\,f(\langle e,1\rangle)\right]$$

Comparing the two formulae for all $f \in C(\mathbb{R})$ with compact support:

$$\begin{aligned} &\frac{1}{2\,\varepsilon^2}\,\mathbb{E}\left[\Phi(\beta)\,1_{K_\varepsilon}(\beta)\,f(\langle\beta,1\rangle)\right] = \int_{\mathbb{R}} \frac{1}{2\,\varepsilon^2}\,\mathbb{E}\left[\Phi(\beta^c)\,1_{K_\varepsilon}(\beta^c)\right] f(c)\,N(0,1/12)(dc) \\ &\to \int_0^\infty 27\sqrt{\frac{6}{\pi^3}}\,\mathbb{E}\left[\Phi\left(V^c\right)\,\rho^c(V^c)\,1_{K_0}(V^c)\right]\,f(c)\,dc = \mathbb{E}\left[\Phi(e)\,f(\langle e,1\rangle)\right] \end{aligned}$$

and (3.5) is proven. $\square$

Proof of Theorem 2.1. It only remains to prove the positivity assertion about the density. Notice that a.s. $V_t^c \geq 0$ for all $t \in [0,1/3] \cup [2/3,1]$, since a.s. $m \geq 0$: therefore a.s.

$$\{V_t^c \geq 0,\ \forall t \in [0,1]\} = \{V_t^c \geq 0,\ \forall t \in [1/3,2/3]\}.$$

The probability of this event is positive for all $c > 0$ while it is 0 for $c = 0$, since $\int_0^1 V_t^0\,dt = 0$. In particular $p_{\langle e,1\rangle}(0) = 0$. Finally, $p_{\langle e,1\rangle}(c) > 0$ yields also $\mathcal{Z}_c > 0$ if $c > 0$. The other results follow from Lemma 3.3. $\square$

4 The linear equation

We start with the linear fourth-order equation, written in abstract form:

$$\begin{cases} dZ_t = -A\,A_D\,Z\,dt + B\,d\mathcal{W}_t, \\ \\ Z_0(x) \;=\; x \in L^2(0,1), \end{cases} \tag{4.1}$$

where $\mathcal{W}$ is a cylindrical white noise in $L^2(0,1)$ and

$$D(B) \;:=\; H^1_0(0,1), \quad B \;:=\; \sqrt{2}\,\frac{d}{d\theta}, \qquad D(B^*) \;:=\; H^1(0,1), \quad B^* \;:=\; -\sqrt{2}\,\frac{d}{d\theta},$$

and we notice that $BB^* = -2A$. We define the strongly continuous contraction semigroups in $L^2(0,1)$:

$$S_t \;:=\; e^{-tAA_D}, \qquad S^*_t \;:=\; e^{-tA_DA}, \qquad t \geq 0. \tag{4.2}$$

We stress that S and S^* are dual to each other with respect to $\langle \cdot, \cdot \rangle$ but not necessarily with respect to $(\cdot,\cdot)_H$. It is well known that Z is equal to:

$$Z_t(x) \;=\; S_t x \;+\; \int_0^t S_{t-s}\,B\,d\mathcal{W}_s$$

and that this process belongs to $C([0,\infty); L^2(0,1))$. Notice that

$$\langle Z_t(x), 1\rangle \;=\; \langle x, S^*_t 1\rangle + \int_0^t \langle B^* S^*_{t-s} 1, d\mathcal{W}_s\rangle \;=\; \langle x, 1\rangle, \tag{4.3}$$

since $S^*_t 1 = 1$ and $B^* S^*_t 1 = B^* 1 = 0$. In particular, the average of Z is constant. Now, the $L^2(0,1)$-valued r.v. $Z_t(x)$ has law:

$$Z_t(x) \sim \mathcal{N}\left(S_t x, Q_t\right), \qquad Q_t \;:=\; \int_0^t S_s BB^* S^*_s\,ds.$$

Notice that:

$$\frac{d}{ds}\,S_s(-A_D)^{-1}S^*_s \;=\; S_s(2A)\,S^*_S \;=\; -\,S_s BB^* S^*_s \;=\; -\,\frac{d}{ds}\,Q_s,$$

so that, recalling that $Q_D := (-A_D)^{-1}$:

$$Q_t \;=\; Q_D \;-\; S_t\,Q_D\,S^*_t, \qquad t \geq 0.$$

In particular, the symmetric operator $Q_\infty : L^2(0,1) \mapsto L^2(0,1)$,

$$Q_\infty \;:=\; \int_0^\infty S_s BB^* S^*_s\,ds,$$

is well defined and we have $0 \leq Q_\infty \leq Q_D$. By Proposition 10.1.4 of [5], Q_∞ is the unique solution among all bounded operators $P : L_0 \mapsto L_0$ (recall (2.4)) of the equation

$$A\,A_D\,P \;+ P\,A_D\,A = BB^* = -2A,$$

and it is easy to check that a solution (and therefore the only one) is given by

$$Q_\infty \,=\, Q_D - \frac{1}{\langle Q_D 1, 1\rangle}\, Q_D 1 \otimes Q_D 1. \tag{4.4}$$

Therefore the law of $Z_t(x)$ converges to the Gaussian measure on $L^2(0,1)$:

$$\mu_c \,:=\, \mathcal{N}(c\cdot \mathbf{a}, Q_\infty),$$

with covariance operator Q_∞ and mean $c\cdot\mathbf{a} \in L^2(0,1)$, where

$$c = \overline{x} = \langle x, 1\rangle, \qquad \mathbf{a}_\theta := 6\,\theta(1-\theta), \quad \theta\in[0,1].$$

Notice that the kernel of Q_∞ is $\{t1 : t\in\mathbb{R}\}$ and $\langle \mathbf{a}, 1\rangle = 1$. Therefore μ_c is concentrated on the affine space L^2_c, defined in (2.4). Finally, we introduce the Gaussian measure on $L^2(0,1)$:

$$\mu \,:=\, \mathcal{N}(0, Q_D), \tag{4.5}$$

recall (2.6). In this case, the kernel of Q_D in $L^2(0,1)$ is the null space, so the support of μ is the full space $L^2(0,1)$. The next result gives a description of μ and μ_c as laws of stochastic processes related to the Brownian bridge $(\beta_\theta, \theta\in[0,1])$.

Lemma 4.1. *Let $(\beta_\theta)_{\theta\in[0,1]}$ a Brownian bridge from 0 to 0. Then μ is the law of β and μ_c is the law of the process β^c defined in (3.3), i.e. of β conditioned on $\{\int_0^1 \beta = c\}$, $c\in\mathbb{R}$.*

Proof. By (2.6), Q_D is given by a symmetric kernel $(\theta\wedge\sigma - \theta\sigma,\ \sigma,\theta\in[0,1])$. Since $\mathbb{E}(\beta_t\beta_s) = t\wedge s - ts$, for all $t,s\in[0,1]$, then it is well known that $\mu = \mathcal{N}(0,Q_D)$ coincides with the law of β. Analogously, the covariance of β^0 is by (3.3)

$$\mathbb{E}(\beta^0_t\beta^0_s) = t\wedge s - ts - 3\,t(1-t)\,s(1-s), \qquad t,s\in[0,1].$$

By the expression for Q_∞ found in (4.4), this is easily seen to be the kernel of Q_∞, so that $\mu_0 = \mathcal{N}(0,Q_\infty)$ is the law of β^0. By the definitions of $\mu_c = \mathcal{N}(c\mathbf{a}, Q_\infty)$, β^c and $\mathbf{a}$, we find that μ_c is the law of $\beta^c = \beta^0 + c\mathbf{a}$. $\square$

In particular, μ_c is a regular conditional distribution of $\mu(dx)$ given $\{\overline{x} = c\}$, i.e.:

$$\mu_c(dx) \,=\, \mu(dx\,|\,\overline{x} = c) \,=\, \mu(dx\,|\,L^2_c).$$

Recall (2.9). Then we have the following result:

Proposition 4.2. *Let $c\in\mathbb{R}$. The bilinear form:*

$$\Lambda^c(\varphi,\psi) \,:=\, \int_H (\nabla_{H_0}\varphi, \nabla_{H_0}\psi)_H\, d\mu_c \,=\, \int_H \langle -A\nabla\varphi, \nabla\psi\rangle\, d\mu_c, \quad \forall\ \varphi,\psi\in \mathrm{Exp}(H),$$

is closable in $L^2(\mu_c)$ and the process $(Z_t(x) : t\geq 0, x\in H_c)$ is associated with the resulting symmetric Dirichlet form $(\Lambda^c, D(\Lambda^c))$. Moreover, $\mathrm{Lip}(H_c)\subset D(\Lambda^c)$ and $\Lambda^c(\varphi,\varphi)\leq [\varphi]^2_{\mathrm{Lip}(H_c)}$.

Proof. The proof is standard, since the process Z is Gaussian: see [5, §10.2]. However we include some details since the interplay between the Hilbert structures of H and $L^2(0,1)$ and the different role of the operators A and A_D can produce some confusion. The starting point is the following integration by parts formula for μ:

$$\int \partial_h \varphi \, d\mu \;=\; \int \langle -A_D h, x\rangle \, \varphi(x) \, \mu(dx) \tag{4.6}$$

for all $\varphi \in C^1_b(H)$ and $h \in D(A_D)$. By conditioning on $\{\overline{x} = c\}$, (4.6) implies:

$$\int \partial_{(h-\overline{h})} \varphi \, d\mu_c \;=\; \int \langle -A_D h, x\rangle \, \varphi(x) \, \mu_c(dx). \tag{4.7}$$

Let now $\varphi(x) := \exp(i\langle x, h\rangle)$ and $\psi(x) := \exp(i\langle x, k\rangle)$, $x \in H$, $h, k \in D(A_D A)$. Then:

$$\mathbb{E}\left[\varphi(Z_t(x))\right] \;=\; \exp\left(i\langle S_t^* h, x\rangle - \frac{1}{2}\langle Q_t h, h\rangle \right)$$

and computing the time derivative at $t = 0$ we obtain the generator of Z:

$$L\varphi(x) \;=\; \varphi(x) \left[-i\langle A_D A h, x\rangle + \langle A h, h\rangle \right]. \tag{4.8}$$

Now we compute the scalar product in $L^2(\mu_c; \mathbb{C})$ between $L\varphi$ and ψ:

$$\begin{aligned}
\int L\varphi \, \overline{\psi} \, d\mu_c &= \int \left[-i\langle A_D A h, x\rangle + \langle A h, h\rangle \right] \exp(i\langle h-k, x\rangle) \, \mu_c(dx) \\
&= \int \left[-\langle A h, h-k\rangle + \langle A h, h\rangle \right] \exp(i\langle h-k, x\rangle) \, \mu_c(dx) \\
&= \int \langle A h, k\rangle \; \exp(i\langle h-k, x\rangle) \, \mu_c(dx) = \int \langle A\nabla\varphi, \nabla\overline{\psi}\rangle \, d\mu_c
\end{aligned}$$

where $\overline{\psi}$ is the complex conjugate of ψ and in the second equality we have used (4.7). It follows that $(L, \mathrm{Exp}(H))$ is symmetric in $L^2(\mu_c)$ and the rest of the proof is standard. $\square$

5 The approximating equation

We consider now the following approximating equation:

$$\begin{cases}
\dfrac{\partial u^{\varepsilon,\alpha}}{\partial t} = -\dfrac{\partial^2}{\partial \theta^2} \left(\dfrac{\partial^2 u^{\varepsilon,\alpha}}{\partial \theta^2} + \dfrac{(u^{\varepsilon,\alpha} + \alpha)^-}{\varepsilon} \right) + \sqrt{2} \, \dfrac{\partial}{\partial \theta} \dot{W}, \\[2ex]
u^{\varepsilon,\alpha}(t,0) = u^{\varepsilon,\alpha}(t,1) = \dfrac{\partial^3 u^{\varepsilon,\alpha}}{\partial \theta^3}(t,0) = \dfrac{\partial^3 u^{\varepsilon,\alpha}}{\partial \theta^3}(t,1) = 0 \\[2ex]
u^{\varepsilon,\alpha}(0,\theta) = x(\theta)
\end{cases} \tag{5.1}$$

where $\varepsilon > 0$. Notice that this is a monotone gradient system in H: see [5, Chapter 12], i.e. (5.1) can be written as follows,

$$dX_t^{\varepsilon,\alpha} = -A \left(A_D X^{\varepsilon,\alpha} - \nabla U_{\varepsilon,\alpha}(X^{\varepsilon,\alpha}) \right) dt + B \, d\mathcal{W}_t, \qquad X_0^{\varepsilon,\alpha}(x) \;=\; x,$$

where $U_{\varepsilon,\alpha} : H \mapsto \mathbb{R}^+$ is defined by

$$U_{\varepsilon,\alpha}(x) := \begin{cases} \dfrac{\|(x+\alpha)^-\|^2}{\varepsilon}, & \text{if } x \in L^2(0,1) \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

We define the probability measure on $L^2(0,1)$:

$$\nu_c^{\varepsilon,\alpha}(dx) := \frac{1}{Z_c^{\varepsilon,\alpha}} \exp\left(-U_{\varepsilon,\alpha}(x)\right) \, \mu_c(dx),$$

where $Z_c^{\varepsilon,\alpha}$ is a normalization constant. Now, recalling (2.9), we introduce the symmetric bilinear form:

$$\mathcal{E}^{\varepsilon,\alpha,c}(\varphi,\psi) := \int_H (\nabla_{H_0}\varphi, \nabla_{H_0}\psi)_H \, d\nu_c^{\varepsilon,\alpha} = \int_H \langle -A\nabla\varphi, \nabla\psi\rangle \, d\nu_c^{\varepsilon,\alpha}, \quad \forall\, \varphi,\psi \in \mathrm{Exp}(H).$$

Notice that this symmetric form is naturally associated with the operator:

$$L^{\varepsilon,\alpha}\varphi(x) := L\varphi(x) + \langle \nabla U_{\varepsilon,\alpha}(x), A\nabla\varphi\rangle, \qquad \forall\, \varphi \in \mathrm{Exp}(H),\ x \in L^2(0,1), \tag{5.2}$$

where $L\varphi$ is defined in (4.8) above. The following proposition states that equation (5.1) has a unique martingale solution, associated with the Dirichlet form arising from the closure of $(\mathcal{E}^{\varepsilon,c}, \mathrm{Exp}(H))$. Moreover, it states that the associated semigroup is Strong Feller.

Proposition 5.1. *Let $c \in \mathbb{R}$ and $\varepsilon > 0$.*

1. *$(L^{\varepsilon,\alpha}, \mathrm{Exp}(H))$ is essentially self-adjoint in $L^2(\nu_c^{\varepsilon,\alpha})$*
2. *$(\mathcal{E}^{\varepsilon,\alpha,c}, \mathrm{Exp}(H))$ is closable in $L^2(\nu_c^{\varepsilon,\alpha})$: we denote by $(\mathcal{E}^{\varepsilon,\alpha,c}, D(\mathcal{E}^{\varepsilon,\alpha,c}))$ the closure. Moreover $\mathrm{Lip}(H_c) \subset D(\mathcal{E}^{\varepsilon,\alpha,c})$ and $\mathcal{E}^{\varepsilon,\alpha,c}(\varphi,\varphi) \le [\varphi]^2_{\mathrm{Lip}(H_c)}$.*

For the proof, see [5] and §9 of [4]. ☐

6 Convergence of the stationary measures

The first technical result is the convergence of $\nu_c^{\varepsilon,\alpha}$ as $\varepsilon \to 0^+$ and then $\alpha \to 0^+$, and in particular the tightness in a suitable Hölder space. By Lemma 4.1, μ_c is the law of β^c defined in (3.3). We set $K_\alpha = \{\omega \in C([0,1]) : \omega \ge -\alpha\}$ and for $\alpha > 0$

$$\nu_c^{0,\alpha} := \mu_c(\,\cdot\,|\,K_\alpha) = \text{law of } \beta^c \text{ conditioned to be greater or equal to } -\alpha.$$

This is well defined, since $\mu_c(K_\alpha) > 0$, and it is easy to see that t

$$\nu_c^{\varepsilon,\alpha} \to \nu_c^{0,\alpha} \quad \text{as } \varepsilon \to 0, \qquad \text{weakly in } C([0,1]). \tag{6.1}$$

Moreover, since β^c has the same path regularity as β, it is easy to see that for all $\alpha > 0$, $\gamma \in (0,1/2)$ and $r \ge 1$:

$$\sup_{\varepsilon>0} \left(\int_H \|x\|^p_{W^{\gamma,r}(0,1)} \, d\nu_c^{\varepsilon,\alpha}(x) \right)^{\frac{1}{p}} \le c_\alpha \left(\int_H \|x\|^p_{W^{\gamma,r}(0,1)} \, d\mu_c(x) \right)^{\frac{1}{p}} < +\infty. \tag{6.2}$$

We also need a similar tightness and convergence result for $(\nu_c^{0,\alpha})_{\alpha>0}$. We recall the definition

$$\nu_c := \mathbb{P}[e \in \cdot \,|\langle e, 1\rangle = c], \quad \text{as defined in Theorem 2.1.}$$

Lemma 6.1. *As $\alpha \to 0^+$, $\nu_c^{0,\alpha}$ converges weakly in $C([0,1])$ to ν_c and*

$$\sup_{\alpha>0} \left(\int_H \|x\|^p_{W^{\gamma,r}(0,1)} \, d\nu_c^{0,\alpha}(x) \right)^{\frac{1}{p}} < +\infty. \tag{6.3}$$

Proof. We use Lemma 3.2. We recall that $\mathbb{E}\left[\Phi(\beta^c)\right] = \mathbb{E}\left[\Phi\left(\Gamma^\beta\right) \, \rho_1\left(\Gamma^\beta\right)\right]$ for all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$. Moreover, as proven in the Proof of Lemma 3.3, the law of Γ^β conditioned on K_α converges to the law of V^c, defined in (2.2), i.e.

$$\lim_{\alpha\to 0^+} \mathbb{E}\left[\Phi(\Gamma^\beta) \,|\, \Gamma^\beta \in K_\alpha\right] = \mathbb{E}[\Phi(V^c)].$$

Notice that ρ_1 is positive, continuous on $C([0,1])$ and bounded by a constant. Then we have

$$\mathbb{E}\left[\|\beta^c\|^p_{W^{\gamma,r}(0,1)} \mid \beta^c \in K_\alpha\right] = \frac{\mathbb{E}\left[\|\Gamma^\beta\|^p_{W^{\gamma,r}(0,1)} \, \rho_1(\Gamma^\beta) \, 1_{(\Gamma^\beta\in K_\alpha)}\right]}{\mathbb{E}\left[\rho_1(\Gamma^\beta) \, 1_{(\Gamma^\beta\in K_\alpha)}\right]}$$

$$\leq \kappa_1 \, \mathbb{E}\left[\|\Gamma^\beta\|^p_{W^{\gamma,r}(0,1)} \,|\, \Gamma^\beta \in K_\alpha\right] \cdot \frac{1}{\mathbb{E}\left[\rho_1(\Gamma^\beta) \,|\, \Gamma^\beta \in K_\alpha\right]} \leq \kappa_2 \mathbb{E}\left[\|\Gamma^\beta\|^p_{W^{\gamma,r}(0,1)} \,|\, \Gamma^\beta \in K_\alpha\right],$$

where the last inequality follows from the convergence $\mathbb{E}\left[\rho_1(\Gamma^\beta) \,|\, \Gamma^\beta \in K_\alpha\right] \to \mathbb{E}\left[\rho_1(V^c)\right] > 0$, $\alpha \to 0^+$. Then it only remains to prove that

$$\sup_{\alpha>0} \frac{\mathbb{E}\left[\|\Gamma^\beta\|^p_{W^{\gamma,r}(0,1)} \, 1_{(\Gamma^\beta\in K_\alpha)}\right]}{\mathbb{P}\left[\Gamma^\beta \in K_\alpha\right]} < +\infty. \tag{6.4}$$

We start with the numerator. We fix three functions $\phi_i : [0,1] \mapsto \mathbb{R}_+$ of class C^∞, such that $\phi_1 + \phi_2 + \phi_3 \equiv 1$, the support of ϕ_1 is in $[0, 1/3)$, the support of ϕ_3 is in $(2/3, 1]$ and the support of ϕ_2 is in $(1/6, 5/6)$. Then it is enough to estimate

$$\mathbb{E}\left[\|\varphi_i \cdot \Gamma^\beta\|^p_{W^{\gamma,r}(0,1)} \, 1_{(\Gamma^\beta\in K_\alpha)}\right], \qquad i = 1, 2, 3.$$

Notice that $\varphi_1\Gamma^\beta = \varphi_1\beta$. We set $I = [0, 1/3]$ and we denote by $(\beta_\theta^{0,a}, \theta \in I)$, resp. $(m_\theta^{b,a}, \theta \in I)$, the Brownian bridge from 0 to a over the interval I, respectively the 3-dimensional Bessel bridge from b to a over the interval I. Then, denoting by p_t the density of $\mathcal{N}(0,t)$,

$$\mathbb{E}(\Phi(\beta_\theta, \theta \in I) \,|\, \beta \geq -\alpha \text{ on } I) = \int_{-\alpha}^{\infty} \mathbb{E}(\Phi(\beta_\theta^{0,a}, \theta \in I) \,|\, \beta^{0,a} \geq -\alpha \text{ on } I) \, p_{2/9}(a) \, da$$

$$= \int_{-\alpha}^{\infty} \mathbb{E}(\Phi(m_\theta^{\alpha,a+\alpha} - \alpha, \theta \in I)) \, p_{2/9}(a) \, da$$

where in the former equality we use the Markov property of β and in latter the equality in law between Brownian bridges conditioned to be positive and 3-dimensional Bessel bridges. Then

$$\mathbb{E}\left[\|\varphi_1 \cdot \Gamma^\beta\|^p_{W^{\gamma,r}(0,1)} \, 1_{(\Gamma^\beta\in K_\alpha)}\right] \leq \mathbb{E}\left[\|\varphi_1 \cdot \beta\|^p_{W^{\gamma,r}(0,1)} \, 1_{(\beta\geq -\alpha \text{ on } [0,1/3]\cup[2/3,1])}\right]$$

$$= \int_{-\alpha}^{\infty} \mathbb{E}\left[\|\varphi_1 \cdot \left(m^{\alpha,a+\alpha} - \alpha\right)\|^p_{W^{\gamma,r}(0,1)}\right] \, p_{2/9}(a) \cdot \gamma_\alpha(a) \, da,$$

where $\gamma_\alpha(a) = \mathbb{P}(\beta \geq -\alpha \text{ on } [0,1/3] \cup [2/3,1] \mid \beta_{1/3} = a)$. Then it is easy to conclude that

$$\sup_{\alpha>0} \frac{1}{\alpha^2}\, \mathbb{E}\left[\|\varphi_1 \cdot \Gamma^\beta\|^p_{W^{\gamma,r}(0,1)}\, 1_{(\Gamma^\beta \in K_\alpha)} \right] < +\infty.$$

By symmetry, the same estimate holds for $\varphi_3 \cdot \Gamma^\beta$. As for $\varphi_2 \cdot \Gamma^\beta$, conditioning on the values of $\beta_{1/3}$ and $\beta_{2/3}$ and using an analogous argument, we find similarly that

$$\sup_{\alpha>0} \frac{1}{\alpha^2}\, \mathbb{E}\left[\|\varphi_2 \cdot \Gamma^\beta\|^p_{W^{\gamma,r}(0,1)}\, 1_{(\Gamma^\beta \in K_\alpha)} \right] < +\infty.$$

We estimate now the denominator of the r.h.s. of (6.4). Recall the definition (3.4) of Γ^ω for $\omega \in C([0,1])$. Notice that

$$\int_0^1 \omega \geq c \implies \Gamma^\omega_t \leq \omega_t, \quad \forall\, t \in [0,1],$$

since $9t(1-t) - 2 \geq 0$ for all $t \in [1/3, 2/3]$. This means that

$$\begin{aligned}\mathbb{P}(\Gamma^\beta \in K_\alpha) &\geq \mathbb{P}\left(\Gamma^\beta \in K_\alpha,\ \int_0^1 \beta \geq c\right) \geq \mathbb{P}\left(\beta \in K_\alpha,\ \int_0^1 \beta \geq c\right) \\ &= \mathbb{P}\left(\int_0^1 \beta \geq c \;\Big|\; \beta \in K_\alpha\right) \cdot \mathbb{P}\left(\beta \in K_\alpha\right) \sim \mathbb{P}\left(\int_0^1 e \geq c\right) 2\alpha^2, \quad \alpha \to 0^+,\end{aligned}$$

since $\mathbb{P}\left(\int_0^1 \beta \geq c \mid \beta \in K_\alpha\right) \to \mathbb{P}\left(\int_0^1 e \geq c\right) > 0$. Then (6.4) is proven.

In order to show that $\nu_c^{0,\alpha}$ indeed converges to ν_c, it is enough to recall formula (3.6) above and the second result of Theorem 2.1. ☐

7 A general convergence result

In this section we recall two results of [2], which we shall apply in section 8 to the convergence in law of the solutions of (5.1) to the solution of (1.1). These processes are reversible and associated with a gradient-type Dirichlet form. Moreover their invariant measures (respectively, $\nu_c^{\varepsilon,\alpha}$ and ν_c), are *log-concave*; a probability measure γ on H is log-concave if for all pairs of open sets B, $C \subset H$

$$\log \gamma\left((1-t)B\right) \geq (1-t)\log \gamma(B) + t \log \gamma(C) \qquad \forall t \in (0,1). \tag{7.1}$$

If $H = \mathbb{R}^k$, then the class of log-concave probability measures contains all measures of the form (here $\mathcal{L}_k$ stands for Lebesgue measure)

$$\gamma := \frac{1}{Z}\, e^{-V} \mathcal{L}_k, \tag{7.2}$$

where $V : H = \mathbb{R}^k \to \mathbb{R}$ is convex and $Z := \int_{\mathbb{R}^k} e^{-V}\, dx < +\infty$, see Theorem 9.4.11 in [1], in particular all Gaussian measures. Notice that the class of log-concave measures is closed under weak convergence. Therefore, it is easy to see by a projection argument that $\nu_c^{\varepsilon,\alpha}$ and ν_c are log-concave.

We denote by $X_t : H^{[0,+\infty[} \to H$ the coordinate process $X_t(\omega) := \omega_t$, $t \geq 0$. Then we recall one of the main results of [2]. We notice that the support of ν_c in H is K_c, the closure in H of $\{h \in L^2(0,1) : h \geq 0,\ \langle h, 1\rangle = c\}$, and the closed affine hull in H of K_c is H_c.

Proposition 7.1 (Markov process and Dirichlet form associated with ν_c and $\|\cdot\|_{H_0}$)**.**

(a) The bilinear form $\mathcal{E} = \mathcal{E}_{\nu_c,\|\cdot\|_{H_0}}$ given by

$$\mathcal{E}(u,v) := \int_{K_c} (\nabla_{H_0} u, \nabla_{H_0} v)_H \, d\nu_c, \qquad u,\, v \in C_b^1(H_c), \tag{7.3}$$

is closable in $L^2(\nu_c)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet Form. Furthermore, the associated semigroup $(P_t)_{t\geq 0}$ in $L^2(\nu_c)$ maps $L^\infty(\nu_c)$ in $C_b(K_c)$.

(b) There exists a unique Markov family $(\mathbb{P}_x : x \in K_c)$ of probability measures on $K_c^{[0,+\infty[}$ associated with $\mathcal{E}$. More precisely, $\mathbb{E}_x[f(X_t)] = P_t f(x)$ for all bounded Borel functions and all $x \in K_c$.

(c) For all $x \in K_c$, $\mathbb{P}_x^*(C(]0,+\infty[; H)) = 1$ and $\mathbb{E}_x[\|X_t - x\|^2] \to 0$ as $t \downarrow 0$. Moreover, $\mathbb{P}_x^*(C([0,+\infty[; H)) = 1$ for ν_c-a.e. $x \in K_c$.

(d) $(\mathbb{P}_x : x \in K_c)$ is reversible with respect to ν_c, i.e. the transition semigroup $(P_t)_{t\geq 0}$ is symmetric in $L^2(\nu_c)$; moreover ν_c is invariant for (P_t), i.e. $\nu_c(P_t f) = \nu_c(f)$ for all $f \in C_b(K_c)$ and $t \geq 0$.

Let $(\mathbb{P}_x^{\varepsilon,\alpha,c} : x \in H_c)$ (respectively $(\mathbb{P}_x : x \in K_c)$) be the Markov process in $[0,+\infty[^{H_c}$ associated to (resp. in $[0,+\infty[^{K_c}$ associated to ν_c) given by Proposition 7.1. We denote by $\mathbb{P}^N_{\nu_c^N} := \int \mathbb{P}_x^N \, d\nu_c^N(x)$ (resp. $\mathbb{P}_{\nu_c} := \int \mathbb{P}_x \, d\nu_c(x)$) the associated stationary measures.

With an abuse of notation, we say that a sequence of measures $(\mathbf{P}_n)$ on $C([a,b]; H)$ converges weakly in $C([a,b]; H_w)$ if, for all $m \in \mathbb{N}$ and $h_1, \ldots, h_m \in H$, the process $(\langle X_\cdot, h_i\rangle_H,\ i = 1, \ldots, m)$ under $(\mathbf{P}_n)$ converges weakly in $C([a,b]; \mathbb{R}^m)$ as $n \to \infty$.

In this setting we have the following stability and tightness result.

Theorem 7.2 (Stability and tightness)**.** *Then, for any $x \in K_c$ and $0 < \varepsilon \leq T < +\infty$,*

$$\lim_{\alpha\to 0^+} \lim_{\varepsilon\to 0^+} \mathbb{P}_x^{\varepsilon,\alpha,c} = \mathbb{P}_x, \quad \textit{weakly in } C([\varepsilon, T]; H_w).$$

Proof. This result follows from Theorem 1.5 in [2], where it is stated that the weak convergence of the invariant measures of a sequence of processes as in Proposition 7.1 implies the weak convergence of the associated processes. Since $\lim_{\alpha\to 0^+} \lim_{\varepsilon\to 0^+} \nu_c^{\varepsilon,\alpha} = \nu_c$, we obtain the result. □

8 Existence of weak solutions of equation (1.1)

In this section we prove the following result on weak existence of solutions to equation (1.1). We define the Polish space $E_T := C(O_T) \times M(O_T) \times C(O_T)$, where $O_T :=]0,T] \times [0,1]$ and $M(O_T)$ is the space of all locally finite positive measures on $]0,T] \times]0,1[$, endowed with the topology of convergence on compacts in $]0,T] \times]0,1[$.

Proposition 8.1. *Let $c > 0$, $u_0 = x \in K_c$ and $u^{\varepsilon,\alpha}$ the solution of (5.1). Set $\eta^{\varepsilon,\alpha} \in M(O_T)$,*

$$\eta^{\varepsilon,\alpha}(dt, d\theta) := \frac{(u^{\varepsilon,\alpha}(t,\theta) + \alpha)^-}{\varepsilon}\, dt\, d\theta.$$

Then $(u^{\varepsilon,\alpha}, \eta^{\varepsilon,\alpha}, W)$ converges in law to (u, η, W), stationary weak solution of (1.1), in E_T, for any $T \geq 0$. The law of u is $\mathbb{P}_x$ and therefore $(u, u_0 = x \in K_c)$ is the Markov process associated with the Dirichlet form (7.3).

We shall use the following easy result:

Lemma 8.2. *Let $\zeta(dt, d\theta)$ be a finite signed measure on $[\delta, T] \times [0,1]$ and $v \in C([\delta,T] \times [0,1])$. Suppose that for all $s \in [\delta, T]$:*

$$\int_{[s,T]\times[0,1]} h_\theta \, \zeta(dt, d\theta) \;=\; 0, \qquad \forall\; h \in C([0,1]),\; \overline{h} = 0, \tag{8.1}$$

and

$$\overline{v}_s \;=\; c > 0, \qquad \int_{[s,T]\times[0,1]} v \, d\zeta \;=\; 0. \tag{8.2}$$

Then $\zeta \equiv 0$.

Proof. Setting $h := k - \overline{k}$, $k \in C([0,1])$, we obtain by (8.1) for all $\delta \leq s \leq t \leq T$:

$$\int_0^1 k_\theta \, \zeta([s,t] \times d\theta) \;=\; \zeta([s,t] \times [0,1]) \int_0^1 k_\theta \, d\theta, \qquad \forall\, k \in C([0,1]).$$

This implies $\zeta(dt, d\theta) = \gamma(dt)\, d\theta$, where $\gamma(t) := \zeta([\delta, t] \times [0,1])$, $t \in [\delta, T]$, is a process with bounded variation. Then by (8.2):

$$0 \;=\; \int_{[s,t]\times[0,1]} v \, d\zeta \;=\; \int_s^t \left(\int_0^1 v_s(\theta)\, d\theta \right) \gamma(ds) \;=\; c\,(\gamma(t) - \gamma(s)),$$

i.e. $\gamma(t) - \gamma(s) = 0$, since $c > 0$. $\square$

Proof of Proposition 8.1. Recall that $\mathbb{P}_x^{\varepsilon,\alpha,c}$ is the law of $u^{\varepsilon,\alpha}$ if $u_0^{\varepsilon,\alpha} = x$. By Theorem 7.2 and Skorohod's Theorem we can find a probability space and a sequence of processes $(v^\varepsilon, w^\varepsilon)$ such that $(v^\varepsilon, w^\varepsilon) \to (v, w)$ in $C(O_T)$ almost surely and $(v^\varepsilon, w^\varepsilon)$ has the same distribution as (u^ε, W) for all $\varepsilon > 0$, where $O_T :=]0,T] \times [0,1]$. Notice that $v \geq 0$ almost surely, since for all t the law of $v_t(\cdot)$ is γ which is concentrated on K and moreover v is continuous on O_T. We set now:

$$\eta^\varepsilon(dt, d\theta) \;:=\; \frac{1}{\varepsilon}\, f\left(v_t^\varepsilon(\theta)\right)\, dt\, d\theta.$$

From (5.1) we obtain that a.s. for all $T \geq 0$ and $h \in D(A^2)$ and $\overline{h} = 0$:

$$\exists \lim_{\varepsilon \to 0^+} \int_{O_T} h_\theta \, \eta^\varepsilon(dt, d\theta). \tag{8.3}$$

The limit is a random distribution on O_T. We want to prove that in fact η^ε converges as a measure in the dual of $C(O_T)$ for all $T \geq 0$. For this, it is enough to prove that the mass $\eta^\varepsilon(O_T)$ converges as $n \to \infty$.

Suppose that $\{\eta^\varepsilon(O_T)\}_n$ is unbounded. We define $\zeta^\varepsilon := \eta^\varepsilon / \eta^\varepsilon(O_T)$. Then ζ^ε is a probability measure on the compact set O_T. By tightness we can extract from any sequence $\varepsilon_n \to 0$ a subsequence along which ζ^ε converges to a probability measure ζ. By the uniform convergence of v^ε we can see that the contact condition $\int_{O_T} v \, d\zeta = 0$ holds. Moreover, dividing (5.1) by $\eta^\varepsilon(O_T)$ for $t \in [0, T]$, we obtain that $\int_{O_t} h_\theta \, \zeta(ds, d\theta) = 0$ for all $h \in D(A^2)$ with $\overline{h} = 0$ and by density for all $h \in C([0,1])$ with $\overline{h} = 0$.

Then ζ and v satisfy (8.1) and (8.2) above, and therefore by Lemma 8.2, $\zeta \equiv 0$, a contradiction since ζ is a probability measure. Therefore $\limsup_{n\to\infty} \eta^\varepsilon(O_T) < \infty$.

By tightness, for any subsequence in $\mathbb{N}$ we have convergence of η^ε to a finite measure η on $[0,T] \times [0,1]$ along some sub-subsequence. Let η_i, $i = 1, 2$, be two such limits and set $\zeta := \eta_1 - \eta_2$. By (8.3) and by density:

$$\int_{O_T} h_\theta \, \eta_1(dt, d\theta) \, = \, \int_{O_T} h_\theta \, \eta_2(dt, d\theta), \qquad \forall \, h \in C([0,1]), \ \overline{h} = 0,$$

i.e. ζ and v satisfy (8.1) and (8.2) above. By Lemma 8.2, $\zeta \equiv 0$, i.e. $\eta_1 = \eta_2$. Therefore, η^ε converges as $n \to \infty$ to a finite measure η on $]0,T] \times [0,1]$. It is now clear that the limit (u, η, W) satisfies (2.7).

Finally, we need to prove that the contact condition holds, i.e. that $\int_{(0,\infty)\times[0,1]} v \, d\eta = 0$. Since $f \geq 0$ and $f(u) > 0$ for $u > 0$, then $u \, f(u) \leq 0$ for all $u \in \mathbb{R}$. Then for any continuous positive $\varphi : (0,1) \mapsto \mathbb{R}$ with compact support

$$0 \, \geq \, \int_{[0,T]\times[0,1]} \varphi \, v^\varepsilon \, d\eta^\varepsilon \, \to \, \int_{[0,T]\times[0,1]} \varphi \, v \, d\eta$$

by the uniform convergence of v^ε to v and the convergence of η^ε to η on compacts. Since $v \geq 0$ and η is a positive measure, then $\int_{[0,T]\times[0,1]} v \, d\eta \leq 0$ is possible only if $\int_{[0,T]\times[0,1]} v \, d\eta = 0$ □

9 Conditioning the Brownian meander to have a fixed time average

In this section we prove an analog of Theorem 2.1 for the standard Brownian meander $(m_t, t \in [0,1])$. We set $\langle m, 1 \rangle := \int_0^1 m_r \, dr$, average of m. Let B a standard Brownian motion such that

$\{m, B\}$ are independent and let $c \geq 0$ be a constant. We introduce the continuous processes:

$$u_t := \begin{cases} \dfrac{1}{\sqrt{2}} m_{2t}, & t \in [0, 1/2] \\[2ex] \dfrac{1}{\sqrt{2}} m_1 + B_{t-\frac{1}{2}}, & t \in [1/2, 1], \end{cases}$$

$$U_t^c := \begin{cases} u_t, & t \in [0, 1/2] \\[2ex] u_t + (12\, t\, (2-t) - 9) \left(c - \displaystyle\int_0^1 u \right), & t \in [1/2, 1]. \end{cases}$$

Notice that $\int_0^1 U_t^c \, dt = c$.

Theorem 9.1. *Setting for all $c \geq 0$*

$$p_{\langle m,1 \rangle}(c) := \sqrt{\frac{24}{\pi}} \, \mathbb{E} \left[e^{-12 \left(\int_0^{1/2} (U_r^c + U_{1/2}^c) \, dr - c \right)^2} 1_{\{U_t^c \geq 0, \ \forall t \in [0,1]\}} \right],$$

and for all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$ and $c > 0$

$$\mathbb{E} \left[\Phi(m) \, | \, \langle m, 1 \rangle = c \right] := \frac{1}{\mathcal{Z}_c} \, \mathbb{E} \left[\Phi(U^c) \, e^{-12 \left(\int_0^{1/2} (U_r^c + U_{1/2}^c) \, dr - c \right)^2} 1_{\{U_t^c \geq 0, \ \forall t \in [0,1]\}} \right],$$

where $\mathcal{Z}_c > 0$ is a normalization factor, we have

1. *$p_{\langle m,1 \rangle}$ is the density of $\langle m, 1 \rangle$, i.e.*

$$\mathbb{P}(\langle m, 1 \rangle \in dc) = p_{\langle m,1 \rangle}(c) \, 1_{\{c \geq 0\}} \, dc.$$

   Moreover $p_{\langle m,1 \rangle}$ is continuous on $[0, \infty)$, $p_{\langle m,1 \rangle}(c) > 0$ for all $c \in (0, \infty)$ and $p_{\langle m,1 \rangle}(0) = 0$.

2. *$(\mathbb{P}[m \in \cdot \, | \, \langle m, 1 \rangle = c], c > 0)$ is a regular conditional distribution of m given $\langle m, 1 \rangle$, i.e.*

$$\mathbb{P}(m \in \cdot \, , \langle m, 1 \rangle \in dc) = \mathbb{P}[m \in \cdot \, | \, \langle m, 1 \rangle = c] \ p_{\langle m,1 \rangle}(c) \, 1_{\{c > 0\}} \, dc.$$

In the notation of section 3.1, we consider $X = (B_t, t \in [0,1])$, standard Brownian motion. It is easy to see that for all $t \in [0,1]$:

$$\mathbb{E} \left[B_t \int_0^1 B_r \, dr \right] = \frac{t\,(2-t)}{2}, \qquad \mathbb{E} \left[\left(\int_0^1 B_r \, dr \right)^2 \right] = \frac{1}{3}.$$

Therefore, it is standard that for all $c \in \mathbb{R}$, B conditioned to $\int_0^1 B = c$ is equal in law to the process:

$$B_t^c := B_t + \frac{3}{2} t\,(2-t) \left(c - \int_0^1 B \right), \qquad t \in [0,1].$$

Lemma 9.2. *Let $c \in \mathbb{R}$. For all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$:*

$$\mathbb{E}\left[\Phi(B) \,\Bigg|\, \int_0^1 B = c\right] \,=\, \mathbb{E}\left[\Phi(B^c)\right] \,=\, \mathbb{E}\left[\Phi\left(S\right)\, \rho(S)\right],$$

where

$$S_t \,:=\, \begin{cases} B_t, & t \in [0,1/2] \\ \\ B_t + (12\,t\,(2-t) - 9)\left(c - \displaystyle\int_0^1 B\right), & t \in [1/2, 1] \end{cases}$$

$$\rho(\omega) \,:=\, \sqrt{8}\, \exp\left(-12\left(\int_0^{\frac{1}{2}} \left(\omega_r + \omega_{\frac{1}{2}}\right) dr - c\right)^2 + \frac{3}{2}\, c^2\right), \quad \omega \in C([0,1]).$$

Proof. We are going to show that we are in the setting of Lemma 3.1 with $X = B$, $Y = B^c$ and $Z = S$. We denote the Dirac mass at θ by δ_θ. In the notation of section 3.1, we consider:

$$\lambda(dt) \,:=\, \sqrt{3}\left(1_{[0,\frac{1}{2}]}(t)\, dt + \frac{1}{2}\,\delta_{\frac{1}{2}}(dt)\right), \qquad \mu(dt) \,:=\, \sqrt{3}\left(1_{[\frac{1}{2},1]}(t)\, dt - \frac{1}{2}\,\delta_{\frac{1}{2}}(dt)\right),$$

and $\kappa := \sqrt{3}\, c$. Then:

$$\gamma(\omega) \,=\, \sqrt{3}\int_0^{\frac{1}{2}} \left(\omega_r + \omega_{\frac{1}{2}}\right) dr, \qquad a(\omega) \,=\, \sqrt{3}\int_{\frac{1}{2}}^1 \left(\omega_r - \omega_{\frac{1}{2}}\right) dr,$$

$$\gamma(\omega) + a(\omega) \,=\, \sqrt{3}\int_0^1 \omega_r\, dr, \qquad I \,=\, 3\int_0^{\frac{1}{2}} (1-r)^2\, dr \,=\, \frac{7}{8}.$$

$$\Lambda_t \,=\, \begin{cases} \sqrt{3}\, t\left(1 - \dfrac{t}{2}\right), & t \in [0,1/2] \\ \\ \dfrac{3\sqrt{3}}{8}, & t \in [1/2,1]. \end{cases} \qquad M_t \,=\, \begin{cases} 0, \;\; t \in [0,1/2] \\ \\ \sqrt{3}\, t\left(1 - \dfrac{t}{2}\right) - \dfrac{3\sqrt{3}}{8}, & t \in [1/2,1]. \end{cases}$$

Tedious but straightforward computations show that with these definitions we have $X = B$, $Y = B^c$ and $Z = S$ in the notation of Lemma 3.1 and (3.1) holds true. Then Lemma 9.2 follows from Lemma 3.1. ☐

Lemma 9.3. *For all bounded Borel $\Phi : C([0,1]) \mapsto \mathbb{R}$ and $f : \mathbb{R} \mapsto \mathbb{R}$:*

$$\mathbb{E}\left[\Phi(m)\, f(\langle m, 1\rangle)\right] = \int_0^\infty \sqrt{\frac{24}{\pi}}\, \mathbb{E}\left[\Phi(U^c)\, e^{-12\left(\int_0^{1/2}(U^c_r + U^c_{1/2})\, dr - c\right)^2} 1_{\{U^c_t \geq 0,\ \forall t \in [0,1]\}}\right] f(c)\, dc.$$

Proof. Recall that m is equal in law to B conditioned to be non-negative (see [7] and (9.1) below). We want to condition B first to be non-negative and then to have a fixed time average. It turns out that Lemma 9.2 allows to compute the resulting law by inverting the two operations: first we condition B to have a fixed average, then we use the absolute continuity between the law of B^c and the law of S and finally we condition S to be non-negative.

We set $K_\varepsilon := \{\omega \in C([0,1]) : \omega \geq -\varepsilon\}$, $\varepsilon \geq 0$. We recall that B conditioned on K_ε tends in law to m as $\varepsilon \to 0$, more generally for all $s > 0$ and bounded continuous $\Phi : C([0,s]) \mapsto \mathbb{R}$, by the Brownian scaling:

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\Phi(B_t,\, t \in [0,s]) \,\middle|\, B_t \geq -\varepsilon,\, \forall\, t \in [0,s]\right] \;=\; \mathbb{E}\left[\Phi\left(\sqrt{s}\, m_{t/s},\, t \in [0,s]\right)\right], \tag{9.1}$$

and this is a result of [7]. By the reflection principle, for all $s > 0$:

$$\mathbb{P}(B_t \geq -\varepsilon,\ \forall\, t \in [0,s]) \;=\; \mathbb{P}(|B_s| \leq \varepsilon) \;\sim\; \sqrt{\frac{2}{\pi s}}\,\varepsilon, \qquad \varepsilon \to 0. \tag{9.2}$$

In particular for all bounded $f \in C(\mathbb{R})$

$$\mathbb{E}\left[\Phi(m)\, f(\langle m, 1\rangle)\right] \;=\; \lim_{\varepsilon \to 0} \sqrt{\frac{\pi}{2}}\, \frac{1}{\varepsilon}\, \mathbb{E}\left[\Phi(B)\, 1_{K_\varepsilon}(B)\, f(\langle B, 1\rangle)\right].$$

We want to compute the limit of $\frac{1}{\varepsilon}\, \mathbb{E}\left[\Phi(B^c)\, 1_{K_\varepsilon}(B^c)\right]$ as $\varepsilon \to 0$. Notice that S, defined in Lemma 9.2, is equal to B on $[0,1/2]$. Therefore, by (9.1) and (9.2) with $s = 1/2$:

$$\sqrt{\frac{\pi}{2}}\, \frac{1}{\varepsilon}\, \mathbb{E}\left[\Phi(B^c)\, 1_{K_\varepsilon}(B^c)\right] \;\to\; \sqrt{2}\, \mathbb{E}\left[\Phi(U^c)\, \rho(U^c)\, 1_{K_0}(U^c)\right].$$

Comparing the last two formulae for all $f \in C(\mathbb{R})$ with compact support:

$$\begin{aligned} &\sqrt{\frac{\pi}{2}}\, \frac{1}{\varepsilon}\, \mathbb{E}\left[\Phi(B)\, 1_{K_\varepsilon}(B)\, f(\langle B, 1\rangle)\right] \;=\; \int_{\mathbb{R}} \sqrt{\frac{\pi}{2}}\, \frac{1}{\varepsilon}\, \mathbb{E}\left[\Phi(B^c)\, 1_{K_\varepsilon}(B^c)\right] f(c)\, N(0,1/3)(dc) \\ &\to \int_0^\infty \sqrt{\frac{24}{\pi}}\, \mathbb{E}\left[\Phi(U^c)\, e^{-12\left(\int_0^{1/2} (U^c_r + U^c_{1/2})\, dr - c\right)^2} 1_{K_0}(U^c)\right] f(c)\, dc = \mathbb{E}\left[\Phi(m)\, f(\langle m, 1\rangle)\right] \end{aligned}$$

and the Lemma is proven. ☐

Proof of Theorem 9.1 The results follow from Lemma 9.3, along the lines of the proof of Theorem 2.1. ☐

It would be now possible to repeat the results of sections 4, 5, 6 and 8, and prove existence of weak solutions of the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial^2}{\partial \theta^2}\left(\dfrac{\partial^2 u}{\partial \theta^2} + \eta\right) + \sqrt{2}\, \dfrac{\partial}{\partial \theta} \dot{W}, \\[2ex] u(t,0) = \dfrac{\partial u}{\partial \theta}(t,1) = \dfrac{\partial^3 u}{\partial \theta^3}(t,0) = \dfrac{\partial^3 u}{\partial \theta^3}(t,1) = 0 \\[2ex] u(0,\theta) = x(\theta) \end{cases} \tag{9.3}$$

and that such weak solutions admit $\mathbb{P}\left[m \in \cdot \,|\, \langle m, 1\rangle = c\right]$ as invariant measures, where $c := \int_0^1 x(\theta)\, d\theta > 0$.

10 Proof of Proposition 3.1.

The result follows if we show that the Laplace transforms of the two probability measures in (3.2) are equal. Notice that Y is a Gaussian process with mean $\kappa\,(\Lambda + M)$ and covariance function:

$$q^Y_{t,s} \,=\, \mathbb{E}\left[(Y_t - \kappa\,(\Lambda_t + M_t))\,(Y_s - \kappa\,(\Lambda_s + M_s))\right] \,=\, q_{t,s} \,-\, (\Lambda_t + M_t)\,(\Lambda_s + M_s),$$

for $t, s \in [0,1]$. Therefore, setting for all $h \in C([0,1])$: $Q_Y h(t) := \int_0^1 q^Y_{t,s}\, h_s\, ds$, $t \in [0,1]$, the Laplace transform of the law of Y is:

$$\mathbb{E}\left[e^{\langle Y,h\rangle}\right] \,=\, e^{\kappa\langle h,\Lambda+M\rangle + \frac{1}{2}\,\langle Q_Y h,h\rangle}.$$

Recall now the following version of the Cameron-Martin Theorem: for all $h \in M([0,1])$

$$\mathbb{E}\left[\Phi(X)\, e^{\langle X,h\rangle}\right] \,=\, e^{\frac{1}{2}\langle Qh,h\rangle}\, \mathbb{E}[\Phi(X + Qh)].$$

Notice that $\gamma(Z) = \gamma(X)$, by (3.1). Therefore $\rho(Z) = \rho(X)$. We obtain, setting $\overline{h} := h - \frac{1}{1-I}\langle M, h\rangle(\lambda + \mu)$:

$$\begin{aligned}
\mathbb{E}\left[e^{\langle Z,h\rangle}\,\rho(Z)\right] &= e^{\frac{\kappa}{1-I}\langle M,h\rangle}\,\mathbb{E}\left[e^{\langle X,\overline{h}\rangle}\,\rho(X)\right] \,=\, e^{\frac{\kappa}{1-I}\langle M,h\rangle + \frac{1}{2}\langle Q\overline{h},\overline{h}\rangle}\,\mathbb{E}\left[\rho\left(X + Q\overline{h}\right)\right] = \\
&= e^{\frac{\kappa}{1-I}\langle M,h\rangle + \frac{1}{2}\langle Q\overline{h},\overline{h}\rangle}\,\frac{1}{\sqrt{1-I}}\,\mathbb{E}\left[e^{-\frac{1}{2}\,\frac{1}{1-I}\,\left(\gamma(X)+\langle\overline{h},\Lambda\rangle-\kappa\right)^2+\frac{1}{2}\,\kappa^2}\right].
\end{aligned}$$

By the following standard Gaussian formula for $\alpha \sim N(0, \sigma^2)$, $\sigma \geq 0$ and $c \in \mathbb{R}$:

$$\mathbb{E}\left[e^{-\frac{1}{2}\,(\alpha+c)^2}\right] \,=\, \frac{1}{\sqrt{1+\sigma^2}}\, e^{-\frac{1}{2}\,\frac{c^2}{1+\sigma^2}},$$

we have now for $\gamma(X) \sim N(0, I)$:

$$\mathbb{E}\left[e^{-\frac{1}{2}\,\frac{1}{1-I}\,\left(\gamma(X)+\langle\overline{h},\Lambda\rangle-\kappa\right)^2}\right] \,=\, \frac{1}{\sqrt{1+\frac{I}{1-I}}}\, e^{-\frac{1}{2}\,\frac{1}{1-I}\,\frac{1}{1+\frac{I}{1-I}}\left(\langle\overline{h},\Lambda\rangle-\kappa\right)^2} \,=\, \sqrt{1-I}\, e^{-\frac{1}{2}\,\left(\langle\overline{h},\Lambda\rangle-\kappa\right)^2}.$$

Therefore, recalling the definition of $\overline{h} := h - \frac{1}{1-I}\langle M, h\rangle(\lambda + \mu)$, we obtain after some trivial computation:

$$\begin{aligned}
\log \mathbb{E}\left[e^{\langle Z,h\rangle}\,\rho(Z)\right] &= \frac{\kappa}{1-I}\langle M,h\rangle + \frac{1}{2}\langle Q\overline{h},\overline{h}\rangle - \frac{1}{2}\,\left(\langle\overline{h},\Lambda\rangle - \kappa\right)^2 + \frac{1}{2}\,\kappa^2 \\
&= \kappa\langle\Lambda + M, h\rangle + \frac{1}{2}\langle Qh,h\rangle - \langle\Lambda + M, h\rangle^2 \,=\, \kappa\langle h,\Lambda + M\rangle + \frac{1}{2}\langle Q_Y h,h\rangle. \quad \square
\end{aligned}$$

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