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Eigenvalues of GUE minors

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Abstract

Consider an infinite random matrix $H = (h_{ij})_{0<i,j}$ picked from the Gaussian Unitary Ensemble (GUE). Denote its main minors by $H_i = (h_{rs})_{1\leq r,s\leq i}$ and let the j:th largest eigenvalue of H_i be μ_j^i. We show that the configuration of all these eigenvalues (i, μ_j^i) form a determinantal point process on $\mathbb{N} \times \mathbb{R}$.
Furthermore we show that this process can be obtained as the scaling limit in random tilings of the Aztec diamond close to the boundary. We also discuss the corresponding limit for random lozenge tilings of a hexagon.

Key words: Random matrices; Tiling problems.

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1 Introduction

The distribution of eigenvalues induced by some measure on matrices has been the study of random matrix theory for decades. These distributions have been found to be universal in the sense that they turn up in various unrelated problems, some of which do not contain a matrix in any obvious way, or contain a matrix that does not look like a random matrix. In this article, we propose to study the eigenvalues of the minors of a random matrix, and argue that this distribution also is universal in some sense by showing that it is the scaling limit of three apparently unrelated discrete models.

The largest eigenvalues of minors of GUE-matrices have been studied in (Bar01), connecting these to a certain queueing model. It is a special case of the very general class of models analysed in (Joh03). The large N limit of this model will yield the distribution of all the eigenvalues of a GUE-matrix and its minors.

This process will turn out also to be the scaling limit of a point process related to random tilings of the Aztec diamond, studied in (Joh05a) and of a process related to random lozenge-tilings of a hexagon, studied in (Joh05b).

1.1 Eigenvalues of the GUE

Consider the following point process on $\Lambda = \mathbb{N} \times \mathbb{R}$. There is a point at (n, μ) iff the n:th main minor of H, i.e. H_n, has an eigenvalue μ. We will call this process the *GUE minor process*. A central result in this article is that this process is a determinantal point process with a certain kernel K^{GUE}.

For details of what it means for a point process to be determinantal, see section 2. An explicit expression for this kernel is given in the next definition.

Definition 1.2. *The* GUE minor kernel *is*

$$K^{\mathrm{GUE}}(r, \xi;\, s, \eta) = -\phi(r, \xi;\, s, \eta) + \sum_{j=-\infty}^{-1} \sqrt{\frac{(s+j)!}{(r+j)!}} h_{r+j}(\xi) h_{s+j}(\eta) e^{-(\xi^2+\eta^2)/2},$$

where $\phi(r, \xi;\, s, \eta) = 0$ when $r \le s$ and

$$\phi(r, \xi;\, s, \eta) = \frac{(\xi - \eta)^{r-s-1}\sqrt{2^{r-s}}}{(r-s-1)!} e^{\frac{1}{2}(\eta^2 - \xi^2)} H(\xi - \eta)$$

for $r > s$.

Here, $h_k(x) = 2^{-k/2}(k!)^{-1/2}\pi^{-1/4}H_k(x)$ are the Hermite polynomials normalised so that

$$\int h_i(x) h_j(x) e^{-x^2} dx = \delta_{ij},$$

$h_k \equiv 0$ when $k < 0$ and H is the Heaviside function defined by

$$H(t) = \begin{cases} 1 & \text{for } t \ge 0 \\ 0 & \text{for } t < 0. \end{cases} \tag{1.1}$$

Theorem 1.3. *The GUE minor process is determinantal with kernel K^{GUE}.*

This will be proved in section 3.

1.4 Aztec Diamond

The Aztec diamond of size N is the largest region of the plane that is the union of squares with corners in lattice points and is contained in the region $|x| + |y| \leq N + 1$, see figure 1.

It can be covered with 2×1 dominoes in $2^{N(N+1)/2}$ ways, (EKLP92a; EKLP92b). Probability distributions on the set of all these possible tilings have been studied in several references, for example (Joh05a; Pro03). Typical samples are characterized by having so called frozen regions in the north, south, east and west, regions where the tiles are layed out like brickwork. In the middle there is a disordered region, the so called tempered region. It is for example known that for large N, the shape of the tempered region tends to a circle, see (JPS98) for precise statement.

The key to analyzing this model is to colour all squares black or white in a checkerboard fashion. Let us chose colour white for the left square on the top row. A horizontal tile is of type N, or north, whenever its left square is white. All other horizontal tiles are of type S, or south. Likewise, a vertical domino is of type W, or west, precisely if its top square is white. Other vertical dominoes are of type E.

In figure 1, tiles of type N and E have been shaded. Notice that along the line $i = 1$, there is precisely one white tile, and its position is a stochastic variable that we denote x_1^1. Along the line $i = 2$ there are precisely two white tiles, at positions x_1^2 and x_2^2 respectively, etc. In general, let x_k^i denote the j-coordinate of the k:th white tile along line i. These white points can be considered a particle configuration, and this particle configuration uniquely determines the tiling. It is shown in (Joh05a) that this process is a determinantal point process on $\mathbf{N}^2 = \{1, 2, \ldots, N\}^2$, and the kernel is computed.

We show that this particle process, properly rescaled, converges weakly to the distribution for eigenvalues of GUE described above. More precisely we have the following theorem that will be proved in section 4.

Theorem 1.5. *Let μ_j^i be the eigenvalues of a GUE matrix and its minors. For each N, let $\{x_j^i\}$ be the position of the particles, as defined above, in a random tiling of the Aztec Diamond of size N. Then for each continuous function of compact support $\phi : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, with $0 \leq \phi \leq 1$,*

$$\mathbb{E}\left[\prod_{i,j}(1 - \phi(i, \mu_j^i))\right] = \lim_{N\to\infty} \mathbb{E}\left[\prod_{i,j}(1 - \phi(i, \frac{x_j^i - N/2}{\sqrt{N/2}}))\right].$$

1.6 Rhombus Tilings

Consider an (a, b, c)-hexagon, i.e. a hexagon with side lengths a, b, c, a, b, c. It can be covered by rhombus-shaped tiles with angles $\pi/3$ and $2\pi/3$ and side length 1, so called *lozenges*. The number of possible such tilings is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i + j + k - 1}{i + j + k - 2}.$$

This formula was proved by Percy MacMahon (1854-1928), see (Sta99, page 401) for historical remarks.

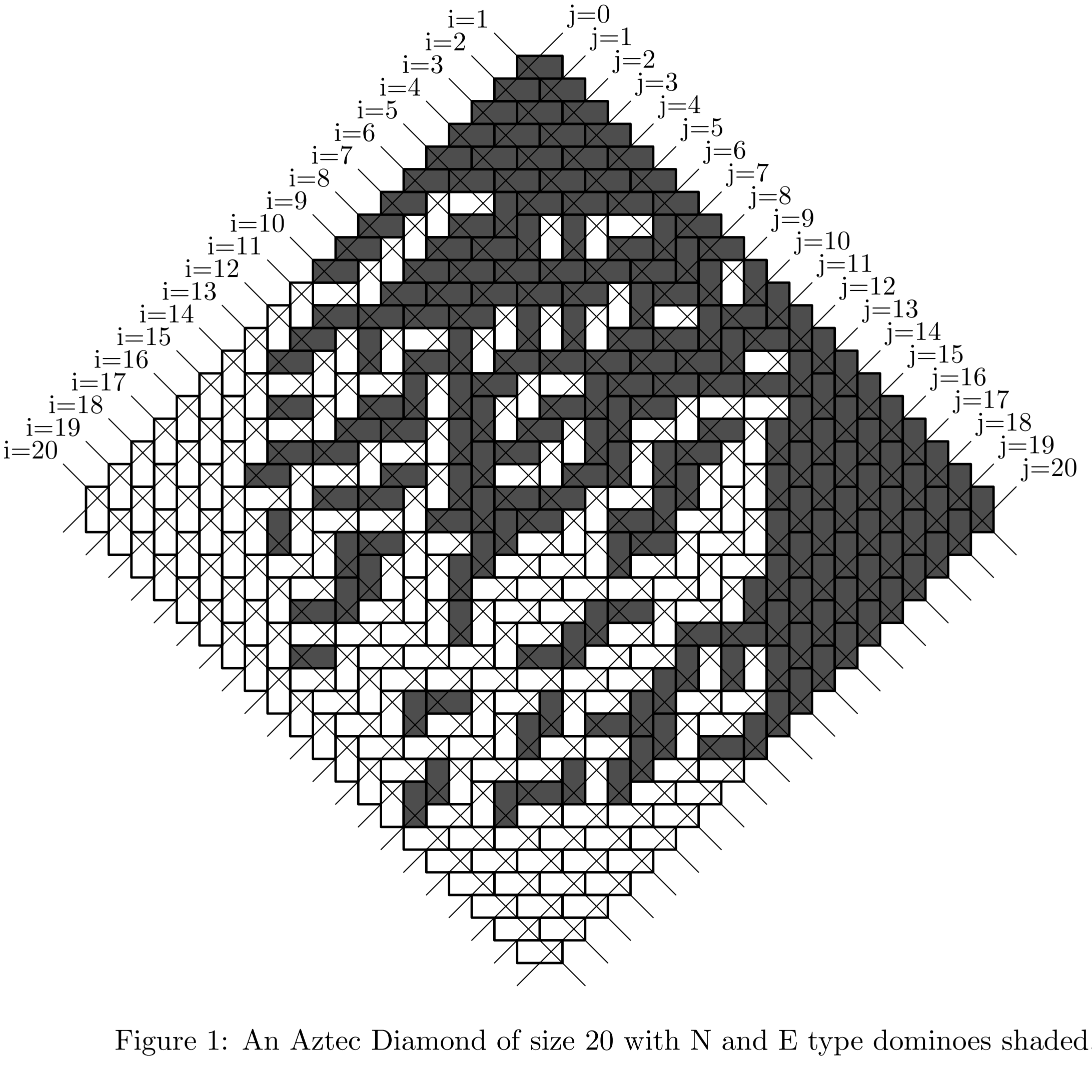

Figure 1: An Aztec Diamond of size 20 with N and E type dominoes shaded.

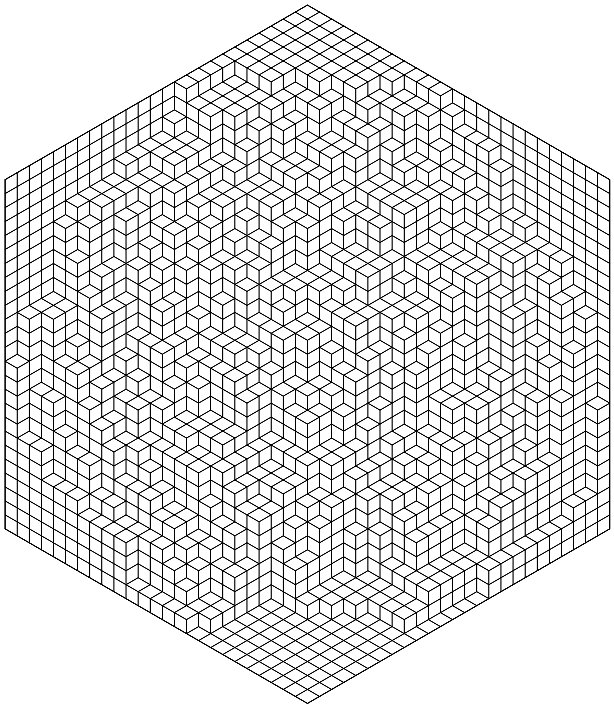

Figure 2: Lozenge tiling of a hexagon.

Thus, we can chose a tiling randomly, each possible tiling assigned equal probability. A typical such tiling is shown in figure 2. Just like in the case of the Aztec diamond, there are frozen regions in the corners of the shape and a disordered region in the middle. It has been shown, that when $a = b = c = N \to \infty$, this so called tempered region, tends to a circle, see (CLP98) for precise statement and other similar results.

Equivalently, consider a simple, symmetric, random walks, started at positions $(0, 2j)$, $1 \le j \le a$. At each step in discrete time, each walker moves up or down, with equal probability. They are conditioned never to intersect and to end at positions $(c + b, c - b + 2j)$. Figure 3, the red lines illustrate such a family of walkers, and shows the correspondence between this process and tilings of the hexagon. These red dots in the figure define a point process. (Joh05b) shows that uniform measure on tilings of the hexagon (or equivalently, uniform measure on the possible configurations of simple, symmetric, random walks) induces a measure on this point process that is determinantal, and computes the kernel.

We will show, in theorem 5.4, that the complement of this process, the blue dots in the figure, is also a determinantal process and compute its kernel.

Let us introduce some notation. Observe that along the line $t = 1$, there is exactly one blue dot. Let its position be x_1^1. Along line $t = 2$ there are two blue dots, at positions x_1^2 and x_2^2 respectively, and so on. All these x_j^i are stochastic variables, and they are of course not independent of each other.

We expect that the scaling limit of the process $\{x_j^i\}_{i,j}$, as the size of the hexagon tends to infinity, is the GUE minor process with kernel K^{GUE}. More precisely, let μ_j^i be the eigenvalues of a GUE matrix and its minors. Then for each continuous function of compact support $\phi : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$,

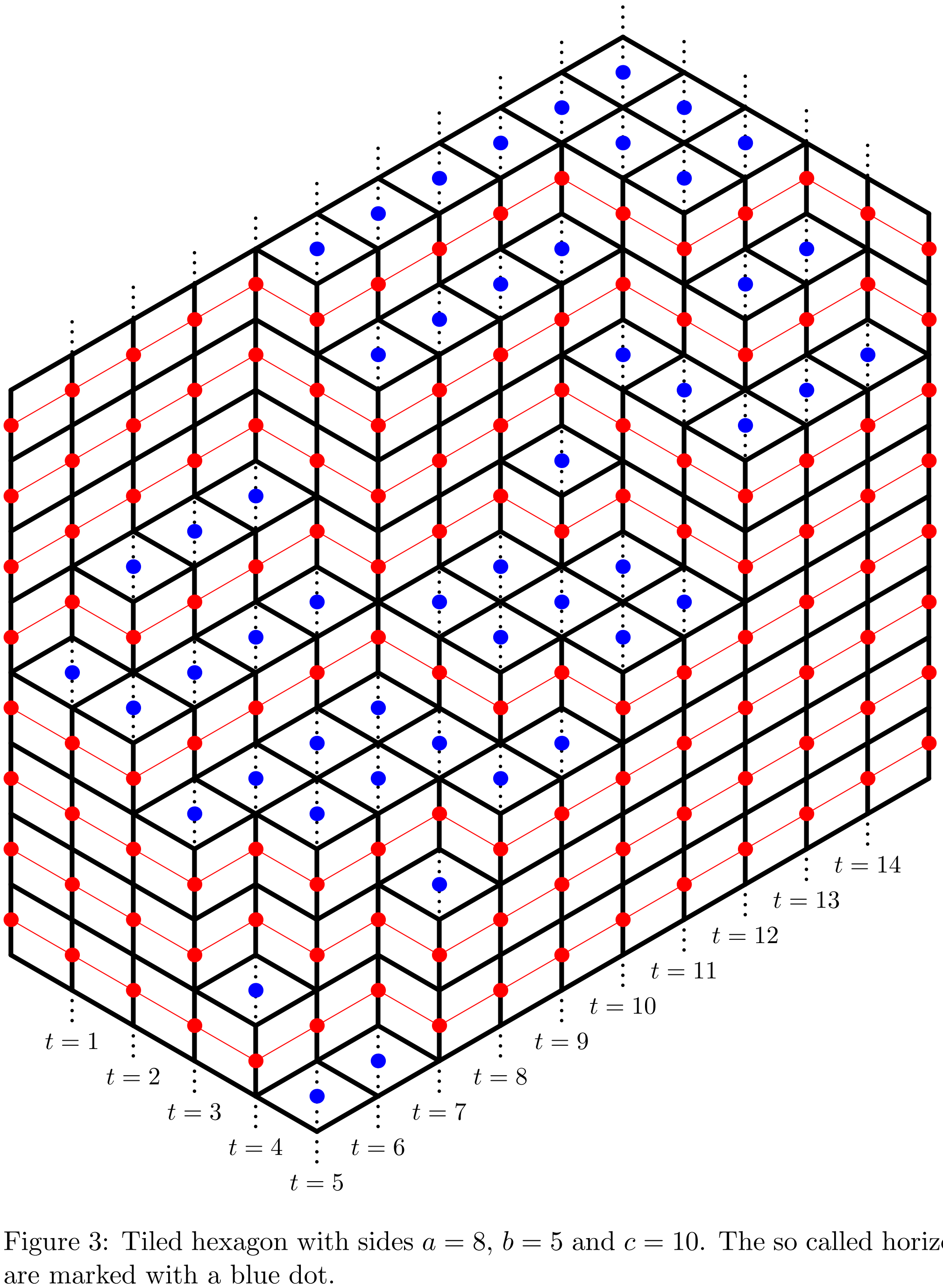

Figure 3: Tiled hexagon with sides $a = 8$, $b = 5$ and $c = 10$. The so called horizontal rhombuses are marked with a blue dot.

$0 \le \phi \le 1$,

$$\mathbb{E}\left[\prod_{i,j}(1-\phi(i,\mu^i_j))\right] = \lim_{N\to\infty} \mathbb{E}\left[\prod_{i,j}(1-\phi(i,\frac{x^i_j - N/2}{\sqrt{N/2}}))\right]. \tag{1.2}$$

We will outline a proof of this result by going to the limit in the formula for the correlation kernel, which involves the Hahn polynomials. A complete proof requires some further estimates of these polynomials.

The GUE minor process has also been obtained as a limit at "turning points" in a 3D partition model by Okounkov and Reshetikhin (OR06). We expect that the GUE minor process should be the universal limit in random tilings where the disordered region touches the boundary.

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2 Determinantal point processes

Let Λ be a complete separable metric space with some reference measure λ. For example $\mathbb{R}$ with Lebesgue measure or $\mathbb{N}$ with counting measure. Let $M(\Lambda)$ be the space of integer-valued and locally finite measures on Λ. A *point process* x is a probability measure on $M(\Lambda)$. For example, let x be a point process. A realisation $x(\omega)$ is an element of $M(\Lambda)$. It will assign positive measure to certain points, $\{x_i(\omega)\}_{1\le i\le N(\omega)}$, sometimes called *particles*, or just points in the process. In the processes that we will study the number of particles in a compact set will have a uniform upper bound.

Many point processes can be specified by giving their *correlation functions*, $\rho_n : \Lambda^n \to \mathbb{R}$, $n = 1, \ldots, \infty$. We will not go into the precise definition of these or when a process is uniquely determined by its correlation functions. For that we refer to any or all of the following references: (DVJ88, Ch. 9.1, A2.1), (Joh05c; Sos00).

Suffice it to say that correlation functions have the following useful property. For any bounded measurable function ϕ with bounded support B, satisfying

$$\sum_{n=1}^{\infty} \frac{||\phi||^n_\infty}{n!} \int_{B^n} \rho_n(x_1,\ldots,x_n)\, d^n x < \infty \tag{2.1}$$

the following holds:

$$\mathbb{E}[\prod_i (1+\phi(x_i(\omega)))] = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \phi(x_1)\cdots\phi(x_n)\rho_n(x_1,\ldots,x_n)\, d^n\lambda. \tag{2.2}$$

Correlation functions are thus useful in computing various expectations. For example, if A is some set and χ_A is the characteristic function of that set, then $1 - \mathbb{E}[\prod(1-\chi_A(x_i))]$ is the probability of at least one particle in the set A. If the correlation functions of a process exist and are known, this probability can then readily be computed with the above formula.

We will study point processes of a certain type, namely those whose correlation functions exist and are of the form

$$\rho_n(x_1,\ldots,x_n) = \det[K(x_i;\, x_j)]_{1\le i,j\le n},$$

i.e. the n:th correlation function is given as a $n \times n$ determinant where $K : \Lambda^2 \to \mathbb{R}$ is some, not necessarily smooth, measurable function. Such a process is called a *determinantal point process* and the function K is called the *kernel* of the point process.

Let $x^1, x^2, \dots, x^N, \dots$ be a sequence of point processes on Λ. Say that x^N assigns positive measure to the points $\{x_i^N(\omega)\}_{1\le i\le N^N(\omega)}$. Then we say that this sequence of point processes *converges weakly* to a point process x, written $x^N \to x$, $N \to \infty$, if for any continuous function ϕ of compact support, $0 \le \phi \le 1$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^{N^N(\omega)} (1-\phi(x_i^N(\omega)))\right] = \mathbb{E}\left[\prod_{i=1}^{N(\omega)} (1-\phi(x_i(\omega)))\right]. \tag{2.3}$$

The next proposition gives sufficient conditions for weak convergence of a sequence of determinantal processes in terms of the kernels.

Proposition 2.1. *Let $x^1, x^2, \dots, x^N, \dots$ be a sequence of determinantal point processes, and let x^N have correlation kernel K^N satisfying*

1. *$K^N \to K$, $N \to \infty$ pointwise, for some function K,*
2. *the K^N are uniformly bounded on compact sets in Λ^2 and*
3. *For C compact, there exists some number $n = n(C)$ such that $\det[K^N(x_i, x_j)]_{1\le i,j\le m} = 0$ if $m \ge n$.*

Then there exists some determinantal point process x with correlation functions K such that $x^N \to x$ weakly.

Proof. We start by showing that there exists such a determinantal point process x. In (Sos00), the following necessary and sufficient conditions for the existence of a random point process with given correlation functions is given.

1. Symmetry.
$$\rho_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \rho_k(x_1, \dots, x_k)$$
2. Positivity. For any finite set of measurable bounded functions $\phi_k : \Lambda^k \to \mathbb{R}$, $k = 0, \dots, M$, with compact support, such that
$$\phi_0 + \sum_{k=1}^{M} \sum_{i_1 \ne \dots \ne i_k} \phi(x_{i_1}, \dots, x_{i_k}) \ge 0 \tag{2.4}$$
for all $(x_1, \dots, x_M) \in I^M$ it holds that
$$\phi_0 + \sum_{k=1}^{N} \int_{I^k} \phi_k(x_1, \dots, x_k)\rho_k(x_1, \dots, x_k)\, dx_1 \dots dx_n \ge 0. \tag{2.5}$$

The first condition is satisfied for all correlation functions coming from determinantal kernels because permuting the rows and the columns of a matrix with the same permutation leaves the determinant unchanged. For the positivity condition consider the kernels K^N. They are kernels of determinantal processes so

$$\phi_0 + \sum_{k=1}^{M} \int_{I^k} \phi_k(x_1, \ldots, x_k) \det[K^N(x_i, x_j)]_{1\leq i,j\leq k}\, dx_1 \ldots dx_n \geq 0. \tag{2.6}$$

As $N \to \infty$, this converges to the same expression with K instead of K^N by Lebesgue's bounded convergence theorem with assumption (2). Positivity of this expression for all N then implies positivity of the limit.

So now we know that x exists. We need to show that $x^N \to x$ Take some test function $\phi : \Lambda \to \mathbb{R}$ with bounded support B. For this function we check the condition in (2.1). The assumption (3) in this theorem implies that the sum is a finite one. Also, $||\phi||_\infty \leq 1$. Assumption (2) is that the functions K^N are uniformly bounded, so in particular they are bounded on B^2, so ρ_k is bounded on B^k. The integral of a bounded function over a bounded set is finite, so this is the finite sum of finite real numbers, which is finite.

Therefore, for each N, by (2.2),

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_i (1 - \phi(x_i^N(\omega)))\right] = \lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^{n} \phi(x_i) \det[K^N(x_i, x_j)]_{1\leq i,j\leq n}\, d^n\lambda(x). \tag{2.7}$$

Condition (3) guarantees that the sum is finite. Lebesgue's bounded convergence theorem applies because the support of ϕ is compact and the correlation functions are bounded on compact sets. Thus the limit exists and is

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \prod_{i=1}^{n} \phi(x_i) \det[K(x_i, x_j)]_{1\leq i,j\leq n}\, d^n\lambda(x) \tag{2.8}$$

$$= \mathbb{E}\left[\prod_i (1 - \phi(x_i(\omega)))\right]. \tag{2.9}$$

This implies that indeed $x^N \to x$, weakly, as $N \to \infty$. □

3 The GUE Minor Kernel

3.1 Performance Table

Consider the following model. Let $\{w(i,j)\}_{(i,j)\in\mathbb{Z}_+^2}$, be independent geometric random variables with parameter q^2. I.e. there is one i.i.d. variable sitting at each integer lattice point in the first quadrant of the plane. Let

$$G(M,N) = \max_{\pi} \sum_{(i,j)\in\pi} w(i,j) \tag{3.1}$$

where the maximum is over all up/right paths from $(1,1)$ to (M,N). The array $[G(M,N)]_{M,N\in\mathbb{N}}$ is called the *performance table*.

Each such up/right path must pass through precisely one of $(M-1,N)$ and $(M,N-1)$, so it is true that $G(M,N)=\max(G(M-1,N),G(M,N-1))+w(M,N)$.

It is known from (Bar01), that $(G(N,1),G(N,2),\dots,G(N,M))$ for fixed M, properly rescaled, jointly tends to the distribution of $(\mu_1^1,\mu_1^2,\dots,\mu_1^M)$ as $N\to\infty$ in the sense of weak convergence of probability measures. We will show that it is possible to define stochastic variables in terms of the values w that jointly converge weakly to the distribution of all the eigenvalues μ_j^i of GUE-matrices.

3.2 Notation

We will use the following notation from (Bar01).

1. $W_{M,N}$ is set of $M\times N$ integer matrices.
2. $W_{M,N,k}$ is set of $M\times N$ integer matrices whose entries sum up to k.
3. $V_M=\mathbb{R}^{M(M-1)/2}$ where the components of each element x are indexed in the following way.
$$x=\begin{array}{cccc} x_1^1 & & & \\ \vdots & \ddots & & \\ x_1^{M-1} & \dots & x_{M-1}^{M-1} & \\ x_1^M & \dots & x_M^{M-1} & x_M^M \end{array}$$
4. $C_{\mathbf{GC}}\subset V_M$ is the subset such that $x_{j-1}^i\geq x_{j-1}^{i-1}\geq x_j^i$.
5. $C_{\mathbf{GC},\mathbb{N}}$ are the integer points of $C_{\mathbf{GC}}$.
6. Let $p:C_{\mathbf{GC}}\to\mathbb{R}^M$ be the projection that picks out the last row of the triangular array, i.e. $p(x)=(x_1^M,\dots,x_M^M)$. Likewise, let $q:C_{\mathbf{GC},\mathbb{N}}\to\mathbb{N}^M$, the projection that picks out the last row of an integer triangular array.

3.3 RSK

Recall that a partition λ of k is a vector of integers $(\lambda_1,\lambda_2,\dots)$, where $\lambda_1\geq\lambda_2\geq\dots$ such that $\sum_i\lambda_i=k$. It follows that only finitely many of the λ_i:s are non-zero.

A partition can be represented by a *Young diagram*, drawn as a configuration of boxes aligned in rows. The i:th row of boxes is λ_i boxes long. A *semi-standard Young tableau* (SSYT) is a filling of the boxes of a Young diagram with natural numbers, increasing from left to right in rows and strictly increasing from the top down in columns. The *Robinsson-Schensted-Knuth algorithm* (RSK algorithm) is an algorithm that bijectively maps $W_{M,N}$ to pairs of semi-standard Young tableau. For details of this algorithm, see for example (Sag01; Sta99).

Fix a matrix $w \in W_{M,N}$. This matrix is mapped by RSK to a pair of SSYT, $(P(w), Q(w))$. The P tableau will contain elements of $\mathbf{M} := \{1, 2, \ldots, M\}$ only. Construct a triangular array

$$x = \begin{array}{cccc} x_1^1 & & & \\ \vdots & \ddots & & \\ x_1^{M-1} & \ldots & x_{M-1}^{M-1} & \\ x_1^M & \ldots & x_M^{M-1} & x_M^M \end{array}$$

where x_j^i is the coordinate of the rightmost box filled with a number at most i in the j-th row of the $P(w)$-tableau. This is a map from $W_{M,N}$ to $C_{\mathbf{GC},\mathbb{N}}$.

3.4 A Measure on Semistandard Young Tableau

Consider the following probability measure on $W_{M,N}$. The elements in the matrix are i.i.d. geometric random variables with parameter q^2. Recall that a variable X is geometrically distributed with parameter q^2, written $X \in \mathrm{Ge}(q^2)$ if $P[X = k] = (1 - q^2)(q^2)^k$, $k \geq 0$. The square here will save a lot of root signs later. Such a stochastic variable has expectation $a = q^2/(1-q^2)$ and variance $b = q^2/(1 - q^2)^2$.

Applying the RSK algorithm to this array induces a measure on SSYT:s, and by the correspondence above, a measure on $C_{\mathbf{GC},\mathbb{N}}$. Call this measure $\pi^{\mathrm{RSK}}_{q^2,M,N}$. The following is shown in (Joh00).

Proposition 3.5. *Let $W_{M,N}$ contain i.i.d. $\mathrm{Ge}(q^2)$ random variables in each position. The probability that the RSK correspondence, when applied to this matrix, will yield Young diagrams of shape $\lambda = (\lambda_1, \ldots, \lambda_M)$ is*

$$\rho^{\mathrm{RSK}}_{q^2,M,N} := \frac{(1-q^2)^{MN}}{M!} \prod_{j=0}^{M-1} \frac{1}{j!(N-M+j)!} \times$$
$$\times \prod_{1 \leq i < j \leq M} (\lambda_i - \lambda_j + j - i)^2 \prod_{i=1}^{M} \frac{(\lambda_i + 1)!}{(\lambda_i + M - i)!} q^{2k}, \quad (3.2)$$

where $k = |\lambda| = \sum_i \lambda_i$.

In other words, the measure $\pi^{\mathrm{RSK}}_{q^2,M,M}$, integrating out all variables not on the last row, is $\rho^{\mathrm{RSK}}_{q^2,M,N}$. This, together with the following result characterizes the measure $\pi^{\mathrm{RSK}}_{q^2,M,M}$ completely.

3.6 Uniform lift

Proposition 3.2 in (Bar01) states that the probability measure $\pi^{\mathrm{RSK}}_{q^2,M,N}$, conditioned on the last row of the triangular array being λ, is uniform on the cone $q^{-1}(\lambda) := \{x \in C_{\mathbf{GC},\mathbb{N}} : q(x) = \lambda\}$. In formulas, this can be formulated as follows.

Proposition 3.7. *For any bounded continuous function $\phi : \mathbf{M} \times \mathbb{Z} \to \mathbb{R}$ of compact support,*

$$\mathbb{E}_{\pi^{\mathrm{RSK}}_{q^2,M,N}}[\prod_{i,j}(1 + \phi(i, x_j^i))] = \sum_{\lambda} \left(\frac{1}{L(\lambda)} \sum_{x \in q^{-1}(\lambda)} \prod_{i,j} (1 + \phi(i, x_j^i)) \right) \rho^{\mathrm{RSK}}_{q^2,M,N}(\lambda).$$

where $L(\lambda)$ is the number of integer points in $q^{-1}(\lambda)$.

The number of such integer points is given by

$$L(\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

3.8 GUE Eigenvalue measure

It is well know, see for example (Meh91), that the probability measure on the eigenvalues induced by GUE measure on $M \times M$ hermitian matrices is

$$\rho_M^{\mathrm{GUE}}(\lambda_1, \ldots, \lambda_M) = \frac{1}{Z_M} \prod_{1 \le i < j \le M} (\lambda_i - \lambda_j)^2 \prod_{1 \le i \le M} \exp(-\lambda_i^2)$$

for some constant Z_M that we need not be concerned with here.

3.9 Uniform lift of GUE measure

(Bar01) shows a result for eigenvalues of minors of GUE matrices that is similar to the above result for partitions. He shows that given the eigenvalues of the whole matrix $\lambda = (\lambda_1 > \cdots > \lambda_M)$, the triangular array of eigenvalues of all the minors are uniformly distributed in $p^{-1}(\lambda) := \{x \in C_{\mathbf{GC}} : p(x) = \lambda\}$. Again we can write this more formally.

Proposition 3.10. *For any bounded continuous function $\phi : \mathbf{M} \times \mathbb{R} \to \mathbb{R}$ of compact support, the measure π_M^{GUE} satisfies*

$$\mathbb{E}[\prod_{i,j}(1 + \phi(i, \mu_j^i))] = \int_\lambda \left(\frac{1}{\mathrm{Vol}(\lambda)} \int_{p^{-1}(\lambda)} \prod_{i,j}(1 + \phi(i, \mu_j^i)) \right) \rho_M^{\mathrm{GUE}}(\lambda)\, d^M \lambda.$$

where $\mathrm{Vol}(\lambda)$ is the volume of the cone $p^{-1}(\lambda)$.

This volume is given by

$$\mathrm{Vol}(\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j}{j - i}.$$

This situation is then very similar to the measure $\pi_{q^2,M,N}^{\mathrm{RSK}}$ above, in the sense that, conditioned the last row, the rest of the variables is uniformly distributed in a certain cone.

3.11 Scaling limit

We are now in a position to see the connection between the measures $\pi_{q^2,M,N}^{\mathrm{RSK}}$ and π_M^{GUE}.

Proposition 3.12. *Let $a := \mathbb{E}[w(1,1)] = q^2/(1-q^2)$ and $b := \mathrm{Var}[w(1,1)] = q^2/(1-q^2)^2$. Then for any bounded continuous function of compact support ϕ,*

$$\mathbb{E}_{\pi_M^{\mathrm{GUE}}}[\prod_{i,j}(1 + \phi(i, \mu_j^i))] = \lim_{N \to \infty} \mathbb{E}_{\pi_{q^2,M,N}^{\mathrm{RSK}}}[\prod_{i,j}(1 + \phi(i, \frac{x_j^i - aN}{\sqrt{bN}}))].$$

Proof. Plug in the expression for the right hand side in proposition 3.7 and for the left hand side in proposition 3.10. Stirling's formula and the convergence of a Riemann sum to an integral proves the theorem. ☐

3.13 Polynuclear growth

The measure $\pi^{\mathrm{RSK}}_{q^2,N,M}$ is a version of the Schur process and is a determinantal process on $\mathbf{M}\times\mathbb{N}$, by (OR03). We will use the following result from (Joh03).

Proposition 3.14. *The process $\{x^i_j\}$ with the measure described in 3.4 is determinantal with kernel*

$$K^{\mathrm{PNG}}_{q^2,N,M}(r,x,s,y)=\frac{1}{(2\pi i)^2}\int\frac{dz}{z}\int\frac{dw}{w}\frac{z}{z-w}\frac{w^{y+N}}{z^{x+N}}\frac{(1-qw)^s}{(1-qz)^r}\frac{(z-q)^{N-M}}{(w-q)^{N-M}}. \tag{3.3}$$

For $r\le s$, the paths of integration for z and w are anticlockwise along circles centred at zero with radii such that $q<|w|<|z|<1/q$. For the case $r>s$, integrate instead along circles such that $q<|z|<|w|<1/q$.

This follows immediately from proposition 3.12 and theorem 3.14 in (Joh03).

Having now introduced the PNG-kernel, we can state the following scaling limit result.

Lemma 3.15. *Let $a=q^2/(1-q^2)$ and $b=q^2/(1-q^2)^2$ as above. The following claims are true for M fixed.*

1. *For $r,\ s\le M$,*

$$\frac{g(r,\xi,N)}{g(s,\eta,N)}\sqrt{2bN}K^{\mathrm{PNG}}_{N,M}(r,\lfloor aN+\xi\sqrt{2bN}\rfloor;\ s,\lfloor aN+\eta\sqrt{2bN}\rfloor)\longrightarrow K^{\mathrm{GUE}}(r,\xi;\ s,\eta)$$

   uniformly on compact sets as $N\to\infty$ for a certain function $g\neq 0$.

2. *The expression*

$$\frac{g(r,\xi,N)}{g(s,\eta,N)}\sqrt{2bN}K^{\mathrm{PNG}}_{N,M}(r,\lfloor aN+\xi\sqrt{2bN}\rfloor;\ s,\lfloor aN+\eta\sqrt{2bN}\rfloor)$$

   is bounded uniformly for $1\le r,s\le M$ and $\xi,\ \eta$ in a compact set.

The proof, given in section 6, is an asymptotic analysis of the integral in (3.3). Now everything is set up so we can prove the main result of this section.

Proof of theorem 1.3. According to proposition 3.12,

$$\mathbb{E}_{\pi^{\mathrm{GUE}}_M}[\prod(1+\phi(i,\mu^i_j))]=\lim_{N\to\infty}\mathbb{E}_{\pi^{\mathrm{RSK}}_{q^2,M,N}}[\prod_{i,j}(1+\phi(i,\frac{x^i_j-aN}{\sqrt{bN}}))]. \tag{3.4}$$

The point processes on the right hand side of this last expression are determinantal. Their kernels can be written

$$K^N(r,\xi,s,\eta):=\frac{g(r,\xi,N)}{g(s,\eta,N)}\sqrt{2bN}K^{\mathrm{PNG}}_{N,M}(r,aN+\xi\sqrt{2bN};\ s,aN+\eta\sqrt{2bN}).$$

for some function g that cancels out in all determinants, and therfore does not affect the correlation functions. By lemma 3.15, these K^N satisfy all the assumptions of proposition 2.1. Thus, the point processes that these define converge weakly to a point process with kernel K^{GUE}. This implies that the measure on the left hand side of equation (3.4), i.e. π_M^{GUE} is determinantal with kernel K^{GUE}. The observation that M was arbitrary completes the proof. □

4 Aztec Diamond

The point-process connected to the tilings of this shape, described in the introduction was thoroughly analyzed in (Joh05a). The following result is shown.

Proposition 4.1. *The process $\{x_j^i\}$ described in section 1.4 is determinantal on $\Lambda = \mathbb{N} \times \mathbb{N}$, with kernel K_N^A given by*

$$K_N^{\mathrm{A}}(2r, x, 2s, y) = \frac{1}{(2\pi i)^2} \int \frac{dz}{z} \int \frac{dw}{w} \frac{w^y (1-w)^s (1+1/w)^{N-s}}{z^x (1-z)^r (1+1/z)^{N-r}} \frac{z}{z-w} \tag{4.1}$$

and reference measure μ which is counting measure on $\mathbb{N}$. The paths of integration are as follows: For $r \leq s$, integrate w along a contour enclosing its pole at -1 anticlockwise, and z along a contour enclosing w and the pole at 0 but not the pole at 1 anticlockwise. For $r > s$, switch the contours of z and w.

Based on this integral formula we can prove the following scaling limit analogous to that in lemma 3.15.

Lemma 4.2. *The following claims hold.*

(1)

$$\frac{g(r,\xi,N)}{g(s,\eta,N)} \sqrt{N/2} K_N^{\mathrm{A}}(2r, \lfloor N/2 + \xi\sqrt{N/2}\rfloor;\, 2s, \lfloor N/2 + \eta\sqrt{N/2}\rfloor) \longrightarrow K^{\mathrm{GUE}}(r,\xi;\, s,\eta)$$

uniformly on compact sets as $N \to \infty$ for an appropriate function $g \neq 0$.

(2) The expression

$$\frac{g(r,\xi,N)}{g(s,\eta,N)} \sqrt{N/2} K_N^{\mathrm{A}}(2r, \lfloor N/2 + \xi\sqrt{N/2}\rfloor;\, 2s, \lfloor N/2 + \eta\sqrt{N/2}\rfloor)$$

is uniformly bounded with respect to N for $(r,\ \xi,\ s,\ \eta)$ contained in any compact set in $\mathbb{N} \times \mathbb{R} \times \mathbb{N} \times \mathbb{R}$.

The proof is based on a saddle point analysis that is presented it section 6. We can now set about proving the main result of this section.

Proof of theorem 1.5. By proposition 4.1, the x_j^i form a determinantal process with kernel K_N^{A}. The rescaled process $(x_j^i - N/2)/\sqrt{N/2}$ has kernel

$$K^N(r,\xi;s,\eta) := \frac{g(r,\xi,N)}{g(s,\eta,N)} \sqrt{N/2} K_N^{\mathrm{A}}(2r, N/2 + \xi\sqrt{N/2};\, 2s, N/2 + \eta\sqrt{N/2}). \tag{4.2}$$

By lemma 4.2, the kernels K^N satisfy all the assumptions of proposition 2.1. So they converge to the process with kernel K^{GUE}. □

5 The Hexagon

Consider an (a,b,c)-hexagon, such as the one in figure 3. First we need some coordinate system to describe the position of the dots. Say that the a simple, symmetric, random walks start at $t = 0$ and $y = 0, 2, \ldots, 2a-2$. In each unit of time, they move one unit up or down, and are conditioned to end up at $y = c-b, c-b+2, \ldots, c-b+2a-2$ at time $t = b+c$ and never to intersect. One realisation of this process is the red dots in figure 3. At time t, the only possible y-coordinates for the red dots are $\{\alpha_t + 2k\}_{0 \le k \le \gamma_t}$, where

$$\gamma_t = \begin{cases} t+a-1 & 0 \le t \le b \\ b+a-1 & b \le t \le c \\ a+b+c-t-1 & c \le t \le b+c, \end{cases} \qquad \alpha_t = \begin{cases} -t & 0 \le t \le b \\ t-2b & b \le t \le b+c. \end{cases}$$

Let $\Lambda_{a,b,c} = \{(t, \alpha_t + 2k) : 0 \le t \le b+c,\ 0 \le k \le \gamma_t\}$ be the set of all the dots, red and blue.

5.1 A determinantal kernel for the hexagon tiling process

We now need to define the normalised associated Hahn polynomials, $\tilde{q}^{(\alpha,\beta)}_{n,N}(x)$. These orthogonal polynomials satisfy

$$\sum_{x=0}^{N} \tilde{q}^{(\alpha,\beta)}_{n,N}(x)\tilde{q}^{(\alpha,\beta)}_{m,N}(x)\tilde{w}^{(\alpha,\beta)}_{N}(x) = \delta_{n,m}, \tag{5.1}$$

where the weight function is

$$\tilde{w}^{(\alpha,\beta)}_{N}(x) = \frac{1}{x!(x+\alpha)!(N+\beta-x)!(N-x)!}.$$

They can be computed as

$$\tilde{q}^{(\alpha,\beta)}_{n,N}(x) = \frac{(-N-\beta)_n(-N)_n}{\tilde{d}^{(\alpha,\beta)}_{n,N}\, n!}\, {}_3F_2\left({}^{-n,n-2N-\alpha-\beta-1,-x}_{-N-\beta,-N}; 1\right),$$

where

$$\left(\tilde{d}^{(\alpha,\beta)}_{n,N}\right)^2 = \frac{(\alpha+\beta+N-1-n)_{N+1}}{(\alpha+\beta+2N+1-2n)n!(\beta+N-n)!(\alpha+N-n)!(N-n)!}.$$

For convenience, let $a_r = |c-r|$ and $b_r = |b-r|$. (Joh05b) shows the following.

Proposition 5.2. *The red dots form a determinantal point process on the space $\Lambda_{a,b,c}$ with kernel*

$$\begin{aligned} \tilde{K}^{\mathrm{L}}_{a,b,c}(r, \alpha_r+2x;\, s, \alpha_s+2y) &= -\phi_{r,s}(\alpha_r+2x, \alpha_s+2y) \\ &+ \sum_{n=0}^{a-1} \sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+s-1-n)!(a+b+c-s-1-n)!}}\, \tilde{q}^{b_r,a_r}_{n,\gamma_r}(x)\tilde{q}^{b_s,a_s}_{n,\gamma_s}(y)\omega_r(x)\tilde{\omega}_s(y), \end{aligned}$$

where $\phi_{r,s}(x,y) = 0$ if $r \ge s$ and

$$\phi_{r,s}(x,y) = \binom{s-r}{\frac{y-x+s-r}{2}}$$

otherwise. Furthermore,

$$\omega_r(x) = \begin{cases} ((b_r + x)!(\gamma_r + a_r - x)!)^{-1} & 0 \le r \le b \\ (x!(\gamma_r + a_r - x)!)^{-1} & b \le r \le c \\ (x!(\gamma_r - x)!)^{-1} & c \le r \le b + c \end{cases}$$

and

$$\tilde{\omega}_s(y) = \begin{cases} (y!(\gamma_s - y)!)^{-1} & 0 \le r \le b \\ ((b_s + y)!(\gamma_s - y)!)^{-1} & b \le r \le c \\ ((b_r + y)!(\gamma_r + a_r - y)!)^{-1} & c \le r \le b + c. \end{cases}$$

It follows that the blue dots also form a determinantal point process. To compute its kernel we need the following lemma.

Lemma 5.3.

$$\binom{s-r}{\frac{s-r+2y+\alpha_s-2x-\alpha_r}{2}} =$$

$$\sum_{n=0}^{\infty} \sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}} \tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(x) \tilde{q}_{n,\gamma_s}^{(b_s,a_s)}(y) \omega_r(x) \tilde{\omega}_s(y)$$

when $s \ge r$.

Proof. This proof uses the results obtained in the proof of 5.2 in (Joh05b, equation 3.25). Define convolution product as follows. For $f, g : \mathbb{Z}^2 \to \mathbb{Z}$, define $(f * g) : \mathbb{Z}^2 \to \mathbb{Z}$ by

$$(f * g)(x, y) := \sum_{z \in \mathbb{Z}} f(x, z) g(z, y).$$

Let $\phi(x, y) := \delta_{x,y+1} + \delta_{x,y-1}$. Also let

$$\begin{aligned} \phi^{*0}(x, y) &:= \delta_{x,y} \\ \phi^{*1}(x, y) &:= \phi(x, y) \\ \phi^{*n}(x, y) &:= (\phi^{*(n-1)} * \phi)(x, y). \end{aligned}$$

Set

$$\begin{aligned} c_{j,k} &:= \frac{1}{(a-k)(j-k)!(a-1-j)!} \\ f_{n,k} &:= \binom{n}{k} \frac{(n-2a-b-c+1)_k}{(-a-b+1)_k(-a)_k} \end{aligned}$$

and finally let

$$\begin{aligned} \psi(n, z) &:= \sum_{m=0}^{n} f_{n,m} \sum_{j=m}^{a-1} c_{j,m} \phi(2j, z) \\ \phi_{0,1}(n, y) &:= \psi(n, y) \\ \phi_{0,r}(n, y) &:= \psi * \phi^{*(r-1)}(n, y). \end{aligned}$$

The dual orthogonality relation to (5.1) is precisely

$$\sum_{n=0}^{\gamma_r} \tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(x)\tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(y)\omega_r(x)\tilde{\omega}_r(y) = \delta_{x,y}. \tag{5.2}$$

By equation (3.25), (3.30) and (3.32) of the above mentioned paper,

$$\phi_{0,r}(n, \alpha_r + 2z) = A(a,b,c,r,n)\tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(z)\tilde{\omega}_r(z),$$

where

$$A(a,b,c,r,n) := \frac{(a+1)_{r-1}\tilde{d}_{n,\gamma_r}^{(b_r,a_r)}n!}{(-a-c+1)_n(-a-b-c+r+1)_n}.$$

Inserting this into the orthogonality relation in (5.2) gives

$$\sum_{n=0}^{\gamma_r} \tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(x)\frac{\omega_r(x)}{A(a,b,c,r,n)}\phi_{0,r}(n,\alpha_r+2z) = \delta_{x,y}.$$

Convolving both sides of the above relation with $\phi^{*(s-r)}$ gives

$$\begin{aligned}\sum_{n=0}^{\gamma_r} \tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(x)\frac{\omega_r(x)}{A(a,b,c,r,n)}\sum_{z\in\mathbb{Z}}\phi_{0,r}(n,\alpha_r+2z)\phi^{*(s-r)}(\alpha_r+2z,\alpha_s+2y)& \\ = \phi^{*(s-r)}(\alpha_r+2x,\alpha_s+2y)&,\end{aligned}$$

which, when the left hand side is simplified, gives

$$\sum_{n=0}^{\gamma_r} \tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(x)\frac{\omega_r(x)}{A(a,b,c,r,n)}\phi_{0,s}(n,\alpha_r+2y) = \phi^{*(s-r)}(\alpha_r+2x,\alpha_s+2y).$$

Invoking equation (5.1) again to simplify the left hand side and explicitly calculating the right hand side gives

$$\sum_{n=0}^{\gamma_r} \frac{A(a,b,c,s,n)}{A(a,b,c,r,n)}\tilde{q}_{n,\gamma_r}^{(b_r,a_r)}(x)\tilde{q}_{n,\gamma_s}^{(b_s,a_s)}(y)\omega_r(x)\tilde{\omega}_s(y) = \binom{s-r}{\frac{s-r+2y+\alpha_s-2x-\alpha_r}{2}}.$$

It is easy to check that

$$\frac{A(a,b,c,s,n)}{A(a,b,c,r,n)} = \sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}},$$

which proves the lemma. $\square$

We now need to introduce the normalized Hahn polynomials $q_{n,N}^{(\alpha,\beta)}(x)$. These satisfy

$$\sum_{x=0}^{N} q_{n,N}^{(\alpha,\beta)}(x)q_{m,N}^{(\alpha,\beta)}(x)w_N^{(\alpha,\beta)}(x) = \delta_{m,n}, \tag{5.3}$$

where

$$w_N^{(\alpha,\beta)}(t) = \frac{(N+\alpha-t)!(\beta+t)!}{t!(N-t)!}. \tag{5.4}$$

Theorem 5.4. *The blue dots form a determinantal point process on the space $\Lambda_{a,b,c}$ with kernel*

$$K^{\mathrm{L}}_{a,b,c}(r,x;\,s,y) = \sum_{n=-\infty}^{-1}\sqrt{\frac{(s+n)!(b+c-r+n)!}{(r+n)!(b+c-s+n)!}}q^{(b_r,a_r)}_{r+n,\gamma_r}(x)q^{(b_s,a_s)}_{s+n,\gamma_s}(y)\sqrt{w^{(b_r,a_r)}_{\gamma_r}(x)w^{(b_s,a_s)}_{\gamma_s}(y)},$$

when $s \geq r$, and

$$K^{\mathrm{L}}_{a,b,c}(r,x;\,s,y) = -\sum_{n=0}^{a-1}\sqrt{\frac{(s+n)!(b+c-r+n)!}{(r+n)!(b+c-s+n)!}}q^{(b_r,a_r)}_{r+n,\gamma_r}(x)q^{(b_s,a_s)}_{s+n,\gamma_s}(y)\sqrt{w^{(b_r,a_r)}_{\gamma_r}(x)w^{(b_s,a_s)}_{\gamma_s}(y)}$$

otherwise.

Proof. It is well known that the complement of a determinantal point processes on a finite set with kernel K is also determinantal with kernel $\tilde{K} = I - K$, i.e. $\tilde{K}(x,y) = \delta_{x,y} - K(x,y)$. Applying this result to our problem, we want to consider $\delta_{x,y}\delta_{r,s} - \tilde{K}^{\mathrm{L}}_{a,b,c}(r,x;\,s,y)$. We now separate two cases. When $s \geq r$ see that

$$K(r,x;\,s,y) = \binom{s-r}{y-x+\frac{s-r+\alpha_s-\alpha_r}{2}} - \sum_{n=0}^{a-1}\sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}}\tilde{q}^{(b_r,a_r)}_{n,\gamma_r}(x)\tilde{q}^{(b_s,a_s)}_{n,\gamma_s}(y)\omega_r(x)\tilde{\omega}_s(y)$$

is a candidate for the kernel for the blue particles. By lemma 5.3 this simplifies to

$$K(r,x;\,s,y) = \sum_{n=a}^{\infty}\sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}}\tilde{q}^{(b_r,a_r)}_{n,\gamma_r}(x)\tilde{q}^{(b_s,a_s)}_{n,\gamma_s}(y)\omega_r(x)\tilde{\omega}_s(y).$$

For $s < r$ we just get

$$K(r,x;\,s,y) = -\sum_{n=0}^{a-1}\sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}}\tilde{q}^{(b_r,a_r)}_{n,\gamma_r}(x)\tilde{q}^{(b_s,a_s)}_{n,\gamma_s}(y)\omega_r(x)\tilde{\omega}_s(y).$$

We now exploit a useful duality result from (Bor02). It states that

$$q^{(\alpha,\beta)}_{n,N}(x)\sqrt{w^{(\alpha,\beta)}_N(x)} = (-1)^x\tilde{q}^{(\alpha,\beta)}_{N-n,N}(x)\sqrt{\tilde{w}^{(\alpha,\beta)}_N(x)}.$$

Insert this into the formulas above and define the new kernel

$$K^{\mathrm{L}}_{a,b,c}(r,x;\,s,y) := (-1)^{y-x}\sqrt{\omega_s(y)\omega_r(x)^{-1}\tilde{\omega}_r(x)\tilde{\omega}_s(y)^{-1}}K(r,x;\,s,y).$$

This kernel gives the same correlation functions as K, since the extra factors cancel out in the determinants. The new kernel can be written as

$$K^{\mathrm{L}}_{a,b,c}(r,x;\ s,y) =$$
$$\sum_{n=a}^{\infty}\sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}}q^{(b_r,a_r)}_{\gamma_r-n,\gamma_r}(x)q^{(b_s,a_s)}_{\gamma_s-n,\gamma_s}(y)\sqrt{w^{(b_r,a_r)}_{\gamma_r}(x)w^{(b_s,a_s)}_{\gamma_s}(y)},$$

when $s \geq r$, and

$$K^{\mathrm{L}}_{a,b,c}(r,x;\ s,y) =$$
$$-\sum_{n=0}^{a-1}\sqrt{\frac{(a+s-1-n)!(a+b+c-r-1-n)!}{(a+r-1-n)!(a+b+c-s-1-n)!}}q^{(b_r,a_r)}_{\gamma_r-n,\gamma_r}(x)q^{(b_s,a_s)}_{\gamma_s-n,\gamma_s}(y)\sqrt{w^{(b_r,a_r)}_{\gamma_r}(x)w^{(b_s,a_s)}_{\gamma_s}(y)}$$

otherwise.

The change of variables $j := a-1-n$ puts these expressions on a simpler form, thereby proving the theorem. □

5.5 Asymptotics

Let $0 < p < 1$ be some real number. Let $\alpha = \gamma pN$, $\beta = \gamma(1-p)N$, $\tilde{x} = \lfloor pN + \sqrt{2p(1-p)N(1+\gamma^{-1})}x\rfloor$. Then

$$\sqrt[4]{2p(1-p)N(1+\gamma^{-1})}\sqrt{w^{(\alpha,\beta)}_{n,N}(\tilde{x})}q^{(\alpha,\beta)}_{n,N}(\tilde{x}) \longrightarrow (-1)^n\sqrt{e^{-x^2}}h_n(x) \tag{5.5}$$

uniformly on compact sets in x as $N\to\infty$. For completeness we give the proof of this result in the appendix.

We would like to apply this with $p=\frac{1}{2}$ and $\gamma=2$ to our kernel K^{L} and letting $a=b=c\to\infty$, i.e. we would like to take the limit

$$K(r,\xi;s,\eta) =$$
$$\lim_{N=a=b=c\to\infty}(-3N)^{r-s}\sqrt{3N/4}K^{\mathrm{L}}_{a,b,c}(r,\lfloor N/2+\xi\sqrt{3N/4}\rfloor;s,\lfloor N/2+\eta\sqrt{3N/4}\rfloor).$$

The factor $(-3N)^{r-s}$ cancels out in all determinants and is thus of no import. For $s\geq r$ we get

$$K(r,\xi;s,\eta) = \sum_{j=-\infty}^{-1}\sqrt{\frac{(s+j)!}{(r+j)!}}h_{r+j}(\xi)h_{s+j}(\eta)e^{-(\xi^2+\eta^2)/2} \tag{5.6}$$

and formally, if we ignore the fact that this turns into an infinite sum, for $s<r$ we get

$$K(r,\xi;s,\eta) = -\sum_{j=0}^{\infty}\sqrt{\frac{(s+j)!}{(r+j)!}}h_{r+j}(\xi)h_{s+j}(\eta)e^{-(\xi^2+\eta^2)/2}. \tag{5.7}$$

This expression can be simplified with the following lemma

Lemma 5.6. *Let H be the Heaviside function defined by equation (1.1) above. Then,*

$$(x-y)^{k-1}H(x-y) = \frac{(k-1)!}{\sqrt{2^k}} \sum_{n=-\infty}^{\infty} \sqrt{\frac{n!}{(n+k)!}} h_n(y) h_{n+k}(x) e^{-y^2}. \tag{5.8}$$

pointwise for $x \neq y$.

The proof is given in section 6.

In view of this result, the infinite series in 5.7 converges and the kernel K is exactly the GUE minor kernel K^{GUE}. The interpretation of this is the following. The distribution of the blue particles, properly rescaled, tends weakly to the distribution of the eigenvalues of GUE minors as the size of the diamond tends to infinity, equation (1.2). The only thing needed to make this a theorem is appropriate estimates of the Hahn polynomials to control the convergence to the infinite sum.

6 Proof of lemmas

Proof of lemma 5.6. As the Hermite polynomials are orthogonal, there is an expansion of the function in the left hand side of (5.8) of the form

$$(x-y)^{k-1}H(x-y) = \sum_{n=-\infty}^{\infty} c_n(y) H_n(x), \tag{6.1}$$

where H_n is the n:th Hermite polynomial, as defined in for example (KS98), and where the coefficients are given by

$$c_n(y) = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} (x-y)^{k-1} H(x-y) H_n(x) e^{-x^2} dx. \tag{6.2}$$

It is known that $e^{-x^2}H_n(x) = -\frac{d}{dx}(e^{-x^2}H_{n-1}(x))$. Integration by parts and limiting the integration interval according to the Heaviside function gives

$$\int_y^{\infty} (x-y)^{k-1} H_n(x) dx = \int_y^{\infty} (k-1)(x-y)^{k-2} H_{n-1}(x) dx$$

Repeating this process $k-1$ times gives

$$c_n(y) = \frac{(k-1)! e^{-y^2} H_{n-k}(y)}{2^n n! \sqrt{\pi}}.$$

Hence,

$$\begin{aligned}(x-y)^{k-1}H(x-y) &= \sum_{n=-\infty}^{\infty} \frac{(k-1)!}{2^n n! \sqrt{\pi}} H_{n-k}(y) H_n(x) e^{-y^2} \\ &= \frac{(k-1)!}{\sqrt{2^k}} \sum_{n=-\infty}^{\infty} \sqrt{\frac{n!}{(n+k)!}} h_n(y) h_{n+k}(x) e^{-y^2}.\end{aligned} \tag{6.3}$$

$\square$

Proof of lemma 3.15. Assume first that $r \le s$. By proposition 3.14 we have to consider the integral

$$\frac{1}{(2\pi i)^2}\int_{\gamma_{r_2}} dz \int_{\gamma_{r_1}} \frac{dw}{w}\frac{1}{z-w}e^{Nf(z)-Nf(w)}\frac{w^{\eta\sqrt{2bN}}}{z^{\eta\sqrt{2bN}}}\frac{(1-qw)^2}{(1-qz)^r}\frac{(w-q)^M}{(z-q)^M},$$

where γ_r is a circle around the origin with radius r oriented anticlockwise, $q < r_1 < r_2 < 1/q$, and

$$f(z) = \log(z-q) - (1-q^2)^{-1}\log z. \tag{6.4}$$

(Here we have ignored the difference between $aN + \xi\sqrt{2bN}$ and its integer part.) Note that $f'(z) = 0$ gives $z = 1/q$. This leads us to choose

$$r_1 = \frac{1}{q} - \frac{2}{a\sqrt{N/2}},$$

and to deform γ_{r_2} to a circle Γ oriented clockwise around $1/q$ with radius $1/a\sqrt{N/2}$. The specific choice of radii are convenient for the computations below. Choose

$$g(r,\xi,N) = 2^{-r/2}e^{-\xi^2/2}q^{-\xi\sqrt{2bN}}\left(\frac{q}{a\sqrt{N/2}}\right)^r.$$

Then,

$$\begin{aligned}
&\frac{g(r,\xi,N)}{g(s,\eta,N)}\sqrt{2bN}K_{N,M}^{\mathrm{PNG}}(r,\lfloor aN+\xi\sqrt{2bN}\rfloor;\, s,\lfloor aN+\eta\sqrt{2bN}\rfloor) \\
&\qquad = \sqrt{2^{s-r}e^{\eta^2-\xi^2}}q^{(\eta-\xi)\sqrt{2bN}}\left(\frac{q}{a\sqrt{N/2}}\right)^{r-s}\frac{\sqrt{2bN}}{(2\pi i)^2} \\
&\qquad \times \int_\Gamma dz \int_{\gamma_{r_1}} \frac{dw}{w}\frac{1}{z-w}e^{Nf(z)-Nf(w)}\frac{w^{\eta\sqrt{2bN}}}{z^{\eta\sqrt{2bN}}}\frac{(1-qw)^2}{(1-qz)^r}\frac{(w-q)^M}{(z-q)^M}.
\end{aligned} \tag{6.5}$$

Parameterize γ_{r_1} by $w(t) = r_1 e^{itE_n}$, $-\pi/E_N \le t \le \pi/E_N$, $E_N = q/a\sqrt{N/2}$. We have

$$\begin{aligned}
\mathrm{Re}(f(w(0)) - f(w(t))) &= \ln\left|\frac{w(0)-q}{w(t)-q}\right| \\
&= -\frac{1}{2}\ln\left(1 + \frac{2r_1 q(1-\cos E_N t)}{(r_1-q)^2}\right) \\
&\le -\frac{1}{2}\ln\left(1 + q^2(1-\cos E_N t)\right),
\end{aligned}$$

for N large enough. Since $\cos x \le 1 + x^2/8$ when $|x| \le \pi$, the last expression is

$$\le -\frac{1}{2}\ln\left(1 + q^2E_N^2t^2/8)\right) \le -Ct^2/N$$

for $|t| \le \pi/E_N$, where $C > 0$ is a constant depending only on q. Hence,

$$\mathrm{Re}\, N(f(w(0)) - f(w(t))) \le -Ct^2 \tag{6.6}$$

for $|t| \leq \pi/E_n$ with $C > 0$.

In the right hand side of (6.5) we make the change of variables

$$z = z(u) = 1/q - u/a\sqrt{N/2} \tag{6.7}$$

with u on the unit circle oriented anticlockwise. We obtain the integral

$$\begin{aligned}\sqrt{2^{s-r}e^{\eta^2-\xi^2}}\,\frac{2iq\sqrt{b}}{(2\pi i)^2 a}\int_{\gamma_1} du \int_{-\pi/E_N}^{\pi/E_N} dt \frac{1}{a\sqrt{N/2}(z(u)-w(t))} e^{N(f(z(u))-f(w(t)))} \\ \times \frac{(qw(t))^{\eta\sqrt{2bN}}}{(qz(u))^{\xi\sqrt{2bN}}}\left(\frac{q}{a\sqrt{N/2}}\right)^{r-s}\frac{(1-qw(t))^s}{(1-qz(u))^r}\frac{(w(t)-q)^M}{(z(u)-q)^M}.\end{aligned} \tag{6.8}$$

Note that $q\sqrt{b}/a = 1$. Also,

$$f(1/q + h) = f(1/q) - a^2h^2/2 + O(h^3) \tag{6.9}$$

for $|h|$ small. Hence, for N sufficiently large,

$$N(f(z(u)) - f(w(0))) = -u^2 + 4 + h_N(u)/\sqrt{N}, \tag{6.10}$$

where $h_N(u)$ is bounded for $|u| = 1$. We have

$$\frac{(qw(t))^{\eta\sqrt{2bN}}}{(qz(t))^{\xi\sqrt{2bN}}} = \left(1 - \frac{2q}{a\sqrt{N/2}}\right)^{\eta\sqrt{2bN}} e^{2i\eta t}\left(1 - \frac{qu}{a\sqrt{a\sqrt{N/2}}}\right)^{-\xi\sqrt{2bN}}. \tag{6.11}$$

By the inequality $(1 + x/n)^n \leq e^x$ for $x > -n$, $n \geq 1$, the right hand side in (6.10) has a bound independent of N. We also have

$$a\sqrt{N/2}|z(u) - w(t)| \geq 1 \tag{6.12}$$

for $u \in \gamma_1$, $|t| \leq \pi/E_N$, and

$$\left|\left(\frac{q}{a\sqrt{N/2}}\right)\frac{(1-qw(t))^s}{(1-qz(t))^r}\frac{(w(t)-q)^M}{(z(u)-q)^M}\right| \leq CN^{s/2} \tag{6.13}$$

for $u \in \gamma_1$, $|t| \leq \pi/E_N$, by (6.7) and the definition of $w(t)$.

It follows from (6.6), (6.10), (6.11), (6.12) and (6.13) that the part of the integral in (6.8) where the t-integration is restricted to $N^{1/3} \leq |t| \leq \pi/E_N$ can be bounded by

$$CN^{s/2}\int_{|t|\geq N^{1/3}} e^{-Ct^2}\,dt,$$

which goes to 0 as $N \to \infty$. When $|t| \leq N^{1/3}$ we have

$$\left|\frac{a\sqrt{N/2}}{q}(1 - qw(t)) - (2 - it)\right| \leq \frac{C}{N^{1/6}}. \tag{6.14}$$

Hence, for $|t| \le N^{1/3}$ we have the bound

$$\left| \left(\frac{1}{a\sqrt{N/2}} \right)^{r-s} \frac{(1-qw(t))^s}{(1-qz(u))^r} \frac{(w(t)-q)^M}{(z(u)-q)^M} \right| \le C, \tag{6.15}$$

and we see that the part of the integral in (6.8) where $|t| \le N^{1/3}$ has a uniform bound for ξ, η in a compact set. This proves claim (2) in lemma 3.15 for $r \le s$.

It also follows from (6.6), (6.7), (6.9), (6.10), (6.11), (6.14), (6.15) and the dominated convergence theorem that the integral in (6.8) converges to

$$\frac{\sqrt{2^{s-r} e^{\eta^2-\xi^2}}}{2\pi^2 i} \int_{\gamma_1} du \int_{\mathbb{R}} dt \, \frac{1}{2-it-u} e^{2\xi u-u^2} e^{(2-it)^2-2\eta(u-it)} \frac{(2-it)^s}{u^r}. \tag{6.16}$$

Now let $v = 2 - it$. Then we should integrate v along the line $\operatorname{Re} v = 2$ from minus to plus infinity. We obtain the integral

$$\frac{\sqrt{2^{s-r} e^{\eta^2-\xi^2}}}{2(\pi i)^2} \int du \, \frac{e^{2\xi u-u^2}}{u^r} \int dt \, \frac{v^s}{v-u} e^{v^2-2\eta v}.$$

Expand $(v-u)^{-1}$ as a geometric series. This turns the expression into

$$\frac{\sqrt{2^{s-r} e^{\eta^2-\xi^2}}}{2(\pi i)^2} \sum_{k=0}^{\infty} \int du \, \frac{e^{2\xi u-u^2}}{u^{r-k}} \int dv \, v^{s-k-1} e^{v^2-2\eta v}$$

and we recognize the classical integral representations of the Hermite polynomials. The expression now becomes

$$\sum_{k=0}^{\infty} \sqrt{\frac{(s-k-1)!}{(r-k-1)!}} h_{r-k-1}(\xi) h_{s-k-1}(\eta) \sqrt{e^{-\xi^2-\eta^2}},$$

which proves claim (1) in the lemma in the case $r \le s$.

We now turn our attention to the case $r > s$. Deforming the w-contour through the z-contour in (3.3), we get the same integral as above save for a residue that we pick up at $z = w$. This is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta} \frac{1}{(1-qe^{i\theta})^{r-s}} \, d\theta. \tag{6.17}$$

We see that the argument above goes through for the remaining integral also when $r > s$. Using the well known formula

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

the integral in (6.17) becomes

$$\frac{1}{2\pi} \sum_{k=0}^{\infty} \binom{r-s+k-1}{k} q^k \int_{-\pi}^{\pi} e^{i(y-x+k)\theta} \, d\theta.$$

It is readily solved as

$$\begin{cases} \binom{r-s+x-y-1}{x-y} q^{x-y} & \text{if } y \le x \\ 0 & \text{if } y > x. \end{cases}$$

With our rescaling, $x = aN + \xi\sqrt{2bN}$ and $y = aN + \eta\sqrt{2bN}$ and the factors $g(r,\xi,N)/g(s,\eta,N)$ we see that the integral in (6.17), this is

$$\sqrt{2^{s-r}e^{\eta^2-\xi^2}}q^{(\eta-\xi)\sqrt{2bN}}\left(\frac{q}{a\sqrt{N/2}}\right)^{r-s}\sqrt{2bN}\frac{\Gamma(r-s+(\xi-\eta)\sqrt{2bN})}{\Gamma((\xi-\eta)\sqrt{2bN})(r-s-1)!}q^{(\xi-\eta)\sqrt{2bN}}H(\xi-\eta) \tag{6.18}$$

where H is the Heaviside function. As $N \to \infty$ we get the limit

$$\sqrt{e^{\eta^2-\xi^2}2^{r-s}}\frac{(\xi-\eta)^{r-s-1}}{(r-s-1)!}H(\xi-\eta), \tag{6.19}$$

at least for $\xi \neq \eta$. The case $\xi = \eta$ is a set of measure zero and is not important. Together with the result for the double integral this completes the proof of claim (1). It remains to show the estimate in claim (2) in this case. But this is easy. The expression in (6.18) is the exact solution of integral (6.17), and since this is bounded in N for ξ, η in a compact set, claim (2) follows. □

Proof of lemma 4.2. Assume first that $r \leq s$. By proposition 4.1 we have to consider the integral

$$\frac{\sqrt{N/2}}{(2\pi i)^2}\int_{\gamma_{r_2}}dz\int_{\gamma_{r_1}}\frac{dw}{w}\frac{1}{z-w}e^{N(f(z)-f(w))}\frac{w^{s+\sqrt{N/2}\eta}}{z^{r+\sqrt{N/2}\xi}}\frac{(1-w)^s}{(1-z)^r}\frac{(1+z)^r}{(1+w)^s},$$

where γ_r is a circle around -1 with radius r oriented anticlockwise, $1 < r_1 < r_2 < 2$ and

$$f(z) = \frac{1}{2}\ln z - \ln(1+z).$$

(Here we have ignored the difference between $N/2 + \xi\sqrt{N/2}$ and its integer part.) In the proof of lemma 3.15 we could chose the contours of integration as circles centred at the origin. This cannot be done here.

Note that $f'(z) = 0$ gives $z = 1$. This leads us to choose

$$r_1 = 2 - \frac{2}{\sqrt{N/8}}$$

and to deform γ_{r_2} to a circle Γ oriented clockwise around 1 with radius $1/\sqrt{N/8}$. The specific choice of radii are convenient for the computations below. Choose

$$g(r,\xi,N) = \sqrt{N^{-r}e^{-\xi^2}}.$$

Then,

$$\frac{g(r,\xi,N)}{g(s,\eta,N)}\sqrt{N/2}K^{\mathrm{A}}(r,\lfloor N/2+\xi\sqrt{N/2}\rfloor;\, s,\lfloor N/2+\eta\sqrt{N/2}\rfloor)$$
$$= \sqrt{N^{s-r}e^{\eta^2-\xi^2}}\frac{\sqrt{N/2}}{(2\pi i)^2}\int_{\Gamma}dz\int_{\gamma_{r_1}}\frac{dw}{w}\frac{1}{z-w}e^{N(f(z)-f(w))}\frac{w^{s+\eta\sqrt{N/2}}}{z^{r+\xi\sqrt{N/2}}}\frac{(1-w)^s}{(1-z)^r}\frac{(1+z)^r}{(1+w)^s}. \tag{6.20}$$

Parameterize γ_{r_1} by

$$w(t) = -1 + r_1 e^{itE_N}, \tag{6.21}$$

for $-\pi/E_N \le t \le \pi/E_N$, $E_N = 1/\sqrt{N/2}$. We have

$$\begin{aligned}\operatorname{Re}(f(w(0)) - f(w(t))) &= \frac{1}{2}\ln\left|\frac{w(0)}{w(t)}\right| \\ &= -\frac{1}{4}\ln\left(1 + \frac{2r_1(1-\cos E_N t)}{(r_1-1)^2}\right) \\ &\le -\frac{1}{4}\ln\left(1 + \frac{1}{2}(1-\cos E_N t)\right)\end{aligned}$$

for large enough N. Again $\cos x \le 1 - x^2/8$ when $|x| \le \pi$, the last expression is

$$\le -\frac{1}{4}\ln\left(1 + E_N^2 t^2/16\right) \le -Ct^2/N$$

for $|t| \le \pi/E_N$, where $C > 0$ is an absolute constant. Hence,

$$\operatorname{Re}(f(w(0)) - f(w(t))) \le -Ct^2/N \tag{6.22}$$

for $|t| \le \pi/E_N$, with $C > 0$.

In the right hand side of (6.20) we make the change of variables

$$z = z(u) = 1 - u/\sqrt{N/8} \tag{6.23}$$

with u on the unit circle oriented anticlockwise, denoted γ. We obtain the integral

$$\begin{aligned}&\sqrt{N^{s-r}e^{\eta^2-\xi^2}}\frac{iE_N}{2\pi^2}\int_\gamma du \int_{-\pi/E_N}^{\pi/E_N} dt \frac{1}{z(u)-w(t)} e^{N(f(z(u))-f(w(t)))} \\ &\qquad \times \frac{(w(t))^{s-1+\eta\sqrt{N/2}}}{(z(u))^{r+\xi\sqrt{N/2}}}\frac{(1-w(t))^s}{(1-z(u))^r}\frac{(1+z(u))^r}{(1+w(t))^{s-1}}.\end{aligned} \tag{6.24}$$

Note that

$$f(1+h) = f(1) - h^2/8 + O(h^3) \tag{6.25}$$

for small $|h|$. Hence, for N sufficiently large

$$N(f(z(u)) - f(w(0))) = -u^2 + 4 + h_N(u)/\sqrt{N}, \tag{6.26}$$

where $h_N(u)$ is bounded for $|u| = 1$. We have

$$\left|\frac{(w(t))^{s-1+\eta\sqrt{N/2}}}{(z(u))^{r+\xi\sqrt{N/2}}}\right| \le 3^{s-1+\eta\sqrt{N/2}}\left(1 - \frac{u}{\sqrt{N/8}}\right)^{-r-\xi\sqrt{N/2}} \le C3^{s+\eta\sqrt{N/2}} \tag{6.27}$$

for some constant $C > 0$ depending on r. We also have

$$\sqrt{N/8}|z(u) - w(t)| \ge 1 \tag{6.28}$$

for $u \in \gamma$ and $|t| \le \pi/E_N$, and

$$\left|\sqrt{N^{s-r}}\frac{(1-w(t))^s}{(1-z(u))^r}\frac{(1+z(u))^r}{(1+w(t))^{s-1}}\right| \le CN^{s/2} \tag{6.29}$$

for $u \in \gamma$, $|t| \leq \pi/E_N$, by (6.23) and (6.21).

It follows from (6.22), (6.26), (6.27), (6.28) and (6.29) that the part of the integral (6.24) where the t-integration is restricted to $N^{1/3} \leq |t| \leq \pi/E_N$ can be bounded by

$$CN^{s/2}3^{s+\eta\sqrt{N/2}} \int_{|t|\geq N^{1/3}} e^{-Ct^2}\, dt,$$

which tends to 0 as $N \to \infty$. When $|t| \leq N^{1/3}$ and $u \in \gamma$, we have

$$\left| \frac{(w(t))^{s-1+\eta\sqrt{N/2}}}{(z(u))^{r+\xi\sqrt{N/2}}} \right| \leq C \tag{6.30}$$

and

$$\left| \sqrt{N^{s-r}} \frac{(1-w(t))^s}{(1-z(u))^r} \frac{(1+z(u))^r}{(1+w(t))^{s-1}} \right| \leq C, \tag{6.31}$$

where C depends on s, r and η but is independent of N.

Hence, we see that the part of the integral in (6.24) where $|t| \leq N^{1/3}$ has a uniform bound for ξ and η in a compact set. This proves claim (2) for $r \leq s$.

It also follows from (6.22), (6.23), (6.25), (6.26), (6.28), (6.29), (6.30), (6.31) and the dominated convergence theorem that the integral in (6.24) converges to

$$\sqrt{2^{s-r}e^{\eta^2-\xi^2}}\, \frac{i}{2\pi^2} \int_\gamma du \int_{\mathbb{R}} dt \frac{1}{(2-it)-u} e^{(2-it)^2-2(2-it)\eta} e^{-u^2+2u\xi} \frac{(2-it)^s}{u^r},$$

which is exactly the integral in (6.16). This proves claim (1) in the lemma in the case $r \leq s$.

For $r > s$ we can deform the contours one through the other to get the same integral as we solved above. On the way we pick up the residue of a pole at $z = w$. It is

$$\frac{1}{2\pi i} \int_\gamma \frac{dw}{w} w^{-(r-s)-(x-y)} \left(\frac{1+w}{1-w} \right)^{r-s}, \tag{6.32}$$

where $x = \lfloor N/2 + \xi\sqrt{N/2} \rfloor$ and $y = \lfloor N/2 + \eta\sqrt{N/2} \rfloor$. The argument above goes through for the remaining integral also when $r > s$. We see that if $\eta > \xi$, then $x - y \to -\infty$ and this last integral is zero. For simplicity, let $k = r - s$ and $\beta = (x-y)/\sqrt{N/2}$. The coefficient in front of w^j in the expansion of $[(1+w)/(1-w)]^k$ is

$$\frac{1}{2\pi i} \int_\gamma \frac{dw}{w} w^{-j} \left(\frac{1+w}{1-w} \right)^k = \sum_{i=0}^{k} \binom{k}{i} \binom{j+i-1}{j} (-1)^{k-i} 2^i.$$

One then sees that the $i = k$ term dominates when N is large.

$$\left| \sum_{i=0}^{k-1} \binom{k}{i} \binom{j+i-1}{j} (-1)^{k-i} 2^i \right| \leq CN^{k-1}. \tag{6.33}$$

Keeping only the $i = k$ term and plugging in our rescaling and the factors $g(r,\xi,N)/g(s,\eta,N)$, we see that the integral in (6.32) is

$$\sqrt{e^{\eta^2-\xi^2}N^{s-r}}\sqrt{N/2}\, 2^{r-s} \binom{\beta\sqrt{N/2} + 2(r-s) - 1}{r-s-1} H(\xi - \eta) \tag{6.34}$$

for $\xi \geq \eta$. When $\xi \neq \eta$ this tends to

$$\sqrt{e^{\eta^2-\xi^2}2^{r-s}}\frac{(\xi-\eta)^{r-s-1}}{(r-s-1)!}H(\xi-\eta)$$

as $N \to \infty$ which together with the corresponding result for the double integral settles claim (1) in the case $r > s$.

Claim (2) in this case follows from the corresponding result for the double integral,(6.33) and the boundedness of the expression in (6.34). $\square$

A Asymptotics for Hahn polynomials

The Hahn polynomials, as they are defined in (KS98), satisfy

$$\sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}Q_m(x;\alpha,\beta,N)Q_n(x;\alpha,\beta,N) = (d_{n,N}^{(\alpha,\beta)})^2\delta_{nm}$$

where

$$(d_{n,N}^{(\alpha,\beta)})^2 = \frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}.$$

The Hermite polynomials are defined as usual:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} H_n(x)H_m(x)\,dx = 2^n n!\delta_{nm}. \tag{A.1}$$

With this notation, the following well known limit theorem holds.

Theorem A.1. *Let $0 < p < 1$ and $\gamma \geq 0$. Let $\tilde{x} = \lfloor pN + x\sqrt{2p(1-p)N(1+\gamma^{-1})}\rfloor$ and*

$$f_{n,N} = (-1)^n\sqrt{\binom{N}{n}2^n n!\left(\frac{p}{1-p}\right)^n\left(\frac{\gamma}{1+\gamma}\right)^n}$$

$$E_n(x) = f_{n,N}Q_n(\tilde{x};\gamma pN,\gamma(1-p)N,N). \tag{A.2}$$

Then

$$E_n(x) - H_n(x) = O(\sqrt{N^{-1}})$$

uniformly on compact sets.

Proof. The idea is induction on n. To start with, $Q_0(y,\alpha,\beta,N) = 1$ and we actually have $E_0(x) = H_0(x)$. For $n = 1$,

$$Q_1(y,\alpha,\beta,N) = 1 - \frac{2+\alpha+\beta}{(\alpha+1)N}x$$

so

$$\begin{aligned} E_1(x) &= -\sqrt{2N\left(\frac{p}{1-p}\right)\left(\frac{\gamma}{1+\gamma}\right)}\left(1 - \frac{2+\gamma N}{(\gamma pN+1)N}\left(pN + \sqrt{2p(1-p)N(1+\gamma^{-1})}x\right)\right) \\ &= \cdots = H_1(x) + O(\sqrt{N^{-1}}). \end{aligned}$$

Now assume that the theorem is true for n and $n-1$. We wish to show that it is true for $n+1$. There are three term recursion formulas for both Hahn and Hermite polynomials. Let

$$A_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$$
$$C_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Then

$$A_n Q_{n+1}(x) = (A_n + C_n - x)Q_n(x) - C_n Q_{n-1}(x) \tag{A.3}$$
$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{A.4}$$

Solving (A.2) for Q_n and inserting into (A.3) gives after some simplification the following:

$$E_{n+1}(x) = \frac{f_{n+1,N}}{f_{n,N}}\left(1 + \frac{C_n}{A_n} - \frac{\tilde{x}}{A_n}\right)E_n(x) - \frac{f_{n+1,N}}{f_{n-1,N}}\frac{C_n}{A_n}E_{n-1}(x). \tag{A.5}$$

Observe that under our scaling,

$$A_n = pN + O(N^{-1})$$
$$C_n = \frac{(1+\gamma)n(1-p)}{\gamma} + O(N^{-1}).$$

Inserting this into equation (A.5) and doing some manipulations gives

$$E_{n+1}(x) = \left(2x + O(N^{-1/2})\right)E_n(x) + \left(2n + O(N^{-1/2})\right)E_{n-1}(x),$$

which with our induction assumption is

$$= 2xH_n(x) + 2nH_{n-1}(x) + O(N^{-1/2})$$
$$= H_{n+1}(x) + O(N^{-1/2}).$$

This completes the proof. □

Applying Stirling's approximation to $d^{\alpha,\beta}_{n,N}$, $f_{n,N}$ and the weight function $w^{(\alpha,\beta)}_N(x)$, it is easy to show that

Corollary A.2. *As before, $\tilde{x} = pN + x\sqrt{2p(1-p)N(1+\gamma^{-1})}$.*

$$\sqrt[4]{2p(1-p)N(1+\gamma^{-1})}\,q^{(\alpha,\beta)}_{n,N}(\tilde{x})\sqrt{w^{\alpha,\beta}_N(\tilde{x})} \longrightarrow (-1)^n h_n(x)e^{-x^2/2} \tag{A.6}$$

as $N \to \infty$ if $\alpha/N \to p\gamma$ and $\beta/N \to (1-p)\gamma$.

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