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Semi-martingales and Rough Paths Theory

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Abstract

We prove that the theory of rough paths, which is used to define path-wise integrals and path-wise differential equations, can be used with continuous semi-martingales. We provide then an almost sure theorem of type Wong-Zakai. Moreover, we show that the conditions UT and UCV, used to prove that one can interchange limits and Itô or Stratonovich integrals, provide the same result when one uses the rough paths theory.

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1 Introduction

The theory of rough paths allows to give a meaning to integrals like

$$z_t = z_0 + \int_0^t g(x_s)\,\mathrm{d}x_s$$

and controlled differential equations like

$$y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}x_s$$

when x is a continuous, irregular path in a Banach space, and g and f are differential forms and vector fields smooth enough (See [21, 19, 16]). But for that, one needs to know the equivalent of the iterated integrals of x, that is $\int_{0\leq s_1\leq\cdots\leq s_k\leq T}\mathrm{d}x_{s_1}\otimes\cdots\otimes\mathrm{d}x_{s_k}$, and to use the topology of p-variation, which is defined using the semi-norm

$$\mathrm{Var}_{p,[0,T]}(x) = \left(\sup_{\text{partition }\Pi\text{ of }[0,T]}\sum_{i=1}^{k-1}|x(t_{i+1})-x(t_i)|^p\right)^{1/p}, \tag{1}$$

where we use the convention that the points of the partitions Π are $t_1 \leq t_2 \leq \cdots \leq t_k$. The real p defines how irregular the path x is, and there is no canonical way to define the iterated integrals of x up to the order $\lfloor p\rfloor$.

However, for a large class of stochastic processes, it is possible to define the equivalent of the iterated integrals for the trajectories. Although the rough path theory is completely deterministic, it has been used for a large class of processes, including Brownian motion, some Gaussian processes in a Banach space [18], fractional Brownian motion [5], free Brownian motion [3], symmetric Markov process [1],...

In this article, we study the use of rough paths theory for general continuous semi-martingales. Let us remark first that any one-dimensional, continuous local martingale M may be written as $M_t = B_{\langle M\rangle_t}$, where B is a Brownian motion and $\langle M\rangle$ the quadratic variation of M. Since almost every trajectory of the Brownian motion is $1/p$-Hölder continuous as soon as $p > 2$, then it is also of finite p-variation for any $p > 2$. Hence, as a M-continuous time-change, see [26], does no change the p-variation, there exists a modification of M whose trajectories are of finite p-variation for any $p > 2$ (See Lemma 2). Moreover, if X is a semi-martingale, the following Itô and Stratonovich integrals exist

$$\Theta_{s,t}(X) = \int_s^t (X_r - X_s)\otimes\,\mathrm{d}X_r \text{ and } \overline{\Theta}_{s,t}(X) = \int_s^t (X_r - X_s)\otimes\circ\mathrm{d}X_r.$$

If $\Theta(X)$ and $\overline{\Theta}(X)$ are of finite $p/2$-variation (See Proposition 1), then $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ are two rough paths of finite p-variations lying above X and one may use the results from the theory of rough paths, especially the continuity of integrals and solutions of differential equations.

The main idea of the proof is to use a X continuous random time-change φ such that $X_\varphi = X_{\varphi(0)} + M_\varphi + V_\varphi$ remains a semi-martingale and such that M_φ and V_φ are Hölder continuous trajectories. But, X_φ is then defined on a random time interval $[0, \widehat{T}]$. Yet, the value of $\widehat{T}$ depends on $\langle M \rangle_T$ and $\mathrm{Var}_{1,[0,T]}(V)$, which are assumed to be controlled.

Basically, we prove two results:

- If $(X^n)_{n\in\mathbb{N}}$ is a family of semi-martingales converging in distribution to X and satisfying the conditions UT (*uniformly tight*) or UCV (*Uniformly controlled variations*), then $(X^n, \Theta(X^n))_{n\in\mathbb{N}}$ and $(X^n, \overline{\Theta}(X^n))_{n\in\mathbb{N}}$ converge respectively in distribution to $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ as rough paths in the topology induced by the p-variation distance. The convergence under the topology of Skorokhod is kwnon since the papers of Mémin and Słomiński [22], and Jakubowski, Mémin and Pagès [11]. Hence, this is fully coherent with the results concerning interchanging limits and stochastic integrals in the semi-martingales theory.

- For almost every trajectory of the continuous semi-martingale X, there exists a family $(X(n))_{n\in\mathbb{N}}$ of piecewise linear approximations of X such that $(X(n), \overline{\Theta}(X(n)))_{n\in\mathbb{N}}$ converges almost surely in the appropriate topology to $(X, \overline{\Theta}(X))$. In other words, one obtains the almost sure convergence of the ordinary integral (resp. the ordinary differential equation)
$$\begin{aligned} Z(n)_t &= Z_0 + \int_0^t f(X(n)_s)\,\mathrm{d}X(n)_s \\ (\text{resp. } Y(n)_t &= Y_0 + \int_0^t g(Y(n)_s)\,\mathrm{d}X(n)_s) \end{aligned}$$
to the Stratonovich integral (resp. the stochastic differential equation) $Z_t = Z_0 + \int_0^t f(X_s) \circ \mathrm{d}X_s$ (resp. $Y_t = Y_0 + \int_0^t g(Y_s) \circ \mathrm{d}X_s$).

A similar convergence result is also given for Itô's integrals and Itô's SDEs.

The particular case of the Brownian motion was already known (See [20, 1],...) and was developed initially in [27]. In this Ph.D. thesis, this method can be used for martingales with α-Hölder continuous trajectories for $\alpha < 1/2$. Thus, these almost-sure Wong-Zakai results are not surprising. Here, we extend this result to a general continuous semi-martingales, and our proof includes recent techniques in the theory of rough path, that leads to some simplification of results.

Besides, these proofs show the role played by Hölder continuous paths among paths of finite p-variation, and that it is not a big deal to use Hölder continuous path instead of path of finite p-variation using a time-change. Moreover, this time-change gives us information on the way to get some uniform control for a family of paths, since we are reduced in controlling the length of the image of the time interval on which the path is initially defined. Then, the conditions UT and UCV appear naturally in this context.

Recent results show that the *enhanced Brownian motion*, that is a multiplicative functional lying above a Brownian motion, could be manipulated as the Brownian motion regarding many of its properties: support theorem, large deviation result, ... See [20, 7, 8].

Hence, one could think that this technique of using a time-change could be applied to generalize almost immediately to continuous semi-martingales some results given for the Brownian motion, as long as only the martingale property of the Brownian motion is involved in their proofs.

2 Rough paths

We refer the reader to [21, 19] or [16] for a detailed insight on the theory of rough paths and the objects we introduce now.

Let N be a fixed integer. Throughout all this article, we consider semi-martingales with values in $\mathbb{R}^N$, and we denote by $|\cdot|$ the norm $|x| = \sup_{k=1,\dots,N}|x^k|$ for $x = (x^1,\dots,x^N)$.

For a continuous function x from $[0,T]$ to $\mathbb{R}^N$, we denote by $\operatorname{Var}_{p,[0,T]}(x)$ its p-variation defined by (1).

Denote by $\mathbb{V}^p([0,T];\mathbb{R}^N)$ the Banach space of continuous functions of finite p-variation with the norm $\operatorname{Var}_{p,[0,T]}(\cdot)+\|\cdot\|_\infty$. Note that $\mathbb{V}^p([0,T];\mathbb{R}^N)$ is not separable.

Set $\Delta^+ = \{\, 0 \leq s \leq t \leq T \,\}$.

For a continuous function x from Δ^+ to $\mathbb{R}^N$, denote also by $\operatorname{Var}_{p,[0,T]}(x)$ its p-variation, that is

$$\operatorname{Var}_{p,[0,T]}(x) = \left(\sup_{\text{partition } \{t_i\}_{i=1,\dots,j} \text{ of } [0,T]} \sum_{i=1}^{j-1} |x(t_i,t_{i+1})|^p \right)^{1/p}.$$

If $x(s,t) = y(t) - y(s)$ for some continuous function y, then $\operatorname{Var}_{p,[0,T]}(x) = \operatorname{Var}_{p,[0,T]}(y)$.

We equip $\mathbb{R}^N \otimes \mathbb{R}^N$ with a norm $\|\cdot\|_{\mathbb{R}^N\otimes\mathbb{R}^N}$ such that $\|x\otimes y\|_{\mathbb{R}^N\otimes\mathbb{R}^N} \leq |x| \times |y|$ for any $x, y \in \mathbb{R}^N$.

We denote by $\mathcal{MV}^p([0,T];\mathbb{R}^N)$ the space of functions $(x_0, x^1_{s,t}, x^2_{s,t})_{(s,t)\in\Delta^+}$ such that

$$x^1_{s,t} = x_t - x_s \text{ for a continuous function } x \text{ from } [0,T] \text{ to } \mathbb{R}^N, \tag{2a}$$

$$x^2 \text{ is continuous from } \Delta^+ \text{ to } \mathbb{R}^N \otimes \mathbb{R}^N, \tag{2b}$$

$$\mathrm{Var}_{p,[0,T]}(x^1) < +\infty, \tag{2c}$$

$$\mathrm{Var}_{p/2,[0,T]}(x^2) < +\infty, \tag{2d}$$

$$x^{2,i,j}_{s,t} = x^{2,i,j}_{s,u} + x^{2,i,j}_{u,t} + x^{1,i}_{s,u} \cdot x^{1,j}_{u,t} \text{ for } i,j \in \{1,\dots,N\} \tag{2e}$$

for all $0 \leq s \leq u \leq t \leq T$. Sometimes, it could be useful to use tensor product notations instead of indexes. This means that $\mathbf{x} = (x^1_{s,t}, x^2_{s,t})$ is seen as $\mathbf{x}_{s,t} = 1 + x^1_{s,t} + x^2_{s,t} \in \mathbb{R} \oplus \mathbb{R}^N \oplus (\mathbb{R}^N)^{\otimes 2}$ (Here, the starting point x_0 of x is not taken into account). Accordingly, (2e) could be rewritten as $x^2_{s,t} = x^2_{s,u} + x^2_{u,t} + x^1_{s,u} \otimes x^1_{u,t}$.

When there is no ambiguity, we identify (x_0, x^1) with x, that is a function on Δ^+ and a starting point with a function on $[0,T]$. Thus, (x_0, x^1, x^2) is also denoted by (x, x^2). The elements of $\mathcal{MV}^p([0,T];\mathbb{R}^N)$ are called *multiplicative functionals*.

The topology we used on $\mathcal{MV}^p([0,T];\mathbb{R}^N)$ is the one induced by the norm

$$\|(x_0, x^1, x^2)\| = |x_0| + \sup_{(s,t)\in\Delta^+} |(x^1, x^2)_{s,t}| + \mathrm{Var}_{p,[0,T]}(x^1) + \mathrm{Var}_{p/2,[0,T]}(x^2).$$

We end this section by a useful Lemma, that allows to estimate the distance in $\mathcal{MV}^p([0,T];\mathbb{R}^N)$ between two rough paths by using the pointwise distance of the increments between dyadic points of the difference of these paths.

Lemma 1 (See for example [9, 1, 18]). *Let $p > 2$ and $\gamma > p/2 - 1$. Let (X^1, X^2) and (Y^1, Y^2) be multiplicative functionals in $\mathcal{MV}^p([0,T];\mathbb{R}^N)$. Then for any partition $0 \leq s_1 \leq \dots \leq s_k \leq T$ of $[0,T]$, there exists a constant C depending only on p and γ such that, for $t^n_j = jT/2^n$*

$$\begin{aligned}\sum_{i=1}^{k-1} |X^2_{s_i,s_{i+1}} - Y^2_{s_i,s_{i+1}}|^{p/2} \leq C\Big(\sum_{n\geq 1} n^\gamma \sum_{j=0}^{2^n-1} |X^1_{t^n_j,t^n_{j+1}} - Y^1_{t^n_j,t^n_{j+1}}|^p\Big)^{1/2} \\ \times \sum_{n\geq 1} n^\gamma \sum_{j=0}^{2^n-1} \Big(|X^1_{t^n_j,t^n_{j+1}}|^p + |Y^1_{t^n_j,t^n_{j+1}}|^p\Big)^{1/2} \\ + \sum_{n\geq 1} n^\gamma \sum_{j=1}^{2^n-1} |X^2_{t^n_j,t^n_{j+1}} - Y^2_{t^n_j,t^n_{j+1}}|^p.\end{aligned}$$

3 The conditions UT and UCV for semi-martingales

Let X be a continuous semi-martingale with respect to a filtration $\mathcal{F}_\cdot = (\mathcal{F}_t)_{t\geq 0}$ with values in $\mathbb{R}^N$, which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_\cdot$ may be assumed to satisfy the usual conditions. There exists a unique decomposition of X as the sum of a local martingale M and a process V of locally of finite variation, both $\mathcal{F}_\cdot$-adapted. The decomposition $X = X_0 + M + V$ is called the *canonical decomposition* of X.

Consider a sequence $(X^n)_{n\in\mathbb{N}}$ of continuous $\mathcal{F}_\cdot^n$-semi-martingales on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. To simplify the notation, denote $\mathbb{P}^n$ by $\mathbb{P}$ when there is no ambiguity.

In all this section, we consider the convergence of semi-martingales in the space of continuous functions with the uniform norm. The convergence in p-variation is considered in the next sections.

The following definitions and results are taken from [22] and the review article [13].

Definition 1 (Condition UT, Uniformly Tight). Let $\mathfrak{H}^n$ be the class of $\mathcal{F}_\cdot^n$-predictable, simple processes bounded by 1 on $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. The sequence $(X^n)_{n\in\mathbb{N}}$ is said to satisfy the *condition UT* if for each $t > 0$ and for any $\varepsilon > 0$, there exists C large enough such that

$$\sup_{n\in\mathbb{N}} \sup_{H\in\mathfrak{H}^n} \mathbb{P}\left[\,|H \cdot X_t^n| > C\,\right] < \varepsilon. \tag{UT}$$

Theorem 1. *Assume that $(X^n)_{n\in\mathbb{N}}$ satisfies the condition UT.*

(i) *The sequence $(X^n)_{n\in\mathbb{N}}$ is tight and any cluster point X of the sequence $(X^n)_{n\in\mathbb{N}}$ is a semi-martingale with respect to the smallest filtration $\mathcal{F}_\cdot^X$ which is right continuous it generates.*

(ii) *Let $(H^n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}_\cdot^n$-progressively measurable càdlàg processes such that $(H^n, X^n)_{n\in\mathbb{N}}$ converges to (H, X) in the Skorohod topology, then X is a semi-martingale with respect to the filtration generated by (H, X), and, when all the stochastic integrals are defined, $(H^n, X^n, H^n \cdot X^n)_{n\in\mathbb{N}}$ converges to $(H, X, H \cdot X)$.*

(iii) *The sequence $(X^n, \langle X^n\rangle)_{n\in\mathbb{N}}$ converges to $(X, \langle X\rangle)$.*

From now, consider a time $T > 0$ and restrict only to processes on $[0, T]$.

Definition 2 (Condition UCV, Uniformly Controlled Variations). The sequence $(X^n)_{n\in\mathbb{N}}$ of semi-martingales with canonical decompositions

$X^n = X_0^n + M^n + V^n$ is said to satisfy the *condition UCV* on $[0,T]$ if

$$(\langle M^n \rangle_T)_{n\in\mathbb{N}} \text{ is tight,} \tag{3}$$

$$(\mathrm{Var}_{1,[0,T]}(V^n))_{n\in\mathbb{N}} \text{ is tight.} \tag{4}$$

Remark 1. This definition is slightly different from the one in [13], but it is easily seen that both definitions are equivalent.

Theorem 2. (i) *Assume that the sequence $(X^n)_{n\in\mathbb{N}}$ converges weakly to X. Then $(X^n)_{n\in\mathbb{N}}$ satisfies the condition UT if and only if it satisfies the condition UCV.*

(ii) *Let $(M^n)_{n\in\mathbb{N}}$ be a sequence of local martingales converging in distribution to M. Then M is a local martingale and $(M^n)_{n\in\mathbb{N}}$ satisfies the conditions UT and UCV. In particular, $(\langle M^n \rangle_T)_{n\in\mathbb{N}}$ is tight.*

(iii) *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of continuous semi-martingales converging in distribution to X and satisfying the equivalent conditions UT or UCV. Assume that $X^n = X_0^n + M^n + V^n$ is the canonical decomposition of X^n. Then there exists a filtration $\mathcal{F}.$ such that X is a $\mathcal{F}.$-semi-martingale and there exists a local $\mathcal{F}.$-martingale M and a term locally of finite variation V $\mathcal{F}.$-adapted such that $(X^n, M^n, V^n)_{n\in\mathbb{N}}$ converges in distribution to (X, M, V).*

Counter-example 1. Of course, any sequence of semi-martingale does not necessarily satisfy the conditions UT and UCV. Let V be a smooth function from $\mathbb{R}$ to $\mathbb{R}$ which is one-periodic, and set

$$X_t^\varepsilon = B_t + \frac{1}{\varepsilon}\int_0^t b(X_s^\varepsilon/\varepsilon)\,\mathrm{d}s, \tag{5}$$

where B is a Brownian motion and $b = \nabla V$. Then, it is well known that X^ε is equal in distribution to $\varepsilon X_{\cdot/\varepsilon}$, where $X_t = B_t + \int_0^t b(X_s)\,\mathrm{d}s$, and that X^ε converges in distribution to $\sigma\beta$, where σ is a constant matrix which is not in general the identity matrix, and β is a Brownian motion (This is the homogenization problem: See for example [2]). Hence $(X^\varepsilon)_{\varepsilon>0}$ cannot satisfy the conditions UT and UCV, since it contradicts Theorem 1-(iii) (See also [15] and [17]).

In facts, it is easy to construct family of semi-martingales that does not satisfy the conditions UT and UCV. For that, one has consider semi-martingales where the term of finite-variation becomes a martingale when one passes to the limit. The example above comes from an homogenization problem and appears in a completely natural situation.

4 Semi-martingales and p-variation

For any one-dimensional, continuous local martingale M on $[0,T]$, there exists a one-dimensional Brownian motion B such that $M_t = B_{\langle M\rangle_t}$ for all $t\in[0,T]$. Let $p>2$ be fixed. Using the scaling property of the Brownian motion, for any $T>0$,

$$\mathbb{E}\left[\operatorname{Var}_{p,[0,T]}(B)\right]\leq C_p T^{1/2} \tag{6}$$

with $C_p=\mathbb{E}[\operatorname{Var}_{p,[0,1]}(B)]$. It follows that for any continuous local martingale M with values in $\mathbb{R}^N$ and any $C>0$, if B^i is such that $B^i_{\langle M^i\rangle_t}=M^i_t$,

$$\begin{aligned}\mathbb{P}\left[\operatorname{Var}_{p,[0,T]}(M)>C\right]=&\mathbb{P}\left[\sup_{i=1,\dots,N}\operatorname{Var}_{p,[0,\langle M^i\rangle_T]}(B^i)>C\right]\\ \leq&\sum_{i=1}^N\mathbb{P}\left[\operatorname{Var}_{p,[0,K]}(B^i)>C;\langle M^i\rangle_T\leq K\right]\\ &+\sum_{i=1}^N\mathbb{P}\left[\operatorname{Var}_{p,[0,\langle M^i\rangle_T]}(B^i)>C;\langle M^i\rangle_T>K\right]\\ \leq&\frac{C_{K,p}}{C}K^{1/2}+\sum_{i=1}^N\mathbb{P}\left[\langle M^i\rangle_T>K\right].\end{aligned} \tag{7}$$

Remark that if V is of finite variations, then for $N=1$ and for all $s\leq t\leq u$,

$$|V_t-V_s|^p+|V_u-V_t|^p\leq\left(\int_s^t|\mathrm{d}V_r|\right)^p+\left(\int_t^u|\mathrm{d}V_r|\right)^p\leq\left(\int_s^u|\mathrm{d}V_r|\right)^p.$$

Clearly,

$$\operatorname{Var}_{p,[0,T]}V\leq\operatorname{Var}_{1,[0,T]}(V). \tag{8}$$

The next lemma follows immediately from (7) and (8).

Lemma 2. *Let X be a $\mathcal{F}_\cdot$-semi-martingale with decomposition $X=X_0+M+V$. Then X is almost surely of finite p-variation as soon as $p>2$. Moreover, if $(X^n)_{n\in\mathbb{N}}$ is a sequence of semi-martingales satisfying the conditions UCV, then for any $\varepsilon>0$ there exists C large enough such that*

$$\mathbb{P}\left[\operatorname{Var}_{p,[0,T]}(X^n)>C\right]<\varepsilon. \tag{9}$$

This implies that if $(X^n)_{n\in\mathbb{N}}$ converges in distribution in $\mathbb{C}([0,T];\mathbb{R}^N)$ to X (which is then a semi-martingale), then $(X^n)_{n\in\mathbb{N}}$ converges in $\mathbb{V}^p([0,T];\mathbb{R}^N)$ to X for any $p>2$.

For a semi-martingale X, define $\Theta^{i,j}_{s,t}(X)$ by

$$\Theta^{i,j}_{s,t}(X) = \int_s^t (X^i_r - X^i_s)\,\mathrm{d}X^j_r$$

and $\Theta(X) = (\Theta^{i,j}_{s,t}(X); 0 \leq s \leq t \leq T, i, j \in \{1, \dots, N\})$.

Define also

$$\begin{aligned}\overline{\Theta}^{i,j}_{s,t}(X) &= \int_s^t (X^i_r - X^i_s)\circ\mathrm{d}X^j_r \\ &= \Theta^{i,j}_{s,t}(X) + \frac{1}{2}\left(\langle X^i, X^j\rangle_t - \langle X^i, X^j\rangle_s\right).\end{aligned}$$

Clearly, $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ satisfy (2a), (2b) and (2e). We have seen in Lemma 2 that they also satisfy (2c). To prove that they belong to $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$, it remains to prove that $\Theta(X)$ and $\overline{\Theta}(X)$ satisfy (2d).

Proposition 1. *Let X be a $\mathcal{F}_\cdot$-semi-martingale. Then $\Theta(X)$ and $\overline{\Theta}(X)$ are almost surely of finite $p/2$-variation for all $p > 2$. Consequently, both $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ belong to $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$ for all $p > 2$.*

Furthermore, if $(X^n)_{n\in\mathbb{N}}$ converges in distribution to X in $\mathbb{C}([0,T];\mathbb{R}^N)$ and satisfies the conditions UCV or UT, then $(X^n, \Theta(X^n))_{n\in\mathbb{N}}$ and $(X^n, \overline{\Theta}(X^n))_{n\in\mathbb{N}}$ converge respectively to $(X, \Theta(X))$ and $(X, \overline{\Theta}(X))$ in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$.

Proof. Let us decompose the semi-martingale X as $X = X_0 + M + V$. Besides, let V^+ and V^- be the increasing, continuous functions such that $V = V^+ - V^-$. We consider now the process (M, V^+, V^-) in $\mathbb{R}^{3N}$, and denote by $\mathrm{Var}_{p,[s,t]}(M, V^+, V^-)$ its p-variation. Associate to the semi-martingale X the function φ defined by

$$\varphi(t) = \inf\left\{ s > 0 \,\middle|\, \mathrm{Var}_{p,[0,s]}(M, V^+, V^-)^p > t \right\}.$$

The process $t \mapsto \mathrm{Var}_{p,[0,t]}(X)$ is continuous and $\mathcal{F}_\cdot$-adapted. Assume that $\varphi(t) = \varphi(s)$ for some $s < t$. Then for $y = M$ or $y = V^\pm$, $\mathrm{Var}_{p,[s,t]}(y) = 0$ since $\mathrm{Var}_{p,[0,s]}(y)^p + \mathrm{Var}_{p,[s,t]}(y)^p \leq \mathrm{Var}_{p,[0,t]}(y)^p$. Consequently, y is constant on $[s,t]$, and thus that X, M and V are constant on the intervals $[\varphi(t-), \varphi(t)]$ for all $t \in [0,T]$.

Let $\mathcal{F}^\varphi_\cdot = (\mathcal{F}^\varphi_t)_{t\geq 0}$ be the filtration defined by $\mathcal{F}^\varphi_t = \mathcal{F}_{\varphi(t)}$. Because M and V are constant on the intervals $[\varphi(t-), \varphi(t)]$, $M_\varphi = (M_{\varphi(t)})_{t\in[0,T]}$ is a local $\mathcal{F}^\varphi_\cdot$-martingale. Moreover, $(V_{\varphi(t)})_{t\in[0,T]}$ is locally of finite variation (See Propositions V.1.4 and V.1.5 in [26]).

Let us set $\widehat{T} = \mathrm{Var}_{p,[0,T]}(M, V^+, V^-)^p$. Then

$$\mathrm{Var}_{p,[0,\varphi(\widehat{T})]}(M, V^+, V^-) = \mathrm{Var}_{p,[0,T]}(M, V^+, V^-) \tag{10}$$

and

$$\operatorname{Var}_{p/2,[0,T]}\Theta(X)=\left(\sup_{t_0\leq\cdots\leq t_n\text{ partition of }[0,\widehat{T}]}\sum_{i=0}^{n-1}\Theta_{\varphi(t_i),\varphi(t_{i+1})}(X)^{p/2}\right)^{2/p}.$$

According to Propositions V.1.4 and V.1.5 in [26],

$$\int_{\varphi(s)}^{\varphi(t)}(X_r^i-X_{\varphi(s)}^i)\,\mathrm{d}X_r^j=\int_s^t(X_{\varphi(r)}^i-X_{\varphi(s)}^i)\,\mathrm{d}X_{\varphi(r)}^j$$

for $i,j\in\{1,\dots,N\}$. Thus, $\Theta_{s,t}(X_\varphi)=\Theta_{\varphi(s),\varphi(t)}(X)$ and

$$\operatorname{Var}_{p/2,[0,T]}\Theta(X)=\operatorname{Var}_{p/2,[0,\widehat{T}]}\Theta(X_\varphi). \tag{11}$$

Assume that the semi-martingale X satisfies

$$\mathbb{E}[\,\widehat{T}^{q/p}\,]=\mathbb{E}\left[\operatorname{Var}_{p,[0,T]}(M,V^+,V^-)^{q/p}\right]<+\infty \tag{12}$$

for some $q>p>2$.

The Burkholder-Davis-Gundy inequality may be applied: There exists some constant C_1 such that

$$\begin{aligned}\mathbb{E}&\left[\left|\int_{\varphi(s)}^{\varphi(t)}(X_r^i-X_{\varphi(s)}^i)\,\mathrm{d}M_r^j\right|^{q/2}\right]\\&\qquad\leq C_1\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)}|X_r^i-X_{\varphi(s)}^i|^2\,\mathrm{d}\langle M^j\rangle_r\right|^{q/4}\right]\\&\leq\frac{C_1}{2}\mathbb{E}\left[\sup_{r\in[\varphi(s),\varphi(t)]}|X_r^i-X_{\varphi(s)}^i|^q\right]+\frac{C_1}{2}\mathbb{E}\left[\,|\langle M^j\rangle_{\varphi(t)}-\langle M^j\rangle_{\varphi(s)}|^{q/2}\,\right].\end{aligned}$$

Applying the other side of the Burkholder-Davis-Gundy inequality, there exists some constant C_2 such that

$$\mathbb{E}\left[\,|\langle M^j\rangle_{\varphi(t)}-\langle M^j\rangle_{\varphi(s)}|^{q/2}\,\right]\leq C_2\mathbb{E}\left[\sup_{r\in[\varphi(s),\varphi(t)]}|M_r^j-M_{\varphi(s)}^j|^q\right].$$

Hence, it follows that for some constant $C>0$,

$$\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)}(X_r^i-X_{\varphi(s)}^i)\,\mathrm{d}M_r^j\right|^{q/2}\right]\leq C\mathbb{E}\left[\operatorname{Var}_{q,[\varphi(s),\varphi(t)]}(M,V^+,V^-)^q\right].$$

Still using the inequality $ab \leq (a^2+b^2)/2$ for any $a,b>0$,

$$\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} (X_r^i - X_{\varphi(s)}^i)\,\mathrm{d}V_r^{\pm,j}\right|^{q/2}\right] \\ \leq \frac{1}{2}\mathbb{E}\left[\sup_{r\in[\varphi(s),\varphi(t)]} |X_r^i - X_{\varphi(s)}^i|^q\right] + \frac{1}{2}\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} |\mathrm{d}V_r^{\pm,j}|\right|^q\right].$$

For any continuous function y and all $0 \leq s \leq t$,

$$\sup_{r\in[s,t]} |y_r - y_s|^q \leq \mathrm{Var}_{q,[s,t]}(y)^q. \tag{13}$$

Moreover, let us remark that $\mathrm{Var}_{q,[s,t]}(X) \leq 3^{1-1/q}\,\mathrm{Var}_{q,[s,t]}(M,V^+,V^-)$.

Besides, for any continuous, increasing function y of finite variation,

$$\left(\int_s^t \mathrm{d}y_s\right)^q \leq \mathrm{Var}_{q,[s,t]}(y)^q. \tag{14}$$

Applying (14) to V^+ and V^- and using (13) to M and X, one gets that for some constant C depending only on q,

$$\mathbb{E}\left[\,|\Theta_{\varphi(s),\varphi(t)}^{i,j}(X)|^{q/2}\,\right] \leq C\mathbb{E}\left[\,\mathrm{Var}_{q,[\varphi(s),\varphi(t)]}(M,V^+,V^-)^q\,\right].$$

Since $\mathrm{Var}_{p,[0,\varphi(s)]}(M,V^+,V^-)^p = s$ for all $s \geq 0$ and

$$\mathrm{Var}_{p,[0,s]}(y)^p + \mathrm{Var}_{p,[s,t]}(y)^p \leq \mathrm{Var}_{p,[0,t]}(y)^p$$

for all $0 \leq s \leq t$ and any continuous function y of finite p-variation,

$$\mathrm{Var}_{p,[\varphi(s),\varphi(t)]}(M,V^+,V^-)^p \leq |t-s|. \tag{15}$$

As $\mathrm{Var}_{q,[s,t]}(y) \leq \mathrm{Var}_{p,[s,t]}(y)$ and $a^q + b^q \leq (a+b)^q$ for any $q \geq 1$ and all $a,b>0$, it follows from (15) that

$$\mathrm{Var}_{q,[\varphi(s),\varphi(t)]}(M,V^+,V^-)^q \leq |t-s|^{q/p}.$$

It follows that

$$\mathbb{E}\left[\left|\int_{\varphi(s)}^{\varphi(t)} (X_r^i - X_{\varphi(s)}^i)\,\mathrm{d}X_r^j\right|^{q/2}\right] \leq K|t-s|^{q/p}$$

for some constant K that depends only on q. It follows that

$$\sum_{k=0}^{2^n-1} \mathbb{E}\left[\left|\int_{\varphi(\widehat{T}k/2^n)}^{\varphi(\widehat{T}(k+1)/2^n)} (X^i_r - X^i_{\varphi(\widehat{T}k/2^n)})\,\mathrm{d}X^j_r\right|^{q/2}\right] \leq 2^{n(1-q/p)}\mathbb{E}[\,\widehat{T}^{q/p}\,]$$

which is finite from (12). From (11) and Lemma 1 with $Y=0$, there exists some constant $c_{p,q,T}$ such that

$$\begin{aligned}\mathbb{E}\left[\,\mathrm{Var}_{q/2,[0,T]}\,\Theta(X)^{q/2}\,\right] &= \mathbb{E}\left[\,\mathrm{Var}_{q/2,[0,\widehat{T}]}\,\Theta(X_\varphi)^{q/2}\,\right]\\ &\leq c_{p,q,T}\mathbb{E}[\,\widehat{T}^{q/p}\,].\end{aligned}$$

Now, to consider the general case, fix K and let $\tau = \varphi(K)$. Then τ is a stopping time and for a $\mathcal{F}_\cdot$-adapted process Y, set $Y^\tau = Y_{\cdot\wedge\tau}$. The continuous process X^τ is a $\mathcal{F}_\cdot$-semi-martingales with canonical decomposition $X^\tau = X_0 + M^\tau + V^\tau$. Remark that $\tau > T$ is equivalent to $\widehat{T} \leq K$. Hence, it follows from (7) that for any $L > 0$,

$$\begin{aligned}&\mathbb{P}\left[\,\mathrm{Var}_{q/2,[0,T]}\,\Theta(X) > C\,\right]\\ &\qquad\leq \mathbb{P}\left[\,\mathrm{Var}_{q/2,[0,T]}\,\Theta(X^\tau) > C\,\right] + \mathbb{P}\left[\,\tau < T\,\right]\\ &\qquad\leq \frac{1}{C^{q/2}}\mathbb{E}\left[\,\mathrm{Var}_{q/2,[0,T]}\,\Theta(X^\tau)^{q/2}\,\right] + \mathbb{P}\left[\,\widehat{T} > K\,\right]\\ &\leq \frac{c_{q,p,T}}{C^{q/2}}K^{q/p} + \frac{C_{K,p}}{K}\sqrt{L} + \sum_{i=1}^N \mathbb{P}\left[\,\langle M^i\rangle_T > L^{1/p}\,\right] + \mathbb{P}\left[\,\mathrm{Var}_{1,[0,T]}(V^+,V^-) > L^{1/p}\,\right].\end{aligned}$$

Now, remark that $V^\pm_t = \frac{1}{2}(V_t \pm \mathrm{Var}_{1,[0,t]}(V))$ (See for example Section I.6 in [24]). Hence, $\mathrm{Var}_{1,[0,t]}(V^\pm) \leq \mathrm{Var}_{1,[0,t]}(V)$. Thus,

$$\mathbb{P}\left[\,\mathrm{Var}_{1,[0,T]}(V^+,V^-) > L^{1/p}\,\right] \leq \mathbb{P}\left[\,\mathrm{Var}_{1,[0,T]}(V) > L^{1/p}\,\right].$$

For any $\varepsilon > 0$, choosing first L, then K and C, it follows that

$$\mathbb{P}\left[\,\mathrm{Var}_{q/2,[0,T]}\,\Theta(X) > C\,\right] < \varepsilon. \tag{16}$$

Hence, $\Theta(X)$ is of finite $q/2$-variation almost surely.

Now, let $(X^n)_{n\in\mathbb{N}}$ be a sequence of semi-martingales satisfying the condition UCV. In (16), when $\varepsilon > 0$ is fixed, the choice of C is uniform in n, since with (16), if L is large enough such that

$$\sup_{n\in\mathbb{N}} \sum_{i=1}^N \mathbb{P}\left[\,\langle M^{i,n}\rangle_T > L^{1/p}\,\right] + \sup_{n\in\mathbb{N}} \mathbb{P}\left[\,\mathrm{Var}_{1,[0,T]}(V^n) > L^{1/p}\,\right] < 2\varepsilon,$$

then there exists C large enough such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left[\, \mathrm{Var}_{q/2,[0,T]}\, \Theta(X^n) > C \,\right] < 3\varepsilon. \tag{17}$$

If $(X^n)_{n\in\mathbb{N}}$ converges in distribution to X, it follows from the condition UT that $\Theta_{0,\cdot}(X^n)$ converges in distribution to $\Theta_{0,\cdot}(X)$ in $\mathbb{C}([0,T];\mathbb{R}^N)$. On the other hand, it follows from (9) and (17) that $(X^n,\Theta(X^n))_{n\in\mathbb{N}}$ is tight in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$, thanks to Corollary 6.1 in [16]. Thus, $(X^n,\Theta(X^n))$ converges to $(X,\Theta(X))$ in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$.

Concerning $\overline{\Theta}(M)$, note that $\langle M^i_\varphi, M^j_\varphi\rangle_t = \langle M^i, M^j\rangle_{\varphi(t)}$ for all $t \geq 0$. Thus, if $\Xi^{i,j}_{s,t}(M) = \frac{1}{2}(\langle M^i, M^j\rangle_t - \langle M^i, M^j\rangle_s)$, then

$$\mathrm{Var}_{p/2,[0,T]}\, \Xi^{i,j}(M) = \mathrm{Var}_{p/2,[0,\widehat{T}]}\, \Xi^{i,j}(M_\varphi).$$

Besides, one knows that

$$\mathbb{E}\left[\, |\Xi^{i,j}_{s,t}(M_\varphi)|^{q/2} \,\right] \leq \mathbb{E}\left[\, |\Xi^{i,i}_{s,t}(M_\varphi)|^{q} \,\right]^{1/2} + \mathbb{E}\left[\, |\Xi^{j,j}_{s,t}(M_\varphi)|^{q} \,\right]^{1/2}.$$

Thus, one has only to act as previously. □

Counter-example 2 *(Counter-example 1 (continuation)).* In the case of $(X^\varepsilon)_{\varepsilon>0}$ defined by (5), $(X^\varepsilon,\overline{\Theta}(X^\varepsilon))_{\varepsilon>0}$ converges in p-variation to $(\sigma\beta_t, \overline{\Theta}_{s,t}(\sigma\beta) + c(t-s))_{0\leq s\leq t\leq T}$, where c is an antisymmetric matrix (See [15, 17]). This implies that the limit of the stochastic integral $\int_0^\cdot f(X^\varepsilon_s)\circ \mathrm{d}X^\varepsilon_s$ is not $\int_0^\cdot f(\sigma\beta_s)\sigma\circ \mathrm{d}\beta_s$, but

$$\int_0^\cdot f(\sigma\beta_s)\sigma\circ\mathrm{d}\beta_s + \frac{1}{2}\int_0^\cdot c_{i,j}\left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}\right)(\sigma\beta_s)\,\mathrm{d}s.$$

5 An almost sure Wong-Zakai theorem

Let X be a $\mathcal{F}_\cdot$-semi-martingale with values in $\mathbb{R}^N$. We prove a result of type Wong-Zakai with respect piecewise-linear approximation of X converging to X in p-variation, together with their Lévy area.

We have seen in the previous section that $(X,\overline{\Theta}(X))$ is a multiplicative functional in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$ for any $p > 2$. From now, we set

$$X^1_{s,t} = X_t - X_s \text{ and } X^2_{s,t} = \overline{\Theta}_{s,t}(X).$$

Notation and hypothesis. Let $p > 2$ and K be a positive random variable. Let φ be a compatible random time-change (that is φ is the right-continuous inverse of a non-decreasing, continuous, $\mathcal{F}_\cdot$-adapted function and X is constant on the intervals $[\varphi(t-),\varphi(t)]$: see [26] for example) such that

$$|M_{\varphi(t)} - M_{\varphi(s)}|^p + |V^+_{\varphi(t)} - V^+_{\varphi(s)}|^p + |V^-_{\varphi(t)} - V^-_{\varphi(s)}|^p \leq K|t-s|. \tag{18}$$

Let us denote by φ^{-1} the right-continuous inverse of φ.

Remark 2. We have seen in the proof of Proposition 1 how to construct a random time change such that (18) is satisfied with $K = 1$.

The properties of φ and the inequality (18) imply that

$$\mathrm{Var}_{p,[0,T]}(X) = \mathrm{Var}_{p,[0,\varphi^{-1}(T)]}\, X_{\varphi(\cdot)} \leq K^{1/p}\varphi^{-1}(T)^{1/p}$$

and

$$X^2_{\varphi(s),\varphi(t)} = \int_s^t (X_{\varphi(r)} - X_{\varphi(s)}) \otimes \circ \mathrm{d}X_{\varphi(r)}.$$

For any $n \in \mathbb{N}$, let $0 \leq s_1^n \leq s_2^n \leq \ldots \leq s_{2^n}^n \leq T$ be a partition of $[0,T]$. For any $n \in \mathbb{N}$ and any $k = 1, 2, \ldots, 2^n$, we set $\Delta_k^n X = X_{s_k^n} - X_{s_{k-1}^n}$ and

$$X(n)_t = X_{s_{k-1}^n} + \frac{t - s_{k-1}^n}{s_k^n - s_{k-1}^n}\Delta_k^n X \text{ for } t \in [s_{k-1}^n, s_k^n]. \tag{19}$$

5.1 A Wong-Zakai theorem for the Stratonovich SDEs

The path $X(n)$ is the piecewise linear approximation of X that coincides with X at the points s_k^n. We could then construct a geometric rough path $(X(n)^1, X(n)^2)$ from $X(n)$ by setting

$$X(n)^1_{s,t} = X(n)_t - X(n)_s$$
$$\text{and } X(n)^2_{s,t} = \int_s^t (X(n)_r - X(n)_s) \otimes \mathrm{d}X(n)_r.$$

Let $0 \leq s \leq t \leq T$, and k and ℓ such that $s_{k-1}^n < s \leq s_k^n \leq s_\ell^n \leq t < s_{\ell+1}^n$,

$$\begin{aligned} X(n)^2_{s,t} = &\int_s^{s_k^n} (X(n)_r - X(n)_s) \otimes \mathrm{d}X(n)_r + \int_{s_\ell^n}^t (X(n)_r - X(n)_{s_\ell^n}) \otimes \mathrm{d}X(n)_r \\ &+ \sum_{j=k}^{\ell-1} \left(\frac{X(n)_{s_{j+1}^n} + X(n)_{s_j^n}}{2} - X(n)_s \right) \otimes (X(n)_{s_{j+1}^n} - X(n)_{s_j^n}). \end{aligned}$$

With this expression, it is easily seen that $X(n)^2_{s,t}$ converges in probability to the Stratonovich integral $X^2_{s,t} = \int_s^t (X_r - X_s) \otimes \circ \mathrm{d}X_r$ for any $s \leq t$ (See for example [26, exercise 2.18, p. 136]).

The following proposition was already known for semi-martingales, but for the convergence in probability with respect to the uniform norm: See [23]. Yet, combining the next proposition and the usual Wong-Zakai result for semi-martingales implies that the integral defined by the rough-path theory agrees with the Stratonovich integral.

Besides, it is important here that the approximation is piecewise linear, since it is known that other approximations could lead to different integrals: See for example [14, Chapter 5.7], [10, Section VI-7], [23, 4] and more specifically [17] regarding the rough paths theory.

Proposition 2. *Let p, K and φ as above. Let $X(n)$ be the piecewise linear approximation of X along the partition $(\varphi^{-1}(\varphi(T)k/2^n))_{k=0,\dots,2^n}$. Then, the geometric multiplicative functional $(X(n)^1, X(n)^2)_{n\in\mathbb{N}}$ converges almost surely in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$ to (X^1, X^2) as $n\to\infty$, and (X^1,X^2) is a* geometric multiplicative functional.

Remark 3. If for all $\beta > 0$, X is a semi-martingale satisfying for some $C_\beta > 0$ the condition

$$\mathbb{E}\left[\,|X_t - X_s|^\beta\,\right] \leq C_\beta |t-s|^{\beta/2}, \text{ for any } 0 \leq s,t \leq T,$$

then one knows from the Kolmogorov lemma, that there exists a α-Hölder continuous version of X for any $\alpha < 1/2$. Moreover, there exists a random variable K such that $|X_t(\omega) - X_s(\omega)|^{1/\alpha} \leq K(\omega)|t-s|$. Thus, Proposition 2 may be applied with the time change $\varphi(t) = t$.

In particular, this proposition is true for the Brownian motion with $\varphi(t) = t$. Hence, we recover a result from [27].

Remark 4. The partition $(\varphi^{-1}(\varphi(T)k/2^n))_{k=0,\dots,2^n}$ is a random partition unless the semi-martingale X has Hölder continuous paths. We have to note that φ^{-1} is not necessarily continuous, but the discontinuities of φ corresponds to the intervals on which X is constant, and are not taken into account in the integrals.

Remark 5. If the partition is deterministic, but no longer related to dyadics, then computations similar to the ones used in the proof of Proposition 2 prove that the convergence in probability of $(X(n)^1, X(n)^2)_{n\in\mathbb{N}}$ to (X^1, X^2) holds in p-variation.

Hypothesis 1. (i) *The function $g = (g_1,\dots,g_N)$ is $\mathcal{C}^1(\mathbb{R}^N,\mathbb{R}^m)^N$ with a derivative which is also α-Hölder continuous with $\alpha > p-2$.*
(ii) *The function $f = (f_1,\dots,f_N)$ is $\mathcal{C}^2(\mathbb{R}^m,\mathbb{R}^m)^N$ with a second-order derivative which is also α-Hölder continuous with $\alpha > p-2$.*

Corollary 1. *Under the hypotheses of Proposition 2 on X and Hypothesis 1 on g and f, then the ordinary integral $Z(n)$ and the solution of the ordinary differential equation $Y(n)$ defined by*

$$Z(n)_t = z + \int_0^t g(X(n)_s)\,\mathrm{d}X(n)_s \textit{ and } Y(n)_t = y + \int_0^t f(Y(n)_s)\,\mathrm{d}X(n)_s$$

converge respectively almost surely in p-variation to

$$Z_t = z + \int_0^t g(X_s) \circ \mathrm{d}X_s \text{ and } Y_t = y + \int_0^t f(Y_s) \circ \mathrm{d}X_s,$$

where the integrals are understood as Stratonovich integrals.

Proof. Denote by $\mathfrak{K}$ (resp. $\mathfrak{I}$) the map that gives the rough path obtained by integrating the differential form $g = \sum_{i=1}^N g_i(x)\mathrm{d}x^i$ (resp. the vector field f) against a rough path. Let also $\mathbf{X}(n)$ be the canonical rough path constructed above $X(n)$. Let us set $\mathbf{Z}(n) = \mathfrak{K}(\mathbf{X}(n))$ and $\mathbf{Y}(n) = \mathfrak{I}(\mathbf{X}(n))$. From the continuity of $\mathfrak{K}$ and $\mathfrak{I}$ from $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$ to $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^m)$, $(\mathbf{Z}(n))_{n\in\mathbb{N}}$ and $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ converge almost surely respectively to the multiplicative functionals $\mathbf{Z} = \mathfrak{K}(\mathbf{X})$ and $\mathbf{Y} = \mathfrak{I}(\mathbf{X})$, that are also geometric. But $Y(n)_t - y = \mathbf{Y}(n)^1_{0,t}$ and $Z(n)_t - y = \mathbf{Z}(n)^1_{0,t}$. Besides, by the Wong-Zakai theorem (See for example [23]), one knows that $(Y(n))_{n\in\mathbb{N}}$ and $(Z(n))_{n\in\mathbb{N}}$ converge in probability to Y and Z. It is now clear that $Y_t = y + \mathbf{Y}^1_{0,t}$ and $Z_t = z + \mathbf{Z}^1_{0,t}$ for any $t \in [0,T]$. $\square$

First part of the proof of Proposition 2. Since X is continuous, it is clear that $(X(n))_{n\in\mathbb{N}}$ converges to X almost surely in the space of continuous functions with the uniform norm.

Let $2 < q < p$. Let $\Pi = \{\, u_i \,|\, 0 \leq u_1 \leq \dots \leq u_k \leq T \,\}$ be a partition of $[0,T]$. We introduce two sets of index (with the convention that $s_0^n = u_0 = 0$ and $u_{k+1} = s^n_{2^n+1} = T$): For $j = 0, \dots, 2^n$, we set

$$\begin{aligned} \Pi_j &= \left\{\, i \,\middle|\, u_i \in [s_j^n, s_{j+1}^n] \,\right\}, \\ \Pi_{\mathrm{left}} &= \left\{\, i \,\middle|\, \exists j, j' \text{ s.t. } u_i \leq s_j^n \leq s_{j'}^n < u_{i+1} \,\right\}. \end{aligned}$$

For each j, we remark that (with the convention that a sum over the empty set is 0),

$$\sum_{i\in\Pi_j,\ i\neq\max\Pi_j} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \leq \sum_{i\in\Pi_j,\ i\neq\max\Pi_j} \frac{|u_{i+1}-u_i|^q}{|s^n_{j+1}-s^n_j|^q} |X_{s^n_{j+1}} - X_{s^n_j}|^q.$$

But

$$\sum_{i\in\Pi_j,\ i\neq\max\Pi_j} |u_{i+1}-u_i|^q \leq |u_{\max\Pi_j} - u_{\min\Pi_j}|^q \leq |s^n_{j+1}-s^n_j|^q.$$

Thus,

$$\sum_{j=0}^{2^n-1} \sum_{i\in\Pi_j,\ i\neq\max\Pi_j} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \leq \sum_{j=0}^{2^n-1} |X_{s^n_{j+1}} - X_{s^n_j}|^q \leq \mathrm{Var}_q(X)^q.$$

For i in Π_{left}, since $|X(n)_s - X_{s^n_j}| \leq |X_{s^n_{j+1}} - X_{s^n_j}|$ and $|X(n)_s - X_{s^n_{j+1}}| \leq |X_{s^n_{j+1}} - X_{s^n_j}|$ for any $s \in [s^n_j, s^n_{j+1}]$,

$$\begin{aligned}|X(n)_{u_i} - X(n)_{u_{i+1}}|^q \leq& 3^{q-1}|X(n)_{u_i} - X_{s^n_j}|^q + 3^{q-1}|X_{s^n_j} - X_{s^n_{j'}}|^q \\ &+ 3^{q-1}|X_{s^n_{j'}} - X(n)_{u_{i+1}}|^q \\ \leq& 3^{q-1}|X_{s^n_{j-1}} - X_{s^n_j}|^q + 3^{q-1}|X_{s^n_j} - X_{s^n_{j'}}|^q \\ &+ 3^{q-1}|X_{s^n_{j'}} - X_{s^n_{j'+1}}|^q.\end{aligned}$$

Hence,

$$\sum_{i\in\Pi_{\text{left}}} |X(n)_{u_{i+1}} - X(n)_{u_i}|^q \leq 3^{q-1}\operatorname{Var}_{q,[0,T]}(X)^q.$$

Let us remark that

$$\begin{aligned}\sum_{i=0}^{k-1}|X(n)_{u_{i+1}} - X(n)_{u_i}|^q =& \sum_{i\in\Pi_{\text{left}}}|X(n)_{u_{i+1}} - X(n)_{u_i}|^q \\ &+ \sum_{j=0}^{2^n-1}\sum_{i\in\Pi_j,\ \text{s.t. } i\neq\max\Pi_j}|X_{u_{i+1}} - X(n)_{u_i}|^q \leq (3^{q-1}+1)\operatorname{Var}_{q,[0,T]}(X)^q.\end{aligned}$$

So, one deduce that

$$\operatorname{Var}_{q,[0,T]}(X(n)) \leq (3^{q-1}+1)^{1/q}\operatorname{Var}_{q,[0,T]}(X). \tag{20}$$

Finally, let us remark that for any $p > q$,

$$\begin{aligned}&\operatorname{Var}_{p,[0,T]}(X - X(n))^p \\ &\quad\leq 2^{q-1}\sup_{t\in[0,T]}|X_t - X(n)_t|^{p-q}\left(\operatorname{Var}_{q,[0,T]}(X)^q + \operatorname{Var}_{q,[0,T]}(X(n))^q\right).\end{aligned}$$

Thus, one obtains the almost sure convergence of $(X(n))_{n\in\mathbb{N}}$ to X in p-variation. □

Let K_0 be fixed, and let T_0 be the $\mathcal{F}^\varphi$-stopping time such that

$$T_0 = \inf\left\{ r > 0 \,\middle|\, \sup_{0\leq s<t\leq r}\frac{|(M,V^+,V^-)_{\varphi(t)} - (M,V^+,V^-)_{\varphi(s)}|^p}{t-s} \geq K_0 \right\}.$$

Let also $\widehat{T}$ be a fixed real number. Let us set $Y_t = X_{\varphi(\cdot)\wedge T_0\wedge\widehat{T}}$, which is a $\mathcal{F}^\varphi$-semi-martingale.

Let $Y(n)$ denotes the piecewise linear approximation of Y along the dyadics at level n, *i.e.*, X and $X(n)$ are replaced by Y and $Y(n)$ in (19), but with the partition $(\widehat{T}k/2^n)_{k=0,\dots,2^n}$. Similarly, one can define $Y(n)^2_{s,t}$. From now, we use the notation $t^n_k = \widehat{T}k/2^n$ for $k = 0, 1, \dots, 2^n$.

The proof relies on the following lemma.

Lemma 3. *For all $n \in \mathbb{N}$ and $j, \ell \in \{0, \dots, 2^n\}$ such that $\ell > j$, we have*

$$Y(n)^2_{t^n_j, t^n_\ell} = Y^2_{t^n_j, t^n_\ell} + [\theta^n_{t^n_j, t^n_\ell}]^{\mathrm{a}},$$

where $[\theta^n_{s,t}]^{\mathrm{a}}$ denotes the anti-symmetric part (i.e., $[A]^{\mathrm{a}} = \frac{1}{2}(A - A^{\mathrm{t}})$ for a $d \times d$-matrix A) of

$$\theta^n_{s,t} = \int_s^t \delta^n_u Y \otimes \mathrm{d}Y_u \text{ with } \delta^n_u Y = \sum_{k=1}^{2^n} (Y_u - Y_{t^n_{k-1}}) \mathbf{1}_{[t^n_{k-1}, t^n_k)}(u).$$

Proof. We set $s = t^n_j$ and $t = t^n_k$. Then, a direct computation shows that

$$Y(n)^2_{s,t} = \sum_{\ell=j+1}^{k} \left((Y_{t^n_{\ell-1}} - Y_s) \otimes \Delta^n_\ell Y + \frac{1}{2} \Delta^n_\ell Y \otimes \Delta^n_\ell Y \right).$$

As the random variable $Y_{t^n_{\ell-1}} - Y_s$ is $\mathcal{F}_{\varphi(t^n_{\ell-1})}$-measurable,

$$\begin{aligned}(Y_{t^n_{\ell-1}} - Y_s) \otimes \Delta^n_\ell Y &= \int_{t^n_{\ell-1}}^{t^n_\ell} (Y_{t^n_{\ell-1}} - Y_s) \otimes \mathrm{d}Y_u \\ &= \int_{t^n_{\ell-1}}^{t^n_\ell} (Y_u - Y_s) \otimes \mathrm{d}Y_u - \int_{t^n_{\ell-1}}^{t^n_\ell} \delta^n_u Y \otimes \mathrm{d}Y_u.\end{aligned}$$

On the other hand, the Itô formula implies that

$$\frac{1}{2} \Delta^n_\ell Y \otimes \Delta^n_\ell Y = \left[\int_{t^n_{\ell-1}}^{t^n_\ell} \delta^n_u Y \otimes \mathrm{d}Y_u \right]^{\mathrm{s}} + \frac{1}{2} \big(\langle Y \rangle_{t^n_\ell} - \langle Y \rangle_{t^n_{\ell-1}} \big),$$

where $[A]^{\mathrm{s}} = \frac{1}{2}(A + A^{\mathrm{t}})$ denotes by symmetric part of a matrix A. Since

$$Y^2_{s,t} = \int_s^t (Y_u - Y_s) \otimes \mathrm{d}Y_u + \frac{1}{2} \big(\langle Y \rangle_t - \langle Y \rangle_s \big),$$

the result is now clear. ☐

Final part of the proof of Proposition 2. Using the Hölder continuity of Y and (20), one get easily that for any $q \geq p$ and any $m \in \mathbb{N}$,

$$|Y^1_{s,t}|^q + |Y(m)^1_{s,t}|^q \leq c_q K_0^q |t - s|^{q/p},$$

where c_q is a constant depending only on q. Besides, as $Y(m)_{s,t} = Y_{s,t}$ if $s = \widehat{T} i / 2^m$, $t = \widehat{T} j / 2^m$ for any $i, j \in \{0, \dots, 2^m\}$,

$$|Y^1_{t^n_i, t^n_{i+1}} - Y(m)^1_{t^n_i, t^n_{i+1}}|^q \leq \begin{cases} 0 & \text{if } m \geq n, \\ c_q \widehat{T}^{q/p} K_0^{q/p} 2^{-nq/p} & \text{if } m < n. \end{cases}$$

We decompose Y as $Y = Y_0 + N + W$, where N is a $\mathcal{F}^\varphi$-martingale and W a term of finite variation.

Lemma 1 asserts that one can evaluate the $p/2$-variation of $Y^2 - Y(m)^2$ if one knows $Y^2_{t_j^n, t_{j+1}^n} - Y(m)^2_{t_j^n, t_{j+1}^n}$, that is the difference between Y^2 and $Y(m)^2$ only at the dyadic points of $[0, \widehat{T}]$.

If $m \geq n$, for any $j = 0, \dots, 2^n$, there exists j' and j'' in $\{0, \dots, 2^m\}$ such such $t_j^n = t_{j'}^m$ and $t_{j+1}^n = t_{j''}^m$. With Lemma 3, $Y^2_{t_j^n, t_{j+1}^n} - Y(m)^2_{t_j^n, t_{j+1}^n} = -[\theta^m_{t^m_{j'}, t^m_{j''}}]^{\mathrm{a}}$. We focus on $\theta^m_{s,t}$. For $m \geq n$,

$$\mathbb{E}\left[|\theta^m_{t_j^n, t_{j+1}^n}|^{q/2} \right] \leq 2^{q-1} \mathbb{E}\left[\left| \int_{t_j^n}^{t_{j+1}^n} \delta_u^m Y \otimes \mathrm{d}N_u \right|^{q/2} \right] \\ + 2^{q-1} \mathbb{E}\left[\left| \int_{t_j^n}^{t_{j+1}^n} \delta_u^m Y \otimes \mathrm{d}W_u \right|^{q/2} \right].$$

We follow the arguments of the proof of Proposition 1. Using the both sides of the Burkholder-Davies-Gundy inequality and the Cauchy-Schwarz inequality, there exists a constant C_q depending only on q such that

$$\mathbb{E}\left[\left| \int_{t_j^n}^{t_{j+1}^n} \delta_u^m Y \otimes \mathrm{d}N_u \right|^{q/2} \right] \\ \leq C_q \mathbb{E}\left[\sup_{t \in [t_j^n, t_{j+1}^n]} |\delta_t^m Y|^q \right]^{1/2} \mathbb{E}\left[\sup_{t \in [t_j^n, t_{j+1}^n]} |N_t - N_{t_j^n}|^q \right]^{1/2}.$$

Using the very definition of T_0 and Y,

$$\sup_{t \in [t_j^n, t_{j+1}^n]} |\delta_t^m Y|^p \leq \frac{K_0 \widehat{T}}{2^m}$$

and that

$$\mathbb{E}\left[\sup_{t \in [t_j^n, t_{j+1}^n]} |N_t - N_{t_j^n}|^q \right] \leq C \left(\frac{K_0 \widehat{T}}{2^n} \right)^{q/p}.$$

From (18), $\mathrm{Var}_{p,[s,t]}(W) \leq 2^{p-1} \mathrm{Var}_{p,[s,t]}(W^+, W^-)^p \leq 2^{p-1} K_0 |t-s|$ and then

$$\mathbb{E}\left[\left| \int_{t_j^n}^{t_{j+1}^n} \delta_u^m Y \otimes \mathrm{d}W_u \right|^{q/2} \right] \leq \left(2^{p-1} \frac{K_0 \widehat{T}}{2^{m/2} 2^{n/2}} \right)^{q/p}.$$

If $m < n$, a direct computation shows that if $[t_j^n, t_{j+1}^n] \subset [t_i^m, t_{i+1}^m]$, then

$$|Y(m)^2_{t_j^n, t_{j+1}^n}| = \frac{(t_{j+1}^n - t_j^n)^2}{(t_{i+1}^m - t_i^m)^2} |Y_{t_{i+1}^m} - Y_{t_i^m}|^2 \leq \frac{2^{2m}}{2^{2n}} \frac{(K_0 \widehat{T})^{2/p}}{2^{2m/p}}.$$

Choose $q > q' > p$. It follows that there exists a constant C depending only on p, q and q' such that

$$\sum_{n>m} n^\gamma \sum_{j=0}^{2^n-1} |Y(m)^2_{t_j^n, t_{j+1}^n}|^{q/2} \leq C \frac{(K_0 \widehat{T})^{q/p}}{1 - 2^{1-q'}} 2^{m(1-q'+q-q/p)} \sup_{n>m} n^\gamma 2^{n(q'-q)}.$$

As $q/p > 1$, if we choose q' so that $q - q' < q/p - 1$, then there exists a constant C depending only on q, q', p and γ such that

$$\sum_{n>m} n^\gamma \sum_{j=0}^{2^n-1} |Y(m)^2_{t_j^n, t_{j+1}^n}|^{q/2} \leq C K_0^{q/p} \widehat{T}^{q/p} 2^{m\delta}$$

with $\delta = 1 - q' + q - q/p < 0$.

On the other hand, we have seen in the proof of Proposition 1 that

$$\sum_{j=0}^{2^{n-1}} \mathbb{E}\left[|Y^2_{t_j^n, t_{j+1}^n}|^{q/2} \right] \leq C 2^{n(1-q/p)} \widehat{T}^{q/p}.$$

Hence, using all these estimates and Lemma 1, one gets for m large enough,

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Var}_{q/2}(Y^2 - Y(m)^2)^{q/2} \right] \\ &\qquad\leq C K_0^{q/p} \widehat{T}^{q/p} \left(2^{m\delta} + \sum_{n=1}^{m} \frac{n^\gamma}{2^{(m+n)q/2p}} + \sum_{n>m} \frac{n^\gamma 2^n}{2^{nq/p}} \right)^{1/2},\end{aligned}$$

where the constant C depends only on p, q and γ.

The Bienaymé-Tchebitchev inequality implies that for all $\alpha_m > 0$ and any $\delta > 0$

$$\mathbb{P}\left[\mathrm{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) \geq \alpha_m \right] \leq \frac{C'}{\alpha_m} \left(2^{m\delta} + \frac{m^{\gamma+1}}{2^{mq/2p}} + S_m \right)$$

with $C' = C K_0^{q/p} \widehat{T}^{q/p}$ and $S_m = \sum_{n>m} n^\gamma 2^{n-nq/p}$. Clearly, $S_m \leq \int_m^{+\infty} t^\gamma \exp(-\zeta t)\, \mathrm{d}t$ for $\zeta = p(1-\beta)\ln 2$. Hence, we deduce easily that $S_m \leq m^{-3}/3$ for m large enough. Setting

$$\alpha_m = m^2 (2^{m\delta} + m^{\gamma+1} 2^{-mq/2p} + m^{-3}/3),$$

we remark that $\alpha_m \xrightarrow[m\to\infty]{} 0$ and that

$$\sum_{m\geq 1} \mathbb{P}\left[\operatorname{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) \geq \alpha_m\right] < +\infty.$$

The Borel-Cantelli Lemma implies that

$$\operatorname{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) \xrightarrow[m\to\infty]{} 0$$

$\mathbb{P}$-almost surely. Now, for a given trajectory, let us choose K_0 and T_0 large enough such that $\varphi(T) \leq \widehat{T}$ and $T_0 \geq \widehat{T}$. Since

$$Y(m)^2_{s,t} = X(m)^2_{\varphi(s),\varphi(t)} \text{ and } Y^2_{s,t} = X^2_{\varphi(s),\varphi(t)},$$

we deduce that

$$\operatorname{Var}_{q/2,[0,\widehat{T}]}(Y^2 - Y(m)^2) = \operatorname{Var}_{q/2,[0,T]}(X^2 - X(m)^2).$$

Besides, since

$$\sup_{0\leq s\leq t\leq T} |X(m)^2_{s,t} - X^2_{s,t}| \leq \operatorname{Var}_{q/2,[0,T]}(X(m)^2 - X^2)^{q/2},$$

one obtains the uniform convergence of $X(m)^2_{s,t}$ to $X^2_{s,t}$ with respect to (s,t) such that $0 \leq s \leq t \leq T$. □

5.2 A Wong-Zakai theorem for the Itô SDEs

The difference between an Itô integral and a Stratonovich integral depends of the cross-variation of the process that is integrated.

Let us recall that we have denoted the family of cross-variations of X by Ξ. For $t \in [0,T]$, we set $j^n(t) = \sup\{\, i \in \{\,0,\dots,2^n\,\} \mid s^n_i \leq t\,\}$ and

$$\Xi(n)_t = \frac{1}{2}\frac{t - s_{j^n(t)}}{s_{j^n(t)+1} - s_{j^n(t)}}(X_{s_{j^n(t)+1}} - X_{s_{j^n(t)}}) \otimes (X_{s_{j^n(t)+1}} - X_{s_{j^n(t)}}) \\ + \frac{1}{2}\sum_{i=1}^{j^n(t)}(X_{s^n_i} - X_{s^n_{i-1}}) \otimes (X_{s^n_i} - X_{s^n_{i-1}}).$$

We set $\Xi(n)_{s,t} = \Xi(n)_t - \Xi(n)_s$. We consider now the (non-geometric) multiplicative functional $(X(n)^1, X(n)^2 + \Xi(n))$.

Proposition 3. *Let p, K and φ as above. Let $X(n)$ be the piecewise linear approximation of X along the partition $(\varphi^{-1}(\varphi(T)k/2^n))_{k=0,\dots,2^n}$. Then, the multiplicative functional $(X(n)^1, X(n)^2 + \Xi(n))$ converges almost surely in $\mathcal{M}\mathbb{V}^p([0,T];\mathbb{R}^N)$ to $(X^1, X^2 + \Xi)$ as $n \to \infty$.*

Proof. Clearly, we have only to prove that $(\Xi(n))_{n\in\mathbb{N}}$ converges in $q/2$-variation to Ξ. The proof is similar to the one of Proposition 2. One has to work with Y instead of X. Let us define $\Xi(n)^Y$ and Ξ^Y by a substitution of X to Y in the definition of $\Xi(n)$ and Ξ. We use the same notations as in the proof of Proposition 2 for K_0, $\widehat{T}$, ... We assume in a first time that $m \geq n$. Then, for $i \in \{0,\dots,2^n\}$ and for $j < k$ in $\{0,\dots,2^m\}$ such that $t_j^m = t_i^n$ and $t_j^m - t_k^m = \widehat{T}2^{-n}$,

$$\begin{aligned}\Xi(m)^Y_{t_i^n,t_i^n} - \Xi^Y_{t_i^n,t_{i+1}^n} &= \Xi(m)^Y_{t_j^n,t_k^n} - \Xi^Y_{t_j^n,t_k^n} \\ &= \sum_{i=j+1}^{k} \frac{1}{2}(Y_{t_{i-1}^m} - Y_{t_i^m})^{\otimes 2} - \Xi^Y_{t_{i-1}^m,t_i^n} = \sum_{i=j+1}^{k} \frac{1}{2}\int_{t_{i-1}^m}^{t_i^n} (Y_r - Y_{t_i^m}) \otimes \mathrm{d}Y_r.\end{aligned}$$

And if $n > m$, for $i \in \{0,\dots,2^n\}$,

$$|\Xi(m)^Y_{t_i^n,t_{i+1}^n} - \Xi^Y_{t_i^n,t_{i+1}^n}| = |(Y_{t_{i-1}^n} - Y_{t_i^n})^{\otimes 2}| \leq \frac{2^{2m}}{2^{2n}} \frac{(K_0\widehat{T})^{2/p}}{2^{2m/p}}.$$

Besides, we have already seen that

$$\mathbb{E}\left[|\Xi^Y_{t_{i-1}^n,t_i^n}|^{q/2} \right] \leq \frac{K_0^{q/p}\widehat{T}^{q/p}}{2^{nq/p}}.$$

We have already used these estimates in the proof of Proposition 2, and the proof of Proposition 3 is similar to the one of Proposition 2. □

Finally, let us remark that the relation between the Itô and the Stratonovich integrals and SDEs together with Corollary 1 allows to deduce that $\mathfrak{K}((X^1, X^2 + \Xi))$ and $\mathfrak{I}((X^1, X^2 + \Xi))$ are multiplicative functionals lying above the Itô integral $z + \int_0^{\cdot} g(X_s)\,\mathrm{d}X_s$ and the solution Y to the Itô's SDE $Y_t = y + \int_0^t f(Y_s)\,\mathrm{d}X_s$, where $\mathfrak{K}$ and $\mathfrak{I}$ have been defined in the proof of Corollary 1.

The proof of the following corollary is now clear.

Corollary 2. *Under the hypotheses of Proposition 2 and Hypothesis 1 on g and f, the ordinary integral*

$$Z(n)_t = z + \int_0^t g(X(n)_s)\,\mathrm{d}X(n)_s + \sum_{i,j=1}^{N} \int_0^t \frac{\partial g_i}{\partial x_j}(X(n)_s)\,\mathrm{d}\Xi^{i,j}(n)_s$$

and the solution of the ordinary differential equation defined by

$$Y(n)_t = y + \int_0^t f(Y(n)_s)\,\mathrm{d}X(n)_s + \sum_{j=1}^{m}\sum_{i,k=1}^{N} \int_0^t f_k^j \frac{\partial f_i}{\partial x_j}(Y(n)_s)\,\mathrm{d}\Xi^{i,k}(n)_s$$

converge almost surely to Z and Y defined above.

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