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ON INFECTION SPREADING AND COMPETITION BETWEEN INDEPENDENT RANDOM WALKS[1]

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Abstract: We study the models of competition and spreading of infection for infinite systems of independent random walks. For the competition model, we investigate the question whether one of the spins prevails with probability one. For the infection spreading, we give sufficient conditions for recurrence and transience (i.e., whether the origin will be visited by infected particles infinitely often a.s.).

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1 Introduction and results

In this paper we deal with systems of independent simple random walks (SRWs) in $\mathbb{Z}^d$, $d \geq 3$. Initially we fix an infinite set S, $0 \notin S$, and we place one particle into each site of S and into the origin; then each particle starts to perform a SRW independently of the others. The models that we are considering here are conservative, i.e., particles do not die and there is no influx of particles to the system (differently from e.g. branching random walks). While the random walks themselves are independent, we consider another types of interaction between the particles. Namely, to each particle is assigned a spin (taking two possible values), and when two particles with different spins meet, their spins may change according to some rules. Below we describe two models which fit into this framework.

A competition model. At time 0 there is a blue particle in each site of S and a red particle at the origin. Each of the particles performs a simple random walk with *continuous time* with rate 1 independently of the others. If a blue particle jumps to a site where there are some red particles, then all those particles become blue. If a red particle jumps to a site that contains some blue particles, then all those become red. The main question that we study here is whether the red particles survive with positive probability.

Theorem 1.1 *Let τ be the moment when the last red particle becomes blue. We have that $\tau < \infty$ a.s. iff $\sum_{x\in S} \|x\|^{-(d-2)} = \infty$.*

An infection model. Here we again consider continuous time simple random walks, and initially the particle from the origin is infected, while the other particles are healthy. Infected particles transmit the infection to all the healthy particles they meet, and there is no recovering.

Note that if the particles do not move until infected, then we obtain the *frog model*, cf. [1, 2, 3, 6, 15, 16, 19]. For the present model (starting from a configuration of constant density), linear growth results and shape theorems were obtained in [8, 9]. For other interacting particle systems on $\mathbb{Z}^d$ modeling the spreading of different infectious diseases see e.g. [10, 17] and references therein. The models investigated there are related to the contact process.

In our infection model, analogously to [15], we are mainly interested in the following question: will the origin be visited infinitely often by infected particles, almost surely? Let us denote by $\mathfrak{G}^N = \{x \in \mathbb{Z}^d : 2^N \leq \|x\| < 2^{N+1}\}$, $N = 1, 2, \dots$ (where $\|x\| = (x_1^2 + \cdots + x_d^2)^{1/2}$). The following theorem gives sufficient conditions for the process to be recurrent (i.e., an infinite number of infected particles visit the origin).

Theorem 1.2 *There exists a constant $\alpha = \alpha(d) > 0$ such that if*

$$|\mathfrak{G}^N \cap S| \geq \alpha 2^{N(d-2)} \tag{1.1}$$

for all N large enough, then

(i) each site of $\mathbb{Z}^d$ will be visited by infected particles infinitely many times a.s.;

(ii) moreover, each particle will be infected a.s.

It is our conjecture that, analogously to Theorem 1.2, for the nonrecurrence of the process it is enough that $|\mathfrak{G}^N \cap S| \leq \alpha' 2^{N(d-2)}$ for a small constant α', for all N large enough. However, at the present time we can only prove a weaker result. Suppose that the initial configuration of healthy particles S is constructed by using the following random procedure: for any $x \neq 0$, we put a particle there with probability $q(x)$, and leave it empty with probability $1 - q(x)$.

Theorem 1.3 *There exists a constant $\alpha' = \alpha'(d) > 0$ such that if*

$$q(x) \leq \frac{\alpha'}{\|x\|^2} \tag{1.2}$$

for all x large enough, then the total number of visits of infected particles to the origin will be finite a.s.

Analogously to Theorem 1.2 (ii), we can conjecture that if α' is small enough, then not every particle will be infected. However, the proof of that is still beyond our reach.

Before going further, let us say a few words about the relationship of Theorems 1.2 and 1.3 with the corresponding results in [15] (besides the discrete time, which is really not important, the only difference of the model of [15] from the present model is that in the former one the particles begin to move at the moment they are infected). The sufficient conditions for the transience (Theorem 1.1 (i) in [15] and Theorem 1.3 in this paper) are similar; however, here the proof is considerably more difficult, since the domination by branching random walk is not trivial. As for Theorem 1.2, it is stronger than the corresponding result in [15], since here we are only interested in the total number of particles inside $\mathfrak{G}^N$, and so we permit that they "accumulate" in some places of $\mathfrak{G}^N$, leaving other regions of $\mathfrak{G}^N$ completely empty (which could not happen in the situation of [15], where the initial configuration was always constructed using the random procedure described before Theorem 1.3 here). In fact, it is not difficult to construct an initial configuration S of healthy particles in such a way that for that S the model with all the particles moving is recurrent, but the frog model is transient. For that it is enough to take S in such a way that (1.1) is satisfied with a large α, but the points of $\mathfrak{G}^N \cap S$ are so grouped together, that the set S is not recurrent (i.e., with positive probability a SRW starting from the origin never hits S). To understand this type of different behaviour of the two models, note that, for the present model, even if initially some healthy particles were grouped together, when the infection will come there they are likely to be much more scattered; that, however, does not happen for the frog model.

One of the main tools in the proof of the above results is the following fact, which may be of independent interest.

Theorem 1.4 *The particle starting from the origin will a.s. meet only a finite number of particles starting from the sites of the set S iff*

$$\sum_{x \in S} \|x\|^{-(d-2)} < \infty. \tag{1.3}$$

Moreover, if (1.3) holds, then with positive probability the particle starting from the origin will not meet any of the particles from S.

When the particles from S do not move, then the question of whether the particle from 0 meets some other particle (in fact, whether it hits S) is answered by the Wiener's criterion (cf. e.g. [18]). This criterion is formulated in terms of capacities, and its verification is usually much more difficult than that of the criterion given by Theorem 1.4 (however, when the "trapping" set S is constructed using a random procedure similar to that of Theorem 1.3, that question can be answered in a more explicit way, cf. [7, 14]). On the other hand, if the particle from the origin is the only one that does not move, then it is quite trivial to verify that the solution given by Theorem 1.4 remains valid.

2 Proofs

This section is organized in the following way. First, we introduce some notations and recall a few well-known facts about SRWs. Then, we prove Theorems 1.4, 1.1, 1.2, 1.3, in that order.

Denote by $\xi^a(t)$ the position at the moment t of the particle that started the continuous time SRW (with rate 1) from the site a. Also, denote by $\hat{\xi}_n$ the discrete time SRW. For $x, y \in \mathbb{Z}^d$, $d \geq 3$, define the Green's function $G(x, y)$ by

$$G(x,y) = \mathbf{E}\Big(|\{n \geq 0 : \hat{\xi}_n = y\}| \mid \hat{\xi}_0 = x\Big) = \sum_{n=0}^{\infty} \mathbf{P}[\hat{\xi}_n = y \mid \hat{\xi}_0 = x].$$

It is known that $G(x,y) = G(0,x-y)$, that $\mathbf{P}[\hat{\xi} \text{ ever hits } y \,|\, \hat{\xi}_0 = x] = G(x,y)/G(0,0)$, and that $G(0,x) \sim \gamma_d \|x\|^{-(d-2)}$ as $x \to \infty$ for some constant γ_d (cf. e.g. [11, 18]).

Note that, as $\xi^a - \xi^b$ is in fact a SRW with rate 2 starting from $a-b$, the following relation holds:

$$\mathbf{P}[\xi^a(t) = \xi^b(t) \text{ for some } t] = \mathbf{P}[\xi^a(t) = b \text{ for some } t] = \frac{G(a,b)}{G(0,0)}. \tag{2.1}$$

Throughout all proofs we denote by $\mathfrak{D}^a(r) = \{x \in \mathbb{Z}^d : \|x - a\| \le r\}$ and $\mathfrak{D}(r) := \mathfrak{D}^0(r)$. Then $\mathfrak{G}^N = \mathfrak{D}(2^{N+1}) \setminus \mathfrak{D}(2^N)$.

Denote $p_t(x,y) = \mathbf{P}[\xi^x(t) = y] = \mathbf{P}[\xi^0(t) = x - y]$. We will need upper and lower estimate on $p_t(x,y)$ in the spirit of the local CLT; however, we will not need statements as strong as the local CLT itself. The following lemma can be deduced from Proposition 1.2.5 and Lemma 1.5.1 of [11]. (The results of [11] are for discrete time, but, for the needs of Lemma 2.1 below, the passage from discrete to continuous time is rather straightforward. Note also that there is nothing mysterious about the constant 8/5 in the lemma below; in Proposition 1.2.5 of [11] there is a parameter α that should be strictly between 1/2 and 2/3, so we just took $\alpha = 5/8$ for concreteness.)

Lemma 2.1 *(i) There exists a constant θ_d such that*

$$p_t(0,x) \le \theta_d t^{-d/2} e^{-\frac{d\|x\|^2}{2t}} \tag{2.2}$$

for $t \ge \|x\|^{8/5}$, and

$$p_t(0,x) \le \theta_d e^{-\|x\|^{1/5}} \tag{2.3}$$

for $t < \|x\|^{8/5}$.

(ii) Suppose that $L_1\|x\|^2 \le t \le L_2\|x\|^2$ for some $L_1, L_2 > 0$. Then there exists a constant $\theta'_d = \theta'_d(L_1, L_2)$ such that

$$p_t(0,x) \ge \theta'_d \|x\|^{-d}. \tag{2.4}$$

Proof of Theorem 1.4. Suppose that $\sum_{x\in S} \|x\|^{-(d-2)} < \infty$. As, by (2.1), $\mathbf{P}[\xi^0(t) = \xi^z(t) \text{ for some t}] \sim \gamma_d \|z\|^{-(d-2)}/G(0,0)$, Borel-Cantelli lemma implies that a.s. the particle from the origin will meet only finite number of other particles. In what follows we prove that in this case it holds also that with positive probability that particle will not meet anyone. Note that $\sum_{x\in S} \|x\|^{-(d-2)} < \infty$ iff $\sum_{x\in S} G(0,x) < \infty$. Choose R in such a way that $\sum_{x\in S\setminus\mathfrak{D}(R)} G(0,x) < G(0,0)/4$, and define the process

$$\zeta_t = \sum_{x\in S\setminus\mathfrak{D}(R)} G(0, \xi^x(t)).$$

Consider the event $B_0 = \{\zeta_t \le G(0,0)/2 \text{ for all } t \ge 0\}$ (it is useful to keep in mind that on the event B_0 none of the particles starting from $S \setminus \mathfrak{D}(R)$ ever enters 0). As

$$G(0,x) - \mathbf{1}\{x=0\} = \frac{1}{2d} \sum_{y:\|y-x\|=1} G(0,y),$$

the process ζ_t is a supermartingale, so we obtain that $\mathbf{P}[B_0] \ge 1/2$.

Consider also the events

$$\begin{aligned} B_1 &= \{\xi^0(t) = 0 \text{ for all } t \in [0,1]\}, \\ B_2 &= \Big\{ \sum_{x \in S \cap \mathfrak{D}(R)} G(0, \xi^x(1)) < \frac{G(0,0)}{2} \\ &\qquad\qquad \text{and } \xi^x(t) \neq 0 \text{ for all } t \in [0,1], x \in S \cap \mathfrak{D}(R) \Big\}. \end{aligned}$$

Clearly, $\mathbf{P}[B_1] > 0$; since the set $S \cap \mathfrak{D}(R)$ is finite, $\mathbf{P}[B_2] > 0$ as well. On the event $B_0 \cap B_1 \cap B_2$ at time 1 the particle from the origin is still the only infected one, and

$$\sum_{x \in S} \frac{G(0, \xi^x(1))}{G(0,0)} < 1. \tag{2.5}$$

By (2.1), the probability that the particle starting from the origin ever meets the particle starting from $\xi^x(1)$ is $\frac{G(0,\xi^x(1))}{G(0,0)}$, so (2.5) implies that, conditioned on $B_0 \cap B_1 \cap B_2$, with positive probability the particle starting from the origin will never meet any of the particles starting from the sites of the set S. Now it only remains to note that the events B_0, B_1, B_2 are independent, so $\mathbf{P}[B_0 \cap B_1 \cap B_2] > 0$.

Now we are going to prove the "only if" implication in Theorem 1.4. First of all we need the following elementary observation:

Lemma 2.2 *If σ is a stopping time independent of the process ξ^x, then for any $a \neq x$*

$$\begin{aligned} &\mathbf{P}[\xi^a(\sigma+t) = \xi^x(\sigma+t) \textit{ for some } t \ge 0 \mid \sigma < \infty] \\ &\qquad \le \mathbf{P}[\xi^a(\sigma+t) = \xi^x(t) \textit{ for some } t \ge 0 \mid \sigma < \infty]. \end{aligned}$$

Proof. We have, by (2.1) and the fact that ξ^x is independent from σ and ξ^a,

$$\begin{aligned} &\mathbf{P}[\xi^a(\sigma+t) = \xi^x(\sigma+t) \text{ for some } t \ge 0 \mid \sigma < \infty] \\ &\quad = \sum_{z,z' \in \mathbb{Z}^d} \mathbf{P}[\xi^a(\sigma) = z,\ \xi^x(\sigma) = z' \mid \sigma < \infty] \mathbf{P}[\xi^z(t) = \xi^{z'}(t) \text{ for some } t \ge 0] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{z,z'\in\mathbb{Z}^d} \mathbf{P}[\xi^a(\sigma)=z,\ \xi^x(\sigma)=z' \mid \sigma<\infty]\frac{G(z,z')}{G(0,0)} \\
&= (\mathbf{P}[\sigma<\infty])^{-1} \sum_{z,z'\in\mathbb{Z}^d}\sum_k \mathbf{P}[\xi^a(k)=z,\ \xi^x(k)=z',\ \sigma=k]\frac{G(z,z')}{G(0,0)} \\
&= (\mathbf{P}[\sigma<\infty])^{-1} \sum_{z,z'\in\mathbb{Z}^d}\sum_k \mathbf{P}[\xi^a(k)=z,\ \sigma=k]\mathbf{P}[\xi^x(k)=z']\frac{G(z,z')}{G(0,0)} \\
&= (\mathbf{P}[\sigma<\infty])^{-1} \sum_{z\in\mathbb{Z}^d}\sum_k \mathbf{P}[\xi^a(k)=z,\ \sigma=k]\sum_{z'\in\mathbb{Z}^d}\mathbf{P}[\xi^x(k)=z']\frac{G(z,z')}{G(0,0)} \\
&= (\mathbf{P}[\sigma<\infty])^{-1} \sum_{z\in\mathbb{Z}^d}\sum_k \mathbf{P}[\xi^a(k)=z,\ \sigma=k]\mathbf{P}[\xi^x(t)=z \text{ for some } t\geq k] \\
&\leq (\mathbf{P}[\sigma<\infty])^{-1} \sum_{z\in\mathbb{Z}^d}\sum_k \mathbf{P}[\xi^a(k)=z,\ \sigma=k]\mathbf{P}[\xi^x(t)=z \text{ for some } t\geq 0] \\
&= (\mathbf{P}[\sigma<\infty])^{-1} \sum_{z\in\mathbb{Z}^d}\sum_k \mathbf{P}[\xi^a(k)=z,\ \sigma=k] \\
&\qquad\qquad \times \mathbf{P}[\xi^x(t)=\xi^z(t) \text{ for some } t\geq 0] \\
&= \mathbf{P}[\xi^a(\sigma+t)=\xi^x(t) \text{ for some } t\geq 0 \mid \sigma<\infty],
\end{aligned}
$$

which proves Lemma 2.2. ■

Using this observation, let us prove the following two lemmas.

Lemma 2.3 *For some fixed integer $\kappa>1$, suppose that $a\in\mathfrak{D}(2^{\kappa(i-1)})$ and $b\in\mathfrak{D}(2^{\kappa i+1})\setminus\mathfrak{D}(2^{\kappa i})$, and denote $\sigma:=\inf\{t:\|\xi^a(t)\|\geq 2^{\kappa(i+1)}\}$. Then there exists κ_0 such that for all $\kappa\geq\kappa_0$ the following inequality holds*

$$
\mathbf{P}[\xi^a(t_0)=\xi^b(t_0) \textit{ for some } t_0<\min\{2^{2\kappa(i+1)},\sigma\}]\geq\frac{\gamma'}{(2^{\kappa i})^{d-2}} \tag{2.6}
$$

for some $\gamma'>0$ and all i.

Proof. Note that the number of jumps of SRW with continuous time with rate 2 until the moment $t=2^{2\kappa(i+1)}$ will be at least $2^{2\kappa(i+1)}$ with probability at least $1/2$ (in fact, it will be so with probability very close to 1 for κ large). Observe also that $\|a-b\|\leq 2^{\kappa i}(2+2^{-\kappa})<\sqrt{2^{2\kappa i+3}}$ for large enough κ. So, using Theorem 2.2 from [1] (it says that the probability that SRW hits a point which is h units away until the time h^2 is of order $O(h^{-(d-2)})$), we get that

$$
\mathbf{P}[\xi^a(t_0)=\xi^b(t_0) \text{ for some } t_0<2^{2\kappa(i+1)}]
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\mathbf{P}[\hat{\xi}_{n_0} = b - a \text{ for some } n_0 < 2^{2\kappa i+3} \mid \hat{\xi}_0 = 0] \\
&\geq \frac{K_1}{(2^{\kappa i})^{d-2}},
\end{aligned}
$$

(recall that $\hat{\xi}$ is the discrete time SRW) where K_1 does not depend on κ. Also, using (2.1), Lemma 2.2 and the fact that

$$
\inf_{z \notin \mathfrak{D}(2^{\kappa(i+1)})} \|b - z\| \geq 2^{\kappa i}(2^\kappa - 2)
$$

we get

$$
\begin{aligned}
&\mathbf{P}[\xi^a(t_0) = \xi^b(t_0) \text{ for the first time for some } t_0 > \sigma] \\
&\quad \leq \mathbf{P}[\xi^a(\sigma + t) = \xi^b(\sigma + t) \text{ for some } t \geq 0 \mid \sigma < \infty] \\
&\quad \leq \mathbf{P}[\xi^a(\sigma + t) = \xi^b(t) \text{ for some } t \geq 0 \mid \sigma < \infty] \\
&\quad \leq \frac{K_2}{(2^{\kappa i}(2^\kappa - 2))^{d-2}} = \frac{K_3}{(2^{\kappa i})^{d-2}},
\end{aligned}
$$

where K_3 can be made arbitrarily small by choosing κ large enough. So,

$$
\begin{aligned}
&\mathbf{P}[\xi^a(t_0) = \xi^b(t_0) \text{ for some } t_0 < \min\{2^{2\kappa(i+1)}, \sigma\} \\
&\quad \geq \mathbf{P}[\xi^a(t_0) = \xi^b(t_0) \text{ for some } t_0 < 2^{2\kappa(i+1)}] \\
&\qquad\quad - \mathbf{P}[\xi^a(t_0) = \xi^b(t_0) \text{ for the first time for some } t_0 > \sigma] \\
&\quad \geq \frac{K_1 - K_3}{(2^{\kappa i})^{d-2}} \geq \frac{\gamma'}{(2^{\kappa i})^{d-2}}
\end{aligned}
$$

for some $\gamma' > 0$. Lemma 2.3 is proved. ■

Lemma 2.4 *If $\|a - b\| \geq x_0$ and $\|a - u\| \geq x_0$, then there exists $\gamma'' > 0$ such that*

$$
\mathbf{P}[\xi^a(t) = \xi^b(t) \textit{ for some } t \textit{ and } \xi^a(t') = \xi^u(t') \textit{ for some } t'] \leq \frac{\gamma''}{x_0^{2d-4}}
$$

for all x_0 large enough.

Proof. We have

$$
\begin{aligned}
&\mathbf{P}[\xi^a(t) = \xi^b(t) \text{ for some } t \text{ and } \xi^a(t') = \xi^u(t') \text{ for some } t'] \\
&\quad = \mathbf{P}[\xi^a(t_0) = \xi^b(t_0) \notin \mathfrak{D}^u(x_0) \text{ for some } t_0 \\
&\qquad\qquad \text{and } \xi^a(t) = \xi^u(t) \text{ for the first time for some } t \geq t_0]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{P}[\xi^a(t_0)=\xi^b(t_0)\in\mathfrak{D}^u(x_0) \text{ for some } t_0\\
&\qquad \text{and } \xi^a(t)=\xi^u(t) \text{ for the first time for some } t\geq t_0]\\
&+\mathbf{P}[\xi^a(t_0)=\xi^u(t_0)\notin\mathfrak{D}^b(x_0) \text{ for some } t_0\\
&\qquad \text{and } \xi^a(t)=\xi^b(t) \text{ for the first time for some } t> t_0]\\
&+\mathbf{P}[\xi^a(t_0)=\xi^u(t_0)\in\mathfrak{D}^b(x_0) \text{ for some } t_0\\
&\qquad \text{and } \xi^a(t)=\xi^b(t) \text{ for the first time for some } t> t_0]\\
= \;& I_1+I_2+I_3+I_4. \qquad (2.7)
\end{aligned}
$$

Clearly, it is sufficient to estimate the terms I_1 and I_2. Define the stopping time σ by $\sigma=\inf\{t:\xi^a(t)=\xi^b(t)\}$ and put $z_\sigma:=\xi^a(\sigma)$. The term I_1 can be estimated as follows. By (2.1), Lemma 2.2, and observing that $\|u-z_\sigma\|\geq x_0$, we have

$$
\begin{aligned}
I_1 &\leq \mathbf{P}[\xi^a(\sigma+t)=\xi^u(\sigma+t) \text{ for some } t\geq 0, z_\sigma\notin\mathcal{D}^u(x_0)\mid \sigma<\infty]\\
&\qquad \times\mathbf{P}[\sigma<\infty]\\
&\leq \mathbf{P}[\xi^{z_\sigma}(t)=\xi^u(t) \text{ for some } t\geq 0, z_\sigma\notin\mathcal{D}^u(x_0)\mid \sigma<\infty]\mathbf{P}[\sigma<\infty]\\
&\leq \frac{K_4}{x_0^{2(d-2)}}, \qquad (2.8)
\end{aligned}
$$

for some $K_4>0$ not depending on x_0. The bound for I_2 is more complicated. Define the random variable $N(z)$ as the number of encounters in z between the particles starting from a and b:

$$
\begin{aligned}
N(z) \;=\; & \sup\{i\geq 0: \text{ there exist } t_1<t_1'<\ldots<t_i<t_i':\\
& \qquad \xi^a(t_j)=\xi^b(t_j)=z,\ \xi^a(t_j')\neq\xi^b(t_j'),\ j=1,\ldots,i\}.
\end{aligned}
$$

Put $\rho_1=\|a-z\|$, $\rho_2=\|b-z\|$, $s:=\frac{(\rho_1^2+\rho_2^2)d}{2t}$. Observe that $\rho_1^2+\rho_2^2\geq x_0^2/2=\|a-b\|^2/2$. By (2.1), Lemma 2.2, and using Lemma 2.1 (i), we get

$$
\begin{aligned}
I_2 &\leq \sum_{z\in\mathfrak{D}^u(x_0)}\mathbf{P}[\xi^u(t)=z \text{ for some } t]\mathbf{P}[\sigma<\infty,\xi^a(\sigma)=z]\\
&\leq \sum_{z\in\mathfrak{D}^u(x_0)}\mathbf{P}[\xi^u(t)=z \text{ for some } t]\mathbf{E}N(z)\\
&\leq K_5\sum_{z\in\mathfrak{D}^u(x_0)}\mathbf{P}[\xi^u(t)=z \text{ for some } t]\mathbf{E}\Big(\int_0^\infty \mathbf{1}\{\xi^a(t)=\xi^b(t)=z\}dt\Big)\\
&\leq \sum_{z\in\mathfrak{D}^u(x_0)}\frac{K_6}{\|u-z\|^{d-2}}\Big(\int_0^\infty t^{-d}e^{-\frac{\rho_1^2 d}{2t}}e^{-\frac{\rho_2^2 d}{2t}}\,dt+O(\|x_0\|^{8/5}e^{-\|x_0\|^{1/5}})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{z\in\mathfrak{D}^u(x_0)} \frac{K_6}{\|u-z\|^{d-2}}\Big(\int_0^\infty \frac{(\rho_1^2+\rho_2^2)s^{d-2}}{2^{1-d}(\rho_1^2+\rho_2^2)^d}e^{-s}\,ds + O(\|x_0\|^{8/5}e^{-\|x_0\|^{1/5}})\Big)\\
&\le \frac{K_7x_0^2}{x_0^{2d}}\sum_{z\in\mathfrak{D}^u(x_0)}\frac{1}{\|u-z\|^{d-2}}. \qquad (2.9)
\end{aligned}
$$

Since

$$
\sum_{z\in\mathfrak{D}^u(x_0)}\frac{1}{\|u-z\|^{d-2}} = O(x_0^2),
$$

using (2.9) we conclude the proof of Lemma 2.4. ■

Now we continue the proof of Theorem 1.4. Let us define the stopping times

$$
\begin{aligned}
\tau_0 &= 0,\\
\tau_i &= \min\{2^{2\kappa i}, \inf\{t : \xi^0(t)\notin \mathfrak{D}(2^{\kappa i}-1)\}\}.
\end{aligned}
$$

Fix $\varepsilon < \frac{\gamma'}{2(\gamma'')^2}$, where γ' is the constant from Lemma 2.3 and γ'' is the constant from Lemma 2.4. Define also the sets $\tilde{S}_i^\varepsilon = S\cap\mathfrak{G}^{\kappa i}$, if $|S\cap\mathfrak{G}^{\kappa i}| < \varepsilon 2^{(\kappa i)(d-2)}$, and if $|S\cap\mathfrak{G}^{\kappa i}| \ge \varepsilon 2^{(\kappa i)(d-2)}$, then to construct the set $\tilde{S}_i^\varepsilon$ we remove some points from $S\cap\mathfrak{G}^{\kappa i}$ in such a way that $|\tilde{S}_i^\varepsilon| = [\varepsilon 2^{(\kappa i)(d-2)}]$.

Define the sequence of events

$$
A_i = \{\xi^0(t)=\xi^b(t) \text{ for some } t\in(\tau_{i-1},\tau_{i+1}] \text{ for at least one } b\in\tilde{S}_i^\varepsilon\},
$$

Let $\mathcal{F}_t^i$ be the sigma-field generated by the collection of random variables $\xi^{a'}(s)$ for $0\le s\le t$ and $a'\in\mathfrak{D}(2^{\kappa i+1})\cap S$.

By virtue of two previous lemmas we show the following result.

Lemma 2.5 *There exists $L_0 = L_0(S)$ such that*

$$
\mathbf{P}[A_i \mid \mathcal{F}_{\tau_{i-1}}^{i-1}] \ge \frac{L_0|\tilde{S}_i^\varepsilon|}{2^{(\kappa i)(d-2)}}.
$$

Proof. By CLT it is elementary to obtain that the SRW originating from $b\in\mathfrak{G}^{\kappa i}$ will still be in $\mathfrak{G}^{\kappa i}$ at the moment τ_{i-1} with probability bounded away from zero by some constant K_8 for all k and all i. We will denote the set of such particles by $\bar{S}_i^\epsilon = \{b : b\in\tilde{S}_i^\epsilon, \xi^b(\tau_{i-1})\in\mathfrak{G}^{\kappa i}\}$. Then there exists a constant K_9 such that

$$
\mathbf{P}(|\bar{S}_i^\epsilon| > K_8|\tilde{S}_i^\epsilon|/2) \ge 1-\exp(-K_9\tilde{S}_i^\epsilon) \ge 1-\exp(-K_9) =: K_{10} \qquad (2.10)
$$

for all $|\tilde{S}_i^\epsilon|\ge 1$.

For all $b \in \tilde{S}_i^\varepsilon$ let us introduce the event $E_b = \{\xi^0(t) = \xi^b(t)$ for some $t \in (\tau_{i-1}, \tau_{i+1}]\}$. Using Lemmas 2.3 and 2.4, we obtain for any $a \in \mathfrak{D}(2^{\kappa(i-1)})$, $0 < m < |\tilde{S}_i^\varepsilon|$, and $B \in \mathcal{F}_{\tau_{i-1}}^{i-1}$ that

$$\begin{aligned}
&\mathbf{P}[A_i \mid B \cap \{\xi^0(\tau_{i-1}) = a\} \cap \{|\bar{S}_i^\epsilon| = m\}] \\
&\quad\geq \mathbf{P}\Big[\bigcup_{b\in\bar{S}_i^\varepsilon} E_b \mid B \cap \{\xi^0(\tau_{i-1}) = a\} \cap \{|\bar{S}_i^\epsilon| = m\}\Big] \\
&\quad\geq \sum_{b\in\bar{S}_i^\varepsilon} \mathbf{P}[E_b \mid B \cap \{\xi^0(\tau_{i-1}) = a\} \cap \{|\bar{S}_i^\epsilon| = m\}] \\
&\qquad - \sum_{b,u\in\bar{S}_i^\varepsilon} \mathbf{P}[E_b \cap E_u \mid B \cap \{\xi^0(\tau_{i-1}) = a\} \cap \{|\bar{S}_i^\epsilon| = m\}] \\
&\quad\geq \sum_{b\in\bar{S}_i^\varepsilon} \frac{\gamma'}{2^{\kappa i(d-2)}} - \sum_{b,u\in\bar{S}_i^\varepsilon} \frac{(\gamma'')^2}{2^{2\kappa i(d-2)}} = m\frac{\gamma'}{2^{\kappa i(d-2)}}\Big(1 - \frac{m(\gamma'')^2}{2^{\kappa i(d-2)}\gamma'}\Big) \\
&\quad\geq \frac{m}{2^{\kappa i(d-2)}}\gamma'\Big(1 - \frac{\varepsilon(\gamma'')^2}{\gamma'}\Big) = \frac{K_{11}m}{2^{(\kappa i)(d-2)}},
\end{aligned}$$

where $1 - \frac{\varepsilon(\gamma'')^2}{\gamma'} > 0$ due to the choice of ε. It follows that

$$\mathbf{P}[A_i \mid B \cap \{|\bar{S}_i^\epsilon| = m\}] \geq \frac{K_{11}m}{2^{(\kappa i)(d-2)}}.$$

Then by (2.10)

$$\mathbf{P}[A_i \mid B] \geq \frac{K_{11}(K_8/2)|\tilde{S}_i^\epsilon|K_{10}}{2^{(\kappa i)(d-2)}}.$$

So, Lemma 2.5 is proved. ∎

Now we are able to finish the proof of Theorem 1.4. If $\sum_{x\in S}\|x\|^{-(d-2)} = \infty$, then

$$\sum_{x\in S}\|x\|^{-(d-2)} \leq \sum_{m=0}^{\kappa-1}\sum_{i=0}^{\infty} \frac{|S \cap \mathfrak{G}^{\kappa i+m}|}{(2^{\kappa i+m})^{d-2}} = \infty$$

for any $\kappa \in \mathbb{N}$. So, let us choose κ according to Lemma 2.3 and observe that there exists $m_0 \in \{0, \dots, \kappa-1\}$ such that

$$\sum_{i=0}^{\infty} \frac{|S \cap \mathfrak{G}^{\kappa i+m_0}|}{(2^{\kappa i+m_0})^{d-2}} = \infty.$$

Without loosing of generality, suppose that $m_0 = 0$ (for the other cases, the proof is quite analogous). Moreover, in this case at least one of the series

$$\sum_{i=0}^{\infty} \frac{|\tilde{S}_{2i}^\varepsilon|}{(2^{2\kappa i})^{d-2}}, \qquad \sum_{i=0}^{\infty} \frac{|\tilde{S}_{2i+1}^\varepsilon|}{(2^{\kappa(2i+1)})^{d-2}}$$

diverges, and we suppose without loss of generality that the first one does.

Observe also that, by Lemma 2.5 and by the fact that A_j only depends on the paths of the particles originating from $\mathfrak{D}(2^{\kappa j})$ until the moment τ_{j+1} it holds that

$$\mathbf{P}[A_{2i} \mid \overline{A}_{2i-2}, \dots, \overline{A}_2] \geq \frac{L_0 |\tilde{S}^{\varepsilon}_{2i}|}{2^{2\kappa i(d-2)}}. \tag{2.11}$$

Now, consider the path ξ^0 of the particle from the origin. Using (2.11), we see that

$$\begin{aligned} & \mathbf{P}[\xi^0(t) = \xi^b(t) \text{ for some } t \geq 0 \text{ for at least one } b \in S] \\ & \geq \ 1 - \prod_{i=1}^{\infty} \mathbf{P}[\overline{A}_{2i} \mid \overline{A}_{2i-2}, \dots, \overline{A}_2] \\ & \geq \ 1 - \prod_{i=1}^{\infty} \Big(1 - \frac{L_0 |\tilde{S}^{\varepsilon}_{2i}|}{2^{2\kappa i(d-2)}}\Big) = 1, \end{aligned} \tag{2.12}$$

which shows that the particle from the origin will a.s. meet someone. It is elementary to get that if initially $\sum_{x\in S} \|x\|^{-(d-2)} = \infty$, then for any $t \geq 0$ and $y \in \mathbb{Z}^d$

$$\sum_{x\in S} \|y - \xi^x(t)\|^{-(d-2)} = \infty \quad \text{a.s.} \tag{2.13}$$

Indeed, for any t and y fixed there exists a constant $h = h(t, y) > 0$ such that for all x with $\|x\|$ large enough $\mathbf{P}[\|y - \xi^x(t)\| > 2\|x\|] \leq e^{-h\|x\|}$. Then

$$\mathbf{P}[\text{there exists } x \in S \cap \mathfrak{G}^i : \|y - \xi^x(t)\| > 2\|x\|] \leq 2^{(i+1)d} e^{-h2^i}.$$

Then, by Borel-Cantelli lemma,

$$\mathbf{P}\Big[\text{there exists } m_0 : \|y - \xi^x(t)\| < 2\|x\| \text{ for all } x \in \bigcup_{i=m_0}^{\infty} S \cap \mathfrak{G}^i\Big] = 1.$$

Hence, the series $\sum_{x\in S} \|y - \xi^x(t)\|^{-(d-2)} \geq \sum_{x\in\bigcup_{i=m_0}^{\infty} S\cap\mathfrak{G}^i} (2\|x\|)^{-(d-2)}$ diverges.

Then (2.13) immediately implies that the particle from the origin will a.s. meet an infinite number of other particles, and so the proof of Theorem 1.4 is concluded. ■

Proof of Theorem 1.1. If $\sum_{x\in S} \|x\|^{-(d-2)} < \infty$, then by Theorem 1.4 the red particle (the one that starts at the origin) will not meet anyone with positive probability, so Theorem 1.1 is immediate in this case.

Now, let $\sum_{x\in S} \|x\|^{-(d-2)} = \infty$. Suppose that with positive probability red particles survive forever. Then by Theorem 1.4 and (2.13) the number of collisions between blue and red particles is infinite with positive probability and then the number of red particles changes infinitely often. Clearly, the number of red particles is a non-negative martingale. It converges a.s. to some random variable. So, from some random moment of time the number of red particles should be constant, which leads to a contradiction. Thus red particles die a.s. ∎

Proof of Theorem 1.2. By Theorem 1.4 a particle starting from the origin will infect infinitely many particles a.s. Then for any $M_0 > 0$ by some random moment of time there will be at least M_0 infected particles. At that moment, those particles are contained in the ball $\mathfrak{D}(2^n)$ for all large enough n. With the help of CLT it is not difficult to see that for any $n > 0$, any $x \in \mathfrak{D}(2^n)$ and any $t < 2^{2n}$

$$\mathbf{P}[\xi^x(t) \in \mathfrak{D}(2^n)] > \delta \tag{2.14}$$

for some $\delta > 0$. Then there is $h^* > 0$ such that with probability greater than $1 - e^{-h^* M_0}$ at least $M := \delta M_0/2$ of these particles will be in $\mathfrak{D}(2^n)$ at the moment 2^{2n}.

In the following lemma we show that if n is chosen large enough compared to M, then with probability exponentially large in M within the time interval $[2^{2n}, 2^{2(n+1)}]$ these M particles will infect at least $(2^{d-2}+1)M$ particles originated from $\mathfrak{G}^n = \{x : 2^n \leq \|x\| < 2^{n+1}\}$ and being in $\mathfrak{D}(2^{n+1})$ at the moment $2^{2(n+1)}$.

For any $H \subset S$ define the *H-restriction* of the process of infection spreading as the process with the initial configuration H of healthy particles. Clearly, the H-restricted process can be coupled with the original process in such a way that at each moment the set of infected particles in the original process contains the set of infected particles in the H-restricted process. Let us introduce the events

$$\begin{aligned} A(L,n) &= \{\text{at moment } 2^{2n} \text{ for the } S \cap \mathfrak{D}(2^n)\text{-restricted process} \\ &\qquad \text{there are at least } L \text{ infected particles in } \mathfrak{D}(2^n)\}, \\ \tilde{A}(L,n) &= \{\text{at moment } 2^{2n} \text{ for the } S \cap \mathfrak{D}(2^n)\text{-restricted process} \\ &\qquad \text{there are at least } L \text{ infected particles} \\ &\qquad \text{originating from } \mathfrak{G}^{n-1} \text{ in } \mathfrak{D}(2^n)\}. \end{aligned}$$

The following fact is the key to the proof of Theorem 1.2.

Lemma 2.6 *Assume that the constant $\alpha > 0$ in (1.1) is fixed large enough. Then there exist $h, \Gamma > 0$ such that for all M and n verifying $M 2^{-n(d-2)} < \Gamma$ we have*

$$\mathbf{P}[\tilde{A}((2^{d-2}+1)M, n+1) \mid A(M,n)] \geq 1 - \exp(-hM). \tag{2.15}$$

Before proving the lemma, let us first finish the proof of the theorem assuming that Lemma 2.6 holds true. Suppose that the event $\tilde{A}(M(2^{d-2}+1), n+1)$ have occurred (by Lemma 2.6, the probability of that is at least $1-\exp(-hM)$), and let us split the set of these $M(2^{d-2}+1)$ particles into two sets $\mathcal{A}_1$ of $M2^{d-2}$ and $\mathcal{B}_1$ of M particles respectively; then apply (2.15) to $\mathcal{A}_1$ in $\mathfrak{D}(2^{n+1})$. Since, by construction, the events $A(\cdot,n), \tilde{A}(\cdot,n)$ do not depend on the particles which were originally outside $\mathfrak{D}(2^n)$, applying Lemma 2.6 (with $A(M,n)$ substituted by $A(M2^{d-2},n+1)$ and $\tilde{A}((2^{d-2}+1)M,n+1)$ substituted by $\tilde{A}((2^{d-2}+1)2^{d-2}M,n+2)$), we obtain the following fact: With probability greater than $1-\exp(-hM2^{d-2})$ within the period $[2^{2(n+1)},2^{2(n+2)}]$ the particles of $\mathcal{A}_1$ will contaminate at least $M2^{d-2}(2^{d-2}+1)$ particles originated from $\mathfrak{G}^{n+1}$ and situated in $\mathfrak{D}(2^{n+2})$ at the moment $2^{2(n+2)}$. We split this set into two subsets $\mathcal{A}_2$ of $M2^{2(d-2)}$ and $\mathcal{B}_2$ of $M2^{(d-2)}$ particles and "send" the particles of the first set to infect those originated from $\mathfrak{G}^{n+2}$. On the ith step of this infection spreading by the time $2^{2(n+i)}$ there will be at least $|\mathcal{A}_i| = M2^{(i-1)(d-2)}2^{d-2}$ plus $|\mathcal{B}_i| = M2^{(i-1)(d-2)}$ infected particles in $\mathfrak{D}(2^{n+i})$ originated from $\mathfrak{G}^{n+i-1}$ with conditional probability at least $1-\exp(-hM2^{(i-1)(d-2)})$. The particles of $\mathcal{A}_i$ are supposed to contaminate the next ones originated from $\mathfrak{G}^{n+i}$. Now let us consider the infected particles from the sets $\mathcal{B}_i$. Note that a particle from $\mathcal{B}_i$ is found in $\mathfrak{D}(2^{n+i})$ (at time $2^{2(n+i)}$) after being infected. Then the probability that it will reach the origin $0\in\mathbb{Z}^d$ after, is at least $O(2^{-(n+i)(d-2)})$. Since the series $\sum_{i=1}^{\infty}|\mathcal{B}_i|2^{-(n+i)(d-2)}$ diverges, then by Borel-Cantelli lemma infinitely many of infected particles from $\cup_i\mathcal{B}_i$ will visit the origin a.s. conditionally that the process of infection spreading by the sets $\mathcal{A}_i$ succeeds. Notice, however, that it fails with probability at most $1-\prod_{i=0}^{\infty}(1-\exp(-hM2^{(d-2)i}))\to 0$ as $M\to\infty$. Since the choice of M is arbitrary, the origin will be visited by infinitely many infected particles a.s. The same argument applies indeed for any point of $\mathbb{Z}^d$ finishing the proof of the statement (i) of the theorem.

Let us also derive from this reasoning the proof of (ii). Let us fix the trajectory $\xi^x(t)$ of the particle starting from the point x. Assume that it is in $\mathfrak{D}(2^{n+i})$ at time $2^{2(n+i)}$. At that moment there are $M2^{(i-1)(d-2)}$ infected particles of $\mathcal{B}_i$ in $\mathfrak{D}(2^{n+i})$. Observe also that all of them are at the distance at most 2^{n+i+1} from $\xi^x(2^{2(n+i)})$. Analogously to Lemma 2.5 one can show that the probability that at least one of those particles will infect the particle starting from x is greater than $1-(1-O(2^{-(n+i)(d-2)}))^{M2^{(i-1)(d-2)}}>\delta>0$ with some $\delta=\delta(M)$, for all i. Hence, if $\xi^x(2^{2(n+i)})\in\mathfrak{D}(2^{n+i})$ happens for infinitely many i and the process of infection spreading by $\mathcal{A}_i$ succeeds, the probability that no particle of $\cup_i\mathcal{B}_i$ infects the particle from x is zero. On the other hand, it is elementary to see that there exists $\delta_0<1$ such that for any $n,i,\tilde{x}$ there exists $m_0\geq 1$ such that

$$\mathbf{P}[\|\xi(2^{2(n+i+m_0)})\|2^{-(n+i+m_0)}\geq 1 \mid \xi(2^{2(n+i)})=\tilde{x}]\leq\delta_0.$$

(Clearly, one can estimate this probability applying CLT to the vector $\xi(2^{2(n+i+m_0)})-\xi(2^{2(n+i)})$.) Then immediately $\liminf_{i\to\infty}\|\xi(2^{2(n+i)})\|2^{-(n+i)}<1$ a.s. Thus, ξ^x will be infected with probability 1, and the proof of Theorem 1.2 is finished. ■

Proof of Lemma 2.6. Let us fix the trajectories of M infected (by the time 2^{2n}) particles starting from $x_1, \dots, x_M$ within the time interval $[2^{2n}, 2^{2(n+1)}]$. We can regard the set of those trajectories as a random space-time set W_M of triples (y, T_1, T_2), $[T_1, T_2] \subset [2^{2n}, 2^{2(n+1)}]$. Formally, the set W_M can be described as follows: suppose that the particle from x_i were at y_i^0 at the moment 2^{2n}. Suppose also that the subsequent evolution of that particle was the following: at moments T_i^j, $j = 1, \dots, n_i - 1$, it jumped from y_i^{j-1} to y_i^j, and there were no jumps in the time interval $(T_i^{n_i-1}, 2^{2(n+1)}]$. Put also $T_i^0 := 2^{2n}$, $T_i^{n_i} := 2^{2(n+1)}$, $i = 1, \dots, M$. Then we define

$$W_M = \{(y_i^j, T_i^{j-1}, T_i^j), i = 1, \dots, M, j = 1, \dots, n_i\}.$$

Next, we modify the set W_M by deleting from there all triples with $y \notin \mathfrak{D}(2^{n+1})$ leaving thus only pieces of trajectories of these particles which are contained in $\mathfrak{D}(2^{n+1})$. We say that a particle originated from x reaches W_M and write $x \to W_M$ if there exists a triple $(y, T_1, T_2) \in W_M$ such that $\xi^x(t) = y$ for some $t \in (T_1, T_2]$. Let us denote by $\mathtt{P}_{W_M}$ the conditional probability when W_M is fixed. The goal is to show that there are some constants $K_1, h_1 > 0$ and $\varepsilon > 0$ such that for any $x \in \mathfrak{G}^n$, any M and n with $M 2^{-n(d-2)} < \varepsilon$

$$\mathbf{P}\Big[W_M : \mathtt{P}_{W_M}[x \to W_M] > K_1 M 2^{-n(d-2)}\Big] \geq 1 - \exp(-h_1 M). \tag{2.16}$$

Before starting the proof of the above inequality, assume first that (2.16) holds. Then it is not difficult to finish the proof of the lemma. Consider a space-time set W_M such that the probability to reach it by any particle originated from $\mathfrak{G}^n$ is at least $K_1 M 2^{-n(d-2)}$. The key observation here is that, for fixed W_M, the events $\{x \to W_M\}$, $x \in S \cap \mathfrak{G}^n$, are conditionally independent. By the assumption of the theorem, initially there are at least $\alpha 2^{n(d-2)}$ particles in $\mathfrak{G}^n$. Then by elementary exponential bounds for sums of independent Bernoulli random variables, with probability greater than $1 - \exp(-h_2 \alpha K_1 M)$ (where $h_2 > 0$ is some constant not depending on M and n) at least $K_1 M 2^{-n(d-2)} \times \alpha 2^{n(d-2)}/2 = \alpha K_1 M/2$ of these particles will reach W_M. Using the observation (2.14) and the same tool again we derive that at least $\delta \alpha K_1 M/4$ of these particles will be located in $\mathfrak{D}(2^{n+1})$ at the moment $2^{2(n+1)}$ with probability at least $1 - \exp(-h_3 \alpha K_1 M)$ where $h_3 > 0$ does not depend on M and n. Then in view of (2.16)

$$\begin{aligned} &\mathbf{P}[\tilde{A}(\delta \alpha K_1 M/4, n+1) \mid A(M, n)] \\ &\quad \geq \ (1 - \exp(-h_1 M))(1 - \exp(-h_2 \alpha K_1 M))(1 - \exp(-h_3 \alpha K_1 M)). \end{aligned}$$

Finally, it suffices to fix the constant α in (1.1) in such a way that $\delta \alpha K_1/4 > 2^{d-1} + 1$, which would conclude the proof of the lemma.

Thus, the proof of Lemma 2.6 is reduced to showing (2.16). To start with, let us split $\mathbb{Z}^d$ into cubes of side $2^{2n/d}$, and consider those that lie fully inside the ball $\mathfrak{D}(2^{n+1})$. Then the volume of each cube is 2^{2n} and there are at most $\ell = O(2^{n(d-2)})$ of them; we will refer to those cubes as $\mathcal{K}_1, \dots, \mathcal{K}_\ell$. Consider the space-time sets W_M satisfying the following property (P) with some constant $0 < \tilde{\gamma} < 1$:

(P): *During at least half of the time within the period* $[2^{2n+1}, 2^{2n+2}]$ *at least* $\tilde{\gamma}M$ *different cubes of* $\mathfrak{D}(2^{n+1})$ *are occupied by points of* W_M.

Now we will show that

(i) There exists a positive constant K_1 (depending on $\tilde{\gamma}$) such that for any $x \in \mathfrak{G}^n$ and any W_M satisfying the property (P) it holds that $\mathtt{P}_{W_M}[x \to W_M] \geq K_1 M 2^{-n(d-2)}$.

(ii) There exists a positive constant h_1 (depending on $\tilde{\gamma}$) such that the probability that the set W_M satisfies the property (P) is at least $1 - \exp(-h_1 M)$.

To proceed with (i), let us take W_M satisfying (P). Let us say that a space-time set $\tilde{W}_M$ is "smaller" than W_M if for any triple $(y, \tilde{T}_1, \tilde{T}_2) \in \tilde{W}_M$ there exists $(y, T_1, T_2) \in W_M$ such that $(\tilde{T}_1, \tilde{T}_2] \subset (T_1, T_2]$. Then

$$\mathtt{P}_{W_M}[x \to W_M] \geq \mathtt{P}_{\tilde{W}_M}[x \to \tilde{W}_M]. \tag{2.17}$$

We construct $\tilde{W}_M$ from W_M by deleting pieces of trajectories of M particles in such a way that *at any moment of time, in each cube containing more than one point of* W_M *it remains exactly one point.* For this purpose let us enumerate M particles and leave in each cube and at each moment of time only the one with the smallest number. In other words, if $(y_1, T_1, T_2), (y_2, T_3, T_4) \in W_M$ such that y_1, y_2 belong to the same cube and if moreover $(T_3, T_4] \subset (T_1, T_2]$ then we delete (y_2, T_3, T_4) from W_M, if $T_1 < T_3 < T_2 < T_4$ then we replace the second triple by (y_2, T_2, T_4). In view of (2.17) it suffices to show (i) for this smaller set $\tilde{W}_M$.

Let us denote by $E_t(x \to \tilde{W}_M)$ the expectation of the total time spent in $\tilde{W}_M$ after the moment t by a particle being in x at time t. Then

$$\begin{aligned} E_t(x \to \tilde{W}_M) &= \sum_{(y,T_1,T_2)\in\tilde{W}_M} \mathbf{E} \int_{(T_1-t)^+}^{(T_2-t)^+} \mathbf{1}[\xi^x(s) = y]\, ds \\ &= \sum_{(y,T_1,T_2)\in\tilde{W}_M} \int_{(T_1-t)^+}^{(T_2-t)^+} \mathbf{P}[\xi^x(s) = y]\, ds. \end{aligned} \tag{2.18}$$

We have

$$E_0(x \to \tilde{W}_M) \le \mathsf{P}_{\tilde{W}_M}[x \to \tilde{W}_M] \sup_{(y,T_1,T_2)\in\tilde{W}_M} E_{T_1}(y \to \tilde{W}_M),$$

so

$$\mathsf{P}_{\tilde{W}_M}[x \to \tilde{W}_M] \ge \frac{E_0(x \to \tilde{W}_M)}{\sup\limits_{(y,T_1,T_2)\in\tilde{W}_M} E_0(y \to \tilde{W}_M)}. \tag{2.19}$$

We will show that the denominator in (2.19) is bounded from above by some constant for all M and n, while the numerator is $O(M2^{-n(d-2)})$.

Let us estimate the denominator. Using the fact that at each moment of time there is at most one point of $\tilde{W}_M$ in each of $\mathcal{K}_i$-s, we can write

$$\begin{aligned} E_0(y \to \tilde{W}_M) &= \sum_{i=1}^{\ell} \sum_{\substack{(z,T_1,T_2)\in\tilde{W}_M:\\ z\in\mathcal{K}_i}} \int_{T_1}^{T_2} \mathbf{P}[\xi^y(s) = z]\, ds \\ &\le \sum_{i=1}^{\ell} \int_0^{+\infty} \sup_{z\in\mathcal{K}_i} \mathbf{P}[\xi^y(s) = z]\, ds. \end{aligned}$$

By Lemma 2.1 (i)

$$\int_0^{+\infty} \sup_{z\in\mathcal{K}_i} \mathbf{P}[\xi^y(s) = z]\, ds \le K_2\big(\min_{z\in\mathcal{K}_i} \|y-z\|\big)^{-(d-2)}$$

for some constant $K_2 > 0$. Then there is a constant $K_3 > 0$ such that for any $j = 1, 2, \dots$, any n and any point z satisfying $j2^{2n/d} \le \|z-y\| < (j+1)2^{2n/d}$, it holds that the mean number of visits to z starting from y does not exceed $K_3(j2^{2n/d})^{2-d}$. The number of cubes of side $2^{2n/d}$ touching $\mathfrak{D}((j+1)2^{2n/d}) \setminus \mathfrak{D}(j2^{2n/d})$ in $\mathbb{Z}^d$ is smaller than $K_4 j^{d-1}$ with some $K_4 > 0$ for all $j = 1, 2, \dots$ all n and all y. Note also that there is at most one particle in $\tilde{W}_M$ at distance smaller than $2^{2n/d}$ and the mean number of visits to it is bounded by some constant $K_5 > 0$. Since there are no particles in $\tilde{W}_M$ outside $\mathfrak{D}(2^{n+1})$, then the maximal distance between any two points of $\tilde{W}_M$ is $2^{n+2} = j2^{2n/d}$ with $j = 2^{n(1-2/d)+2}$. Thus the denominator in (2.19) is bounded by the constant

$$K_5 + \sum_{j=1}^{2^{n(1-2/d)+2}} K_3 K_4 j^{d-1}(j2^{2n/d})^{2-d} = K_5 + K_3K_4 2^{-n(2-4/d)} \sum_{j=1}^{2^{n(1-2/d)}} j \le K_6$$

for all n. Let us remark that it was essential to obtain this estimate that in each cube there is at most one particle of $\tilde{W}_M$. Otherwise, if we allowed particles to accumulate in one cube in big amount (e.g. growing with M), then a particle starting from this cube would reach such a space-time set for a mean number of times rather large (growing with M).

Now let us deal with the numerator in (2.19). Note that the validity of the property (P) for $\tilde{W}_M$ ensures that in $\tilde{W}_M$ there are at least $\tilde{\gamma}M$ points during a large part of time. This will be crucial in the estimate of the numerator. First, applying Lemma 2.1 (ii), we deduce from the property (P) that

$$\begin{aligned}
& E_0(x \to \tilde{W}_M) \\
& \quad \geq \sum_{(z,T_1,T_2)\in\tilde{W}_M} \int_{T_1}^{T_2} \mathbf{P}[\xi^x(s) = z]\, ds \\
& \quad \geq K_7 2^{-2dn} \int_{2^{2n+1}}^{2^{2n+2}} |\{(z,T_1,T_2) \in \tilde{W}_M : T_1 \leq s \leq T_2\}|\, ds \\
& \quad \geq K_7 2^{-2dn} \int_{2^{2n+1}}^{2^{2n+2}} \tilde{\gamma} M \mathbf{1}\Big\{|\{(z,T_1,T_2) \in \tilde{W}_M : T_1 \leq s \leq T_2\}| \geq \tilde{\gamma}M\Big\}\, ds \\
& \quad \geq K_8 M 2^{-(d-2)n}
\end{aligned}$$

with some positive constants K_7, K_8 as $\|x-z\|^2/t$ is bounded uniformly for all $x, z \in \mathfrak{D}(2^{n+1})$ and $t \in [2^{2n+1}, 2^{2n+2}]$. So, the probability (2.19) is limited from below by $K_8\tilde{\gamma}M2^{-n(d-2)}/K_6$ for all W_M verifying (P). This finishes the proof of (i).

Finally, we concentrate on (ii). From Lemma 2.1 (ii) it follows that there exists $\beta > 0$ such that for any n, any cube $\mathcal{K}$ of side $2^{2n/d}$ in $\{(3/2)2^n \leq \|x\| \leq 2^{n+1}\}$ the probability to be in this cube at time t for a particle starting at time 2^{2n} from any point in $\mathfrak{D}(2^n)$ is greater than $\beta 2^{-n(d-2)}$ for all $t \in [2^{2n+1}, 2^{2n+2}]$. Thus at any moment of time $t \in [2^{2n+1}, 2^{2n+2}]$, M particles of W_M are distributed throughout $K_9 2^{n(d-2)}$ cells (cubes) ($K_9 > 0$ is some constant) of $\{(3/2)2^n \leq \|x\| \leq 2^{n+1}\}$; the probability for each particle to be in a given cell is greater than $\beta 2^{-n(d-2)}$. Assume that n is such that $K_9 2^{n(d-2)} = \hat{\Gamma}M$ with some $\hat{\Gamma} > 0$. Then we have to estimate the number of non-empty cells in an easy combinatorial problem: each of M particles goes to each of $\hat{\Gamma}M$ cells independently of the others with probability $\beta K_9/\hat{\Gamma}M$, or disappears with probability $1 - \beta K_9$. It is elementary to verify that for all small enough $\tilde{\gamma} = \tilde{\gamma}(\beta, K_9, \hat{\Gamma}) > 0$ there exists $h = h(\tilde{\gamma}) > 0$ such that at least $\tilde{\gamma}M$ cells are occupied with probability greater than $1 - \exp(-hM)$. Furthermore, if the number of cubes $K_9 2^{n(d-2)} > \hat{\Gamma}M$ then we can group some cubes together in order to form "exactly" $\hat{\Gamma}M$ cubes of the same volume and then apply the previous estimate. Indeed, if the number of non-empty cubes in this coarser partition is bigger than $\tilde{\gamma}M$ than the same is true for the initial partition. Therefore for any M and n such that $K_9 2^{n(d-2)} \geq \hat{\Gamma}M$ and at any moment of time $t \in [2^{2n+1}, 2^{2n+2}]$ at least $\tilde{\gamma}M$ cubes are occupied by particles with probability greater than $1 - \exp(-hM)$. Now let us construct a random variable $\zeta_{W_M}(t)$ taking the value 1 if at time t at least $\tilde{\gamma}M$ cubes contain particles of W_M and the value 0 otherwise. Then $\mathbf{E}\zeta_{W_M}(t) \geq 1 - \exp(-hM)$ for any $t \in [2^{2n+1}, 2^{2n+2}]$. Let $\tilde{\zeta}_{W_M} = 2^{-2n-1} \int_{2^{2n+1}}^{2^{2n+2}} \zeta(s)\, ds$ be the proportion of time within the interval $[2^{2n+1}, 2^{2n+2}]$ when at least $\tilde{\gamma}M$ cubes contain particles of W_M. It is elementary to see from the facts $\tilde{\zeta}_{W_M} \leq 1$ and $\mathbf{E}\tilde{\zeta}_{W_M} \geq 1 - \exp(-hM)$ that

$$\mathbf{P}[\tilde{\zeta}_{W_M} > 1/2] \geq 1 - 2\exp(-hM).$$

Now it is straightforward to notice that the event $\{\tilde{\zeta}_{W_M} > 1/2\}$ means that the property (P) is verified for W_M. Thus, (ii) is proven, and so the proof of Lemma 2.6 is finished. ∎

Proof of Theorem 1.3. Similarly to the proof of the corresponding statement in [15], the idea is to dominate the process of spreading of infection by a branching random walk. However, here this comparison is not so straightforward as in [15]. The key to the proof of Theorem 1.3 is the following fact:

Lemma 2.7 *Suppose that the initial configuration of particles is sampled accordingly to product measure $\mathfrak{P}_q = \otimes_{x\in\mathbb{Z}^d}\mathcal{P}(q(x))$, where $\mathcal{P}(\lambda)$ stands for the Poisson distribution with parameter λ. Let $\tilde{\mathcal{F}}_t$ be the σ-field generated by the process of infection spreading up to time t and $\eta_t(x)$ be the number of healthy particles at site x at time t. Then, at each t, the field $\{\eta_t(x)\}_{x\in\mathbb{Z}^d}$ is dominated by a field $\{\hat{\eta}_t(x)\}_{x\in\mathbb{Z}^d}$ which is independent of $\tilde{\mathcal{F}}_t$ and has the law $\mathfrak{P}_{q_t}$, where $q_t(x) = \sum_y p_t(y,x)q(y)$.*

Proof. Let $\mathfrak{W}$ be the set of all finite trajectories until time t. We can represent the elements of $\mathfrak{W}$ as $w = (x_1, \dots, x_n; t_1, \dots, t_{n-1})$ where $x_i \in \mathbb{Z}^d$, $\|x_i - x_{i-1}\| = 1$, and t_i-s are the moments of jumps from x_i to x_{i+1}, $0 < t_1 < t_2 < \cdots < t_{n-1} \leq t$. If $w = (x_1, \dots, x_n; t_1, \dots, t_{n-1})$, then we say that $|w| = n$. For $0 \leq s \leq t$ let $w(s)$ be the position of the trajectory at time s (i.e., $w(s) = x_i$, if $t_{i-1} \leq s < t_i$, $w(s) = x_1$, if $s < t_1$, $w(s) = x_n$, if $s \geq t_{n-1}$). Let also

$$\mathfrak{W}_{x,n} = \{w \in \mathfrak{W} :\ w(0) = x, |w| = n\} = \mathfrak{S}_{x,n} \times \mathfrak{T}_n,$$

where $\mathfrak{S}_{x,n}$ are all discrete-time trajectories of length n beginning in x, and $\mathfrak{T}_n = \{(t_1, \dots, t_{n-1}) :\ 0 < t_1 < t_2 < \cdots < t_{n-1} \leq t\} \subset \mathbb{R}^{n-1}$ for $n > 1$, and $\mathfrak{T}_1 = \emptyset$. Clearly, any measure Λ on $\mathfrak{W} = \cup_{x,n}\mathfrak{W}_{x,n}$ has the property $\Lambda(A) = \sum_{x,n}\Lambda(A\cap\mathfrak{W}_{x,n})$, $A \in \mathfrak{W}$. So, it is sufficient to define Λ on $A = \nu \times A_0$, where $\nu \in \mathfrak{S}_{x,n}$, $A_0 \subset \mathfrak{T}_n$. We set then, for $\nu \in \mathfrak{S}_{x,n}$, $A_0 \subset \mathfrak{T}_n$, and $n > 1$

$$\Lambda(\nu \times A_0) := \frac{q(x)e^{-t}|A_0|}{(2d)^{n-1}} \tag{2.20}$$

(here $|\cdot|$ stands for the Lebesgue measure in $\mathbb{R}^{n-1}$), and $\Lambda(\nu \times A_0) = q(x)e^{-t}$ for $n = 1$.

Now we go back to the process of infection spreading and let us denote by $\tilde{\eta}_t(x)$ the number of (both healthy and infected) particles at site x at time t. Then it is not difficult to see that the process $\{\tilde{\eta}_s(x)\}_{x\in\mathbb{Z}^d, s\in(0,t)}$ can be viewed as a Poisson point process on $\mathfrak{W}$ with parameter measure $\Lambda(\cdot)$ (cf. e.g. Section 2.4 of [4] for the theory of general Poisson processes). To give a heuristic explanation of that fact, note that the quantity in (2.20) should be equal to the mean number of particles starting in x that follow the same fixed trajectory ν of (discrete) length n and have the vector of times of jumps belonging to A_0. Since the moments of jumps of a given particle form *one-dimensional* Poisson process with rate 1, the probability that there are $n-1$ jumps within the time interval $[0,t]$ is $\frac{t^{n-1}}{(n-1)!}e^{-t}$. The factor $1/(2d)^{n-1}$ corresponds to the probability of the fixed discrete-time trajectory ν. Then, recall the fact that conditioned on the event that a one-dimensional Poisson process has k points on some interval, those points are independent and have uniform distribution. Since in $\mathfrak{T}_n$ the coordinates are ordered, this gives rise to the factor $\frac{(n-1)!|A_0|}{t^{n-1}}$, so, gathering the pieces, we arrive to (2.20). Then, for $A \subset \mathfrak{W}$ let $\mathfrak{N}(A)$ be the random set of the points of the Poisson process which lie inside the set A, and $N(A)$ stands for the cardinality of the set $\mathfrak{N}(A)$.

Now, to prove Lemma 2.7 we use the method similar to the generations method in percolation (cf. e.g. [12]). Denote

$$T(w_1, w_2; u) = \inf\{s : \ u < s \le t, \ w_1(s) = w_2(s)\},$$

and $T(w_1, w_2; u) = \infty$ if such s does not exist, so $T(w_1, w_2; u)$ can be interpreted as the first moment in the interval $[u, t]$ when the trajectory w_1 meets w_2.

Let $\mathcal{B}_0 = \mathfrak{W}_0 := \cup_n \mathfrak{W}_{0,n}$. If $N(\mathcal{B}_0) = 0$, then we stop.

If $N(\mathcal{B}_0) = k_0 \ge 1$, write $\mathfrak{N}(\mathcal{B}_0) = \{w_1^0, \dots, w_{k_0}^0\}$. Define

$$\mathcal{B}_1 = \bigcup_{i=1}^{k_0} \{w \in \mathfrak{W} \setminus \mathcal{B}_0 : \ T(w, w_i^0; 0) < \infty\}.$$

If $N(\mathcal{B}_1) = 0$, then stop.

If $N(\mathcal{B}_1) = k_1 \ge 1$, then suppose that $\mathfrak{N}(\mathcal{B}_1) = \{w_1^1, \dots, w_{k_1}^1\}$ and define $u_i^1 := \min_j T(w_i^1, w_j^0; 0)$. Define also

$$\mathcal{B}_2 = \bigcup_{i=1}^{k_1} \{w \in \mathfrak{W} \setminus \{\mathcal{B}_0 \cup \mathcal{B}_1\} : \ T(w, w_i^1; u_i^1) < \infty\}.$$

If $N(\mathcal{B}_2) = 0$, then stop.

If $N(\mathcal{B}_2) = k_2 \geq 1$, then suppose that $\mathfrak{N}(\mathcal{B}_2) = \{w_1^2, \dots, w_{k_2}^2\}$ and define $u_i^2 := \min_j T(w_i^2, w_j^1; u_i^1)$. This construction is then repeated until certain instant n^* when it stops (in fact, an elementary percolation argument, similar to that of [5], implies that n^* is finite a.s., but we do not really need that).

Denote $\mathcal{B} = \cup_n \mathcal{B}_n$. By construction, $\mathfrak{N}(\mathcal{B})$ is the set of the trajectories infected until time t, and the key observation here is that $\mathcal{B}$ is independent of $\mathfrak{N}(\mathfrak{W} \setminus \mathcal{B})$. Let $\mathfrak{N}'(\mathfrak{W})$ be another Poisson point process with parameter measure Λ, independent of the first one, and $N'(\cdot) := |\mathfrak{N}'(\cdot)|$. Define

$$\hat{\eta}_t(x) = N(\{w: \ w(t) = x, w \notin \mathcal{B}\}) + N'(\{w: \ w(t) = x, w \in \mathcal{B}\}).$$

Since $\eta_t(x) = N(\{w: \ w(t) = x, w \notin \mathcal{B}\})$, it is clear that $\{\eta_t(x)\}_{x\in\mathbb{Z}^d}$ is dominated by $\{\hat{\eta}_t(x)\}_{x\in\mathbb{Z}^d}$; by the above observation, $\{\hat{\eta}_t(x)\}_{x\in\mathbb{Z}^d}$ is independent of $\tilde{\mathcal{F}}_t$ and is Poisson with parameters

$$\mathbf{E}\hat{\eta}_t(x) = \Lambda(\{w: \ w(t) = x\}) = \sum_y p_t(y,x)q(y),$$

thus concluding the proof of Lemma 2.7. ■

Lemma 2.8 *Suppose that $q(x) \leq \frac{U}{(\|x\|+1)^2}$ for some $U > 0$ and for all $x \in \mathbb{Z}^d$. Then there exists a constant $U' > 0$ such that $q_t(x) \leq \frac{UU'}{(\|x\|+1)^2}$ for all $x \in \mathbb{Z}^d$.*

Proof. Using Lemma 2.1 (i), we write

$$\begin{aligned} q_t(x) &= \sum_{y:\|y\|\leq\frac{\|x\|}{2}} p_t(y,x)q(y) + \sum_{y:\|y\|>\frac{\|x\|}{2}} p_t(y,x)q(y) \\ &\leq U \sum_{i=1}^{\left[\frac{\|x\|}{2}\right]} \frac{K_1}{\|x\|^d}\frac{i^{d-1}}{i^2} + \frac{K_2 U}{(\|x\|+1)^2} \sum_{y:\|y\|>\frac{\|x\|}{2}} p_t(y,x) \\ &\leq U\frac{K_3}{(\|x\|+1)^2} + \frac{K_2 U}{(\|x\|+1)^2} \end{aligned}$$

where the constants K_1, K_2, K_3 do not depend on t and x. Lemma 2.8 is proved. ■

Now we are able to finish the proof of Theorem 1.3. Note that for all x large enough, a Bernoulli random variable with parameter $\alpha'/\|x\|^2$ is dominated by a Poisson random variable with parameter $2\alpha'/\|x\|^2$. By Lemmas 2.7 and 2.8, the process of infection spreading can be dominated by a branching random walk for which the mean number of newly generated particles at site x is at most $2U'\alpha'/\|x\|^2$. To complete the proof of Theorem 1.3, we note that, by Theorem 5.1 of [13] (the condition on variance of offspring distribution is not essential for the proof of the transience there), that branching random walk is transient for α' small enough. ■

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