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Occupation times of branching systems with initial inhomogeneous Poisson states and related superprocesses*

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Abstract

The (d,α,β,γ)-branching particle system consists of particles moving in $\mathbb{R}^d$ according to a symmetric α-stable Lévy process ($0<\alpha\le 2$), splitting with a critical $(1+\beta)$-branching law ($0<\beta\le 1$), and starting from an inhomogeneous Poisson random measure with intensity measure $\mu_\gamma(dx)=dx/(1+|x|^\gamma), \gamma\ge 0$. By means of time rescaling T and Poisson intensity measure $H_T\mu_\gamma$, occupation time fluctuation limits for the system as $T\to\infty$ have been obtained in two special cases: Lebesgue measure ($\gamma=0$, the homogeneous case), and finite measures ($\gamma>d$). In some cases $H_T\equiv 1$ and in others $H_T\to\infty$ as $T\to\infty$ (high density systems). The limit processes are quite different for Lebesgue and for finite measures. Therefore the question arises of what kinds of limits can be obtained for Poisson intensity measures that are intermediate between Lebesgue measure and finite measures. In this paper the measures $\mu_\gamma, \gamma\in(0,d]$, are used for investigating this question. Occupation time fluctuation limits are obtained which interpolate in some way between the two previous extreme cases. The limit processes depend on different arrangements of the parameters d,α,β,γ. There are two thresholds for the dimension d. The first one, $d=\alpha/\beta+\gamma$, determines the need for high density or not in order to obtain non-trivial limits, and its relation with a.s. local extinction of the system is discussed. The second one, $d=[\alpha(2+\beta)-\gamma\vee\alpha]/\beta$ (if $\gamma<d$), interpolates between the two extreme cases, and it is a critical dimension which separates different qualitative behaviors of the limit processes, in particular long-range dependence in "low" dimensions, and independent increments in "high" dimensions.

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In low dimensions the temporal part of the limit process is a new self-similar stable process which has two different long-range dependence regimes depending on relationships among the parameters. Related results for the corresponding $(d, \alpha, \beta, \gamma)$-superprocess are also given.

Key words: Branching particle system, superprocess, occupation time fluctuation, limit theorem, stable process, long-range dependence.

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1 Introduction

Occupation time fluctuation limits have been proved for the so-called (d,α,β)-branching particle systems in $\mathbb{R}^d$ with initial Poisson states in two special cases, namely, if the Poisson intensity measure is either Lebesgue measure, denoted by λ, or a finite measure [BGT1], [BGT2], [BGT3], [BGT4], [BGT6]. Those cases are quite special, as explained below, and the limit processes are very different. Therefore the question arises of what happens with Poisson intensity measures that are intermediate between Lebesgue measure and finite measures. That is the main motivation for the present paper, and our aim is to obtain limit processes that interpolate in some way between those of the two special cases. One of our objectives is to find out when the limits have long-range dependence behavior and to describe it. Another motivation is to derive analogous results for the corresponding superprocesses.

In a (d,α,β)-branching particle system the particles move independently in $\mathbb{R}^d$ according to a standard spherically symmetric α-stable Lévy process, $0<\alpha\leq 2$, the particle lifetime is exponentially distributed with parameter V, and the branching law is critical with generating function

$$s+\frac{1}{1+\beta}(1-s)^{1+\beta},\quad 0<s<1, \tag{1.1}$$

where $0<\beta\leq 1$ (called $(1+\beta)$-branching law), which is binary branching for $\beta=1$. The parameter V is not particularly relevant, but it is convenient to use it. The empirical measure process $N=(N_t)_{t\geq 0}$ is defined by

$$N_t(A)=\text{ number of particles in the Borel set } A\subset\mathbb{R}^d \text{ at time } t. \tag{1.2}$$

A common assumption for the initial distribution N_0 is to take a Poisson random measure with locally finite intensity measure μ. The corresponding (d,α,β)-superprocess $Y=(Y_t)_{t\geq 0}$ is a measure-valued process, which is a high-density/short-life/small-particle limit of the particle system, with $Y_0=\mu$. See [D], [E], [P2] for background on those particle systems and superprocesses. In this paper we investigate (the limiting behavior of) the corresponding occupation time processes, i.e.,

$$\int_0^t N_s ds,\ t\geq 0,\ \text{and}\ \int_0^t Y_s ds,\ t\geq 0.$$

We recall that the distributions of these processes are characterized by their Laplace functionals as follows [GLM], [DP2]:

$$E\exp\left\{-\int_0^t \langle N_s,\varphi\rangle ds\right\}=\exp\{-\langle\mu,v_\varphi(t)\rangle\},\ \ \varphi\in\mathscr{S}(\mathbb{R}^d), \tag{1.3}$$

where $v_\varphi(x,t)$ is the unique (mild) solution of the non-linear equation

$$\begin{aligned}\frac{\partial}{\partial t}v_\varphi &= \Delta_\alpha v_\varphi-\frac{V}{1+\beta}v_\varphi^{1+\beta}+\varphi(1-v_\varphi),\\ v_\varphi(x,0) &= 0,\end{aligned} \tag{1.4}$$

and

$$E\exp\left\{-\int_0^t \langle Y_s,\varphi\rangle ds\right\}=\exp\{-\langle\mu,u_\varphi(t)\rangle\},\ \ \varphi\in\mathscr{S}(\mathbb{R}^d), \tag{1.5}$$

where $u_\varphi(x,t)$ is the unique (mild) solution of the non-linear equation

$$\begin{aligned} \frac{\partial}{\partial t} u_\varphi &= \Delta_\alpha u_\varphi - \frac{V}{1+\beta} u_\varphi^{1+\beta} + \varphi, \\ u_\varphi(x,0) &= 0, \end{aligned} \tag{1.6}$$

and Δ_α is the infinitesimal generator of the α-stable process. (See the end of the Introduction for the standard notations $\langle \quad , \quad \rangle, \mathscr{S}(\mathbb{R}^d)$.)

For $\mu = \lambda$, N_0 is homogeneous Poisson. This case is special (and technically simpler) because λ is invariant for the α-stable process (which implies in particular that $EN_t = \lambda$ for all t), and there is the following persistence/extinction dichotomy [GW], which heuristically explains the need for high density in some cases in order to obtain non-trivial occupation time fluctuation limits, and anticipates the situation we will encounter in this paper:

(i) Persistence: If $d > \alpha/\beta$, then N_t converges in law to an equilibrium state N_∞ as $t \to \infty$, such that $EN_\infty = \lambda$.

(ii) Extinction: If $d \le \alpha/\beta$, then N_t becomes locally extinct in probability as $t \to \infty$, i.e., for any bounded Borel set $A, N_t(A) \to 0$ in probability.

An analogous persistence/extinction dichotomy holds for the corresponding superprocess (with $Y_0 = \lambda$) [DP1]. For $\alpha = 2$ (Brownian motion) and $d < 2/\beta$ a stronger extinction holds: the superprocess becomes locally extinct in finite time a.s. [I2], and we shall see that so does the particle system.

The case of μ finite is special because the particle system goes to extinction globally in finite time a.s. for every dimension d, and so does the superprocess [P2].

The time-rescaled occupation time fluctuation process $X_T = (X_T(t))_{t\ge 0}$ of the particle system is defined by

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - EN_s) ds, \tag{1.7}$$

where N_s is given by (1.2) and F_T is a norming. The problem is to find F_T such that X_T converges in distribution (in some way) as $T \to \infty$, and to identify the limit process and study its properties. This was done for $\mu = \lambda$ in the persistence case, $d > \alpha/\beta$, [BGT3], [BGT4]. For the extinction case, $d \le \alpha/\beta$, in [BGT6] we introduced high density, meaning that the initial Poisson intensity measure was taken of the form $H_T\lambda$, with $H_T \to \infty$ as $T \to \infty$, so as to counteract the local extinction, and we obtained similarly high-density limits with μ finite. For $\mu = \lambda$ and $d > \alpha/\beta$ the same results hold with or without high density (with different normings) [BGT6]. The limit processes are different for $\mu = \lambda$ and μ finite (some differences are mentioned below), and the results for any finite measure are essentially the same.

In order to study asymptotics of X_T as $T \to \infty$ with measures μ that are intermediate between the two previous cases, we consider Poisson intensity measures of the form

$$\mu_\gamma(dx) = \frac{1}{1+|x|^\gamma} dx, \quad \gamma \ge 0. \tag{1.8}$$

We call the model so defined a (d,α,β,γ)- branching particle system, and a (d,α,β,γ)-superprocess the corresponding measure-valued process. To obtain non-trivial limits we multiply μ_γ by H_T, which is suitably chosen in each case. Since $\mu_0 = \lambda$ and μ_γ is finite for $\gamma > d$, by varying γ in the interval

$(0,d]$ we obtain limits that are between those of the two previous cases, which are extreme in this sense, in a way that interpolates between them. The substantial role of γ was already noted in the simpler model of particle systems without branching [BGT5].

The above mentioned behaviors of N_t in the cases $\gamma = 0$ and $\gamma > d$ raise the following questions on what happens for $\gamma \in (0,d]$, and on its effect on asymptotics of X_T: When does N_t suffer *a.s. local extinction* in the sense that for each bounded Borel set A there is a finite random time τ_A such that $N_t(A) = 0$ for all $t \geq \tau_A$ a.s.? In this case the total occupation time $\int_0^\infty N_t(A)dt$ is finite a.s., and therefore high density is needed in order to obtain non-trivial limits for X_T. For $\gamma > 0, N_t(A)$ converges to 0 in probability as $t \to \infty$ for any bounded Borel set A and every dimension d, so local extinction in probability occurs, but the total occupation time may or may not be finite. It turns out that the threshold between the need for high density or not is given by $d = \alpha/\beta + \gamma$, and then a natural question is whether $d = \alpha/\beta + \gamma$ is also the border to a.s. local extinction of the particle system. We will come back to this question.

The limits for X_T in [BGT6] are of three different kinds for both $\mu = \lambda$ and μ finite. In the first case there is a critical dimension, $d_c = \alpha(1+\beta)/\beta$. For the "low" dimensions, $d < d_c$, the limit has a simple spatial structure (the measure λ) and a complex temporal structure (with long-range dependence). For the "high" dimensions, $d > d_c$, the limit has a complex spatial structure (distribution-valued) and a simple temporal structure (with stationary independent increments). For the "critical" dimension, $d = d_c$, the spatial and the temporal structures are both simple, but the order of the fluctuations (F_T) is larger, as is typical in phase transitions. The limit processes are always continuous for $d < d_c$, and for $d \geq d_c$ they are continuous if and only if $\beta = 1$ (when the limits are Gaussian). For μ finite, an analogous trichotomy of results holds, with a new critical dimension, $d_c = \alpha(2+\beta)/(1+\beta)$, another difference being that the limits for the critical and high dimensions are constant in time for $t > 0$.

In this paper we show analogous limits of X_T for (d,α,β,γ)-branching particle system; the critical dimension changes between the ones above, $\alpha(1+\beta)/\beta$ for $\gamma = 0$ (Lebesgue measure case), and $\alpha(2+\beta)/(1+\beta)$ for $\gamma > d$ (finite measure case), and they are linked with a unified formula,

$$d_c(\gamma) = \alpha\frac{2+\beta}{\beta} - \frac{\gamma \vee \alpha}{\beta}, \tag{1.9}$$

which interpolates between the two cases (see Remark 2.2(a) for a precise statement). There are several limit processes depending on different arrangements of d,α,β,γ. Some are analogous to those for $\mu = \lambda$, and some are similar to those for μ finite (or even essentially the same). For $\gamma < d$ there are six different cases that include the three ones recalled above for $\gamma = 0$. For $\gamma > d$ there are the three cases obtained in [BGT6] (generally for finite μ). In the case $\gamma < d$ and $d < d_c(\gamma)$, the temporal structure of the limit is a new real, stable, self-similar, continuous, long-range dependence process ξ, defined in (2.1) below, which has two different long-range dependence regimes if $\alpha < 2$ (Theorem 1(a) and Proposition 2.3). This strange type of long-range dependence behavior already appears in the homogeneous case, $\gamma = 0$ [BGT3], [BGT6]. An analogous phenomenon occurs with $0 < \gamma < d$, the border between the two long-range dependence regimes changes continuously with γ, and it disappears in the limit $\gamma \nearrow d$ (see formula (2.13)). For $\gamma > d$ there is only one long-range dependence regime, not depending on γ, β.

In [BGT1], [BGT2], [BGT3], [BGT4], [BGT5], [BGT6] we have given the convergence results for X_T in a strong form (functional convergence when it holds), but in the present article our main objective is identifying the limits, so we have not attempted to prove the strongest form of convergence in each case, nevertheless we expect that convergence in law in a space of continuous functions holds in all cases where the limit is continuous. We prove functional convergence only in the case of the above mentioned long-range dependence process ξ because of its special properties. A technical difficulty for the tightness proof is the lack of moments if $\beta < 1$.

The time-rescaled occupation time fluctuation process for the (d,α,β,γ)-superprocess Y is defined analogously as (1.7),

$$X_T(t) = \frac{1}{F_T}\int_0^{Tt}(Y_s - EY_s)ds, \tag{1.10}$$

and the limits are obtained from (the proofs of) the results for the (d,α,β,γ)-branching particle systems, as a consequence of the fact that the log-Laplace equation of the occupation time of the superprocess is simpler than that of the particle system (see (1.3), (1.4), and (1.5), (1.6).)

Our results on the fluctuation limits of superprocesses generalize those of Iscoe [I1], who considered the homogeneous case ($\gamma = 0$) only.

Let us come back to the question of high density and local extinction for the (d,α,β,γ)-branching particle system. From Theorems 2.1, 2.5 and 2.6 it follows immediately (see Corollary 2.10) that in all cases where high density is not necessary (i.e., we may take $H_T \equiv 1$) there is no a.s. local extinction (in spite of the fact that local extinction in probability occurs if $\gamma > 0$). For instance, condition (2.6) in Theorem 2.1(a) holds automatically if $d > \alpha/\beta + \gamma$ (with $\gamma < d$), hence high density is not necessary for a non-trivial limit of X_T in this case. On the other hand, high density is indispensable to obtain a non-trivial limit if either $d \leq \alpha/\beta + \gamma$ or $\alpha < \gamma \leq d$. In the latter case the total occupation time of any bounded Borel set by the process N is finite a.s. (it has finite mean). We prove a.s. local extinction for $\alpha = 2$ and $d < 2/\beta + \gamma$, and we conjecture that a.s. local extinction holds generally for $d < \alpha/\beta + \gamma$ also if $\alpha < 2$. This conjecture is supported by the fact that for $d = \alpha/\beta + \gamma, \gamma < \alpha$, there is an ergodic result (Proposition 2.9). For $\alpha = 2$ and $d < 2/\beta + \gamma$ it follows from [I2] (Theorem 3_β) that the $(d,2,\beta,\gamma)$-superprocess suffers a.s. local extinction. The method of [Z] can also be used to prove this (private communication). The proof of a.s. local extinction of the particle system consists in showing that a.s. local extinction of the $(d,2,\beta,\gamma)$-superprocess implies a.s. local extinction of the $(d,2,\beta,\gamma)$-branching particle system (Theorem 2.8). This implication is not as simple as it might seem because the well-known Cox relationship between the particle system and the superprocess (i.e., for each t, N_t is a doubly stochastic Poisson random measure with random intensity measure given by Y_t) is not enough to relate the long time behaviors of the two processes. But the argument does not work for $\alpha < 2$. In this case the superprocess Y has the instantaneous propagation of support property, i.e., with probability 1 for each $t > 0$ if the closed support of Y_t is not empty, then it is all of $\mathbb{R}^d$. This follows from the result proved in [P1] for finite initial measure and finite variance branching ($\beta = 1$ in our model), which is extended in [LZ] for more general superprocesses and branching mechanisms (including $\beta < 1$ in our case). It follows that for the (d,α,β,γ)-superprocess with $\alpha < 2$, a.s. local extinction and global extinction are equivalent, and it is known that if the initial measure has infinite total mass, the probability of global extinction in finite time is 0. Nevertheless, the total occupation time of a bounded set for the superprocess with $\alpha < 2$ may or may not be finite, and this is what is directly relevant for us (see the proof of Theorem 2.8). Iscoe [I1] showed that for initial Lebesgue measure and $\alpha = 2$ the total occupation time of a bounded set is finite if and only if $d < 2/\beta$, and he conjectured that an analogous result holds for $\alpha < 2$ and $d < \alpha/\beta$. So far as we know, this conjecture has not been proved.

Summarizing, if $\gamma < d$ and $\gamma < \alpha$, there are two thresholds for the asymptotics of X_T, namely, $\alpha/\beta + \gamma$, and $d_c(\gamma)$ given by (1.9). The first one, which is smaller than the second one, appears to be the border to a.s. local extinction (we know that it is for $\alpha = 2$), and it determines the need for high density. The second one is the critical dimension between changes of behavior of the limit processes, in particular the change from long-range dependence to independent increments, and from continuity to discontinuity if $\beta < 1$. An interpretation of $d_c(\gamma)$ in terms of the model seems rather mysterious, even in the case $\gamma = 0$ (see [BGT4], Section 4, for several questions on the meaning of results).

The general methods of proof developed in [BGT3], [BGT4], [BGT6], and the special cases for $\beta = 1$, where the limits are Gaussian [BGT1], [BGT2], can be used for the proofs involving μ_γ. However, a considerable amount of technical work is unavoidable in order to deal with $\gamma > 0$. Moreover, each case requires different calculations. We will abbreviate the proofs as much as possible.

Related work appears in [BZ], [M1], [M2] for the special case $\gamma = 0$, $\beta = 1$. [BZ] studies occupation time fluctuations of a single point for a system of binary branching random walks on the lattice with state dependent branching rate. [M1], [M2] consider general critical finite variance branching laws. [BZ], [M1], [M2] and [M3] also study the systems in equilibrium. We have already mentioned the paper [I1] on occupation time fluctuation limits of (d, α, β)-superprocesses. Some other papers regarding extinction, ergodicity and occupation times of branching particle systems and superprocesses are [BZ], [CG], [DGW], [DR], [DF], [EK], [FG], [FVW], [H], [IL], [K], [LR], [MR], [M3], [Sh], [Ta], [VW], [Zh] (and references therein).

We have given special attention to the long-range dependence stable process ξ and its properties because long-range dependence is currently a subject of much interest (see e.g. [DOT], [H1], [H2], [S], [T] for discussions and literature), hence it is worthwhile to study different types of stochastic models where it appears. Other types of long-range dependence processes have been found recently (e.g. [CS], [GNR], [HJ], [HV], [KT], [MY], [LT], [PTL]), in particular in models involving heavy-tailed distributions.

The following notation is used in the paper.

$\mathscr{S}(\mathbb{R}^d)$: space of C^∞ rapidly decreasing functions on $\mathbb{R}^d$.

$\mathscr{S}'(\mathbb{R}^d)$: space of tempered distributions (topological dual of $\mathscr{S}(\mathbb{R}^d)$).

$\langle\ ,\ \rangle$: duality on $\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)$, or on $\mathscr{S}'(\mathbb{R}^{d+1})\times\mathscr{S}(\mathbb{R}^{d+1})$, in particular, integral of a function with respect to a tempered measure.

$\Rightarrow_C$: weak convergence on the space of continuous functions $C([0,\tau], \mathscr{S}'(\mathbb{R}^d))$ for each $\tau > 0$.

$\Rightarrow_f$: weak convergence of finite-dimensional distributions of $\mathscr{S}'(\mathbb{R}^d)$-valued processes.

$\Rightarrow_i$: integral convergence of $\mathscr{S}'(\mathbb{R}^d)$-valued processes, i.e., $X_T \Rightarrow_i X$ if, for any $\tau > 0$, the $\mathscr{S}'(\mathbb{R}^{d+1})$-valued random variables $\widetilde{X}_T$ converge in law to $\widetilde{X}$ as $T \to \infty$, where $\widetilde{X}$ (and, analogously $\widetilde{X}_T$) is defined as a space-time random field by

$$\langle \widetilde{X}, \Phi\rangle = \int_0^\tau \langle X(t), \Phi(\cdot, t)\rangle dt, \quad \Phi \in \mathscr{S}(\mathbb{R}^{d+1}). \tag{1.11}$$

$\Rightarrow_{f,i}$: $\Rightarrow_f$ and $\Rightarrow_i$ together.

Recall that in general $\Rightarrow_f$ and $\Rightarrow_i$ do not imply each other, and either one of them, together with tightness of $\{\langle X_T, \varphi\rangle\}_{T\geq 1}$ in $C([0,\tau], \mathbb{R})$, $\tau > 0$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$, implies $\Rightarrow_C$ [BGR].

The transition probability density, the semigroup, and the potential operator of the standard symmetric α-stable Lévy process on $\mathbb{R}^d$ are denoted respectively by $p_t(x), \mathcal{T}_t$ (i.e., $\mathcal{T}_t\varphi = p_t * \varphi$) and (for $d > \alpha$)

$$G\varphi(x) = \int_0^\infty \mathcal{T}_t\varphi(x)dt = C_{\alpha,d}\int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-\alpha}}dy, \tag{1.12}$$

where

$$C_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})}. \tag{1.13}$$

Generic constants are written, C, C_i, with possible dependencies in parenthesis.

Section 2 contains the results, and Section 3 the proofs.

2 Results

Given $\beta \in (0,1]$, let M be an independently scattered $(1+\beta)$-stable measure on $\mathbb{R}^{d+1}$ with control measure λ_{d+1} (Lebesgue measure) and skewness intensity 1, i.e., for each Borel set $A \subset \mathbb{R}^{d+1}$ such that $0 < \lambda_{d+1}(A) < \infty, M(A)$ is a $(1+\beta)$-stable random variable with characteristic function

$$\exp\left\{-\lambda_{d+1}(A)|z|^{1+\beta}\left(1 - i(\operatorname{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\}, \quad z \in \mathbb{R},$$

the values of M are independent on disjoint sets, and M is σ-additive a.s. (see [ST], Definition 3.3.1).

For $\alpha \in (0,2], \gamma \geq 0$, we define the process

$$\xi_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r)\left(\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dy\right)^{1/(1+\beta)} \int_r^t p_{u-r}(x)du\right) M(drdx), \quad t \geq 0, \tag{2.1}$$

which is well defined provided that

$$\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dy\left(\int_r^t p_{u-r}(x)du\right)^{1+\beta} drdx < \infty, \tag{2.2}$$

(see [ST]). For $\gamma = 0$, ξ is the same as the process ξ defined by (2.1) in [BGT6]. We also recall the following process defined by (2.2) in [BGT6],

$$\zeta_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r)p_r^{1/(1+\beta)}(x)\int_r^t p_{u-r}(x)du\right) M(drdx), \quad t \geq 0, \tag{2.3}$$

which is well defined if $d < \alpha(2+\beta)/(1+\beta)$.

We consider the (d,α,β,γ)-branching particle system described in the Introduction with X_T defined by (1.7). Recall that the initial Poisson intensity measure is $H_T\mu_\gamma$. We formulate the results for low, critical and high dimensions separately, since, as mentioned in the Introduction, the qualitative behaviors of the limit processes are different in each one of these cases. In the theorems below K is a positive number depending on $d, \alpha, \beta, \gamma, V$, which may vary from case to case and it is possible to compute it explicitly.

The results for the low dimensions are contained in the following theorem.

Theorem 2.1. *(a) Assume $\gamma < d$ and*

$$d < \alpha \frac{2+\beta}{\beta} - \frac{\gamma \vee \alpha}{\beta}. \tag{2.4}$$

Then the process ξ given by (2.1) is well defined, and for

$$F_T^{1+\beta} = H_T T^{2+\beta-(d\beta+\gamma)/\alpha} \tag{2.5}$$

with $H_T \geq 1$ and

$$\lim_{T\to\infty} \frac{T^{1-(d-\gamma)\beta/\alpha}}{H_T^\beta} = 0, \tag{2.6}$$

we have $X_T \Rightarrow_C K\lambda\xi$.

(b) Let $\gamma \geq d$,

$$d < \alpha \frac{2+\beta}{1+\beta}, \tag{2.7}$$

and put

$$k(T) = \begin{cases} \log T & \text{if} \quad \gamma = d, \\ 1 & \text{if} \quad \gamma > d. \end{cases}$$

Then for

$$F_T^{1+\beta} = H_T T^{2+\beta-(1+\beta)d/\alpha} k(T) \tag{2.8}$$

with

$$\lim_{T\to\infty} \frac{T}{H_T^\beta k(T)^\beta} = 0, \tag{2.9}$$

we have $X_T \Rightarrow_{f,i} K\lambda\zeta$, where ζ is defined by (2.3).

Remark 2.2. (a) For $\gamma = 0$, Theorem 2.1(a) is the same as Theorem 2.2(a) in [BGT6]. For $\gamma \leq \alpha$ and $\gamma < d$, the bound on the dimension remains the same, equal to $\alpha(1+\beta)/\beta$ (see (2.4)), and for $\alpha < \gamma < d$, it changes continuously, tending to the threshold (2.7) as $\gamma \nearrow d$.
(b) For d satisfying (2.4) and additionally $d > \alpha/\beta + \gamma$, condition (2.6) holds with $H_T = 1$, so in this case high density is not needed, and the limit of X_T is the same as for the high-density model.

(c) The case $\gamma > d$ is included for completeness only, since it is contained in Theorem 2.7 of [BGT6], where a general finite intensity measure was considered. The same remark applies also to the theorems for critical and high dimensions (Theorems 2.5 and 2.6 below).

(d) Note that the limit process is the same (up to constant) for the infinite intensity measure $H_T dx/(1+|x|^d)$ $\quad (\gamma = d)$ as for finite measures.

(e) In Theorem 2.1(b) we consider convergence $\Rightarrow_{f,i}$ only. We are sure that functional convergence holds (in fact, for the case $\gamma > d$ this was proved in [BGT6]), but, as stated in the Introduction, we are mainly interested in the identification of limits and we do not attempt to give convergence results in the strongest forms. The same applies to the theorems that follow.

In the next proposition we gather some basic properties of the process ξ defined by (2.1), in particular its long-range dependence property. (The process ζ was discussed in [BGT6]). In [BGT3] we introduced a way of measuring long-range dependence in terms of the *dependence exponent*, defined by

$$\kappa = \inf_{z_1,z_2\in\mathbb{R}} \inf_{0\leq u<v<s<t} \sup\{\theta>0 : D_T(z_1,z_2;u,v,s,t) = o(T^{-\theta}) \text{ as } T\to\infty\}, \tag{2.10}$$

where

$$\begin{aligned} &D_T(z_1,z_2;u,v,s,t) \\ &\quad= |\log Ee^{i(z_1(\xi_v-\xi_u)+z_2(\xi_{T+t}-\xi_{T+s})} - \log Ee^{iz_1(\xi_v-\xi_u)} - \log Ee^{iz_2(\xi_{T+t}-\xi_{T+s})}|. \end{aligned} \tag{2.11}$$

(see also [RZ]).

Proposition 2.3. *Assume $\gamma<d$ and (2.4). Then*

(a) ξ is $(1+\beta)$-stable, totally skewed to the right if $\beta<1$.

(b) ξ is self-similar with index $(2+\beta-(d\beta-\gamma)/\alpha)/(1+\beta)$.

(c) ξ has continuous paths.

(d) ξ has the long-range dependence property with dependence exponent

$$\kappa = \begin{cases} \frac{d}{\alpha} & \text{if either} \quad \alpha=2, \quad \text{or} \quad \alpha<2 \quad \text{and} \quad \beta>\frac{d-\gamma}{d+\alpha}, \\ \frac{d}{\alpha}(1+\beta-\frac{d-\gamma}{d+\alpha}) & \text{if} \quad \alpha<2 \quad \text{and} \quad \beta\leq\frac{d-\gamma}{d+\alpha}. \end{cases}$$

Remark 2.4. (a) Here, as in the case $\gamma=0$ (Theorem 2.7 of [BGT3]), the intriguing phenomenon of two long-range dependence regimes occurs for $\alpha<2$. It seems also interesting to note that putting formally $\gamma\geq d$ in (2.12) we obtain $\kappa=d/\alpha$ (with no change of regime), which is indeed the dependence exponent of the process ζ (Proposition 2.9 of [BGT6]). On the other hand, the process ζ itself is not obtained from ξ by putting $\gamma\geq d$.

(b) If $\gamma=0$ and $\beta=1$, then ξ is the sub-fractional Brownian motion (multiplied by a constant) considered in [BGT], [BGT1].

We now turn to the critical dimensions, i.e., the cases where the inequalities in (2.4) and (2.7) are replaced by equalities. It turns out that in spite of different conditions on the normings, the limits have always the same form as for finite intensity measure, with the only exception of the case given in Theorem 2.5(a) below.

Theorem 2.5. *(a) Assume $\gamma<d,\gamma<\alpha$,*

$$d=\alpha\frac{1+\beta}{\beta} \tag{2.12}$$

and

$$F_T^{1+\beta} = H_T T^{1-\gamma/\alpha}\log T, \tag{2.13}$$

with $H_T \geq 1$. Then $X_T \Rightarrow_{f,i} K\lambda\eta$, where η is a $(1+\beta)$-stable process with independent, non-stationary increments (for $\gamma > 0$) whose laws are determined by

$$Ee^{iz(\eta_t-\eta_s)} = \exp\left\{-(t^{1-\gamma/\alpha} - s^{1-\gamma/\alpha})|z|^{1+\beta}\left(1 - i(\mathrm{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\}, \quad z \in \mathbb{R}, \quad t \geq s \geq 0,$$

$\eta_0 = 0$.

(b) In all the remaining critical cases, i.e.,

(i) $\gamma = \alpha, \gamma < d$ with d satisfying (2.12),

$$F_T^{1+\beta} = H_T(\log T)^2, \tag{2.14}$$

and

$$\lim_{T\to\infty} \frac{(\log T)^{1-\beta}}{H_T^{\beta}} = 0, \tag{2.15}$$

(ii) $\alpha < \gamma < d$ with

$$d = \alpha\frac{2+\beta}{\beta} - \frac{\gamma}{\beta}, \tag{2.16}$$

$$F_T^{1+\beta} = H_T \log T, \tag{2.17}$$

and

$$\lim_{T\to\infty} \frac{T^{(1+\beta)(\gamma/\alpha-1)}}{H_T^{\beta}(\log T)^{1+\beta}} = 0, \tag{2.18}$$

(iii)

$$\gamma = d = \alpha\frac{2+\beta}{1+\beta}, \tag{2.19}$$

with F_T satisfying (2.14) and

$$\lim_{T\to\infty} \frac{T}{H_T^{\beta}(\log T)^{1+2\beta}} = 0, \tag{2.20}$$

(iv) $\gamma > d = \alpha(2+\beta)/(1+\beta), F_T^{1+\beta} = H_T \log T$ and $\lim_{T\to\infty} TH_T^{-\beta} = 0$,

we have $X_T \Rightarrow_{f,i} K\lambda\vartheta$, where ϑ is a real process such that $\vartheta_0 = 0$ and for $t > 0, \vartheta_t = \vartheta_1 = (1+\beta)$-stable random variable totally skewed to the right, i.e.,

$$Ee^{iz\vartheta_1} = \exp\left\{-|z|^{1+\beta}\left(1 - i(\mathrm{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\}.$$

It remains to consider the high dimensions.

Theorem 2.6. *(a) Assume $\gamma < d, \gamma < \alpha$,*

$$d > \alpha\frac{1+\beta}{\beta}, \tag{2.21}$$

and

$$F_T^{1+\beta} = H_T T^{1-\gamma/\alpha}, \tag{2.22}$$

with $H_T \geq 1$. Then $X_T \Rightarrow_{f,i} X$, where X is an $\mathscr{S}'(\mathbb{R}^d)$-valued $(1+\beta)$-stable process with independent, non-stationary increments (for $\gamma > 0$) determined by

$$Ee^{i\langle X_t - X_s, \varphi\rangle} = \exp\Big\{ -K(t^{1-\gamma/\alpha} - s^{1-\gamma/\alpha}) \int_{\mathbb{R}^d} \Big(V|G\varphi(x)|^{1+\beta}(1 - i(\mathrm{sgn} G\varphi(x)) \tan\frac{\pi}{2}(1+\beta)) + 2c_\beta \varphi(x) G\varphi(x)\Big) dx \Big\}, \quad \varphi \in \mathscr{S}(\mathbb{R}^d), \quad t \geq s \geq 0, \tag{2.23}$$

$X_0 = 0$, where

$$c_\beta = \begin{cases} 0 & \text{if} \quad 0 < \beta < 1, \\ 1 & \text{if} \quad \beta = 1, \end{cases}$$

and G is defined by (1.12).

(b) Assume $\gamma < d, \gamma = \alpha$, and d satisfying (2.21) with

$$F_T^{1+\beta} = H_T \log T, \tag{2.24}$$

and $H_T \geq 1$. Then $X_T \Rightarrow_{f,i} X$, where X is an $\mathscr{S}'(\mathbb{R}^d)$-valued process such that $X_0 = 0$, and for $t > 0, X_t = X_1 = (1+\beta)$-stable random variable determined by

$$Ee^{i\langle X_1, \varphi\rangle} = \exp\Big\{ -K \int_{\mathbb{R}^d} \Big(V|G\varphi(x)|^{1+\beta}(1 - i(\mathrm{sgn} G\varphi(x)) \tan\frac{\pi}{2}(1+\beta)) + 2c_\beta \varphi(x) G\varphi(x)\Big) dx \Big\}, \quad \varphi \in \mathscr{S}(\mathbb{R}^d), \tag{2.25}$$

with c_β given by (2.24).

(c) Assume $\gamma > \alpha$,

$$d > \alpha \frac{2+\beta}{\beta} - \frac{\gamma \wedge d}{\beta}, \tag{2.26}$$

$$F_T^{1+\beta} = H_T, \tag{2.27}$$

and

$$\lim_{T\to\infty} T H_T^{-\beta} = 0. \tag{2.28}$$

Then $X_T \Rightarrow_{f,i} X$, where X is an $\mathscr{S}'(\mathbb{R}^d)$-valued process such that $X_0 = 0$, and for $t > 0, X_t = X_1 = (1+\beta)$-stable random variable determined by

$$Ee^{i\langle X_1, \varphi\rangle} = \exp\Big\{ -K \int_{\mathbb{R}^d} \Big(V|G\varphi(x)|^{1+\beta}(1 - i(\mathrm{sgn} G\varphi(x)) \tan\frac{\pi}{2}(1+\beta)) + 2c_\beta \varphi(x) G\varphi(x)\Big) G\mu_\gamma(dx) \Big\}, \quad \varphi \in \mathscr{S}(\mathbb{R}^d), \tag{2.29}$$

with c_β given by (2.24).

Remark 2.7. (a) As in all the cases studied previously [BGT3], [BGT4], [BGT6], we observe the same phenomenon that in low dimensions the limit processes are continuous with a simple spatial structure and a complicated temporal structure (with long-range dependence), while in high dimensions they are truly $\mathscr{S}'(\mathbb{R}^d)$-valued with independent increments, and not necessarily continuous.

(b) For low dimensions the forms of the limits depend on the relation between d and γ only, whereas for critical and high dimensions only the relationship between α and γ is relevant. More precisely, in critical dimensions we have different forms of the limits for $\gamma < \alpha$ and $\gamma \geq \alpha$, and in high dimensions the forms are different for $\gamma < \alpha, \gamma = \alpha$ and $\gamma > \alpha$. In the case $\gamma > \alpha$ even the normings are the same, depending only on β.

(c) For $\beta = 1$ the limits are centered Gaussian. In high dimensions there is no continuous transition between the cases $\beta < 1$ and $\beta = 1$; in the latter case an additional term appears. The coefficient c_β defined in (2.24) was introduced in order to present the results in unified forms.

(d) We have assumed that the initial intensity measure is determined by μ_γ of the form (1.8). It will be clear that the same results are obtained for the measure $\mu_\gamma(dx) = |x|^{-\gamma}dx$ if $d > \gamma$. Analogously as in the non-branching case [BGT5], other generalizations are also possible.

Let us look further into the need for high density (i.e., to assume $H_T \to \infty$) and the question of a.s. local extinction. For $\gamma > d$ the Poisson intensity measure is finite, so there is only a finite number of particles at time $t = 0$ and the system becomes globally extinct in finite time a.s. due to the criticality of the branching. Also, for $\gamma \wedge d > \alpha$ it is not difficult to see that the total occupation time $\int_0^\infty N_s ds$ is finite a.s. on bounded sets (see [BGT5], Proposition 2.1, because EN_s is the same for the systems with and without branching), so high density is also necessary. We have a more delicate situation in the remaining cases where the threshold is $d = \alpha/\beta + \gamma$. Concerning extinction, the situation is completely clear for $\alpha = 2$. In Theorem 2.8 below we state that for $\alpha = 2$ and $d < 2/\beta + \gamma$ there is a.s. local extinction, hence the total occupation time of any bounded set is finite a.s. We conjecture that the same is true for $d < \alpha/\beta + \gamma$ if $\alpha < 2$, but we have not been able to prove it.

Theorem 2.8. *Assume $\alpha = 2$. If $d < 2/\beta + \gamma$, then for each bounded Borel set A,*

$$P[\text{there exists} \quad \tau_A < \infty \quad \text{such that} \quad N_t(A) = 0 \quad \text{for all} \quad t \geq \tau_A] = 1.$$

The proof of this theorem relies on Iscoe's a.s. local extinction result for the superprocess [I2], by showing that in general (i.e., for $0 < \alpha \leq 2$) a.s. local finiteness of the total occupation time of the $(d, \alpha, \beta, \gamma)$-superprocess implies a.s. local extinction of the $(d, \alpha, \beta, \gamma)$-branching particle system. On the other hand, as explained in the Introduction, for $\alpha < 2$ the a.s. local extinction for the superprocess cannot occur, and we do not know how to prove directly the a.s. local finiteness of its total occupation time.

The next ergodic-type result, which is a direct generalization of [Ta], shows that $\alpha/\beta + \gamma$ is indeed a natural threshold.

Proposition 2.9. *Assume*

$$\gamma < \alpha, \quad d = \frac{\alpha}{\beta} + \gamma, \quad F_T = T^{1-\gamma/\alpha}, \tag{2.30}$$

and denote

$$Z_T(t) = \frac{1}{F_T} \int_0^{Tt} N_s ds, \quad t \geq 0. \tag{2.31}$$

Then $Z_T \Rightarrow_C \lambda\xi$, where ξ is a real non-negative process with Laplace transform

$$E\exp\{-\theta_1\xi_{t_1} - \cdots - \theta_n\xi_{t_n}\} = \exp\left\{-\int_{\mathbb{R}^d} v(x,\tau)|x|^{-\gamma}dx\right\}, \tag{2.32}$$

for any $\tau > 0$, where $\theta_1, \dots, \theta_n \geq 0, 0 \leq t_1 < \cdots < t_n \leq \tau$, and $v(x,t)$ is the unique non-negative solution of the equation

$$v(x,t) = \int_0^t p_{t-s}(x)\psi(\tau - s)ds - \frac{V}{1+\beta}\int_0^t \mathscr{T}_{t-s}v^{1+\beta}(\cdot,s)(x)ds, \tag{2.33}$$

with

$$\psi(s) = \sum_{k=1}^n \theta_k 1\!\!1_{[0,t_k]}(s). \tag{2.34}$$

To complete the discussion on a.s. local extinction we formulate a corollary which follows immediately from our results but which, nevertheless, seems worth stating explicitly.

Corollary 2.10. *If $\gamma < \alpha$ and $d \geq \alpha/\beta + \gamma$, then the (d,α,β,γ)-branching particle system does not have the a.s. local extinction property.*

Indeed, for $d > \alpha/\beta + \gamma$, from Theorems 2.1, 2.5 and 2.6 it follows that one may take $H_T \equiv 1$. By Proposition 2.1 of [BGT5], $E\int_0^T \langle N_s, \varphi\rangle ds$ (for $\int_{\mathbb{R}^d}\varphi(x)dx \neq 0$) is of larger order than F_T as $T \to \infty$ hence, by (1.7), for any bounded Borel set A, $\int_0^\infty N_s(A)ds = \infty$ a.s., which excludes a.s. local extinction. For $d = \alpha/\beta + \gamma$ the result follows immediately from Proposition 2.9.

We end with the results for the superprocess.

Theorem 2.11 *Let Y be the (d,α,β,γ)-superprocess and X_T its occupation time fluctuation process defined by (1.10). Then the limit results for X_T as $T \to \infty$ are the same as those in Theorems 2.1, 2.5 and 2.6, with the same normings, and $c_\beta = 0$ in all cases in Theorem 2.6.*

3 Proofs

3.1 Scheme of proofs

The proofs of Theorems 2.1, 2.5 and 2.6 follow the general scheme presented in [BGT6]. For completeness we recall the main steps.

As explained in [BGT3; BGT4; BGT6], in order to prove convergence $\Rightarrow_i$ it suffices to show

$$\lim_{T\to\infty} Ee^{-\langle \widetilde{X}_T, \Phi\rangle} = Ee^{-\langle \widetilde{X}, \Phi\rangle} \tag{3.1}$$

for each $\Phi \in \mathscr{S}(\mathbb{R}^{d+1}), \Phi \geq 0$, where X is the corresponding limit process and $\widetilde{X}_T, \widetilde{X}$ are defined by (1.11). To prove convergence $\Rightarrow_C$ according to the space-time approach [BGR] it is enough to show additionally that the family $\{\langle X_T, \varphi\rangle\}_{T\geq 1}$ is tight in $C((0,\tau],\mathbb{R}), \varphi \in \mathscr{S}(\mathbb{R}^d), \tau > 0$. Without loss of generality we may fix $\tau = 1$ (see (1.11)). To simplify slightly the calculations we consider Φ of the form

$$\Phi(x,t) = \varphi \otimes \psi(x,t) = \varphi(x)\psi(t), \ \varphi \in \mathscr{S}(\mathbb{R}^d), \psi \in \mathscr{S}(\mathbb{R}), \varphi, \psi \geq 0.$$

Denote

$$\varphi_T = \frac{1}{F_T}\varphi,\ \chi(t) = \int_t^1 \psi(s)ds,\ \chi_T(t) = \chi\left(\frac{t}{T}\right). \tag{3.2}$$

We define

$$v_T(x,t) = 1 - E\exp\left\{-\int_0^t \langle N_r^x, \varphi_T\rangle \chi_T(T-t+r)dr\right\},\ 0 \le t \le T, \tag{3.3}$$

where N^x is the empirical process of the branching system started from a single particle at x. The following equation for v_T was derived in [BGT3] (formula (3.8), see also [BGT1]) by means of the Feynman-Kac formula:

$$v_T(x,t) = \int_0^t \mathcal{T}_{t-u}\left[\varphi_T\chi_T(T-u)(1-v_T(\cdot,u)) - \frac{V}{1+\beta}v_T^{1+\beta}(\cdot,u)\right](x)du,\ 0 \le t \le T. \tag{3.4}$$

This equation (with $T=1$) is the space-time version of equation (1.4). It is the log-Laplace equation for $\widetilde{L}$ (as in (1.11)), where L is the occupation time $L_t = \int_0^t N_s ds$. Formulas (3.3) and (3.4) imply

$$0 \le v_T \le 1, \quad v_T(x,t) \le \int_0^t \mathcal{T}_{t-u}\varphi_T(x)\chi_T(T-u)du. \tag{3.5}$$

For brevity we denote

$$\nu_T(dx) = H_T\mu_\gamma(dx) = \frac{H_T}{1+|x|^\gamma}dx. \tag{3.6}$$

By the Poisson property and (3.4) we have

$$\begin{aligned} Ee^{-\langle \widetilde{X}_T, \varphi\otimes\psi\rangle} &= \exp\left\{-\int_{\mathbb{R}^d} v_T(x,T)\nu_T(dx) + \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_u\varphi_T(x)\chi_T(u)du\nu_T(dx)\right\} && (3.7)\\ &= \exp\left\{\frac{V}{1+\beta}I_1(T) + I_2(T) - \frac{V}{1+\beta}I_3(T)\right\}, && (3.8)\end{aligned}$$

where

$$\begin{aligned} I_1(T) &= \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left[\left(\int_0^s \mathcal{T}_{s-u}\varphi_T\chi_T(T-u)du\right)^{1+\beta}\right](x)ds\nu_T(dx), && (3.9)\\ I_2(T) &= \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}(\varphi_T\chi_T(T-s)v_T(\cdot,s))(x)ds\nu_T(dx), && (3.10)\\ I_3(T) &= \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left[\left(\int_0^s \mathcal{T}_{s-u}\varphi_T\chi_T(T-u)du\right)^{1+\beta} - v_T^{1+\beta}(\cdot,s)\right](x)ds\nu_T(dx). && (3.11)\end{aligned}$$

In the proofs of Theorems 2.1, 2.5 and in Theorem 2.6 for $\beta < 1$ we show

$$\lim_{T\to\infty}\exp\left\{\frac{V}{1+\beta}I_1(T)\right\} = Ee^{-\langle \widetilde{X}, \varphi\otimes\psi\rangle}, \tag{3.12}$$

and

$$\lim_{T\to\infty} I_2(T) = 0, \tag{3.13}$$

where (3.13) is obtained from

$$I_2(T) \le \frac{C}{F_T^2} \int_{\mathbb{R}^d} \int_0^T \int_0^T \mathcal{T}_s(\varphi \mathcal{T}_u \varphi)(x) du ds \nu_T(dx) \tag{3.14}$$

(see (3.5)). In Theorem 2.6 for $\beta = 1$ the limit of $I_2(T)$ is non-trivial and corresponds to the expressions involving c_β (see (2.23), (2.24), (2.25),(2.29)). In all the cases

$$\lim_{T\to\infty} I_3(T) = 0 \tag{3.15}$$

By the argument in [BGT6], in order to prove (3.15) we show

$$\lim_{T\to\infty} J_1(T) = 0 \tag{3.16}$$

and

$$\lim_{T\to\infty} J_2(T) = 0, \tag{3.17}$$

where

$$\begin{aligned} J_1(T) &= \int_{\mathbb{R}^d} \int_0^T \mathcal{T}_{T-s} \left[\left(\int_0^s \mathcal{T}_{s-u} \left(\varphi_T \int_0^u \mathcal{T}_{u-r} \varphi_T dr \right) du \right)^{1+\beta} \right] (x) ds \nu_T(dx) \\ &\le \frac{1}{F_T^{2+2\beta}} \int_{\mathbb{R}^d} \int_0^T \mathcal{T}_s \left[\left(\int_0^T \mathcal{T}_u \left(\varphi \int_0^T \mathcal{T}_r \varphi dr \right) du \right)^{1+\beta} \right] (x) ds \nu_T(dx), \end{aligned} \tag{3.18}$$

$$\begin{aligned} J_2(T) &= \int_{\mathbb{R}^d} \int_0^T \mathcal{T}_{T-s} \left[\left(\int_0^s \mathcal{T}_{s-u} \left(\int_0^u \mathcal{T}_{u-r} \varphi_T dr \right)^{1+\beta} du \right)^{1+\beta} \right] (x) ds \nu_T(dx) \\ &\le \frac{1}{F_T^{(1+\beta)(1+\beta)}} \int_{\mathbb{R}^d} \int_0^T \mathcal{T}_s \left[\left(\int_0^T \mathcal{T}_u \left(\int_0^T \mathcal{T}_r \varphi dr \right)^{1+\beta} du \right)^{1+\beta} \right] (x) ds \nu_T(dx) \end{aligned} \tag{3.19}$$

We remark that the proof of (3.15) is the only place where the high density (with specific conditions on H_T) is required in some cases.

Finally, the $\Rightarrow_f$ convergence is obtained by an analogous argument as explained in the proof of Theorem 2.1 in [BGT4].

3.2 Auxiliary estimates

Recall that the transition density p_t of the standard α-stable process has the self-similarity property

$$p_{at}(x) = a^{-d/\alpha} p_t(a^{-1/\alpha} x), \quad x \in \mathbb{R}^d, \ a > 0, \tag{3.20}$$

and it satisfies

$$\frac{C_1}{1+|x|^{d+\alpha}} \le p_1(x) \le \frac{C_2}{1+|x|^{d+\alpha}}, \tag{3.21}$$

where the lower bound holds for $\alpha < 2$.

Denote

$$f(x) = \int_0^1 p_s(x)ds. \tag{3.22}$$

The following estimate can be easily deduced from (3.20) and (3.21):

$$f(x) \le \frac{C}{|x|^{d+\alpha}}, \tag{3.23}$$

$$f(x) \le \begin{cases} C \quad \text{if} \quad d < \alpha, \\ C(1 \vee \log|x|^{-1}) \quad \text{if} \quad d = \alpha, \\ C/|x|^{d-\alpha} \quad \text{if} \quad d > \alpha. \end{cases} \tag{3.24}$$

We will also use the following elementary estimates: Let $0 < a, b < d$. If $a + b > d$, then

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^a|x|^b}dx \le \frac{C}{|y|^{a+b-d}}. \tag{3.25}$$

If $a + b = d$, then

$$\int_{|x|\le 1} \frac{1}{|x-y|^a|x|^b}dx \le C(1 \vee \log|y|^{-1}). \tag{3.26}$$

If $a + b < d$, then

$$\int_{|x|\le 1} \frac{1}{|x-y|^a|x|^b}dx \le C. \tag{3.27}$$

Now, let $a > d, 0 < b < d$, then

$$\int_{\mathbb{R}^d} \frac{1}{1+|x-y|^a} \frac{1}{|x|^b}dx \le \frac{C}{|y|^b}. \tag{3.28}$$

For $d > \gamma$, denote

$$f_\gamma(y) = \int_{\mathbb{R}^d} f(y-x)|x|^{-\gamma}dx, \tag{3.29}$$

where f is defined in (3.22). From the estimates above we obtain

$$\sup_{|y|>1} |y|^\gamma f_\gamma(y) < \infty, \tag{3.30}$$

and

$$f_\gamma(y) \le \begin{cases} C \quad \text{if} \quad \gamma < \alpha, \\ C(1 \vee \log|y|^{-1}) \quad \text{if} \quad \gamma = \alpha, \\ C/|y|^{\gamma-\alpha} \quad \text{if} \quad \gamma > \alpha. \end{cases} \tag{3.31}$$

3.3 Proof of Theorem 2.1(a)

According to the scheme sketched above, in order to prove $\Rightarrow_{f,i}$ convergence we show (3.1). By (3.7)-(3.11)and (2.1) it is enough to prove (3.12), which amounts to

$$\lim_{T\to\infty} I_1(T) = \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_s(x-y)\left(\int_s^1 p_{u-s}(y)\chi(u)du\right)^{1+\beta} dyds\frac{dx}{|x|^\gamma}\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}, \quad (3.32)$$

(see (3.9)) and, additionally, (3.13) and (3.15). To simplify the notation we will carry out the proof for μ_γ of the form $\mu_\gamma(dx) = |x|^{-\gamma}dx$ instead of (1.8). It will be clear that in the present case ($d < \gamma$) this will not affect the result.

By (3.9), (3.2), (3.6), the definition of $\mathscr{T}_t$, substituting $s' = 1 - s/T, u' = 1 - u/T$, we have

$$\begin{aligned} & I_1(T) \\ = & \frac{T^{2+\beta}H_T}{F_T^{1+\beta}}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_{Ts}(x-y)\left(\int_s^1\int_{\mathbb{R}^d} p_{T(u-s)}(y-z)\varphi(z)\chi(u)dzdu\right)^{1+\beta}|x|^{-\gamma}dydsdx. \end{aligned} \quad (3.33)$$

Denote

$$\widetilde{\varphi}_T(x) = T^{d/\alpha}\varphi(T^{1/\alpha}x) \quad (3.34)$$

and

$$g_s(x) = \int_s^1 p_{u-s}(x)\chi(u)du, \quad 0 \le s \le 1. \quad (3.35)$$

Observe that

$$g_s \le Cf, \quad (3.36)$$

where f is defined by (3.22). By (3.20) and (2.5), making obvious spatial substitutions in (3.33), we obtain

$$I_1(T) = \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_s(x-y)(g_s * \widetilde{\varphi}_T(y))^{1+\beta}|x|^{-\gamma}dxdsdy. \quad (3.37)$$

Note that if we consider the measure μ_γ of the form (1.8), then in (3.37) instead of $|x|^{-\gamma}$ we have $(T^{-\gamma/\alpha} + |x|^\gamma)^{-1}$. Since $g_s \in L^1(\mathbb{R}^d)$, by (3.34) it is clear that $g_s * \widetilde{\varphi}_T(y)$ converges to $g_s(y)\int_{\mathbb{R}^d}\varphi(z)dz$ almost everywhere in y. Hence, to prove (3.32) it remains to justify the passage to the limit under the integrals in (3.37). Denote

$$h_T(y) = \int_0^1\int_{\mathbb{R}^d} p_s(x-y)(g_s * \widetilde{\varphi}_T(y))^{1+\beta}|x|^{-\gamma}dxds. \quad (3.38)$$

First we prove pointwise convergence of h_T, which amounts to showing that the integrand is majorized by an integrable function independent of T. Fix $y \neq 0$. We use (3.36) and observe that

$$\begin{aligned} f * \widetilde{\varphi}_T(y) &= \int_{|z|\le|y|/2} f(y-z)\widetilde{\varphi}_T(z)dz + \int_{|z|>|y|/2} f(y-x)\frac{(T^{1/\alpha}|x|)^d\varphi(T^{d/\alpha}x)}{|x|^d}dx \\ &\le f\left(\frac{y}{2}\right)\int_{|z|\le|y|/2}\widetilde{\varphi}_T(z)dz + \frac{C}{|y|^d}\int_{|z|>|y|/2} f(y-x)dz \\ &\le f\left(\frac{y}{2}\right)\int_{\mathbb{R}^d}\varphi(z)dz + \frac{C}{|y|^d}, \end{aligned} \quad (3.39)$$

by the unimodal property of the α-stable density and since $\varphi \in \mathscr{S}(\mathbb{R}^d)$. We conclude by noting that

$$\int_{\mathbb{R}^d} \int_0^1 p_s(x-y)|x|^{-\gamma} ds dx < \infty \quad \text{for} \quad y \neq 0,$$

by (3.31).

Since (see (3.29))

$$h_T(y) \leq C f_\gamma(y)((f * \widetilde{\varphi}_T)(y))^{1+\beta}, \tag{3.40}$$

to prove convergence of $I_1(T)$ it suffices to show that the right-hand side of (3.40) (denoted by h_T^*) converges in $L^1(\mathbb{R}^d)$ as $T \to \infty$.

If $\gamma < \alpha$, then f_γ is bounded by (3.31), and the assumption (2.4) implies that

$$f \in L^{1+\beta}(\mathbb{R}^d), \tag{3.41}$$

so $(f * \widetilde{\varphi}_T)^{1+\beta}$ converges in $L^1(\mathbb{R}^d)$.

Next assume $\gamma \geq \alpha$. It is easily seen that $h_T^* 1\!\!1_{\{|y| \geq 1\}}$ converges in $L^1(\mathbb{R}^d)$, by (3.41) and (3.30). To prove that $h_T^*(y) 1\!\!1_{\{|y|<1\}}$ converges in $L^1(\mathbb{R}^d)$ too, it suffices to find $p, q > 1, 1/p + 1/q = 1$, such that

$$f_\gamma(y) 1\!\!1_{\{|y|<1\}} \in L^p(\mathbb{R}^d) \tag{3.42}$$

and

$$f^{1+\beta} \in L^q(\mathbb{R}^d). \tag{3.43}$$

If $\gamma = \alpha$, then (3.31) implies that (3.42) holds for any $p > 1$, and by (2.4) it is clear that (3.43) is satisfied for q sufficiently close to 1.

If $\gamma > \alpha$, condition (2.4) is equivalent to

$$\frac{\gamma - \alpha}{d} + \frac{(1+\beta)(d-\alpha)}{d} < 1,$$

so we can take p and q such that $1/p + 1/q = 1$,

$$\frac{1}{p} > \frac{\gamma - \alpha}{d} \quad \text{and} \quad \frac{1}{q} > \frac{(1+\beta)(d-\alpha)}{d}.$$

For such p and q we have (3.42) and (3.43) by (3.31), (3.23) and (3.24).

This completes the proof of (3.32).

We proceed to the proof of (3.13). By (3.14), applying the same substitutions as for $I_1(T)$ and using the notation (3.34) we have

$$I_2(T) \leq C \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} \int_{\mathbb{R}^d} f_\gamma(y) \widetilde{\varphi}_T(y)(f * \widetilde{\varphi}_T)(y) dy. \tag{3.44}$$

Assume $\gamma < \alpha$. By (3.31),

$$\begin{aligned} I_2(T) &\leq C_1 \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} ||\widetilde{\varphi}_T(f * \widetilde{\varphi}_T)||_1 \\ &\leq C_1 \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} ||\varphi||_1 ||f||_{1+\beta} ||\widetilde{\varphi}_T||_{\frac{1+\beta}{\beta}} \to 0 \end{aligned}$$

by (2.4), and since

$$\|\widetilde{\varphi}_T\|_p = \|\varphi\|_p T^{(d/\alpha)(p-1)/p} \quad \text{for} \quad p \geq 1. \tag{3.45}$$

Next, let $\gamma \geq \alpha$. By (3.44) and (3.30) we have

$$I_2(T) \leq C_2(I_2'(T) + I_2''(T)), \tag{3.46}$$

where

$$I_2'(T) = \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} \|\widetilde{\varphi}_T(f * \widetilde{\varphi}_T)\|_1, \tag{3.47}$$

and

$$I_2''(T) = \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} \int_{|y|\leq 1} f_\gamma(y)\widetilde{\varphi}_T(y)(f * \widetilde{\varphi}_T)(y)dy. \tag{3.48}$$

By the Hölder and Young inequalities,

$$I_2'(T) \leq \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} \|\widetilde{\varphi}_T\|_p \|f\|_q \|\varphi\|_1 \tag{3.49}$$

for any $p, q \geq 1, 1/p + 1/q = 1$. If $1/q > (d-\alpha)/d$, then $\|f\|_q < \infty$, and if $1/q$ is sufficiently close to $(d-\alpha)/d$, then by (3.45), (2.5) and (2.4) the right-hand side of (3.49) converges to 0 as $T \to \infty$.

We estimate $I_2''(T)$ using the generalized Hölder inequality

$$I_2''(T) \leq \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2} \|f_\gamma 1\!\!1_{\{|\cdot|\leq 1\}}\|_r \|\widetilde{\varphi}_T\|_p \|f\|_q \|\varphi\|_1$$

for $r, p, q \geq 1, 1/p + 1/r + 1/q = 1$. We take r, q such that $1/r > (\gamma - \alpha)/d$ (then the r-norm will be finite by (3.31)) and $1/q > (d-\alpha)/d$. By (2.4), it is easily seen that one can choose r, q as above and $p = \alpha/(2\alpha - \gamma) + \varepsilon$ for $\varepsilon > 0$ sufficiently small (note that by (2.4), $2\alpha > \gamma$). Then by (3.45) and (2.5) we obtain that $I_2''(T) \to 0$ as $T \to \infty$. Thus, we have proved (3.13).

According to the general scheme, in order to obtain (3.15) it suffices to show (3.16) and (3.17). The proofs are quite similar to the argument presented above, therefore we omit the proof of (3.16) and we give an outline of the proof of (3.17), since this is the only place where the condition (2.6) is needed.

By (3.19), (2.5) and the usual substitutions we have

$$J_2(T) \leq C \frac{T^{1-(d/\alpha)\beta+(\gamma/\alpha)\beta}}{H_T^\beta} R(T), \tag{3.50}$$

where

$$R(T) = \int_{\mathbb{R}^d} f_\gamma(y)(f * (f * \widetilde{\varphi}_T)^{1+\beta})^{1+\beta}(y)dy. \tag{3.51}$$

By (2.6), to prove (3.17) it remains to show that

$$\sup_T R(T) < \infty. \tag{3.52}$$

If $\gamma < \alpha$ then, by (3.31),

$$R(T) \leq C_1 ||f * (f * \widetilde{\varphi}_T)^{1+\beta}||_{1+\beta}^{1+\beta} \leq C_1 ||f||_{1+\beta}^{1+\beta} ||f||_{1+\beta}^{(1+\beta)(1+\beta)} ||\varphi||_1^{(1+\beta)(1+\beta)} < \infty, \tag{3.53}$$

by (3.41).

If $\gamma \geq \alpha$, then we write

$$R(T) = \int_{|y|>1} \ldots + \int_{|y|\leq 1} \ldots .$$

By (3.30) the first integral can be estimated as in (3.53), and the second one is bounded by

$$||f_\gamma 1\!\!1_{\{|\cdot|\leq 1\}}||_p ||f||_{q(1+\beta)}^{1+\beta} ||f||_{1+\beta}^{(1+\beta)(1+\beta)} ||\varphi||_1^{(1+\beta)(1+\beta)},$$

where $1/p + 1/q = 1$. We already know that there exist such p and q that this expression is finite (see (3.42) and (3.43)).

We have thus established the convergence

$$X_T \Rightarrow_{f,i} K\lambda\xi.$$

In order to obtain $\Rightarrow_C$ convergence it suffices to show that the family $\{\langle X_T, \varphi\rangle\}_{T\geq 1}$ is tight in $C([0,1], \mathbb{R})$ for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$ ([Mi]). One may additionally assume that $\varphi \geq 0$. We apply the method presented in [BGT3] and [BGT6]. We start with the inequality

$$P(|\langle \widetilde{X}_T, \varphi \otimes \psi\rangle| \geq \delta) \leq C\delta \int_0^{1/\delta} (1 - \mathrm{Re}(E \exp\{-i\theta \langle \widetilde{X}_T, \varphi \otimes \psi\rangle\}))d\theta, \tag{3.54}$$

valid for any $\psi \in \mathscr{S}(\mathbb{R}), \delta > 0$. Fix $0 \leq t_1 < t_2 \leq 1$ and take ψ approximating $\delta_{t_2} - \delta_{t_1}$ such that $\chi(t) = \int_t^1 \psi(s)ds$ satisfies

$$0 \leq \chi \leq 1\!\!1_{[t_1,t_2]}. \tag{3.55}$$

Then the left hand side of (3.54) approximates

$$P(|\langle X_T(t_2), \varphi\rangle - \langle X_T(t_1), \varphi\rangle| \geq \delta).$$

So, in order to show tightness one should prove that the right hand side of (3.54) is estimated by

$$C(t_2^h - t_1^h)^{1+\sigma} \text{ for some } h, \sigma > 0.$$

By the argument in [BGT6] this reduces to showing that

$$A(T) \leq C(t_2^h - t_1^h)^{1+\sigma} \tag{3.56}$$

and

$$I_1(T) \leq C(t_2^h - t_1^h)^{1+\sigma}, \tag{3.57}$$

where I_1 is defined by (3.9), and

$$A(T) = \frac{H_T}{F_T^2} \int_{\mathbb{R}^d} \int_0^T \int_0^s \mathscr{T}_{T-s}(\varphi \mathscr{T}_{s-u}\varphi)(x) \chi\left(1 - \frac{s}{T}\right) \chi\left(1 - \frac{u}{T}\right) |x|^{-\gamma} du ds dx. \tag{3.58}$$

The proofs of (3.56) and (3.57) are quite involved and lengthy, therefore, as an example we present only the argument for the case $\gamma < \alpha$, which, together with (2.4) implies

$$d < \alpha \frac{1+\beta}{\beta}. \tag{3.59}$$

We start with (3.57). By self-similarity of p_s we have

$$\int_{\mathbb{R}^d} p_s(x-y)|x|^{-\gamma} dx \leq C s^{-\gamma/\alpha}, \quad y \in \mathbb{R}^d. \tag{3.60}$$

Using this, (3.37), (3.35) and (3.55) we obtain

$$I_1(T) \leq C(W_1(T) + W_2(T)), \tag{3.61}$$

where

$$W_1(T) = \int_{\mathbb{R}^d} \int_0^{t_1} s^{-\gamma/\alpha} \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^d} p_{u-s}(y-z) \widetilde{\varphi}_T(z) dz du \right)^{1+\beta} ds dy, \tag{3.62}$$

$$W_2(T) = \int_{\mathbb{R}^d} \int_{t_1}^{t_2} s^{-\gamma/\alpha} \left(\int_{s}^{t_2} \int_{\mathbb{R}^d} p_{u-s}(y-z) \widetilde{\varphi}_T(z) dz du \right)^{1+\beta} ds dy. \tag{3.63}$$

Fix any ρ such that

$$\max\left\{ \frac{d}{\alpha} - \frac{1}{\beta}, 0 \right\} < \rho < 1 \tag{3.64}$$

(see (3.59). For any fixed $s \in [0, t_1]$ we apply the Jensen inequality to the measure

$$\frac{(u-s)^{-\rho}}{\int_{t_1}^{t_2} (r-s)^{-\rho} dr} 1\!\!1_{[t_1,t_2]}(u) du,$$

obtaining

$$W_1(T) \leq \int_{\mathbb{R}^d} \int_0^{t_1} s^{-\gamma/\alpha} \left(\int_{t_1}^{t_2} (r-s)^{-\rho} dr \right)^{\beta} \int_{t_1}^{t_2} (u-s)^{-\rho} ((u-s)^{\rho} p_{u-s} * \widetilde{\varphi}_T(y))^{1+\beta} du ds dy.$$

We have

$$||p_{u-s} * \widetilde{\varphi}_T||_{1+\beta} \leq ||p_{u-s}||_{1+\beta} ||\varphi||_1 = (u-s)^{-(d/\alpha)\beta/(1+\beta)} ||p_1||_{1+\beta} ||\varphi||_1, \tag{3.65}$$

hence

$$W_1(T) \leq C(t_2 - t_1)^{(1-\rho)\beta} \int_{t_1}^{t_2} \int_0^{t_1} s^{-\gamma/\alpha} (u-s)^{\rho\beta - (d/\alpha)\beta} ds du,$$

which, after the substitution $s' = s/u$, by (3.64) and $\gamma < \alpha$, yields

$$W_1(T) \leq C_1 (t_2 - t_1)^{(1-\rho)\beta} (t_2^h - t_1^h) \leq C_2 (t_2^{h\wedge 1} - t_1^{h\wedge 1})^{1+(1-\rho)\beta}, \tag{3.66}$$

where $h = 2 - \gamma/\alpha + \rho\beta - (d/\alpha)\beta > 0$ by assumptions.

Next, by (3.63) we have

$$W_2(T) \le \int_{\mathbb{R}^d}\int_{t_1}^{t_2} s^{-\gamma/\alpha}\left(\left(\left(\int_0^{t_2-t_1} p_u du\right) * \widetilde{\varphi}_T(y)\right)^{1+\beta} dsdy.\right.$$

The Young inequality, substitution $u' = u/(t_2 - t_1)$, and self-similarity imply

$$\begin{aligned} W_2(T) &\le C(t_2^{1-\gamma/\alpha} - t^{1-\gamma/\alpha})(t_2 - t_1)^{1+\beta-(d/\alpha)\beta}||f||_{1+\beta}^{1+\beta}||\varphi||_1^{1+\beta} \\ &\le C_1(t_2^{1-\gamma/\alpha} - t_1^{1-\gamma/\alpha})^{2+\beta-(d/\alpha)\beta}, \end{aligned} \tag{3.67}$$

by (3.41).

Combining (3.66), (3.67), (3.61) and using (3.59), we obtain (3.57).

It remains to prove (3.56).

Applying the usual substitutions to $A(T)$ given by (3.58) we obtain

$$\begin{aligned} &A(T) \\ &= \frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2}\int_0^1\int_s^1\int_{\mathbb{R}^{3d}} |x|^{-\gamma} p_s(x-y)\widetilde{\varphi}_T(y)p_{u-s}(y-z)\widetilde{\varphi}_T(z)\chi(s)\chi(u)dxdzdyduds, \end{aligned}$$

hence, by (3.60), (3.55) and the Hölder inequality,

$$A(T) \le C\frac{H_T T^{2-d/\alpha-\gamma/\alpha}}{F_T^2}\int_{t_1}^{t_2}\int_s^{t_2} s^{-\gamma/\alpha}||\widetilde{\varphi}_T||_{\frac{1+\beta}{\beta}}||p_{u-s} * \widetilde{\varphi}_T||_{1+\beta}duds.$$

We use (3.65), (3.45) and (2.5), obtaining

$$A(T) \le C_1(t_2^{1-\gamma/\alpha} - t_1^{1-1/\alpha})(t_2 - t_1)^{1-(d/\alpha)\beta/(1+\beta)},$$

which implies (3.56) by (3.59). This completes the proof of tightness. □

3.4 Proof of Theorem 2.1(b)

We prove the theorem for

$$\gamma = d < \frac{2+\beta}{1+\beta}\alpha \tag{3.68}$$

(see Remark 2.2(c)). Recall that in this case $k(T)$ occurring in (2.8) and (2.9) is $\log T$.

According to the discussion in Section 3.1 it suffices to prove (3.12), (3.13) and (3.15). By the form of the limit process (see (2.3)), (3.12) is equivalent to

$$\lim_{T\to\infty} I_1(T) = \frac{\sigma(S_{d-1})}{\alpha}\int_{\mathbb{R}^d}\int_0^1 p_s(y)\left(\int_s^1 p_{u-s}(y)\chi(u)du\right)^{1+\beta} dsdy\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}, \tag{3.69}$$

where $\sigma(S_{d-1})$ is the measure of the unit sphere in $\mathbb{R}^d (= 2$ if $d = 1)$.

By (3.9), (3.2), (3.6), using similar substitutions as in the previous section, we obtain

$$I_1(T)$$
$$= \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_s(xT^{-1/\alpha}-y)\left(\int_s^1\int_{\mathbb{R}^d} p_{u-s}(y-z)\chi(u)\widetilde{\varphi}_T(z)dzdu\right)^{1+\beta}\frac{1}{1+|x|^d}dydsdx, \tag{3.70}$$

where $\widetilde{\varphi}_T$ is given by (3.34). We write

$$I_1(T) = I_1'(T) + I_1''(T) + I_1'''(T), \tag{3.71}$$

where

$$I_1'(T) = \frac{1}{\log T}\int_{1<|x|<T^{1/\alpha}}\int_0^1\int_{\mathbb{R}^d}\dots \tag{3.72}$$
$$I_1''(T) = \frac{1}{\log T}\int_{|x|\geq T^{1/\alpha}}\int_0^1\int_{\mathbb{R}^d}\dots \tag{3.73}$$
$$I_1'''(T) = \frac{1}{\log T}\int_{|x|\leq 1}\int_0^1\int_{\mathbb{R}^d}\dots. \tag{3.74}$$

Passing to polar coordinates in the integral with respect to x we have

$$I_1'(T) = \frac{1}{\log T}\int_1^{T^{1/\alpha}}\int_{S_{d-1}}\int_0^1\int_{\mathbb{R}^d} p_s(wrT^{-1/\alpha}-y)(g_s*\widetilde{\varphi}_T(y))^{1+\beta}\frac{r^{d-1}}{1+r^d}dyds\sigma(dw)dr,$$

where g is defined by (3.35). The crucial step is the substitution

$$r' = \frac{\log r}{\log T}, \tag{3.75}$$

which gives

$$I_1'(T) = \int_0^{1/\alpha}\int_{S_{d-1}}\int_0^1\int_{\mathbb{R}^d} p_s(wT^{r-1/\alpha}-y)(g_s*\widetilde{\varphi}_T(y))^{1+\beta}\frac{T^{rd}}{1+T^{rd}}dyds\sigma(dw)dr.$$

It is now clear that if one could pass to the limit under the integrals as $T\to\infty$, then $I_1'(T)$ would converge to the right hand side of (3.69). This procedure is indeed justified by the fact that for f defined by (3.22) we have

$$f \in L^{2+\beta}(\mathbb{R}^d), \tag{3.76}$$

which follows from (3.68), (3.23) and (3.24). We omit the details, which are similar to the argument in [BGT6] (see (3.51) therein).

Next we show that $I_1''(T)$ and $I_1'''(T)$ tend to zero. In $I_1''(T)$ (see (3.73)) we substitute $x' = xT^{-1/\alpha}$ and we use (3.36), obtaining

$$\begin{aligned} I_1''(T) &\leq \frac{C}{\log T}\int_{|x|>1}\int_0^1\int_{\mathbb{R}^d} p_s(x-y)(f*\widetilde{\varphi}_T(y))^{1+\beta}\frac{T^{d/\alpha}}{1+|x|^dT^{d/\alpha}}dydsdx \\ &\leq \frac{C_1}{\log T}||f*\widetilde{\varphi}_T||_{1+\beta}^{1+\beta} \leq \frac{C_1}{\log T}||f||_{1+\beta}^{1+\beta}||\varphi||_1^{1+\beta}\to 0. \end{aligned}$$

$I_1^{'''}(T)$ (see (3.74)) is estimated as follows:

$$\begin{aligned}
I_1^{'''}(T) &\leq \frac{C}{\log T}\int_{|x|\leq 1} f*(f*\widetilde{\varphi}_T)^{1+\beta}(xT^{-1/\alpha})dx \\
&\leq \frac{C_2}{\log T}||f*(f*\widetilde{\varphi}_T)^{1+\beta}||_\infty \\
&\leq \frac{C_2}{\log T}||f||_{2+\beta}||(f*\widetilde{\varphi}_T)^{1+\beta}||_{(2+\beta)/(1+\beta)} \\
&\leq \frac{C_2}{\log T}||f||_{2+\beta}^{2+\beta}||\varphi||_1^{1+\beta} \to 0,
\end{aligned}$$

by (3.76). This and (3.71) prove (3.69).

To prove (3.13) we use (3.14) and easily obtain

$$I_2(T) \leq \frac{CH_T T^{2-2d/\alpha}}{F_T^2}\int_{\mathbb{R}^2} f*(\widetilde{\varphi}_T(f*\widetilde{\varphi}_T))(xT^{-1/\alpha})\frac{1}{1+|x|^d}dx.$$

We write the right-hand side as the sum of integrals over $\{|x|\leq T^{1/\alpha}\}$ and $\{|x|>T^{1/\alpha}\}$. To estimate the integral over $\{|x|\leq T^{1/\alpha}\}$ we use

$$\sup_{T>2}\frac{1}{\log T}\int_{|x|\leq T^{1/\alpha}}\frac{1}{1+|x|^d}dx<\infty, \tag{3.77}$$

and in the second integral we apply $1/(1+|x|^d)\leq T^{-d/\alpha}$. For each of the integrals we use appropriately the Hölder inequality, properties of the convolution and (3.45), obtaining the estimates $C_1T^{2((d/\alpha)1/(2+\beta)-1/(1+\beta))}$ and $C_2T^{(d/\alpha)1/(2+\beta)-2/(1+\beta)}$, respectively (the factors involving negative powers of H_T and $\log T$ have been estimated by constants). These bounds tend to zero as $T\to\infty$ by (3.68). We omit details. This proves (3.13).

To prove (3.16) and (3.17) we use (3.18) and (3.19). Again, we consider separately the integrals over $\{|x|\leq T^{1/\alpha}\}$ and $\{|x|>T^{1/\alpha}\}$, and apply the same tricks as for $I_2(T)$.

For $J_1(T)$ we obtain the estimate

$$J_1(T)\leq CT^{(d/\alpha)(1+\beta)/(2+\beta)-1}\to 0$$

($\log T$ and H_T appear with negative powers only), whereas

$$J_2(T)\leq C_1\frac{T}{H_T^\beta(\log T)^\beta}+C_2\frac{T}{H_T^\beta(\log T)^{1+\beta}}\to 0$$

by assumption (2.9). The proof of Theorem 2.1 is complete □

3.5 Proof of Proposition 2.3

Properties (a)-(c) are clear, following from (2.1) and Theorem 2.1(a). Recall that the index of self-similarity is defined as $a\in\mathbb{R}$ such that the process $(\xi_{ct})_{t\in\mathbb{R}_+}$ has the same distribution as $(c^a\xi_t)_{t\in\mathbb{R}_+}$ for any $c>0$.

To calculate the dependence exponent of ξ (see (2.10), (2.11)) first note that by (2.1) and Proposition 3.4.2 of [ST] the finite-dimensional distributions of ξ are given by

$$
\begin{aligned}
&E\exp\{i(z_1\xi_{t_1}+\cdots+z_k\xi_{t_k})\}\\
=\;& \exp\Bigg\{-\int_{\mathbb{R}^{d+1}}\Bigg[\Bigg|\sum_{j=1}^{k} z_j\left(\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dy\right)^{1/(1+\beta)} 1\!\!1_{[0,t_j]}(r)\int_r^{t_j} p_{u-r}(x)du\Bigg|^{1+\beta}\\
&\times\Bigg(1-i\ \mathrm{sgn}\Bigg(\sum_{j=1}^{k} z_j\left(\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dy\right)^{1/(1+\beta)}\\
&\times\ 1\!\!1_{[0,t_j]}(r)\int_r^{t_j} p_{u-r}(x)du\Bigg)\tan\frac{\pi}{2}(1+\beta)\Bigg)\Bigg]\,drdx\Bigg\}. \qquad (3.78)
\end{aligned}
$$

The argument goes along the lines of the proof of Theorem 2.7 of [BGT3]. For fixed $z>0$ and $0\le u<v<s<t$ we define $D_T^+=D_T(1,z;u,v,s,t)$ and $D_T^-=D_T(1,-z;u,v,s,t)$ (the formulas for D^+,D^- are obtained from (2.11) and (3.78)), and we prove

$$
D_T^{\pm}\le\begin{cases} CT^{-d/\alpha} \text{ if either } \quad \alpha=2 \text{ or } \beta>(d-\gamma)/(d+\alpha),\\ CT^{-(d/\alpha)\delta} \text{ for any } \quad \beta<\delta<1+\beta+(d-\gamma)/(d+\alpha) \text{ if } \alpha<2, \beta\le(d-\gamma)/(d+\alpha),\end{cases}
$$

and for T sufficiently large,

$$
\begin{aligned}
D_T^+ &\ge CT^{-d/\alpha},\\
D_T^+ &\ge CT^{-(d/\alpha)\delta} \text{ for any } \delta>1+\beta-(d-\gamma)/(d+\alpha) \text{ if } \alpha<2, \beta\le(d-\gamma)/(d+\alpha). \qquad (3.79)
\end{aligned}
$$

The upper estimates are obtained similarly as (4.3), (4.4) in [BGT3] and (3.108) in [BGT6]. The only difference is that in formulas (4.9) and (4.10) in [BGT3] a new factor, $\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dy$, appears (which corresponds to $p_r(x)$ in (3.101) in [BGT6]). This factor is responsible for the new long-range dependence threshold and the form of the dependence exponent (2.12). In the estimates we use (3.30).

The first of the lower estimates is obtained exactly as (4.18) in [BGT3]. The new expression,

$$
\int_u^{(u+v)/2}\int_{|x|\le 1}\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dydxdr,
$$

that appears at the right-hand side is finite by (3.31).

To derive (3.79) we argue as in (4.22), (4.24) of [BGT3] and we apply estimates (4.21) (which holds for $|x|\le T^{1/\alpha}$) and (4.23) therein, obtaining

$$
D_T^+\ge CT^{-(d/\alpha)(1+\beta)+\varepsilon\beta(d+\alpha)}\int_{u+(v-u)/4}^{(u+v)/2}\int_{1\le|x|\le T^{d/(d+\alpha)\alpha-\varepsilon}}\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma}dydxdr, \qquad (3.80)
$$

where $\varepsilon>0$ is sufficiently small.

For $1 \le |x| \le T^{d/(d+\alpha)\alpha-\varepsilon}$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} p_r(x-y)|y|^{-\gamma} dy &\ge C|x|^{-\gamma} \int_{|x-y|\le 1/2} p_r(x-y)dy \\
&\ge C_1|x|^{-\gamma} \inf_{\frac{v-u}{4}+u<r<\frac{u+v}{2}} \inf_{|z|\le\frac{1}{2}} p_r(z) \ge C_2 T^{-d\gamma/d(d+\alpha)\alpha+\varepsilon\gamma}.
\end{aligned}
$$

Putting this into (3.80) we obtain (3.79). □

3.6 Proof of Theorem 2.5

Each of the cases requires a different proof, and none of them is straightforward. We will present a detailed proof of the part (a) only. In the remaining cases we will confine ourselves to explaining why the limit processes have the forms given in the theorem (recall that part (b)(iv) has been proved in [BGT6]). It seems instructive to compare the proofs for this theorem to the argument given in the proof of Theorem 2.1(b) for $\gamma = d$. Although the critical cases are of different kinds, some of the technical tricks repeat in all cases, nevertheless they are applied in a slightly different way and are far from being identical.

Proof of case (a) To simplify calculations we again consider the measure μ_γ of the form $\mu_\gamma(dx) = |x|^{-\gamma}dx$ instead of (1.8).

In (3.9) we substitute $u' = s-u$ and then $s' = (T-s)/T$, obtaining

$$
I_1(T) = \frac{H_T T}{F_T^{1+\beta}} \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_{sT}(x-y)\left(\int_0^{T(1-s)}\int_{\mathbb{R}^d} p_u(y-z)\varphi(z)\chi\left(s+\frac{u}{T}\right)dzdu\right)^{1+\beta} |x|^{-\gamma}dydsdx.
$$

Using (2.13), (3.20) and substitution $x' = x(sT)^{-1/\alpha}$ we have

$$
\begin{aligned}
I_1(T) &= \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_1(x-y(sT)^{-1/\alpha}) \\
&\quad\times\left(\int_0^{T(1-s)}\int_{\mathbb{R}^d} p_u(y-z)\varphi(z)\chi\left(s+\frac{u}{T}\right)dzdu\right)^{1+\beta} s^{-\gamma/\alpha}|x|^{-\gamma}dydsdx \\
&= I_1'(T)+I_1''(T)+I_1'''(T), \qquad (3.81)
\end{aligned}
$$

where

$$
I_1'(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^1\int_{1\le|y|\le T^{1/\alpha}} \cdots \qquad (3.82)
$$

$$
I_1''(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^1\int_{|y|>T^{1/\alpha}} \cdots \qquad (3.83)
$$

$$
I_1'''(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^1\int_{|y|<1} \cdots \qquad (3.84)
$$

Passing to polar coordinates in the integral with respect to y and making substitution (3.75) we obtain

$$\begin{aligned}
&I_1'(T) \\
&\quad = \int_{\mathbb{R}^d}\int_0^1\int_0^{1/\alpha}\int_{S_{d-1}} p_1(x - wT^{r-1/\alpha}s^{-1/\alpha}) \\
&\qquad \times\left(\int_0^{T(1-s)}\int_{\mathbb{R}^d} p_u(wT^r - z)\varphi(z)\chi\left(\frac{u}{T}+s\right)dzdu\right)^{1+\beta} s^{-\gamma/\alpha}|x|^{-\gamma}T^{rd}\sigma(dw)drdsdx.
\end{aligned}$$

We substitute $z' = T^{-r}z, u' = uT^{-r\alpha}$, use (3.20) and (2.12), arriving at

$$I_1'(T) = \int_{\mathbb{R}^d}\int_0^1\int_0^{1/\alpha}\int_{S_{d-1}} p_1(x - ws^{-1/\alpha}T^{r-1/\alpha})\left(h_T(r,s,w)\right)^{1+\beta} s^{-\gamma/\alpha}|x|^{-\gamma}\sigma(dw)drdsdx, \quad (3.85)$$

where

$$h_T(r,s,w) = \int_0^{T^{1-r\alpha}(1-s)}\int_{\mathbb{R}^d} p_u(w-z)T^{rd}\varphi(zT^r)\chi(s+uT^{r\alpha-1})dzdu. \quad (3.86)$$

It is clear that on the set of integration one should have

$$\begin{aligned}
\lim_{T\to\infty} h_T(r,s,w) &= \int_{\mathbb{R}^d}\varphi(z)dz\int_0^\infty p_u(w)du\chi(s) \\
&= C_{d,\alpha}\int_{\mathbb{R}^d}\varphi(z)dz\chi(s),
\end{aligned} \quad (3.87)$$

where $C_{d,\alpha}$ is given by (1.13), which should yield

$$\lim_{T\to\infty} I_1'(T) = C_{d,\alpha}^{1+\beta}\frac{1}{\alpha}\sigma(S_{d-1})\int_{\mathbb{R}^d} p_1(x)|x|^{-\gamma}dx\int_0^1 s^{-\gamma/\alpha}\chi^{1+\beta}(s)ds\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}. \quad (3.88)$$

However, (3.87) and (3.88) need a justification. It is easy to see that the first integral in (3.86) can be replaced by $\int_0^\infty du$. Since

$$\lim_{T\to\infty}\int_{\mathbb{R}^d} p_u(w-z)T^{rd}\varphi(zT^r)dz = p_u(w)\int_{\mathbb{R}^d}\varphi(z)dz,$$

it is clear that in order to prove (3.87) it suffices to show that

$$\sup_{T>2}\sup_{w\in S_{d-1}}\int_{\mathbb{R}^d} p_u(w-z)T^{rd}\varphi(zT^r)dz$$

is integrable in u. This is clear for $u \geq 1$ since $d > \alpha$, and for $u < 1$ we argue similarly as in (3.39) obtaining an integrable bound $C_1 p_u(w/2) + C_2$. In the same way one shows that $h_T(r,s,w) \leq C$. This together with (3.87) easily implies (3.88).

Next, it is easy to see that for I_1''' defined by (3.84) we have

$$I_1'''(T) \leq \frac{C}{\log T} \to 0. \quad (3.89)$$

A little more work is needed to prove that also

$$\lim_{T\to\infty} I_1''(T)=0. \tag{3.90}$$

By (3.83),

$$\begin{aligned} I_1''(T) &\le \frac{C}{\log T}\int_{|y|>T^{1/\alpha}}\left(\int_0^T\int_{\mathbb{R}^d} p_u(y-z)\varphi(z)dzdu\right)^{1+\beta} dy \\ &\le C_1(R_1(T)+R_2(T)), \end{aligned}$$

where

$$\begin{aligned} R_1(T) &= \frac{1}{\log T}\int_{|y|>T^{1/\alpha}}\left(\int_0^T\int_{|z|\le\frac{T^{1/\alpha}}{2}} p_u(y-z)\varphi(z)dzdu\right)^{1+\beta} dy, \\ R_2(T) &= \frac{1}{\log T}\int_{|y|>T^{1/\alpha}}\left(\int_0^T\int_{|z|>\frac{T^{1/\alpha}}{2}} p_u(y-z)\varphi(z)dzdu\right)^{1+\beta} dy. \end{aligned}$$

We have

$$\begin{aligned} R_1(T) &\le \frac{1}{\log T}\int_{|y|>T^{1/\alpha}}\left(\int_0^T p_u\left(\frac{y}{2}\right)du\right)^{1+\beta} dy\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta} \\ &= \frac{C}{\log T}\int_{|y|>1}\left(\int_0^1 p_u\left(\frac{y}{2}\right)du\right)^{1+\beta} dy, \end{aligned}$$

after obvious substitutions and using (2.12). Hence $\lim_{T\to\infty} R_1(T)=0$ by (3.23). Furthermore,

$$\begin{aligned} R_2(T) &\le \frac{C}{\log T}\int_{|y|>T^{1/\alpha}}\left(\int_0^T\int_{|z|>\frac{T^{1/\alpha}}{2}} p_u(y-z)\frac{|z|^2\varphi(z)}{T^{2/\alpha}}dzdu\right)^{1+\beta} dy \\ &\le \frac{C_1}{T^{(2/\alpha)(1+\beta)}}, \end{aligned}$$

since, under (2.12),

$$\sup_T \frac{1}{\log T}\int_{\mathbb{R}^d}\left(\int_0^T\int_{\mathbb{R}^d} p_u(y-z)\varphi_1(z)dzdu\right)^{1+\beta} dy<\infty,$$

for any $\varphi_1\in\mathscr{S}(\mathbb{R}^d)$ by (3.33) in [BGT4] (in our case $\varphi_1(z)=|z|^2\varphi(z)$). This proves (3.90), and by (3.81)-(3.84), (3.88) and (3.89) we have established (3.12).

To prove (3.13) we use (3.14), which, after standard substitutions, gives

$$\begin{aligned} & I_2(T) \\ &\le CH_T^{-1/(1+\beta)}T^{-2/(1+\beta)+2\gamma/\alpha(1+\beta)+2-d/\alpha-\gamma/\alpha}(\log T)^{-2/(1+\beta)}\int_{\mathbb{R}^d} f_\gamma(y)\widetilde{\varphi}_T(y)(f*\widetilde{\varphi}_T)(y)dy \end{aligned}$$

(see (2.13), (3.22), (3.29), (3.34)). By (3.31) we have

$$I_2(T) \le C_1 T^{-2/(1+\beta)+2\gamma/\alpha(1+\beta)+2-d/\alpha-\gamma/\alpha} \sup_y f * \widetilde{\varphi}_T(y). \tag{3.91}$$

The assumptions (2.12) and $\gamma < \alpha$ imply that

$$\frac{2}{1+\beta} - \frac{2\gamma}{\alpha(1+\beta)} - 2 + \frac{d}{\alpha} + \frac{\gamma}{\alpha} = \frac{1}{\beta}\theta$$

for some $\theta > 1$. Observe that

$$f \in L^q(\mathbb{R}^d),\ 1 \le q < 1+\beta, \tag{3.92}$$

by (3.23), (3.24) and (2.12). Fix $q > 1$ such that $(1+\beta)/\theta < q < 1+\beta$ and $p = q/(q-1)$. By the Hölder inequality and (3.45) we obtain

$$I_2(T) \le C_1 T^{-\theta/\beta+((1+\beta)/\beta)(1/q)} ||\varphi||_p \to 0$$

as $T \to \infty$, by assumption on q.

To prove (3.16) we use (3.18) which, by an analogous argument as for I_2, gives

$$J_1(T) \le C T^{-1+\gamma/\alpha-(1+\beta)/\beta} ||f * (\widetilde{\varphi}_T(f * \widetilde{\varphi}_T))||_{1+\beta}^{1+\beta}. \tag{3.93}$$

Taking $q < 1+\beta$ sufficiently close to $1+\beta$ and using (3.92), the Young and Hölder inequalities can be applied to conclude that, by (3.45),

$$||f * (\widetilde{\varphi}_T(f * \widetilde{\varphi}_T))||_{1+\beta}^{1+\beta} \le O(T^r),$$

where $0 < r < 1 - \gamma/\alpha + (1+\beta)/\beta$; we omit details. This and (3.93) yield (3.16).

To prove (3.17) we write an estimate similar to (3.50), namely,

$$J_2(T) \le C T^{\beta(\gamma/\alpha-1)} R(T), \tag{3.94}$$

where $R(T)$ is defined by (3.51). In the present case (3.52) does not hold, but similarly as before, using the Young inequality, (3.92) and (3.45) it can be shown that $R(T) = O(T^\varepsilon)$ for any $\varepsilon > 0$. This and (3.43) imply (3.17) since $\gamma < \alpha$. The proof of part (a) of the theorem is complete.

Sketch of the proof of case (b)

(i) We repeat the argument as in (3.81) -(3.86) with F_T given by (2.14). Again, it can be shown that the only significant term is $I_1'(T)$, i.e., $\lim_{T\to\infty} I_1(T) = \lim_{T\to\infty} I_1'(T)$. In order to derive this limit, in (3.85) we substitute $s' = sT^{1-r\alpha}$, obtaining

$$I_1'(T) = \frac{1}{\log T} \int_{\mathbb{R}^d} \int_0^{1/\alpha} \int_0^{T^{1-r\alpha}} \int_{S_{d-1}} p_1(x - ws^{-1/\alpha})(h_T(r, sT^{-1+r\alpha}, w))^{1+\beta} s^{-1} |x|^{-\alpha} \sigma(dw) ds dr dx.$$

It is easy to see that the limit remains the same if the integral $\int_0^{T^{1-r\alpha}} \dots ds$ is replaced by $\int_1^{T^{1-r\alpha}} \dots ds$. Let $\widetilde{I}_1'(T)$ denote $I_1'(T)$ after this change. Next, we substitute $s' = \log s / \log T$ and we have

$$\widetilde{I}_1'(T) = \int_0^{1/\alpha} \int_{S_{d-1}} \int_0^{1-r\alpha} \int_{\mathbb{R}^d} p_1(x - wT^{-s/\alpha}) |x|^{-\alpha} (h_T(r, T^{s-1+r\alpha}, w))^{1+\beta} dx ds \sigma(dw) dr.$$

By (3.86), it is clear that on the set of integration

$$\lim_{T\to\infty} h_T(r, T^{s-1+r\alpha}, w) = C_{d,\alpha}\int_{\mathbb{R}^d}\varphi(z)dr\chi(0).$$

This shows that we should have

$$\begin{aligned}\lim_{T\to\infty} I_1(T) &= \lim_{T\to\infty}\widetilde{I}_1'(T)\\ &= C_{d,\alpha}^{1+\beta}\sigma(S_{d-1})\int_{\mathbb{R}^d}p_1(x)|x|^{-d}dx\int_0^{1/\alpha}(1-r\alpha)dr\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}\chi^{1+\beta}(0),\end{aligned}\tag{3.95}$$

and this passage to the limit can be indeed justified. The right-hand side of (3.95) is equal to $\log E\exp\{-C\langle\widetilde{X},\varphi\otimes\psi\rangle\}$, where $X(=K\lambda\vartheta)$ is the limit process defined in the theorem. We skip the remaining parts of the proof.

(ii) In (3.9) we substitute $u' = s-u$ and then $s' = T-s$, obtaining

$$I_1(T) = I_1'(T) + I_1''(T),$$

where

$$\begin{aligned}I_1'(T) &= \frac{1}{\log T}\int_1^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_1((x-y)s^{-1/\alpha})s^{-d/\alpha}|x|^{-\gamma}\\ &\times\left(\int_0^{T-s}\int_{\mathbb{R}^d}p_u(y-z)\varphi(z)\chi\left(\frac{u+s}{T}\right)dzdu\right)^{1+\beta}dydxds\end{aligned}$$

and

$$I_1''(T) = \frac{1}{\log T}\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\ldots dydxds.$$

It can be shown that

$$\lim_{T\to\infty} I_1''(T) = 0.$$

In $I_1'(T)$ we substitute $x' = xs^{-1/\alpha}, y' = ys^{-1/\alpha}, u' = u/s$, and use (3.20), which gives

$$\begin{aligned}I_1'(T) &= \frac{1}{\log T}\int_1^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_1(x-y)|x|^{-\gamma}s^{-(d/\alpha)\beta-\gamma/\alpha+(1+\beta)}\\ &\times\left(\int_0^{T/s-1}\int_{\mathbb{R}^d}p_u(y-zs^{-1/\alpha})\varphi(z)\chi\left(\frac{s(u+1)}{T}\right)dzdu\right)^{1+\beta}dydxds.\end{aligned}$$

By (2.16), $s^{-(d/\alpha)\beta-\gamma/\alpha+1+\beta} = s^{-1}$, so, the substitution $s' = \log s/\log T$ yields

$$\begin{aligned}I_1'(T) &= \int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_1(x-y)|x|^{-\gamma}\\ &\times\left(\int_0^{T^{1-s}-1}\int_{\mathbb{R}^d}p_u(y-zT^{-s/\alpha})\varphi(z)\chi\left(((u+1)T^{s-1}\right)dzdu\right)^{1+\beta}dydxds.\end{aligned}\tag{3.96}$$

It is now seen that one should have

$$
\begin{aligned}
\lim_{T\to\infty} I_1(T) = \lim_{T\to\infty} I_1'(T) \\
= \quad C_{\alpha,d}^{1+\beta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_1(x-y)|x|^{-\gamma}|y|^{-(d-\alpha)(1+\beta)}dxdy\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}\chi^{1+\beta}(0) \quad (3.97)
\end{aligned}
$$

Note that the integrals are finite by (3.21), (3.28) and (2.16). The justification of (3.97) requires some work, but we omit it for brevity.

(iii) As $d=\gamma$, we must keep the measure μ_γ in its original form (1.8).

Arguing as in the proof of (ii) and taking into account (2.14), instead of (3.96) we obtain

$$
\begin{aligned}
I_1'(T) = \frac{1}{\log T}\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_1(x-y)\frac{T^{sd/\alpha}}{1+|xT^{s/\alpha}|^d} \\
\times\left(\int_0^{T^{1-s}-1}\int_{\mathbb{R}^d} p_u(y-zT^{-s/\alpha})\varphi(z)\chi((u+1)T^{s-1})dzdu\right)^{1+\beta} dydxds.
\end{aligned}
$$

Since

$$
\lim_{T\to\infty}\frac{1}{\log T}\int_{\mathbb{R}^d} p_1(x-y)\frac{T^{sd/\alpha}}{1+|xT^{s/\alpha}|^d}dx = s\frac{1}{\alpha}\sigma(S_{d-1})p_1(y),
$$

it can be shown, with some effort, that

$$
\begin{aligned}
\lim_{T\to\infty} I_1(T) &= \lim_{T\to\infty} I_1'(T) \\
&= C_{\alpha,d}^{1+\beta}\frac{1}{2\alpha}\sigma(s_{d-1})\int_{\mathbb{R}^d} p_1(y)|y|^{-(d-\alpha)(1+\beta)}dy\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}\chi^{1+\beta}(0).
\end{aligned}
$$

Again, we omit the remaining parts of the proof.

3.7 Proof of Theorem 2.6

We only give an outline of the proof. The following lemma is constantly used.

Lemma *Let* $\varphi\in\mathscr{S}(\mathbb{R}^d), \varphi\geq 0$.

(a) If $d>\alpha(2+\beta)/(1+\beta)$*, then the functions* $G\varphi, G(G\varphi)^{1+\beta}$ *and* $G(G\varphi)^2$ *are bounded.*

(b) If $d>\alpha(1+\beta)/\beta$*, then additionally* $(G\varphi)^{1+\beta}$ *and* $(G(G\varphi)^{1+\beta})^{1+\beta}$ *are integrable (and bounded).*

(c) If $\alpha<\gamma\leq d$ *and* $d>\alpha(2+\beta)/\beta-\gamma/\beta$*, then additionally to the properties in (a),*

$$
\int_{\mathbb{R}^d} G(G\varphi)^{1+\beta}(x)\frac{1}{1+|x|^\gamma}dx<\infty.
$$

This Lemma follows easily from (1.12) and (3.25)-(3.28).

Proof of part (a) of the theorem. As before we consider $\mu_\gamma(dx)=|x|^{-\gamma}dx$. In (3.9) we substitute $u'=s-u$, then $s'=(T-s)/T$ and, finally, $x'=xT^{1/\alpha}s^{-1/\alpha}$, obtaining

$$
\begin{aligned}
&I_1(T)\\
&= \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_1(x-ys^{-1/\alpha}T^{-1/\alpha})\left(\int_0^{T(1-s)}\mathcal{T}_u\varphi(y)\chi\left(s+\frac{u}{T}\right)du\right)^{1+\beta} s^{-\gamma/\alpha}|x|^{-\gamma}dydsdx
\end{aligned}
$$

(see (2.22)). It is easily seen that by part (b) of the Lemma we have

$$\lim_{T\to\infty} I_1(T)=\int_{\mathbb{R}^d} p_1(x)|x|^{-\gamma}dx\int_0^1 s^{-\gamma/\alpha}\chi^{1+\beta}(s)ds\int_{\mathbb{R}^d}(G\varphi(y))^{1+\beta}dy. \tag{3.98}$$

For $\beta<1$ this is exactly $\log E\exp\{-C\langle\widetilde{X},\varphi\otimes\psi\rangle\}$, where X is the limit process described in the theorem. Moreover, in this case (3.14) and boundedness of $G\varphi$ easily imply

$$I_2(T)\le C\quad T^{(1-\gamma/\alpha)(1-2/(1+\beta))}\to 0.$$

For $\beta=1$ we use (3.10) and (3.4), obtaining

$$I_2(T)=I_2'(T)-I_2''(T)-\frac{V}{2}I_2'''(T),$$

where

$$
\begin{aligned}
I_2'(T) &= \frac{H_T}{F_T^2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_s(x-y)|x|^{-\gamma}dx\varphi(y)\chi(s)\int_s^T\mathcal{T}_{u-s}\varphi(y)\chi\left(\frac{u}{T}\right)dudyds,\\
I_2''(T) &= \frac{H_T}{F_T^2}\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d} p_{T-s}(x-y)\varphi(y)\chi_T(T-s)\\
&\quad\times\int_0^s\mathcal{T}_{s-u}(\varphi\chi(T-u)v_T(\cdot,u))(y)|x|^{-\gamma}dudydsdx\\
I_2'''(T) &= \frac{H_T}{F_T^2}\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d} p_{T-s}(x-y)\varphi(y)\chi_T(T-s)\int_0^s\mathcal{T}_{s-u}(v_T^2(\cdot,u))(y)|x|^{-\gamma}dudydsdx.
\end{aligned}
$$

Substituting $u'=u-s, s'=s/T$, and then $x'=xT^{1/\alpha}s^{1/\alpha}$ and using part (a) of the Lemma and (2.22) we have

$$\lim_{T\to\infty} I_2'(T)=\int_{\mathbb{R}^d} p_1(x)|x|^{-\gamma}dx\int_0^1 s^{-\gamma/\alpha}\chi^2(s)ds\int_{\mathbb{R}^d}\varphi(y)G\varphi(y)dy. \tag{3.99}$$

Applying (3.5), (2.22) and the Lemma above we get

$$
\begin{aligned}
I_2''(T) &\le \frac{CH_T}{F_T^3}\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d} p_s(x-y)|x|^{-\gamma}\varphi(y)G(\varphi G\varphi)(y)dydsdx\\
&\le C_1T^{-(1/2)(1-\gamma/\alpha)}\to 0,
\end{aligned}
$$

and, analogously,

$$\begin{aligned} I_2'''(T) &\le \frac{CH_T}{F_T^3}\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d} p_s(x-y)|x|^{-\gamma}\varphi(y)G((G\varphi)^2)(y)dydsdx \\ &\le C_2T^{-(1/2)(1-\gamma/\alpha)} \to 0. \end{aligned}$$

This and (3.98), (3.99) imply that for $\beta = 1$ the limit of $(V/2)I_1(T) + I_2(T)$ is exactly $\log E\exp\{-C\langle\widetilde{X}, \varphi\otimes\psi\rangle\}$. Similar estimations, together with the Lemma, yield (3.16) and (3.17). This completes the proof of part (a) of the theorem.

Proof of part (b) of the theorem. Following the general scheme one can show

$$\begin{aligned} \lim_{T\to\infty} I_1(T) &= \int_{\mathbb{R}^d} p_1(x)|x|^{-\alpha}dx\chi^{1+\beta}(0)\int_{\mathbb{R}^d}(G\varphi)^{1+\beta}(y)dy, \\ \lim_{T\to\infty} I_2(T) &= c_\beta\int_{\mathbb{R}^d} p_1(x)|x|^{-\alpha}dx\chi^2(0)\int_{\mathbb{R}^d}\varphi(y)G\varphi(y)dy, \end{aligned}$$

and (3.16) and (3.17) (recall that c_β is defined by (2.24)). This is accomplished by an argument similar to the one used in part (a). Due to the criticality ($\gamma = \alpha$), the integrals $\int_0^T \ldots ds$ in (3.9)-(3.11) require a different treatment. They are split into $\int_0^1 \ldots ds + \int_1^T \ldots ds$; the first summand converges to zero, and in the second one we use the substitution $s' = \log s/\log T$. Here, again, we use repeatedly the Lemma above together with the easily checked fact that

$$\sup_{T>2}\frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_s h(x)|x|^{-\alpha}dsdx < \infty$$

for any integrable and bounded function h (recall that $d > \alpha$). We omit details.

Proof of part (c) of the theorem. Recall that the case $\gamma > d$ has been proved in [BGT6]. For $\alpha < \gamma \le d$ we use the Lemma (part (c) is particularly important). We show

$$\begin{aligned} \lim_{T\to\infty} I_1(T) &= \int_{\mathbb{R}^d} G(G\varphi)^{1+\beta}(x)\mu_\gamma(dx)\chi^{1+\beta}(0), \\ \lim_{T\to\infty} I_2(T) &= c_\beta\int_{\mathbb{R}^d} G(\varphi G\varphi)(x)\mu_\gamma(dx)\chi^2(0), \end{aligned}$$

(3.16) and (3.17). Here μ_γ is either given by (1.8) or, for $\gamma < d$ one can take $\mu_\gamma(dx) = |x|^{-\gamma}dx$. Again, the details are omitted.

3.8 Proof of Theorem 2.8

The proof is based on two general lemmas which, hopefully, are of interest by themselves.

Lemma A. *Let Y be an (d,α,β)-superprocess with $Y_0 = \mu$ and N be the empirical process of the corresponding branching particle system. If for any bounded Borel set $A \subset \mathbb{R}^d$,*

$$P\left[\int_0^\infty Y_t(A)dt < \infty\right] = 1, \tag{3.100}$$

then also

$$P\left[\int_0^\infty N_t(A)dt < \infty\right] = 1. \tag{3.101}$$

Proof of Lemma A. Let ζ denote the standard α-stable Lévy process in $\mathbb{R}^d$, and let ξ be a Markov process with semigroup

$$\mathscr{S}_t\varphi(x) = E_x\left[\exp\left\{-\int_0^t \psi(\zeta_s)ds\right\}\varphi(\zeta_t)\right],$$

where ψ is a fixed element of $C_K^\infty(\mathbb{R}^d)$ (bounded support), $\psi \geq 0$. The process ξ takes values in $\mathbb{R}^d \cup \{\dagger\}$, where $\dagger$ is a cemetery point where it remains after killing by $\exp\{-\int_0^t \psi(\zeta_s)ds\}$. The infinitesimal generator of ξ is

$$A\varphi(x) = (\Delta_\alpha - \psi(x))\varphi(x).$$

Let Y^ψ be a superprocess in $\mathbb{R}^d$ constructed from ξ and $(1+\beta)$-branching, with $Y_0^\psi = \mu$. The Laplace functional of its occupation time is given by

$$E\exp\left\{-\int_0^t \langle Y_s^\psi, \varphi\rangle ds\right\} = \exp\{-\langle \mu, u_\varphi^\psi(t)\rangle\},\ \varphi \in C_K^\infty(\mathbb{R}^d),\ \varphi \geq 0, \tag{3.102}$$

where $u_\varphi^\psi(x,t)$ is the unique (mild) solution of

$$\begin{aligned}
\frac{\partial}{\partial t}u_\varphi^\psi(x,t) &= (\Delta_\alpha - \psi(x))u_\varphi^\psi(x,t) - \frac{V}{1+\beta}(u_\varphi^\psi(x,t))^{1+\beta} + \varphi(x), \\
u_\varphi^\psi(x,0) &= 0
\end{aligned} \tag{3.103}$$

(cf. (1.6)). The Laplace functional of the occupation time of the process N is given by

$$E\exp\left\{-\int_0^t \langle N_s, \varphi\rangle ds\right\} = \exp\{-\langle \mu,\ v_\varphi(t)\rangle\},\ \ \varphi \in C_K^\infty(\mathbb{R}^d), \quad \varphi \geq 0, \tag{3.104}$$

where $v_\varphi(x,t)$ is the unique solution of

$$\begin{aligned}
\frac{\partial}{\partial t}v_\varphi(x,t) &= \Delta_\alpha v_\varphi(x,t) - \frac{V}{1+\beta}(v_\varphi(x,t))^{1+\beta} + \varphi(x)(1 - v_\varphi(x,t)) \\
&= (\Delta_\alpha - \varphi(x))v_\varphi(x,t) - \frac{V}{1+\beta}(v_\varphi(x,t))^{1+\beta} + \varphi(x), \\
v_\varphi(x,0) &= 0
\end{aligned} \tag{3.105}$$

(cf. (1.4)). Equations (3.103) and (3.105) coincide for $\varphi = \psi$, hence, from (3.102), (3.104),

$$E\exp\left\{-\int_0^t \langle Y_s^\psi, \psi\rangle ds\right\} = E\exp\left\{-\int_0^t \langle N_s, \psi\rangle ds\right\} = \exp\{-\langle \mu, u_\psi^\psi(t)\rangle\}. \tag{3.106}$$

The superprocesses $Y \equiv Y^0$ and Y^ψ are obtained as (high-density/short-life/small-particle) limits of the same process N, removing first the killed particles in the case of Y^ψ. Hence

$$E\exp\left\{-\int_0^t \langle Y_s^\psi, \varphi\rangle ds\right\} \geq E\exp\left\{-\int_0^t \langle Y_s^0, \varphi\rangle ds\right\} \quad \text{for all } t, \tag{3.107}$$

for any $\varphi \geq 0$, in particular for $\varphi = \psi$, hence, from (3.106), (3.107), and (3.102) with $\psi = 0$,

$$\exp\{-\langle \mu, u_\psi^\psi(t)\rangle\} \geq \exp\{-\langle \mu, u_\psi^0(t)\rangle\} \quad \text{for all } t. \tag{3.108}$$

Taking $\mu = \delta_x$ in (3.102) we see that

$$u_\psi^\psi(x,t) \nearrow u_\psi^\psi(x) \quad \text{and} \quad u_\psi^0(x,t) \nearrow u_\psi^0(x) \quad \text{as} \quad t \nearrow \infty,$$

hence, from (3.108),

$$\langle \mu, u_\psi^\psi\rangle \leq \langle \mu, u_\psi^0\rangle. \tag{3.109}$$

From (3.100), (3.106), (3.109),

$$\begin{aligned} 1 &= P\left[\int_0^\infty \langle Y_t, \psi\rangle dt < \infty\right] = \lim_{\theta\searrow 0} \exp\{-\langle \mu, u_{\theta\psi}^0\rangle\} \\ &\leq \lim_{\theta\searrow 0} \exp\{-\langle \mu, u_{\theta\psi}^{\theta\psi}\rangle\} = P\left[\int_0^\infty \langle N_t, \psi\rangle dt < \infty\right], \end{aligned}$$

so (3.101) is satisfied for any bounded set $A \subset \mathbb{R}^d$ and the lemma is proved. □

Lemma B. *Let N be the empirical process of the (d,α,β)-branching particle system with locally finite initial intensity measure μ. If (3.101) is satisfied for any bounded set $A \subset \mathbb{R}^d$, then*

$$P[\sup\{t : N_t(A) > 0\} < \infty] = 1 \tag{3.110}$$

for any bounded set A.

Proof of Lemma B. Let B_R be a closed ball in $\mathbb{R}^d$ with radius R centered at the origin. Let $(t_i, x_i, \tau_i), i = 1, 2, \ldots,$ be a sequence of random vectors defined as follows for any realization of the branching particle system. First we exclude all the particles which start inside B_R at time 0 and their progenies. Let t_1 be the first time any of the remaining particles enters B_R, x_1 is the entry point, and τ_1 is the occupation time of the closed ball $B_1(x_1)$ of radius 1 centered at x_1 by the tree generated by the entered particle. We exclude this tree from further consideration. Let t_2 be the first time after t_1 that any of the remaining particles enters B_R, with x_2 and τ_2 defined analogously as above; and so on. Let η denote the total number of first entries $(t_i, x_i, \tau_i), i = 1, \ldots, \eta$. We will show that $\eta < \infty$ a.s.. Suppose to the contrary that $P[\eta = \infty] > 0$. By construction, $\sum_{i=1}^\eta \tau_i \leq \int_0^\infty N_t(B_{R+1})dt$, hence $\sum_{i=1}^\eta \tau_i < \infty$ a.s. by (3.101). By the strong Markov property and homogeneity of the motion, conditioned on $\{\eta = \infty\}$ the random variables τ_i are i.i.d.. Hence

$$P\left[\sum_{i=1}^\eta \tau_i = \infty | \eta = \infty\right] = 1,$$

and this is a contradiction since, as observed above, $P[\sum_{i=1}^{\infty}\tau_i=\infty]=0$.

Going back to the particles that start inside B_R, there are only finitely many of them since $\mu(B_R)<\infty$.

In conclusion, with probability 1 only finitely many initial particles generate trees that contribute to the occupation time of any given bounded set, and all those trees become extinct a.s. in finite time by criticality of the branching. So (3.110) is proved. $\square$

Now, to prove Theorem 2.8 it suffices to observe that under its assumptions the corresponding superprocess Y suffers local extinction by Theorem 3_β of [I2], hence (3.100) is clearly satisfied and the theorem follows immediately from the lemmas.

3.9 Proof of Proposition 2.9

First observe that it suffices to prove convergence of finite-dimensional distributions. Indeed, in the proof of Theorem 2.1(a) we have shown tightness of $X_T=Z_T-EZ_T$, and the presence of high density was not relevant in that proof. On the other hand, from Proposition 2.1 of [BGT5] it follows easily that the family of deterministic processes $(E\langle Z_T,\varphi\rangle)_{T\geq 1}$ is tight in $C([0,\tau],\mathbb{R}),\tau>0$. Hence tightness of Z_T follows.

Without loss of generality we assume that $\tau=1$. Fix $0\leq t_1<t_2<\ldots<t_n\leq 1,\varphi_1,\ldots,\varphi_n\in\mathscr{S}(\mathbb{R}^d)$, and we may additionally assume that $\varphi_1,\ldots,\varphi_n\geq 0$. In order to show $\Rightarrow_f$ convergence we prove that

$$\lim_{T\to\infty}E\exp\left\{-\sum_{k=1}^{n}\langle Z_T(t_k),\varphi_k\rangle\right\}=\exp\left\{-\int_{\mathbb{R}^d}v(x,1)|x|^{-\gamma}dx\right\},\tag{3.111}$$

where v satisfies (2.36) with ψ given by (2.37) for $\theta_k=\int_{\mathbb{R}^d}\varphi_k(y)dy$ (as explained in [Ta], the solution of (2.36) is unique.)

For simplicity we consider $\mu_\gamma(dx)=|x|^{-\gamma}dx$ (it will be clear that for μ_γ given by (1.8) the limit is the same). Also, to simplify the notation we take $\varphi_1=\ldots=\varphi_n=\varphi$. Essentially the same argument can be carried out in the general case.

As in [Ta] and [BGT4] (the possibility to pass from space-time random variable to the present situation) we have

$$E\exp\left\{-\sum_{k=1}^{n}\langle Z_T(t_k),\varphi\rangle\right\}=\exp\left\{-\int_{\mathbb{R}^d}v_T(x,T)|x|^{-\gamma}dx\right\},\tag{3.112}$$

where v satisfies (3.4) with $\chi(t)=\sum_{k=1}^{n}1\!\!1_{[0,t_k]}(t),\chi_T(t)=\chi(t/T)$. Formula (3.112) is an analogue of (3.7), and its form is simpler since now we do not subtract the mean.

The right-hand side of (3.112) can be written as

$$\exp\left\{-\int_{\mathbb{R}^d}h_T(x,1)|x|^{-\gamma}dx\right\},$$

where

$$h_T(x,t)=T^{d/\alpha-\gamma/\alpha}v_T(xT^{1/\alpha},Tt),\ 0\leq t\leq 1.\tag{3.113}$$

To prove (3.111) it suffices to show that $h_T(\cdot,1)$ converges to $v(\cdot,1)$ in $L^1(\mathbb{R}^d,|x|^{-\gamma}dx)$; in fact, we will prove that

$$h_T \to v \quad \text{in} \quad C([0,1],L^1(\mathbb{R}^d,|x|^{-\gamma}dx)). \tag{3.114}$$

By (3.113), (3.4), (2.30) and (3.20) we have

$$\begin{aligned} h_T(x,t) &= \int_0^t \mathcal{T}_{t-s}T^{d/\alpha}\varphi(T^{1/\alpha}\cdot)(x)\chi(1-s)ds \\ &- T^{\gamma/\alpha}\int_0^t \mathcal{T}_{t-s}(\varphi(T^{1/\alpha}\cdot)h_T(\cdot,s))(x)\chi(1-s)ds \\ &- \frac{V}{1+\beta}\int_0^t \mathcal{T}_{t-s}(h_T(\cdot,s))^{1+\beta}(x)ds. \end{aligned} \tag{3.115}$$

In particular, this implies that

$$h_T(x,t) \le CT^{d/\alpha}\int_0^t \mathcal{T}_s\varphi(T^{1/\alpha}\cdot)ds. \tag{3.116}$$

Let

$$\begin{aligned} &R_T(x) \\ &= \sup_{t\le 1}\left|\int_0^t \mathcal{T}_{t-s}(T^{d/\alpha}\varphi(T^{1/\alpha}\cdot))(x)\chi(1-s)ds - \int_0^t p_{t-s}(x)\chi(1-s)ds\int_{\mathbb{R}^d}\varphi(y)dy\right|. \end{aligned} \tag{3.117}$$

(compare with the first formula on page 851 of [Ta]). We will show that

$$\lim_{T\to\infty}\int_{\mathbb{R}^d} R_T(x)|x|^{-\gamma}dx = 0. \tag{3.118}$$

Applying the usual substitutions and the fact that $p_u(x)$ is a decreasing function of $|x|$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} R_T(x)|x|^{-\gamma}dx &\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(y)\left|\int_0^1 (p_u(x-T^{-1/\alpha})-p_u(x))du\right||x|^{-\gamma}dydx \\ &\le C\int_{\mathbb{R}^d}\varphi(y)\int_{\mathbb{R}^d}|f(x-T^{-1/\alpha}y)-f(x)|dxdy \\ &+ C\left(\int_{|x|\le 1}\int_{\mathbb{R}^d}\varphi(y)|x|^{-\gamma p}dydx\right)^{1/p}\left(\int_{|x|\le 1}\int_{\mathbb{R}^d}\varphi(y)|f(x-T^{-1/\alpha}y)-f(x)|^q dydx\right)^{1/q}, \end{aligned}$$

where f is defined by (3.22), and $p,q>1$ are such that $\gamma p<d,(d-\alpha)q<d$,
$1/p+1/q=1$ (such p and q exist since $\gamma<\alpha<d$). Hence (3.118) easily follows from (3.23) and (3.24).

Next, we will show that

$$\lim_{T\to\infty}\int_{\mathbb{R}^d}\sup_{t\le 1}\left|T^{\gamma/\alpha}\int_0^t \mathcal{T}_{t-s}(\varphi(T^{1/\alpha}\cdot)\chi(1-s)h_T(\cdot,s)(x)\right||x|^{-\gamma}dx = 0. \tag{3.119}$$

(3.116) and (3.31) imply that the expression under the lim can be estimated by

$$\begin{aligned} T^{\gamma/\alpha}\int_{\mathbb{R}^d} f_\gamma(y)\varphi(T^{1/\alpha}y)(f*\widetilde{\varphi}_T)(y)dy &\leq T^{\gamma/\alpha}\|\varphi(T^{1/\alpha}\cdot)\|_{\frac{1+\beta}{\beta}}\|f\|_{1+\beta}\|\varphi\|_1 \\ &\leq CT^{\gamma/\alpha-(d/\alpha)\beta/(1+\beta)} \to 0 \end{aligned}$$

(we have used (2.30) and $f \in L^{1+\beta}$ by (3.23) and (3.24)).

To prove (3.114) we check the Cauchy condition, i.e.,

$$J(T_1,T_2) := \int_{\mathbb{R}^d} \sup_{t\leq 1} |h_{T_1}(x,t)-h_{T_2}(x,t)||x|^{-\gamma}dx \to 0 \text{ as } T_1,T_2\to\infty. \tag{3.120}$$

Using (3.115), (3.118) and (3.119) we have

$$J(T_1,T_2) \leq J_1(T_1,T_2) + \frac{V}{1+\beta}J_2(T_1,T_2), \tag{3.121}$$

where $\lim\limits_{T_1,T_2\to\infty} J_1(T_1,T_2) = 0$ and

$$J_2(T_1,T_2) = \int_{\mathbb{R}^d} \sup_{t\leq 1}\int_0^t \mathscr{T}_{t-s}|h_{T_1}^{1+\beta}(\cdot,s)-h_{T_2}^{1+\beta}(\cdot,s)|(x)ds|x|^{-\gamma}dx.$$

By (3.31), the inequality $|a^{1+\beta}-b^{1+\beta}| \leq (1+\beta)|a-b|(a^\beta+b^\beta)$, $a,b\geq 0$, and the Hölder inequality,

$$\begin{aligned} J_2(T_1,T_2) &\leq C\int_{\mathbb{R}^d}\sup_{t\leq 1}|h_{T_1}^{1+\beta}(y,t)-h_{T_2}^{1+\beta}(y,t)|dy \\ &\leq C_1\left\|\sup_{t\leq 1}|h_{T_1}(\cdot,t)-h_{T_2}(\cdot,t)|\right\|_{1+\beta}\left(\left\|\sup_{t\leq 1}|h_{T_1}(\cdot,t)|\right\|_{1+\beta}^\beta + \left\|\sup_{t\leq 1}|h_{T_2}(\cdot,t)|\right\|_{1+\beta}^\beta\right) \end{aligned}$$

Using (3.116) and the fact that $f\in L^{1+\beta}$ it is easily seen that

$$\sup_{T\geq 1}\left\|\sup_{t\leq 1} h_T(\cdot,t)\right\|_{1+\beta} < \infty.$$

To show (3.120) it suffices to prove that

$$\lim_{T_1,T_2\to\infty}\left\|\sup_{t\leq 1}|h_{T_1}(\cdot,t)-h_{T_2}(\cdot,t)|\right\|_{1+\beta} = 0. \tag{3.122}$$

This can be derived in a similar way as in [Ta] (see (2.19) and subsequent estimates therein). The only difference is that the term corresponding to $I_2(T)$ in [Ta] requires a slightly more delicate treatment; in our case it has the form

$$T^{(\gamma/\alpha)(1+\beta)}\int_{\mathbb{R}^d}\sup_{s\leq 1}\left|\int_0^s \mathscr{T}_{s-u}(\varphi(T^{1/\alpha}\cdot)h_{T_1}(\cdot,u))(x)du\right|^{1+\beta}dx.$$

Using (3.116) and the Hölder inequality this is estimated by

$$T^{(d/\alpha)(1+\beta)}||f*(\varphi(T^{1/\alpha}\cdot)(f*\widetilde{\varphi}_T))||_{1+\beta}^{1+\beta} \leq T^{(\gamma/\alpha)(1+\beta)}||f||_{1+\beta}^{1+\beta}||\varphi(T^{1/\alpha}\cdot)||_p^{1+\beta}||f||_q^{1+\beta}||\varphi||_1^{1+\beta},$$

where $q=d/(d-\alpha+\varepsilon), p=d/(\alpha-\varepsilon)$, and $\varepsilon>0$ is such that $\gamma<\alpha-\varepsilon$. Then the right-hand side is not bigger than $CT^{((1+\beta)/\alpha)(\gamma+\varepsilon-\alpha)}$, which tends to zero as $T\to\infty$.

Combining (3.120), (3.118), (3.119) and (3.122), it is seen that one can pass to the limit in (3.115) letting $T\to\infty$, thus obtaining that the limit of h_T satisfies (2.33). This proves (3.114) and completes the proof of the Proposition. □

Proof of Theorem 2.11

The proof is similar to those for the particle system, starting from an equation analogous to (3.7)-(3.8), where $\widetilde{X}_T$ is now defined for the occupation time fluctuation process (1.10) corresponding to the (d,α,β,γ) superprocess Y, and equation (3.4) is replaced by

$$v_T(x,t)=\int_0^t \mathcal{T}_{t-u}\left[\varphi_T\chi_T(T-u)-\frac{V}{1+\beta}v_T^{1+\beta}(\cdot,u)\right](x)du, \tag{3.123}$$

where the term $I_2(T)$ in equation (3.8), given by (3.10), does not appear. This reflects the fact that comparing the log-Laplace equations (1.4) for the particle system and (1.6) for the superprocess, the term $-\varphi v_\varphi$ is missing in (1.6). (Equation (3.123) can be obtained from (3.4) by the same limiting procedure that yields the superprocess from the branching particle system. An equation analogous to (3.7) for the superprocess can be derived from continuous dependence of the occupation time process with respect to the superprocess, and continuity of the mapping $C([0,\tau],\mathcal{S}'(\mathbb{R}^d)\ni x\mapsto \widetilde{x}\in\mathcal{S}'(\mathbb{R}^{d+1})$ in (1.11) [BGR].) It follows that the results for the superprocess are the same as those for the particle system, except in the cases where $I_2(T)$ has a non-zero limit, and to obtain the results in those cases it suffices to delete those non-zero limits. Therefore the limits in Theorems 2.1 and 2.5 are the same for the superprocess, and for those in Theorem 2.6, $c_\beta=0$ in all cases. □

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