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Concentration inequalities for s-concave measures of dilations of Borel sets and applications

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Abstract

We prove a sharp inequality conjectured by Bobkov on the measure of dilations of Borel sets in the Euclidean space by a s-concave probability measure. Our result gives a common generalization of an inequality of Nazarov, Sodin and Volberg and a concentration inequality of Guédon. Applying our inequality to the level sets of functions satisfying a Remez type inequality, we deduce, as it is classical, that these functions enjoy dimension free distribution inequalities and Kahane-Khintchine type inequalities with positive and negative exponent, with respect to an arbitrary s-concave probability measure.

Key words: dilation; localization lemma; Remez type inequalities; log-concave measures; large deviations; small deviations; Khintchine type inequalities; sublevel sets.

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1 Introduction

The main purpose of this paper is to establish a sharp inequality, conjectured by Bobkov in [B3], comparing the measure of a Borel set in $\mathbb{R}^n$ with a s-concave probability measure and the measure of its dilation. Among the s-concave probability measures are the log-concave ones ($s = 0$) and thus the Gaussian ones, so that it is expected that they satisfy good concentration inequalities and large and small deviations inequalities. This is indeed the case and these inequalities as well as Kahane-Khintchine type inequalities with positive and negative exponent are deduced. By using a localization theorem in the form given by Fradelizi and Guédon in [FG], we exactly determine among s-concave probability measures μ on $\mathbb{R}^n$ and among Borel sets F in $\mathbb{R}^n$, with fixed measure $\mu(F)$, what is the smallest measure of the t-dilation of F (with $t \geq 1$). This infimum is reached for a one-dimensional measure which is s-affine (see the definition below) and $F = [-1, 1]$. In other terms, it gives a uniform upper bound for the measure of the complement of the dilation of F in terms of t, s and $\mu(F)$.

The resulting inequality applies perfectly to sublevel sets of functions satisfying a Remez inequality, *i.e.* functions such that the t-dilation of any of their sublevel sets is contained in another of their sublevel set in a uniform way (see section 2.3 below). The main examples of such functions f are the seminorms ($f(x) = \|x\|_K$, where K is a centrally symmetric convex set in $\mathbb{R}^n$), the real polynomials in n-variables ($f(x) = P(x) = P(x_1, \dots, x_n)$, with $P \in \mathbb{R}[X_1, \dots, X_n]$) and more generally the seminorms of vector valued polynomials in n-variables ($f(x) = \|\sum_{j=1}^N P_j(x)e_j\|_K$, with $P_1, \dots, P_N \in \mathbb{R}[X_1, \dots, X_n]$ and $e_1, \dots, e_N \in \mathbb{R}^n$). Other examples are given in section 3. For these functions we get an upper bound for the measures of their sublevel sets in terms of the measure of other sublevel sets. This enables to deduce that they satisfy large deviation inequalities and Kahane-Khintchine type inequalities with positive exponent. But the main feature of the inequality obtained is that it may also be read backward. Thus it also implies small deviation inequalities and Kahane-Khintchine type inequalities with negative exponent.

Before going in more detailed results and historical remarks, let us fix the notations. Given non-empty subsets A, B of the Euclidean space $\mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we set $A + B = \{x + y;\ x \in A, y \in B\}$, $\lambda A = \{\lambda x;\ x \in A\}$, $\overline{A}$ to be the closure of A and $A^c = \{x \in \mathbb{R}^n;\ x \notin A\}$. For all $s \in [-\infty, 1]$, we say that a measure μ in $\mathbb{R}^n$ is s-concave if the inequality

$$\mu(\lambda A + (1-\lambda)B) \geq [\lambda \mu^s(A) + (1-\lambda)\mu^s(B)]^{1/s}$$

holds for all compact subsets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and all $\lambda \in [0, 1]$. The limit cases are interpreted by continuity, thus the right hand side of this inequality is equal to $\min(\mu(A), \mu(B))$ for $s = -\infty$ and $\mu(A)^\lambda \mu(B)^{1-\lambda}$ for $s = 0$. Notice that an s-concave measure is t-concave for all $t \in [-\infty, s]$. In particular, all these measures belong to the class of convex measures (the $-\infty$-concave measures in the terminology of Borell). For a probability measure μ, $\mathrm{supp}(\mu)$ denotes its support. For $\gamma \in [-1, +\infty]$, a function $\psi : \mathbb{R}^n \to \mathbb{R}^+$ is γ-concave if the inequality

$$\psi(\lambda x + (1-\lambda)y) \geq [\lambda \psi^\gamma(x) + (1-\lambda)\psi^\gamma(y)]^{1/\gamma} \tag{1}$$

holds for all x and y such that $\psi(x)\psi(y) > 0$ and all $\lambda \in [0, 1]$, where the limit cases $\gamma = 0$ and $\gamma = +\infty$ are also interpreted by continuity, for example the $+\infty$-concave functions are constant on their support, which is convex. The link between the s-concave probability measures and the γ-concave functions is described in the work of Borell [Bor2].

Theorem [Bor2] *Let μ be a measure in $\mathbb{R}^n$, let G be the affine hull of* $\operatorname{supp}(\mu)$*, set $d = \dim G$ and m the Lebesgue measure on G. Then for $s \in [-\infty, 1/d]$, μ is s-concave if and only if $d\mu = \psi dm$, where $0 \leq \psi \in L^1_{loc}(\mathbb{R}^n, dm)$ and ψ is γ-concave with $\gamma = s/(1-sd) \in [-1/d, +\infty]$.*

According to this theorem, we say that a measure μ is s-affine when its density ψ is γ-affine, with $\gamma = s/(1-sd)$, i.e. when ψ satisfies equality in (1), for all x and y such that $\psi(x)\psi(y) > 0$ and all $\lambda \in [0,1]$. Let us give some examples of convex measures. A Dirac measure is s-concave for any s, the uniform measure on a convex set K in $\mathbb{R}^n$ is $1/n$-concave and the Cauchy distribution on $\mathbb{R}^n$ is -1-concave, since its density

$$c_n \left(1 + \|x\|_2^2\right)^{-\frac{n+1}{2}},$$

where $\|\cdot\|_2$ denotes the Euclidean norm, is $\frac{-1}{n+1}$-concave.

In [Bor1], Borell started the study of concentration properties of s-concave probability measures. He noticed that for any centrally symmetric convex set K the inclusion $K^c \supset \frac{2}{t+1}(tK)^c + \frac{t-1}{t+1}K$ holds true. From the definition of s-concavity he deduced that for every s-concave measure μ

$$\mu(K^c) \geq \left(\frac{2}{t+1}\mu\big((tK)^c\big)^s + \frac{t-1}{t+1}\mu(K)^s\right)^{1/s}. \tag{2}$$

From this very easy but non-optimal concentration inequality, Borell showed that seminorms satisfy large deviation inequalities and Kahane-Khintchine type inequalities with positive exponent. The same method was pushed forward in 1999 by Latała [L] to deduce a small ball probability for symmetric convex sets which allowed him to get a Kahane-Khintchine inequality until the geometric mean.

In 1991, Bourgain [Bou] used the Knothe map [K] to transport sublevel sets of polynomials. He deduced that, with respect to $1/n$-concave measure on $\mathbb{R}^n$ (*i.e.* uniform measure on convex bodies), the real polynomials in n-variables satisfy some distribution and Kahane-Khintchine type inequalities with positive exponent. The same method was used by Bobkov in [B2] and recently in [B3] to generalize the result of Bourgain to s-concave measures and arbitrary functions, by using a "modulus of regularity" associated to the function. But the concentration inequalities obtained in all these results using Knothe transport map are not optimal.

In 1993, Lovász and Simonovits [LS] applied the localization method (using bisection arguments) to get the sharp inequality between the measure of a symmetric convex body K and the measure of its dilation, for a log-concave probability measure μ

$$\mu\big((tK)^c\big) \leq \mu(K^c)^{\frac{t+1}{2}}. \tag{3}$$

This improves inequality (2) of Borell in the case $s = 0$. The localization method itself was further developped in 1995 by Kannan, Lovász and Simonovits [KLS] in a form more easily applicable. In 1999, Guédon [G] applied the localization method of [LS] to generalize inequality (3) to the case of s-concave probability measures, getting thus a full extension of inequality (2). Guédon proved that if $\mu(tK) < 1$ then

$$\mu(K^c) \geq \left(\frac{2}{t+1}\mu\big((tK)^c\big)^s + \frac{t-1}{t+1}\right)^{1/s} \tag{4}$$

and deduced from it the whole range of sharp inequalities (large and small deviations and Kahane-Khintchine) for symmetric convex sets. In 2000, Bobkov [B1] used the localization in the form given in [KLS] and the result of Latała [L] to sharpen the result of Bourgain on polynomials, with log-concave measures and proved that polynomials satisfy a Kahane-Khintchine inequality until the geometric mean. In 2000 (published in 2002 [NSV1]), Nazarov, Sodin and Volberg used the same bisection method to prove a "geometric Kannan-Lovász-Simonovits lemma" for log-concave measures. They generalized inequality (3) to arbitrary Borel set

$$\mu(F_t^c) \leq \mu(F^c)^{\frac{t+1}{2}}, \tag{5}$$

where F_t^c is the complement of F_t, the t-dilation of F, which is defined by

$$F_t = F \cup \left\{ x \in \mathbb{R}^n;\ \text{there exists an interval } I \ni x \text{ s.t. } |I| < \frac{t+1}{2}|F \cap I| \right\},$$

where $|\cdot|$ denotes the (one dimensional) Lebesgue measure and $t \geq 1$. Notice that this definition of t-dilation is not the original definition of Nazarov, Sodin and Volberg [NSV1]. In the later, they introduced an auxiliary compact convex set K and used λ instead of $\frac{t+1}{2}$. The definition given above is the complement of their original one inside K. Our definition is more intrinsic because the auxiliary set has disappeared. See section 2.1 for a more detailed comparison between the two definitions and the analysis of the topological properties of the dilation.

In [NSV1], Nazarov, Sodin and Volberg also noticed that t-dilation is well suited for sublevel sets of functions satisfying a Remez type inequality and deduced from the concentration inequality (5) that these functions satisfy the whole range of sharp inequalities (large and small deviations and Kahane-Khintchine). The paper [NSV1] had a large diffusion (already as a preprint) and interested many people. For example, Carbery and Wright [CW] and Alexander Brudnyi [Br3] directly applied the localization as presented in [KLS] to deduce distributional inequalities and Kahane-Khintchine type inequalities for the norm of vector valued polynomials in n-variables and functions with bounded Chebyshev degree, respectively.

Our main result is the following theorem which extends inequality (4) of Guédon to arbitrary Borel sets (since as we shall see in section 2, if F is a centrally symmetric convex set K then $F_t = tK$) and inequality (5) of Nazarov, Sodin and Volberg to the whole range of s-concave probability measures. It establishes a conjecture of Bobkov [B3] (who also proved in [B3] a weaker inequality). After we had proven these results, we learned from Bobkov that, using a different method, Bobkov and Nazarov [BN] simultaneously and independently proved Theorem 1.

Theorem 1. *Let F be a Borel set in $\mathbb{R}^n$ and $t \geq 1$. Let $s \in (-\infty, 1]$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Let*

$$F_t = F \cup \left\{ x \in \mathbb{R}^n;\ \textit{there exists an interval } I \ni x \textit{ s.t. } |I| < \frac{t+1}{2}|F \cap I| \right\}.$$

If $\mu(F_t) < 1$ then

$$\mu(F^c) \geq \left(\frac{2}{t+1}\mu(F_t^c)^s + \frac{t-1}{t+1} \right)^{1/s}. \tag{6}$$

Inequality (6) is sharp. For example, there is equality in (6) if $n=1$, $F=[-1,1]$ and μ is of density

$$\psi(x)=\frac{(a-sx)_+^{\frac{1}{s}-1}}{(a+s)^{\frac{1}{s}}}\mathbf{1}_{[-1,+\infty)}(x),\ \text{with } a>\max(-s,st),$$

with respect to the Lebesgue measure on $\mathbb{R}$, where $a_+=\max(a,0)$, for every $a\in\mathbb{R}$. Notice that this measure μ is s-affine on its support (which is $[-1,a/s]$ if $s>0$ and $[-1,+\infty)$ if $s\leq 0$).

As noticed by Bobkov in [B3], in the case $s\leq 0$, the right hand side term in inequality (6) vanishes if $\mu(F_t^c)=0$, hence the condition $\mu(F_t)<1$ may be cancelled. But in the case $s>0$, the situation changes drastically. This condition is due to the fact that a s-concave probability measure measure, with $s>0$, has necessarily a bounded support. From this condition we directly deduce the following corollary, which was noticed by Guédon [G] in the case where F is a centrally symmetric convex set.

Corollary 1. *Let F be a Borel set in $\mathbb{R}^n$. Let $s\in(0,1]$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Denote by V the (convex compact) support of μ. Then*

$$V\subset\overline{F_t}\quad\textit{for every}\quad t\geq\frac{1+\mu(F^c)^s}{1-\mu(F^c)^s}.$$

Proof: From Theorem 1, if $\mu(F_t)<1$ then

$$\mu(F^c)\geq\left(\frac{2}{t+1}\mu(F_t^c)^s+\frac{t-1}{t+1}\right)^{1/s}>\left(\frac{t-1}{t+1}\right)^{1/s},$$

which contradicts the hypothesis on t. Hence $\mu(F_t)=1$, thus $\mu(\overline{F_t})=1$. It follows that $V\subset\overline{F_t}$.

In section 2, we study some general properties of dilation and determine its effect on examples. The case of convex sets is treated in section 2.2, the case of sublevel sets of the seminorm of a vector valued polynomial in section 2.3 and the case of sublevel sets of a Borel measurable function in section 2.4. In section 2.4, we also give a functional version of Theorem 1 and we investigate the relationship between Remez inequality and inclusion of sublevel sets. In section 3, we deduce distribution and Kahane-Khintchine inequalities for functions of bounded Thebychev degree. Section 4 is devoted to the proof of Theorem 1. The main tool for the proof is the localization theorem in the form given by Fradelizi and Guédon in [FG].

2 Properties and examples of the dilation

2.1 General properties and comparison of definitions

We first establish some basic properties of the dilation of a Borel set F,

$$F_t=F\cup\left\{x\in\mathbb{R}^n;\ \text{there exists an interval } I\ni x \text{ s.t. } |I|<\frac{t+1}{2}|F\cap I|\right\},$$

with $t \geq 1$, then we study some topological properties of the dilation, finally we compare our definition with the one given in [NSV1].

Let us start with basic properties. For any $t \geq 1$, one has $F_t \supset F$, with equality for $t = 1$. In the definition of the dilation, we may assume that the interval I has x as an endpoint, because if the end points of I are a and b and neither $[a, x]$ nor $[x, b]$ satisfies the inequality then I could not satisfy it. Moreover the t-dilation is affine invariant, *i.e.* for any affine transform $A : \mathbb{R}^n \to \mathbb{R}^n$, we have $(AF)_t = A(F_t)$. The definition of the t-dilation is one-dimensional in the sense that, if we denote by $\mathscr{D}$ the set of affine lines in $\mathbb{R}^n$, then

$$F_t = \bigcup_{D \in \mathscr{D}} (F \cap D)_t.$$

We establish now some topological properties of the dilation. For any x, y in $\mathbb{R}^n$, we denote by $\nu_{x,y}$ the Lebesgue measure on the interval $[x, y]$ normalized so that its total mass is $\|x - y\|_2$ and by φ_F, the function defined on $\mathbb{R}^2$ by

$$\varphi_F(x, y) = \nu_{x,y}(F) = |F \cap [x, y]|.$$

With these notations and the previous observation, one has

$$F_t = F \cup \left\{ x \in \mathbb{R}^n;\ \exists\ y \in \mathbb{R}^n\ \nu_{x,y}(F) > \frac{2}{t+1}\|x - y\|_2 \right\} = F \cup \Pi_1\left(\Phi_t(F)\right),$$

where $\Pi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Pi_1(x, y) = x$, for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ denotes the first projection and

$$\Phi_t(F) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n;\ \varphi_F(x, y) > \frac{2}{t+1}\|x - y\|_2 \right\}.$$

If F is open in $\mathbb{R}^n$, then $\mathbf{1}_F$ is lower semi-continuous on $\mathbb{R}^n$, hence there exists an increasing sequence of continuous functions $(f_k)_k$ on $\mathbb{R}^n$ such that $\mathbf{1}_F = \sup_k f_k$. By the monotone convergence theorem, we get

$$\varphi_F(x, y) = \int \mathbf{1}_F d\nu_{x,y} = \int \sup_k f_k d\nu_{x,y} = \sup_k \int f_k d\nu_{x,y}.$$

Since, for every continuous function f on $\mathbb{R}^n$, the function $(x, y) \mapsto \int f\, d\nu_{x,y}$ is continuous on $\mathbb{R}^{2n}$, we deduce that φ_F is lower semi-continuous on $\mathbb{R}^{2n}$. This implies that the set $\Phi_t(F)$ is open in $\mathbb{R}^{2n}$, thus its projection $\Pi_1\left(\Phi_t(F)\right)$ is open and the dilation $F_t = F \cup \Pi_1\left(\Phi_t(F)\right)$ is open in $\mathbb{R}^n$.
Notice also that for every $x \in F$, if one chooses y in a neighborhood of x so that $[x, y] \subset F$, then $\nu_{x,y}(F) = \|x - y\|_2 > \frac{2}{t+1}\|x - y\|_2$, for every $t > 1$. Hence, for an open set F and $t > 1$, the definition of F_t can be simplified to

$$F_t = \left\{ x \in \mathbb{R}^n;\ \exists\ y \in \mathbb{R}^n\ |[x, y]| < \frac{t+1}{2}|F \cap [x, y]| \right\}. \tag{7}$$

If F is closed in $\mathbb{R}^n$, then φ_F is upper semi-continuous on $\mathbb{R}^{2n}$. Writing

$$\Phi_t(F) = \bigcup_{k \geq 1} \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n;\ \varphi_F(x, y) \geq \frac{2}{t+1}\|x - y\|_2 + \frac{1}{k} \right\}$$

we see that $\Phi_t(F)$ is an F_σ set in $\mathbb{R}^{2n}$. Thus the dilation $F_t = F \cup \Pi_1\left(\Phi_t(F)\right)$ is an F_σ set in $\mathbb{R}^n$. Moreover, if $(F_k)_k$ is an increasing sequence of Borel sets in $\mathbb{R}^n$, then

$$\varphi_{\cup_k F_k} = \sup_k \varphi_{F_k} \quad \text{hence} \quad \Phi_t\left(\cup_k F_k\right) = \cup_k \Phi_t(F_k).$$

And if $(F_k)_k$ is a decreasing sequence of Borel sets in $\mathbb{R}^n$, then

$$\varphi_{\cap_k F_k} = \inf_k \varphi_{F_k} \quad \text{hence} \quad \Phi_t\left(\cap_k F_k\right) = \cap_k \Phi_t(F_k).$$

Using either transfinite induction on the Baire class of F or the monotone class theorem we deduce that for every Borel set F, φ_F is Borel measurable, hence $\Phi_t(F)$ is a Borel set in $\mathbb{R}^{2n}$ thus $F_t = F \cup \Pi_1\left(\Phi_t(F)\right)$ is analytic, therefore Lebesgue measurable in $\mathbb{R}^n$.

In dimension 1, we have more regularity. For any Lebesgue measurable set $F \subset \mathbb{R}$, the function φ_F is continuous on $\mathbb{R}^2$, hence the set $\Phi_t(F)$ is open in $\mathbb{R}^2$, thus its projection $\Pi_1\left(\Phi_t(F)\right)$ is open and its dilation $F_t = F \cup \Pi_1\left(\Phi_t(F)\right)$ is Lebesgue measurable in $\mathbb{R}$.

Now we compare the definition of the dilation as given above with the original definition of [NSV1]. Given a Borel subset A of a (Borel) convex set K in R^n and a number $\lambda > 1$, they define

$$A(\lambda) = \left\{ x \in A;\ |A \cap I| \geq \left(1 - \frac{1}{\lambda}\right)|I| \text{ for any interval } I \text{ s.t. } x \in I \subset K \right\}$$

and they prove that for any log-concave measure μ supported in K one has $\mu(A(\lambda)) \leq \mu(A)^\lambda$. The relationship with our definition is the following: if we define $F = K \setminus A$ then $A(\lambda) = K \setminus F_{2\lambda-1}$. Let us establish this relationship. For every interval $I \subset K$, one has $|I| = |A \cap I| + |F \cap I|$ hence

$$A(\lambda) = \left\{ x \in A;\ |I| \geq \lambda |F \cap I| \text{ for any interval } I \text{ s.t. } x \in I \subset K \right\}.$$

Let $x \in A(\lambda)$ and $I \ni x$ be any interval, then $J := I \cap K$ is an interval such that $x \in J \subset K$, hence $|I| \geq |J| \geq \lambda |F \cap J| = \lambda |F \cap I|$, since $F \subset K$. Thus

$$A(\lambda) = \left\{ x \in A;\ |I| \geq \lambda |F \cap I| \text{ for any interval } I \text{ s.t. } x \in I \right\}.$$

On the other hand, with our definition of dilation, one gets

$$\begin{aligned} K \setminus F_{2\lambda-1} &= \left\{ x \in K;\ x \notin F \text{ and } |I| \geq \lambda |F \cap I| \text{ for any interval } I \ni x \right\} \\ &= \left\{ x \in A;\ |I| \geq \lambda |F \cap I| \text{ for any interval } I \text{ s.t. } x \in I \right\} \\ &= A(\lambda). \end{aligned}$$

Since μ is supported on K, one deduces that $\mu(A(\lambda)) = \mu(F^c_{2\lambda-1})$, hence the statement of Theorem 1 may equivalently be stated in terms of $A(\lambda)$. As said before, we prefer our definition because it is more intrinsic, the dilation of a set doesn't depend of any auxilliary set. For example, as seen below, with our definition, the dilation of an (open) symmetric convex set F is $F_t = tF$. But, with the definition of [NSV1], the corresponding relation is the following: if one has $A \subset K$, with K convex and $K \setminus A$ open and convex then $A(\lambda) = K \setminus (2\lambda - 1)A$.

2.2 Dilation of convex sets

Fact 1. *Let K be an open convex set then, for every $t \geq 1$,*

$$K_t = K + \frac{t-1}{2}(K-K) = \frac{t+1}{2}K + \frac{t-1}{2}(-K) \tag{8}$$

and if moreover K is centrally symmetric then $K_t = tK$.

Proof: For $t = 1$, the equalities are obvious, so we assume $t > 1$. The second equality in (8) deduces from the convexity of K. To prove the equality of the sets in (8), we prove both inclusions:

Let $x \in K_t$. Since K is open and $t > 1$, from (7), there exists a point $a \in \mathbb{R}^n$ such that $|[a,x]| < \frac{t+1}{2}|K \cap [a,x]|$. Since K is convex it follows that $K \cap [a,x]$ is an interval. Denote by b and c its endpoints. We may assume that $c \in (b,x]$ and $b \in [a,c)$. Hence there is $\lambda \in (0,1]$ such that $c = (1-\lambda)b + \lambda x$. This gives

$$\frac{\|c-b\|_2}{\lambda} = \|x-b\|_2 \leq \|x-a\|_2 < \frac{t+1}{2}\|c-b\|_2 \,.$$

Thus $\frac{1}{\lambda} < \frac{t+1}{2}$. Therefore

$$x = c + \left(\frac{1}{\lambda} - 1\right)(c-b) \in K + \left(\frac{1}{\lambda} - 1\right)(K-K) \subset K + \frac{t-1}{2}(K-K) \,.$$

Conversely, let $x \in \frac{t+1}{2}K + \frac{t-1}{2}(-K)$. If $x \in K$, the result is obvious so we assume that $x \notin K$. There exists $b, c \in K$ such that $x = \frac{t+1}{2}c + \frac{t-1}{2}(-b)$. Since K is convex we deduce that the set $[b,x] \cap K$ is an interval with b as an endpoint. Since K is open there exists $d \in \mathbb{R}^n$ such that $[b,x] \cap K = [b,d)$ and we have $c \in [b,d)$. Then

$$|[b,x]| = \|x-b\|_2 = \frac{t+1}{2}\|c-b\|_2 < \frac{t+1}{2}\|d-b\|_2 = \frac{t+1}{2}\,|K \cap [b,x]| \,.$$

Therefore $x \in K_t$.

If moreover K is centrally symmetric it is obvious that $K_t = tK$. □

Remarks:

1) It is not difficult to see that if we only assume that K is convex (and not necessarily open) then the same proof shows actually that

$$K_t = \mathrm{relint}\left(K + \frac{t-1}{2}(K-K)\right) = \mathrm{relint}\left(\frac{t+1}{2}K + \frac{t-1}{2}(-K)\right),$$

where relint(A) is the relative interior of A, *i.e.* the interior of A relative to its affine hull.

2) The family of convex sets described by (8) where introduced by Hammer [H], they may be equivalently defined in the following way. Let us recall that the support function of a convex set K in the direction $u \in S^{n-1}$ is defined by $h_K(u) = \sup_{x\in K}\langle x,u\rangle$ and that an open convex set K is equal to the intersection of the open slabs containing it:

$$K = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n;\ -h_K(-u) < \langle x,u\rangle < h_K(u)\} \,.$$

The width of K in direction $u \in S^{n-1}$ is defined by $w_K(u) = h_K(u) + h_K(-u)$. Then for every $t \geq 1$,

$$K_t = \bigcap_{u \in S^{n-1}} \left\{ x;\ -h_K(-u) - \frac{t-1}{2} w_K(u) < \langle x, u \rangle < h_K(u) + \frac{t-1}{2} w_K(u) \right\}.$$

Moreover, since this definition can be extended to the values $t \in (0,1)$, it enables thus to define the t-dilation of an open convex set for $0 < t < 1$ and in the symmetric case, the equality $K_t = tK$ is still valid for $t \in (0,1)$. Using that the family of convex sets $(K_t)_{t>0}$ is absorbing, Minkowski defined what is now called the "generalized Minkowski functional" of K:

$$\alpha_K(x) = \inf\{t > 0;\ x \in K_t\}$$

Notice that α_K is convex and positively homogeneous. If moreover K is centrally symmetric then $K_t = tK$, which gives $\alpha_K(x) = \|x\|_K$. We shall see in the next section how this notion was successfully used in polynomial approximation theory (see for example [RS]).

From Fact 1, Theorem 1 and Corollary 1, we deduce the following corollary.

Corollary 2. *Let K be a closed convex set in $\mathbb{R}^n$ and $t \geq 1$. Let $s \in (-\infty, 1]$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Denote by V the support of μ.*

i) If $\mu\big(K + \frac{t-1}{2}(K-K)\big) < 1$ then

$$\mu(K^c) \geq \left(\frac{2}{t+1} \mu\Big(\big(K + \frac{t-1}{2}(K-K)\big)^c \Big)^s + \frac{t-1}{t+1} \right)^{1/s}.$$

ii) If $s > 0$ then

$$V \subset K + \frac{\mu(K^c)^s}{1 - \mu(K^c)^s}(K-K)$$

iii) If $s > 0$ and K is centrally symmetric then

$$V \subset \frac{1 + \mu(K^c)^s}{1 - \mu(K^c)^s} K$$

Applying $iii)$ to the uniform probability measure on V we deduce that for every closed convex sets V and K in $\mathbb{R}^n$, with K symmetric

$$V \subset \frac{|V|^{\frac{1}{n}} + |V \cap K^c|^{\frac{1}{n}}}{|V|^{\frac{1}{n}} - |V \cap K^c|^{\frac{1}{n}}} K.$$

2.3 Dilation of sublevel sets of the seminorm of vector valued polynomials

Let P be a polynomial of degree d, with n variables and with values in a Banach space E, that is

$$P(x_1, \ldots, x_n) = \sum_{k=1}^{N} P_k(x_1, \ldots, x_n) e_k\ ,$$

where $e_1, ..., e_N \in E$ and $P_1, ..., P_N$ are real polynomials with n variables and degree at most d. Let K be a centrally symmetric convex set in E, and denote by $\|\cdot\|_K$ the seminorm defined by K in E and let $c > 0$ be any constant. The following fact was noticed and used by Nazarov, Sodin and Volberg in [NSV1], in the case of real polynomials.

Fact 2. *Let P be a polynomial of degree d, with n variables and with values in a Banach space E and let $t \geq 1$. Let K be a centrally symmetric convex set in E and $c > 0$. Then*

$$\{x \in \mathbb{R}^n;\ \|P(x)\|_K < c\}_t \subset \{x \in \mathbb{R}^n;\ \|P(x)\|_K < c\, T_d(t)\},$$

where T_d is the Chebyshev polynomial of degree d, i.e.

$$T_d(t) = \frac{\left(t + \sqrt{t^2 - 1}\right)^d + \left(t - \sqrt{t^2 - 1}\right)^d}{2},$$

for every $t \in \mathbb{R}$ such that $|t| \geq 1$.

This fact is actually a reformulation, in terms of dilation, of the Remez inequality [R] which asserts that for every real polynomial Q of degree d and one variable, for every interval I in $\mathbb{R}$ and every Borel subset J of I,

$$\sup_I |Q| \leq T_d\left(2\frac{|I|}{|J|} - 1\right) \sup_J |Q|.$$

Let us prove the inclusion. Let $F = \{x \in \mathbb{R}^n;\ \|P(x)\|_K < c\}$ and let $x_0 \in F_t$. Notice that F is open. There exists an interval $I = [a, b]$ containing x_0 such that $|I| < \frac{t+1}{2}|F \cap I|$. The key point is that

$$\|P\big((1-\lambda)a + \lambda b\big)\|_K = \sup_{\xi \in K^*} \xi\Big(P\big((1-\lambda)a + \lambda b\big)\Big) = \sup_{\xi \in K^*} Q_\xi(\lambda),$$

where $K^* = \{\xi \in E^*; \forall\, x \in K, \xi(x) \leq 1\}$ is the polar of K and

$$Q_\xi(\lambda) = \xi\Big(P\big((1-\lambda)a + \lambda b\big)\Big)$$

is a real polynomial of one variable and degree at most d. Let $J := \{\lambda \in [0,1];\ (1-\lambda)a + \lambda b \in F\}$, then $|J| = |F \cap I|/|I|$. Applying Remez inequality to Q_ξ we have

$$\sup_{\lambda \in [0,1]} Q_\xi(\lambda) \leq T_d\left(\frac{2}{|J|} - 1\right) \sup_{\lambda \in J} |Q_\xi(\lambda)| = T_d\left(\frac{2|I|}{|F \cap I|} - 1\right) \sup_{x \in F \cap I} |\xi\big(P(x)\big)|.$$

Taking the supremum, using that T_d is increasing on $[1, +\infty)$ and the definition of F, we get

$$\begin{aligned}
\|P(x_0)\|_K &\leq \sup_{[0,1]} \|P\big((1-\lambda)a + \lambda b\big)\|_K = \sup_{[0,1]} \sup_{\xi \in K^*} Q_\xi(\lambda) \\
&\leq T_d\left(\frac{2|I|}{|F \cap I|} - 1\right) \sup_{\xi \in K^*} \sup_{x \in F \cap I} \xi\big(P(x)\big) \\
&< T_d(t) \sup_{x \in F \cap I} \|P(x)\|_K \leq c T_d(t).
\end{aligned}$$

Remark: Notice that the Chebyshev polynomial of degree one is $T_1(t) = t$. Hence if we take the polynomial $P(x) = x = \sum x_i e_i$, where $(e_1, ..., e_n)$ is the canonical orthonormal basis of $\mathbb{R}^n$, we see that the case of vector valued polynomials generalizes the case of symmetric convex sets.

Fact 2 has an interesting reformulation in terms of polynomial inequalities in real approximation theory. It may be written in the following way. Denote by $\mathscr{P}_d^n(E)$ the set of polynomials of degree d, with n variables and with values in a Banach space E. Let $P \in \mathscr{P}_d^n(E)$ and K be a symmetric convex set in E. Let F be a Borel set in $\mathbb{R}^n$ and $t > 1$. For $x \in F_t$

$$\|P(x)\|_K \leq T_d(t) \sup_{z \in F} \|P(z)\|_K.$$

Let us assume that the Borel set F in $\mathbb{R}^n$ has the property that, for each x in R^n, there is an affine line D containing x such that $|F \cap D| > 0$, which is the case if F has non-empty interior. Then $\bigcup_{t>1} F_t = \mathbb{R}^n$. In this case, we may define for every $x \in \mathbb{R}^n$ the "generalized Minkowski functional" of F at x as

$$\alpha_F(x) = \inf\{t > 1;\ x \in F_t\}.$$

Using this quantity, we get the following reformulation of Fact 2.

Corollary 3. *Let F be a Borel set in $\mathbb{R}^n$. Let $P \in \mathscr{P}_d^n(E)$ and K be a centrally symmetric convex set in E. For every x in $\mathbb{R}^n$,*

$$\|P(x)\|_K \leq T_d\left(\alpha_F(x)\right) \sup_{z \in F} \|P(z)\|_K.$$

Let us introduce the notations coming from approximation theory. With the notations of the corollary, we define

$$C_d(F, x, K) = \sup\{\|P(x)\|_K; P \in \mathscr{P}_d^n(E), \sup_{x \in F} \|P(x)\|_K \leq 1, n \geq 1\}.$$

Then the inequality may be written in the following form.

$$C_d(F, x, K) = T_d\left(\alpha_F(x)\right).$$

For F being convex and the polynomial P being real valued, this is a theorem of Rivlin-Shapiro [RS] (see also an extension in [RSa1] and [RSa2]). We get thus an extension of their theorem to non-convex sets F.

Applying Theorem 1 to the level set of a polynomial we get the following corollary, which was proved in the case $s = 0$ by Nazarov, Sodin and Volberg in [NSV1] and in the case $d = 1$ and $P(x) = x$ by Guédon in [G].

Corollary 4. *Let P be a polynomial of degree d, with n variables and with values in a Banach space E and let $t \geq 1$. Let K be a centrally symmetric convex set in E and $c > 0$. Let $s \leq 1$ and μ be a s-concave probability measure on $\mathbb{R}^n$. If $\mu(\{x;\ \|P(x)\|_K \geq cT_d(t)\}) > 0$, then*

$$\mu(\{x;\ \|P(x)\|_K \geq c\}) \geq \left(\frac{2}{t+1}\mu(\{x;\ \|P(x)\|_K \geq cT_d(t)\})^s + \frac{t-1}{t+1}\right)^{1/s}.$$

For $s = 0$,

$$\mu(\{x;\ \|P(x)\|_K \geq cT_d(t)\}) \leq \mu(\{x;\ \|P(x)\|_K \geq c\})^{\frac{t+1}{2}}.$$

Applying Corollary 1, we get the following extension of a theorem of Brudnyi and Ganzburg [BG] (which treats the case of probability measures μ which are uniform on a convex body). It is a multi-dimensional version of Remez inequality.

Corollary 5. *Let P be a polynomial of degree d, with n variables and with values in a Banach space E. Let K be a centrally symmetric convex set in E. Let $s \in (0,1]$, μ be a s-concave probability measure on $\mathbb{R}^n$ and let V be the support of μ. Then, for every measurable $\omega \subset V$*

$$\sup_{x\in V}\|P(x)\|_K \le T_d\left(\frac{1+\mu(\omega^c)^s}{1-\mu(\omega^c)^s}\right)\sup_{x\in\omega}\|P(x)\|_K \le \left(\frac{4}{s\mu(\omega)}\right)^d \sup_{x\in\omega}\|P(x)\|_K.$$

Proof: We apply Corollary 1 to $F=\{x;\ \|P(x)\|_K \le \sup_{x\in\omega}\|P(x)\|_K\}$ and Fact 2 to deduce that

$$V \subset \overline{F_t} \subset \{x\in\mathbb{R}^n;\ \|P(x)\|_K \le T_d(t)\sup_{x\in\omega}\|P(x)\|_K\},\quad \forall\ t \ge \frac{1+\mu(F^c)^s}{1-\mu(F^c)^s}.$$

Since $\omega \subset F$, we may apply the preceding inclusion to $t=\frac{1+\mu(\omega^c)^s}{1-\mu(\omega^c)^s}$ and this gives the first inequality. The second one follows using that $T_d(t)\le(2t)^d$ for every $t\ge 1$ and easy computations.

2.4 Dilation of sublevel sets of Borel measurable functions

In Fact 1 and Fact 2 we saw the effect of dilation on convex sets and level sets of vector valued polynomials. We want to describe now the most general case of level sets of Borel measurable functions. As in Fact 2, we shall see in the following proposition that for any Borel measurable function, an inclusion between the dilation of the level sets is equivalent to a Remez type inequality.

Proposition 1. *Let $f:\ \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function and $t>1$. Let $u_f(t)\in[1,+\infty)$. The following are equivalent.*
i) For every interval I in $\mathbb{R}^n$ and every Borel subset J of I such that $|I|<t|J|$,

$$\sup_I |f| \le u_f(t)\sup_J |f|.$$

ii) For every $\lambda>0$,

$$\{x\in\mathbb{R}^n;\ |f(x)|\le\lambda\}_{2t-1} \subset \left\{x\in\mathbb{R}^n;\ |f(x)|\le \lambda u_f(t)\right\}.$$

We shall say that a non-decreasing function $u_f:\ (1,+\infty)\to[1,+\infty)$ is *a Remez function* of f if it satisfies i) or ii) of the previous proposition, for every $t>1$ and that it is *the optimal Remez function* of f if it is the smallest Remez function of f.

For example, using *i)*, the Remez inequality asserts that if we take $f(x)=\|P(x)\|_K$ where P is a polynomial of degree d, with n variables and with values in a Banach space E and K is a symmetric convex set then $t\mapsto T_d(2t-1)$ is a Remez function of f. Using *ii)* and Fact 1, we get that $u_f(t)=2t-1$ is the optimal Remez function of $f(x)=\|x\|_K$.

Proof of Proposition 1:
$i)\Longrightarrow ii)$: Let $F=\{x\in\mathbb{R}^n;\ |f(x)|\le\lambda\}$ and let $x\in F_{2t-1}$. There exists an interval I containing x such that $|I|<t|F\cap I|$. Hence

$$|f(x)| \le \sup_I |f| \le u_f(t)\sup_{F\cap I}|f| \le \lambda u_f(t).$$

$ii) \Longrightarrow i)$: Let I be an interval in $\mathbb{R}^n$ and J be a Borel subset of I such that $|I| < t|J|$. Let $\lambda = \sup_J |f|$ and let $x \in I$, then $J \subset \{|f| \le \lambda\} \cap I$ hence

$$|I| < t|J| \le t|\{|f| \le \lambda\} \cap I|,$$

thus $x \in \{|f| \le \lambda\}_{2t-1}$. From $ii)$ we get $|f(x)| \le \lambda u_f(t)$. This gives $i)$. $\square$

Applying Theorem 1 to the level set of a Borel measurable function, we get the following.

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function and $u_f : (1, +\infty) \to [1, +\infty)$ be a Remez function of f. Let $s \in (-\infty, 1]$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Let $t > 1$ and $\lambda > 0$. If $\mu(\{x;\ |f(x)| \ge \lambda u_f(t)\}) > 0$, then*

$$\mu(\{x;\ |f(x)| > \lambda\}) \ge \left(\frac{1}{t}\mu(\{x;\ |f(x)| > \lambda u_f(t)\})^s + 1 - \frac{1}{t}\right)^{1/s}. \tag{9}$$

For $s = 0$,

$$\mu(\{x;\ |f(x)| > \lambda u_f(t)\}) \le \mu\left(\{x;\ |f(x)| > \lambda\}\right)^t.$$

Remark: Theorem 2 improves a theorem given by Bobkov in [B3]. As in [B3], notice that Theorem 2 is a functional version of Theorem 1. As a matter of fact, we may follow the proof given by Bobkov. If a Borel subset F of $\mathbb{R}^n$ and $u > 1$ are given, we apply Theorem 2 to $t = \frac{u+1}{2}$, $\lambda = 1$ and

$$f = 1 \quad \text{on } F, \quad f = 2 \quad \text{on } F_u \setminus F \quad \text{and} \quad f = 4 \quad \text{on } F_u^c.$$

Using ii) of Proposition 1 it is not difficult to see that $u_f(t) = 2$. Then inequality (6) follows from inequality (9).

Applying Corollary 1, in the similar way as in Corollary 5 and using Proposition 1 instead of Fact 2, we get the following.

Corollary 6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function. Let $u_f : (1, +\infty) \to [1, +\infty)$ be a Remez function of f. Let $s \in (0, 1]$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Let $\omega \subset \mathbb{R}^n$ be measurable, then*

$$\|f\|_{L^\infty(\mu)} \le \sup_\omega |f|\, u_f\left(\frac{1}{1 - \mu(\omega^c)^s}\right) \le \sup_\omega |f|\, u_f\left(\frac{1}{s\mu(\omega)}\right).$$

Instead of using u_f, Bobkov in [B2] and [B3] introduced a related quantity, the "modulus of regularity" of f,

$$\delta_f(\varepsilon) = \sup_{x,y} |\{\lambda \in [0,1];\ |f((1-\lambda)x + \lambda y)| \le \varepsilon |f(x)|\}|, \quad \text{for } 0 < \varepsilon \le 1.$$

It is not difficult to see that

$$\delta_f(\varepsilon) = \sup_{x,y} \frac{|\{z \in [x,y];\ |f(z)| \le \varepsilon \sup_{[x,y]} |f|\}|}{|[x,y]|}$$

and thus

$$\delta_f(\varepsilon) = \sup\left\{\frac{|J|}{|I|};\ J \subset I, \text{where } I \text{ is an interval and} \sup_J |f| \le \varepsilon \sup_I |f|\right\}.$$

Hence δ_f is the smallest function satisfying that for every interval I and every Borel subset J of I

$$\frac{|J|}{|I|} \le \delta_f\left(\frac{\sup_J |f|}{\sup_I |f|}\right),$$

which is a Remez-type inequality. For smooth enough functions, the relationship between u_f, the optimal Remez function of f and δ_f is given by

$$\delta_f(\varepsilon) = \frac{1}{u_f^{-1}(1/\varepsilon)},$$

where u_f^{-1} is the reciprocal function of u_f. Hence if $f(x) = \|P(x)\|_K$ where P is a polynomial of degree d, with n variables and with values in a Banach space E and K is a symmetric convex set then, using that $u_f(t) \le T_d(2t-1)$ and $T_d(t) \le 2^{d-1}t^d$, for every $|t| \ge 1$, we get

$$u_f(t) \le T_d(2t-1) \le 2^{d-1}(2t-1)^d \le (4t)^d$$

and

$$\delta_f(\varepsilon) \le \frac{2}{T_d^{-1}(1/\varepsilon)+1} \le 4\left(\frac{\varepsilon}{2}\right)^{1/d} \le 4\varepsilon^{1/d}, \tag{10}$$

for every $|t| \ge 1$. For $f(x) = \|x\|_K$, we get $\delta_f(\varepsilon) = \frac{2\varepsilon}{\varepsilon+1}$ as noticed by Bobkov in [B2]. Notice that inequalities (10) improve the previous bound given by Bobkov in [B2] and [B3].

The interest of the quantity δ_f comes from the next corollary, which was conjectured by Bobkov in [B3] (for $s = 0$, it deduces from [NSV1] as noticed in [B2]).

Corollary 7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function and $0 < \varepsilon \le 1$. Let $s \le 1$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Let $\lambda < \|f\|_{L^\infty(\mu)}$, then*

$$\mu(\{|f| \ge \lambda\varepsilon\}) \ge \left(\delta_f(\varepsilon)\mu(\{|f| \ge \lambda\})^s + 1 - \delta_f(\varepsilon)\right)^{1/s} \tag{11}$$

and if μ is log-*concave (*i.e. *for $s = 0$) then*

$$\mu(\{|f| \ge \lambda\varepsilon\}) \ge \mu(\{|f| \ge \lambda\})^{\delta_f(\varepsilon)}.$$

Proof: We apply Theorem 1 to the set $F = \{|f| < \lambda\varepsilon\}$ and $t = \frac{2}{\delta_f(\varepsilon)} - 1$. Let $x \in F_t$, there exists an interval I containing x such that

$$|I| < \frac{t+1}{2}|F \cap I| = \frac{|F \cap I|}{\delta_f(\varepsilon)}.$$

From the definition of δ_f, this implies that

$$f(x) \le \sup_I |f| < \frac{1}{\varepsilon}\sup_{F\cap I} |f| \le \lambda.$$

Hence $F_t \subset \{|f| < \lambda\}$. This gives the result. □

3 Distribution and Kahane-Khintchine type inequalities

It is classical that from an inequality like inequality (9) (or in its equivalent form (11)), it is possible to deduce distribution and Kahane-Khintchine type inequalities. Due to its particular form, this type of concentration inequality may be read forward or backward and thus permits to deduce both small and large deviations inequalities.

3.1 Functions with bounded Chebyshev degree

Before stating these inequalities, let us define an interesting set of functions, the functions f such that their Remez function u_f is bounded from above by a power function, *i.e.* there exists $A > 0$ and $d > 0$ satisfying $u_f(t) \le (At)^d$, for every $t > 1$ which means that for every interval I in $\mathbb{R}^n$ and every Borel subset J of I

$$\sup_I |f| \le \left(\frac{A|I|}{|J|}\right)^d \sup_J |f|.$$

In this case, the smallest power satisfying this inequality is called *the Chebyshev degree* of f and denoted by d_f and the best constant corresponding to this degree is denoted by A_f. This is also equivalent to assume that $\delta_f(\varepsilon) \le A_f \varepsilon^{1/d_f}$, for every $0 < \varepsilon < 1$. Notice that if f has bounded Chebyshev degree (*i.e.* $d_f < +\infty$) then $|f|^{1/d_f}$ has Chebyshev degree one and $A_{|f|^{1/d_f}} = A_f$. For such functions inequality (9) becomes, for every $t > 1$,

$$\mu(\{|f|^{1/d_f} > \lambda\}) \ge \left(\frac{1}{t}\mu(\{x;\ |f(x)|^{1/d_f} > \lambda A_f t\})^s + 1 - \frac{1}{t}\right)^{1/s} \tag{12}$$

and for $s = 0$

$$\mu(\{x;\ |f(x)|^{1/d_f} > \lambda A_f t\}) \le \mu(\{|f|^{1/d_f} > \lambda\})^t. \tag{13}$$

For example if $f(x) = \|x\|_K$ then $u_f(t) = 2t - 1$ hence $d_f = 1$ and $A_f = 2$. If $f(x) = \|P(x)\|_K$ where P is a polynomial of degree d, with n variables and with values in a Banach space E and K is a symmetric convex set then $d_f = d$ and $A_f = 4$. More generally, following [NSV1] and [CW], if $f = e^u$, where $u : \mathbb{R}^n \to \mathbb{R}$ is the restriction to $\mathbb{R}^n$ of a plurisubharmonic function $\tilde{u} : \mathbb{C}^n \to \mathbb{R}$ such that $\limsup_{|z| \to +\infty} \frac{\tilde{u}(z)}{\log |z|} \le 1$, then $d_f = 1$ and $A_f = 4$. Another type of example was given by Nazarov, Sodin and Volberg in [NSV1]: if

$$f(x) = \sum_{k=1}^{d} c_k e^{i\langle x_k, x\rangle},$$

with $c_k \in \mathbb{C}$ and $x_k \in \mathbb{R}^n$ then $d_f = d$. Finally, Alexander Brudnyi in [Br1], [Br2], [Br3] (see also Nazarov, Sodin and Volberg [NSV2]) proved also that for any $r > 1$, for any holomorphic function f on $B_{\mathbb{C}}(0, r) \subset \mathbb{C}^n$, the open complex Euclidean of radius r centered at 0, the Chebyshev degree of f is bounded.

3.2 Small deviations and Kahane-Khintchine type inequalities for negative exponent

Let us start with the following small deviation inequality, which was proved by Guédon [G] in the case where $f = \|\cdot\|_K$ and by Nazarov, Sodin and Volberg [NSV1] in the case where $s = 0$. It was proved in a weaker form and conjectured in this form by Bobkov in [B3]. This type of inequality is connected to small ball probabilities.

Corollary 8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function and $0 < \varepsilon \leq 1$. Let $s \leq 1$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Let $\lambda < \|f\|_{L^\infty(\mu)}$, then*

$$\mu(\{|f| \leq \lambda\varepsilon\}) \leq \delta_f(\varepsilon) \times \frac{1-\mu(\{|f| \geq \lambda\})^s}{s}. \tag{14}$$

In particular, if μ is log-*concave (*i.e. *for $s=0$) then*

$$\mu(\{|f| \leq \lambda\varepsilon\}) \leq \delta_f(\varepsilon) \times \log\left(1/\mu(\{|f| \geq \lambda\})\right).$$

Proof: Let $s \neq 0$. The proof given by Guédon in [G] works here also. We reproduce it here for completeness. Since $s \leq 1$ the function $x \mapsto (1-x)^{1/s}$ is convex on $(-\infty, 1]$, hence

$$(1-x)^{1/s} \geq 1 - \frac{x}{s}.$$

The result follows from inequality (11) and the inequality above applied to $x = \delta_f(\varepsilon)(1-\mu(\{|f| \geq \lambda\})^s)$. For $s = 0$ the result follows by taking limits or can be proved along the same lines. □

In the case of functions with bounded Chebyshev degree, inequality (14) take a simpler form and, by integrating on level sets, it immediately gives an inverse Hölder Kahane-Khintchine type inequality for negative exponent. Thus, we get the following corollary, generalizing a theorem of Guédon [G] (for $f = \|.\|_K$) and Nazarov, Sodin and Volberg [NSV1] (for $s = 0$).

Corollary 9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function with bounded Chebyshev degree. Let $s \leq 1$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Denote by M_f the μ-median of $|f|^{1/d_f}$ and denote $c_s := (1-2^{-s})/s$, for $s \neq 0$ and $c_0 = \ln 2$. Then for every $0 < \varepsilon < 1$*

$$\mu(\{|f|^{\frac{1}{d_f}} \leq M_f\,\varepsilon\}) \leq A_f c_s \varepsilon, \tag{15}$$

and for every $-1 < q < 0$,

$$\||f|^{\frac{1}{d_f}}\|_{L^q(\mu)} \geq M_f \left(1 - \frac{qA_f c_s}{q+1}\right)^{1/q} \geq M_f e^{-\frac{A_f c_s}{q+1}}. \tag{16}$$

Proof: Inequality (15) deduces from inequality (14) by taking $\lambda = M_f$. The proof of inequality (16) is then standard, we apply inequality (15)

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^{\frac{q}{d_f}} d\mu(x) &= -q\int_0^{+\infty} t^{q-1}\mu\left(\{x;\ |f(x)|^{\frac{1}{d_f}} \leq t\}\right)dt \\
&\leq -q\int_0^{M_f} t^q \frac{A_f c_s}{M_f} dt - q\int_{M_f}^{+\infty} t^{q-1} dt \\
&= M_f^q\left(1 - \frac{qA_f c_s}{q+1}\right).
\end{aligned}$$

Then we take the q-th root (recall that $q < 0$) to get inequality (16).

3.3 Large deviations and Kahane-Khintchine type inequalities for positive exponent

On the contrary to the small deviations case, the behaviour of large deviations of a function with bounded Chebyshev degree with respect to a s-concave probability measure heavily depends on the range of s, mainly on the sign of s. But all behaviours follow from inequality (12) applied to $\lambda = M_f$, the μ-median of $|f|^{1/d_f}$, which gives, for every $s \leq 1$, $s \neq 0$,

$$\mu(\{|f|^{\frac{1}{d_f}} \geq A_f M_f t\}) \leq \left(1 - t(1-2^{-s})\right)_+^{\frac{1}{s}} \tag{17}$$

and for $s = 0$,

$$\mu(\{|f|^{\frac{1}{d_f}} \geq A_f M_f t\}) \leq 2^{-t}.$$

For $s \geq 0$, it follows from inequality (17) that $|f|^{1/d_f}$ has exponentially decreasing tails and a standard argument implies an inverse Hölder inequality.

Corollary 10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function with bounded Chebyshev degree. Let $0 \leq s \leq 1$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Denote by M_f the μ-median of $|f|^{1/d_f}$ and denote $c_s := (1-2^{-s})/s$, for $s > 0$ and $c_0 = \ln 2$. Then for every $t > 1$*

$$\mu(\{|f|^{\frac{1}{d_f}} \geq A_f M_f t\}) \leq \left(1 - sc_s t\right)_+^{\frac{1}{s}} \leq e^{-c_s t} \leq e^{-\frac{t}{2}} \tag{18}$$

and for every $p > 0$,

$$\||f|^{\frac{1}{d_f}}\|_{L^p(\mu)} \leq A_f M_f \left(1 + \frac{pB(p, 1+\frac{1}{s})}{(sc_s)^p}\right)^{\frac{1}{p}} \leq A_f M_f \left(1 + 2^p \Gamma(p+1)\right)^{\frac{1}{p}}. \tag{19}$$

Proof: Inequality (18) deduces from inequality (17). The proof of inequality (19) is then standard, we write

$$\int_{\mathbb{R}^n} |f(x)|^{\frac{p}{d_f}} d\mu(x) = p \int_0^{+\infty} t^{p-1} \mu\left(\{x;\ |f(x)|^{\frac{1}{d_f}} \geq t\}\right) dt$$

and we apply inequality (18) as in the proof of Corollary 9. □

For $s < 0$ the situation changes drastically, inequality (17) only implies that the tail of $|f|^{1/d_f}$ decreases as $t^{1/s}$, which is the sharp behaviour if we take the example of measure μ on $\mathbb{R}$ given after Theorem 1 and $f(x) = |x|$.

Corollary 11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function with bounded Chebyshev degree. Let $s \leq 0$ and μ be a s-concave probability measure on $\mathbb{R}^n$. Denote by M_f the μ-median of $|f|^{1/d_f}$ and denote $d_s := (2^{-s}-1)^{1/s}$. Then for every $t > 1$*

$$\mu(\{|f|^{1/d_f} \geq A_f M_f t\}) \leq t^{\frac{1}{s}} \left(2^{-s} - 1 + \frac{1}{t}\right)^{\frac{1}{s}} \leq d_s t^{\frac{1}{s}}. \tag{20}$$

and for every $0 < p < -\frac{1}{s}$,

$$\||f|^{1/d_f}\|_{L^p(\mu)} \leq A_f M_f \left(1 + d_s \frac{p}{p + \frac{1}{s}}\right)^{\frac{1}{p}}. \tag{21}$$

Proof: Inequality (20) deduces from inequality (17). The proof of inequality (21) is then standard.

4 Proof of Theorem 1

While in [B2] and [B3], Bobkov used an argument based on a transportation argument, going back to Knothe [K] and Bourgain [Bou], our proof follows the same line of argument as Lovász and Simonovits in [LS], Guédon in [G], Nazarov, Sodin and Volberg in [NSV1], Brudnyi in [Br3] and Carbery and Wright in [CW], the geometric localization theorem, which reduces the problem to the dimension one. The main difference with these proofs is that the geometric localization is used here in the presentation given by Fradelizi and Guédon in [FG] which don't use an infinite bisection method but prefers to see it as an optimization theorem on the set of s-concave measures satisfying a linear constraint and the application of the Krein-Milman theorem. Let us recall the main theorem of [FG].

Theorem [FG] *Let n be a positive integer, let K be a compact convex set in $\mathbb{R}^n$ and denote by $\mathscr{P}(K)$ the set of probability measures in $\mathbb{R}^n$ supported in K. Let $f : K \to \mathbb{R}$ be an upper semi-continuous function, let $s \in [-\infty, \frac{1}{2}]$ and denote by P_f the set of s-concave probability measures λ supported in K satisfying $\int f\, d\lambda \geq 0$. Let $\Phi : \mathscr{P}(K) \to \mathbb{R}$ be a convex $w*$-upper semi-continuous function. Then*

$$\sup_{\lambda \in P_f} \Phi(\lambda)$$

is achieved at a probability measure ν which is either a Dirac measure at a point x such that $f(x) \geq 0$, or a probability measure ν which is s-affine on a segment $[a, b]$, such that $\int f\, d\nu = 0$ and $\int_{[a,x]} f\, d\nu > 0$ on (a, b) or $\int_{[x,b]} f\, d\nu > 0$ on (a, b).

Remarks:

1) In Theorem [FG] and in the following, we say that a measure ν is s-affine on a segment $[a, b]$ if its density ψ satisfies that ψ^γ is affine on $[a, b]$, where $\gamma = \frac{s}{1-s}$.

2) Notice that in Theorem [FG] it is assumed that $s \leq \frac{1}{2}$. If $\frac{1}{2} < s \leq 1$, as follows from Theorem [Bor1], the set of s-concave measures contains only measures whose support is one-dimensional and the Dirac measures. Moreover, a quick look at the proof of Theorem [FG] shows that the conclusions of the theorem remain valid except the fact that the measure ν is s-affine. It would be interesting to know if Theorem [FG] may be fully extended to $\frac{1}{2} < s \leq 1$.

The proof of Theorem 1 splits in two steps. The first step consists in the application of Theorem [FG] to reduce to the one-dimensional case and the second step is the proof of the one-dimensional case:

Step 1: Reduction to the dimension 1.

Let F be a Borel set in $\mathbb{R}^n$ and $t > 1$. Let $s \in (-\infty, 1]$ and μ be a s-concave probability measure on $\mathbb{R}^n$ such that $\mu(F_t^c) > 0$. Our aim is to prove that

$$\mu(F^c) \geq \left(\frac{2}{t+1}\mu(F_t^c)^s + \frac{t-1}{t+1}\right)^{1/s} \quad i.e.\ \mu(F) \leq 1 - \left(\frac{2}{t+1}\mu(F_t^c)^s + \frac{t-1}{t+1}\right)^{1/s}.$$

By a standard approximation, we may assume that μ is compactly supported. We denote by K its support which is a convex set in $\mathbb{R}^n$ and by G, the affine subspace generated by K. Notice that in the proof of this inequality, we always may assume that $F \subset K$ (if we replace F by $\tilde{F} := F \cap K$, then $\mu(\tilde{F}) = \mu(F)$ and $\tilde{F}_t \subset F_t$, hence $\mu(\tilde{F}_t^c) \geq \mu(F_t^c)$).

From Theorem [Bor1] of Borell stated in the introduction, μ is absolutely continuous with respect to the Lebesgue measure on G. Using the regularity of the measure, we may assume that F is compact in K. To satisfy the other semi-continuity hypothesis, we would need F_t to be open. Since this is not necessarily the case, we introduce an auxiliary open set O such that $F_t \subset O$ and $\mu(O^c) > 0$. Define $\theta \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$ and $\Phi : \mathscr{P}(K) \to \mathbb{R}$ by

$$\theta = \mu(O^c) > 0, \quad f = \mathbf{1}_{O^c} - \theta \quad \text{and} \quad \Phi(\lambda) = \lambda(F),\ \forall \lambda \in \mathscr{P}(K).$$

Since F is closed and O is open, the functions f and Φ are upper semi-continuous. With these definitions, the set P_f defined in the statement of the preceding theorem is

$$P_f = \{\lambda \in \mathscr{P}(K);\ \lambda \text{ is } s\text{−concave and } \lambda(O^c) \geq \theta\}.$$

Since $\mu \in P_f$, if we prove that

$$\sup_{\lambda \in P_f} \Phi(\lambda) \leq 1 - \left(\frac{2}{t+1}\theta^s + \frac{t-1}{t+1}\right)^{1/s}, \tag{22}$$

we will get that for any open set O containing F_t such that $\mu(O^c) > 0$

$$\mu(F) \leq 1 - \left(\frac{2}{t+1}\mu(O^c)^s + \frac{t-1}{t+1}\right)^{1/s}.$$

Taking the supremum on such open set O and using the regularity of μ, it will give the result. From Theorem [FG], to establish inequality (22) it is enough to prove it for two types of particular measure ν:

- the measure ν is a Dirac measure at a point x such that $f(x) \geq 0$. It implies that $\mathbf{1}_{O^c}(x) \geq \theta > 0$, thus $x \notin O$, hence $x \notin F$, since $F \subset F_t \subset O$. Therefore $\Phi(\delta_x) = \delta_x(F) = 0$. This proves inequality (22) in this case.

- the measure ν is s-concave on a segment $[a,b]$, such that $\int f\,d\nu = 0$ and $\int_{[a,x]} f\,d\nu > 0$ on (a,b) or $\int_{[x,b]} f\,d\nu > 0$ on (a,b). Without loss of generality we may assume that $\int_{[x,b]} f\,d\nu > 0$ on (a,b). Hence these conditions give

$$\nu(O^c) = \theta \quad \text{and} \quad \nu(O^c \cap [x,b]) > \nu(O^c)\nu([x,b]),\ \forall x \in (a,b)\,.$$

As explained at the beginning of the proof, we may assume that $F \subset [a,b]$. We also assume that $[a,b] \subset \mathbb{R}$, with $a < b$. It is easy to see that for $F \subset \mathbb{R}$, its dilation F_t is open. Hence we may choose $O = F_t$ and get rid of the auxiliary set O. So we have

$$\nu(F_t^c) = \theta \quad \text{and} \quad \nu(F_t^c \cap [x,b]) > \nu(F_t^c)\nu([x,b]),\ \forall x \in (a,b)\,. \tag{23}$$

Letting x tends to b and using that F_t is open, the second condition implies that $b \notin F_t$.

Let us see now why we may assume that $a \in F$. Since F is closed, if $a \notin F$, then $a' := \inf F > a$. Let ν' be the probability measure, which is the restriction of ν to the interval $[a', b]$, *i.e.* $\nu' = \nu_{|[a',b]}/\nu([a',b])$. Then

$$\nu'(F^c) = \frac{\nu(F^c \cap [a',b])}{\nu([a',b])} = \frac{\nu(F^c) - \nu([a,a'])}{1 - \nu([a,a'])} \leq \nu(F^c)$$

and from the second condition in (23)

$$\nu'(F_t^c) = \frac{\nu(F_t^c \cap [a',b])}{\nu([a',b])} > \nu(F_t^c).$$

This ends the first step. We showed that to prove Theorem 1 for any s-concave measure μ and any Borel set F, it is enough to prove it for the s-concave probability measures ν which are supported on a segment $[a,b] \subset \mathbb{R}$, with $b \notin F_t$, $a \in F$ and $F \subset [a,b]$. Moreover for $s \leq \frac{1}{2}$, we also may assume that ν is s-affine.

Step 2: Proof in dimension 1.

Let us start with a joint remark with Guédon:
In the case where F is convex, it is now easy to conclude, which enables us to recover the result of Guédon [G]. From the convexity of F and F_t there exists $c < d$ such that $F = [a,c]$ and $F_t \cap [a,b] = [a,d)$ and we have $a < c < d < b$. Using that $d \notin F_t$ and the definition of F_t, for any interval I containing d, we have $|I| \geq \frac{t+1}{2}|F \cap I|$. For $I = [a,d]$, this gives $d - a \geq \frac{t+1}{2}(c-a)$ and so

$$c \leq \frac{2}{t+1}d + \left(\frac{t-1}{t+1}\right)a \quad \text{hence} \quad [c,b] \supset \frac{2}{t+1}[d,b] + \frac{t-1}{t+1}[a,b].$$

Since ν is s-concave, we get

$$\nu([c,b]) \geq \left(\frac{2}{t+1}\nu([d,b])^s + \frac{t-1}{t+1}\nu([a,b])^s\right)^{1/s}.$$

This ends the proof in this case since $\nu(F^c) = \nu([c,b])$, $\nu(F_t^c) = \nu([d,b])$ and $\nu([a,b]) = 1$.

The general case is more complicate. The proof of Nazarov, Sodin and Volberg [NSV1], to treat the log-concave ($s = 0$) one-dimensional case, extends directly to the case $s \leq 1$, with some suitable adaptations in the calculations, so we don't reproduce it here. But for $s \leq \frac{1}{2}$, using that ν may be assumed s-affine, we can shorten the proof (in fact, we only use the monotonicity of the density of ν).

Since F_t is open in $\mathbb{R}$, it is the countable union of disjoint intervals. By approximation, we may assume that there are only a finite number of them. Since $a \in F \subset F_t$ and $b \notin F_t$, we can write

$$F_t \cap [a,b] = [a_0, b_0) \cup \left(\bigcup_{i=1}^{N}(a_i, b_i)\right) \quad \text{with } a_i < b_i < a_{i+1},\ 0 \leq i \leq N-1,$$

where $a_0 = a$. Let $F_i = F \cap (a_i, b_i)$. Denote by ψ the density of ν with respect to the Lebesgue measure. There are two cases:

- If ψ is non-decreasing: this is the easiest case. Let $0 \le i \le N$. Since $b_i \notin F_t$, using the definition of F_t, it follows that for every interval I containing b_i, we have $|I| \ge \frac{t+1}{2}\|F \cap I|$. Let $x \in (a_i, b_i)$, if we apply it to $I = [x, b_i]$ we get

$$|[x, b_i]| \ge \frac{t+1}{2}|[x, b_i] \cap F|.$$

Hence the function $\rho := 1 - \frac{t+1}{2}\mathbf{1}_F$ satisfies $\int_x^{b_i} \rho(u)du \ge 0$. Integrating by parts this gives

$$\int_{a_i}^{b_i} \rho(u)\psi(u)du = \psi(a_i)\int_{a_i}^{b_i} \rho(x)dx + \int_{a_i}^{b_i}\left(\int_x^{b_i} \rho(u)du\right)\psi'(x)dx \ge 0.$$

Hence $\nu\big((a_i, b_i)\big) \ge \frac{t+1}{2}\nu(F_i)$ and since $F_t = \cup(a_i, b_i)$ and $F = \cup F_i$, it follows that $\nu(F_t) \ge \frac{t+1}{2}\nu(F)$. Therefore, using the comparison between the s-mean (with $s \le 1$) and the arithmetic mean, we conclude that

$$\nu(F^c) \ge \frac{2}{t+1}\nu(F_t^c) + \frac{t-1}{t+1} \ge \left(\frac{2}{t+1}\nu(F_t^c)^s + \frac{t-1}{t+1}\right)^{1/s}.$$

- If ψ is non-increasing: We first prove that, for each $0 \le i \le N$

$$\nu(F_i^c) \ge \left(\frac{2}{t+1}\nu((a_i, b_i)^c)^s + \frac{t-1}{t+1}\right)^{1/s}. \tag{24}$$

For $i \ge 1$, we have $a_i \notin F_t$ and it is similar as the previous case. Indeed, for every $x \in (a_i, b_i)$, $|[a_i, x]| \ge \frac{t+1}{2}|[a_i, x] \cap F|$ and an integration by parts gives that $\nu\big((a_i, b_i)\big) \ge \frac{t+1}{2}\nu(F_i)$. From the comparison of the means, inequality (24) follows.
For $i = 0$, we have $a_0 = a \in F$. We define $F_0' = [a_0, c_0]$, where c_0 is chosen such that $|F_0'| = |F_0|$. Since ψ is non-increasing, we have $\nu(F_0') \ge \nu(F_0)$ and since $b_0 \notin F_t$,

$$|[a_0, b_0]| \ge \frac{t+1}{2}|[a_0, b_0] \cap F| = \frac{t+1}{2}|F_0| = \frac{t+1}{2}|F_0'| = \frac{t+1}{2}|[a_0, c_0]|.$$

Hence $b_0 - a_0 \ge \frac{t+1}{2}(c_0 - a_0)$. As in the joint remark with Guédon given before, we get that

$$\nu([c_0, b]) \ge \left(\frac{2}{t+1}\nu([b_0, b])^s + \frac{t-1}{t+1}\nu([a_0, b])^s\right)^{1/s}.$$

Therefore we get inequality (24) for $i = 0$:

$$\nu(F_0^c) \ge \nu(F_0'^c) = \nu([c_0, b]) \ge \left(\frac{2}{t+1}\nu((a_0, b_0)^c)^s + \frac{t-1}{t+1}\right)^{1/s}.$$

The inequality (24) may be written $\nu(F_i) \le \varphi(\nu\big(a_i, b_i\big))$, for $0 \le i \le N$, where $\varphi : [0, 1] \to [0, 1]$ is defined by

$$\varphi(x) = 1 - \left(\frac{2}{t+1}(1-x)^s + \frac{t-1}{t+1}\right)^{1/s}.$$

From Minkowski inequality for the s-mean, with $s \le 1$, the function φ is convex on $[0, 1]$. Denote $\lambda_i = \nu(\big(a_i, b_i\big))/\nu(F_t)$. Using that $\varphi(0) = 0$ and the convexity of φ we get

$$\nu(F_i) \le \varphi(\nu\big(a_i, b_i\big)) = \varphi\big(\lambda_i \nu(F_t)\big) \le \lambda_i \varphi\big(\nu(F_t)\big).$$

Summing on i and using that $\sum_{i=1}^{N} \lambda_i = 1$, we conclude that

$$\nu(F) \leq \varphi\big(\nu(F_t)\big).$$

This is the result. $\square$

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[1]The name of this author is really Remes and not Remez, hence the reference given here is correct. In the rest of the article, we chose to spell it Remez to agree with the usual literature.