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THICK POINTS FOR TRANSIENT SYMMETRIC STABLE PROCESSES

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Keywords Stable process, occupation measure, multifractal spectrum

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Thick Points for Transient Symmetric Stable Processes

Amir Dembo* Yuval Peres† Jay Rosen‡ Ofer Zeitouni§

Abstract

Let $\mathcal{T}(x,r)$ denote the total occupation measure of the ball of radius r centered at x for a transient symmetric stable processes of index $\beta < d$ in $I\!R^d$ and $\Lambda_{\beta,d}$ denote the norm of the convolution with its 0-potential density, considered as an operator on $L^2(B(0,1),dx)$. We prove that $\sup_{|x|\le 1} \mathcal{T}(x,r)/(r^\beta |\log r|) \to \beta\Lambda_{\beta,d}$ a.s. as $r \to 0$. Furthermore, for any $a \in (0, \beta\Lambda_{\beta,d})$, the Hausdorff dimension of the set of "thick points" x for which $\limsup_{r\to 0} \mathcal{T}(x,r)/(r^\beta |\log r|) = a$, is almost surely $\beta - a/\Lambda_{\beta,d}$; this is the correct scaling to obtain a nondegenerate "multifractal spectrum" for transient stable occupation measure. We also show that the lim inf scaling of $\mathcal{T}(x,r)$ is quite different: we exhibit positive, finite, non-random $c'_{\beta,d}, C_{\beta,d}$, such that $c'_{\beta,d} < \sup_x \liminf_{r\to 0} \mathcal{T}(x,r)/r^\beta < C_{\beta,d}$ a.s.

1 Introduction

The symmetric stable process $\{X_t\}$ of index $\beta < d$ in $I\!R^d$ does not hit fixed points, hence does not have local times. Nevertheless, the path will be "thick" at certain points in the sense of having larger than "usual" occupation measure in the neighborhood of such points. Our first result tells us just how "thick" a point can be.

Recall that the *occupation measure* μ_T^X is defined as

$$\mu_T^X(A) = \int_0^T \mathbf{1}_A(X_t)\, dt$$

for all Borel sets $A \subseteq I\!R^d$. As usual, we let

$$u^0(x) = \frac{c_{\beta,d}}{|x|^{d-\beta}} \tag{1.1}$$

denote the 0-potential density for $\{X_t\}$, where $c_{\beta,d} = 2^{-\beta}\pi^{-d/2}\Gamma(\frac{d-\beta}{2})/\Gamma(\frac{\beta}{2})$. Let $\Lambda_{\beta,d}$ denote the norm of

$$K_{\beta,d}f(x) = \int_{B(0,1)} u^0(x-y) f(y)\, dy$$

considered as an operator from $L^2(B(0,1),\, dx)$ to itself. Throughout, $B(x,r)$ denotes the ball in $I\!R^d$ of radius r centered at x.

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Theorem 1.1 *Let $\{X_t\}$ be a symmetric stable process of index $\beta < d$ in $\mathbb{R}^d$. Then, for any $R \in (0, \infty)$ and any $T \in (0, \infty]$,*

$$\lim_{\epsilon \to 0} \sup_{|x| \leq R} \frac{\mu_T^X(B(x, \epsilon))}{\epsilon^\beta |\log \epsilon|} = \beta \Lambda_{\beta,d} \quad \text{a.s.} \tag{1.2}$$

Remarks:

- Our proof shows that for any $T \in (0, \infty]$,

$$\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \frac{\mu_T^X(B(X_t, \epsilon))}{\epsilon^\beta |\log \epsilon|} = \beta \Lambda_{\beta,d} \quad \text{a.s.} \tag{1.3}$$

- The scaling behavior of stable occupation measure around any fixed time t is governed by the stable analogue of the LIL of Ciesielski-Taylor, see [11]; for any $T \in (0, \infty]$ and $t \leq T$,

$$\limsup_{\epsilon \to 0} \frac{\mu_T^X(B(X_t, \epsilon))}{\epsilon^\beta \log |\log \epsilon|} = \frac{\beta \Lambda_{\beta,d}}{2} \quad \text{a.s.} \tag{1.4}$$

  (1.2) and (1.3) are our analogues of Lévy's uniform modulus of continuity. The proof of such results for Brownian occupation measure was posed as a problem by Taylor in 1974 (see [12, Pg. 201]) and solved by us in [3, Theorem 1.2].

Our next result is related to (1.4) in the same way that the formula of Orey and Taylor [5] for the dimension of Brownian fast points is related to the usual LIL. It describes the multifractal nature, in a fine scale, of "thick points" for the occupation measure of $\{X_t\}$ (We call a point $x \in \mathbb{R}^d$ a *thick point* if x is in the set considered in (1.5) for some $a > 0$; similiarly, $t > 0$ is called a $\mathsf{thick\ time}$ if it is in the set Thick_a considered in (1.6) for some $a > 0$ and $T > 0$.)

Theorem 1.2 *Let $\{X_t\}$ be a symmetric stable process of index $\beta < d$ in $\mathbb{R}^d$. Then, for any $T \in (0, \infty]$ and all $a \in (0, \beta\Lambda_{\beta,d}]$,*

$$\dim\{x \in \mathbb{R}^d \,\Big|\, \limsup_{\epsilon \to 0} \frac{\mu_T^X(B(x, \epsilon))}{\epsilon^\beta |\log \epsilon|} = a\} = \beta - a\Lambda_{\beta,d}^{-1} \quad \text{a.s.} \tag{1.5}$$

Equivalently, for any $T \in (0, \infty]$ and all $a \in (0, \beta\Lambda_{\beta,d}]$,

$$\dim\{0 \leq t < T \,\Big|\, \limsup_{\epsilon \to 0} \frac{\mu_T^X(B(X_t, \epsilon))}{\epsilon^\beta |\log \epsilon|} = a\} = 1 - a\Lambda_{\beta,d}^{-1}/\beta \quad \text{a.s.} \tag{1.6}$$

Denote the set in (1.6) by Thick_a. Then $\mathsf{Thick}_a \neq \emptyset$ at the critical value $a = \beta\Lambda_{\beta,d}$.

For all $a \in (0, \beta\Lambda_{\beta,d})$, the union $\mathsf{Thick}_{\geq a} := \cup_{b \geq a} \mathsf{Thick}_b$ has the same Hausdorff dimension as Thick_a a.s., but its packing dimension a.s. satisfies $\dim_{\mathrm{P}}(\mathsf{Thick}_{\geq a}) = 1$. Equivalently,

$$\dim_{\mathrm{P}}\{x \in \mathbb{R}^d \,\Big|\, \limsup_{\epsilon \to 0} \frac{\mu_T^X(B(x, \epsilon))}{\epsilon^\beta |\log \epsilon|} \geq a\} = \beta \quad \text{a.s.} \tag{1.7}$$

Remark:

- Combining (1.5) and (1.2) we see that

$$\sup_{x\in\mathbb{R}^d}\limsup_{\epsilon\to 0}\frac{\mu_\infty^X(B(x,\epsilon))}{\epsilon^\beta|\log\epsilon|}=\beta\Lambda_{\beta,d}\qquad\text{a.s.}$$

  In particular, the sets in (1.5) and (1.6) are a.s. empty for any $a>\beta\Lambda_{\beta,d}$, $T\in(0,\infty]$.

- For any $x\notin\overline{\{X_t\,\big|\,0\le t\le T\}}$ and ϵ small enough, $\mu_T^X(B(x,\epsilon))=0$. Hence, the equivalence of (1.5) and (1.6) is a direct consequence of the equivalence between spatial and temporal Hausdorff dimensions for stable motion due to Perkins-Taylor (see [7, Eqn. 0.1]), together with the fact that $\overline{\{X_t\,\big|\,0\le t\le T\}}-\{X_t\,\big|\,0\le t\le T\}$ is countable.

Our next theorem gives a precise estimate of the total duration in $[0,1]$ that the stable motion spends in balls of radius ϵ that have unusually high occupation measure. Such an estimate (which is an analogue in our setting of the "coarse multifractal spectrum", cf. Riedi [10]), cannot be inferred from Theorem 1.2.

Theorem 1.3 *Let $\{X_t\}$ be a symmetric stable process of index $\beta<d$ in $\mathbb{R}^d$, and denote Lebesgue measure on $\mathbb{R}$ by $\mathcal{L}eb$. Then, for any $a\in(0,\beta\Lambda_{\beta,d})$,*

$$\lim_{\epsilon\to 0}\frac{\log\mathcal{L}eb\Big\{0\le t\le 1\,\Big|\,\mu_1^X(B(X_t,\epsilon))\ge a\epsilon^\beta|\log\epsilon|\Big\}}{\log\epsilon}=a\Lambda_{\beta,d}^{-1}\qquad\text{a.s.}$$

Theorem 1.3 will be derived as a corollary of the following theorem which provides a pathwise asymptotic formula for the moment generating function of the ratio $\mu_1^X\left(B(X_t,\epsilon)\right)/\epsilon^\beta$.

Theorem 1.4 *Let $\{X_t\}$ be a symmetric stable process of index $\beta<d$ in $\mathbb{R}^d$. Then, for each $\theta<\Lambda_{\beta,d}^{-1}$,*

$$\lim_{\epsilon\to 0}\int_0^\infty e^{\theta\mu_1^X(B(X_t,\epsilon))/\epsilon^\beta}\,dt=\left(\mathbb{E}\left(e^{\theta\mu_\infty^X(B(0,1))}\right)\right)^2\qquad\text{a.s.}\tag{1.8}$$

It follows from the proof that both sides of (1.8) are infinite if $\theta\ge\Lambda_{\beta,d}^{-1}$.
The thick points considered thus far are centers of balls $B(x,\epsilon)$ with unusually large occupation measure for infinitely many radii, but these radii might be quite rare. The next theorem shows that for "consistently thick points" where the balls $B(x,\epsilon)$ have unusually large occupation measure for *all* small radii ϵ and the same center x, what constitutes "unusually large" must be interpreted more modestly.

Theorem 1.5 *Let $\{X_t\}$ be a symmetric stable process of index $\beta<d$ in $\mathbb{R}^d$. Then for some non-random $0<c'_{\beta,d}<C_{\beta,d}<\infty$ we have,*

$$c'_{\beta,d}\le\sup_{x\in\mathbb{R}^d}\liminf_{\epsilon\to 0}\frac{\mu_\infty^X(B(x,\epsilon))}{\epsilon^\beta}\le C_{\beta,d}\qquad\text{a.s.}\tag{1.9}$$

Remarks:

- In particular, replacing the lim sup by lim inf in (1.5) and (1.6) results with an a.s. empty set for all $a > 0$.
- The new assertion in (1.9) is the right hand inequality; the inequality on the left is an immediate consequence of the existence of "stable slow points", see [6, Theorem 22].

It is an open problem to determine the precise asymptotics in (1.9), even in the special case of Brownian motion.

This paper generalizes the results of our paper [3] on thick points of spatial Brownian motion to all transient symmetric stable processes. There are several sources of difficulty in this extension:

- Ciesielski-Taylor [2] provide precise estimates for the tail of the Brownian occupation measure of a ball in $\mathbf{R}^d$; the existing estimates for stable occupation measure are not as precise.
- The Lévy modulus of continuity for Brownian motion was used in [3] to obtain long time intervals where the process does not exit certain balls.

We overcome these difficulties by

- A spectral analysis of the convolution operator defined by the potential density.
- Conditioning on the absence of large jumps in certain time intervals; this creates strong dependence between different scales, and our general results on "random fractals of limsup type" are designed to handle that dependence.

Sections 2 and 3 state and prove the two new ingredients which are needed in order to establish the results in the generality stated here, that is the Localization Lemma 2.2 and the Exponential Integrability Lemma 3.1. Applying these lemmas, Section 4 goes over the adaptations of the proofs of [3] which allows us to establish the results stated above for all transient stable occupation measures.

2 Localization for stable occupation measures

We start by providing a convenient representation of the law of the total occupation measure $\mu_\infty^X(B(0,1))$. This representation is the counterpart of the Ciesielski-Taylor representation for the total occupation measure of spatial Brownian motion in [2, Theorem 1].

Lemma 2.1 *Let $\{X_t\}$ be a symmetric stable process of index $\beta < d$ in $I\!R^d$. Then, for any $u > 0$,*

$$\mathbf{P}\left(\mu_\infty^X(B(0,1)) > u\right) = \sum_{j=1}^{\infty} \psi_j e^{-u/\lambda_j} \ , \tag{2.1}$$

where $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_j \geq \cdots > 0$ are the eigenvalues of the operator $K_{\beta,d}$ with the corresponding orthonormal eigenvectors $\phi_j(y)$, $\psi_j := \phi_j(0)(1,\phi_j)_{B(0,1)}$ and the infinite sum in (2.1) converges uniformly in u away from 0.

Proof of Lemma 2.1: We use $p_t(x)$ to denote the transition density of $\{X_t\}$ and for ease of exposition write K, Λ for $K_{\beta,d}, \Lambda_{\beta,d}$ respectively, and let $\mathcal{J} = \mu_\infty^X(B(0,1))$. While u^0 is in general only in $L^1(B(0,1), dx)$, each application of K lowers the degree of divergence by β (this is easily seen by scaling), so $v := K^{m-1}u^0$, the convolution kernel of K^m is bounded and in fact continuous for m large enough. Fix such m and note that for any $n \geq m$,

$$
\begin{aligned}
\mathbb{E}\left(\mathcal{J}^n\right) &= \mathbb{E}\left(\left\{\int_0^\infty \mathbf{1}_{B(0,1)}(X_s)\,ds\right\}^n\right) \\
&= n! \int_{B(0,1)^n} \int_{0 \leq t_1 \leq \cdots \leq t_n} \prod_{j=1}^n p_{t_j - t_{j-1}}(x_j - x_{j-1})\,dt_1 \cdots dt_n\,dx_1 \cdots dx_n. \\
&= n! \int_{B(0,1)^n} \prod_{j=1}^n u^0(x_j - x_{j-1})\,dx_1 \cdots dx_n \\
&= n!(1, K^{n-1}u^0)_{B(0,1)} = n!(1, K^{n-m}v)_{B(0,1)}\,. \qquad (2.2)
\end{aligned}
$$

Thus, for $g(z,u) := \sum_{n=m}^\infty z^{n-m}u^n/n!$, by the standard Neumann series for the resolvent, for any $z \in \mathbb{C}$ such that $|z| < \Lambda^{-1} := \theta^*$,

$$
\mathbb{E}\left(g(z,\mathcal{J})\right) = \sum_{\ell=0}^\infty z^\ell (1, K^\ell v)_{B(0,1)} = (1, (\mathbf{I} - zK)^{-1}v)_{B(0,1)} \qquad (2.3)
$$

Taking $z \in [-3\theta^*/4, -\theta^*/4]$, we find after m integrations by parts that

$$
\mathbb{E}\left(g(z,\mathcal{J})\right) = \int_0^\infty g(z,u)\,d\mathbf{P}(\mathcal{J} > u) = \int_0^\infty e^{zu} f_m(u)du\ , \qquad (2.4)
$$

where $f_m(u)$ is the $(m-1)$-fold integral from u to ∞ of $\mathbf{P}(\mathcal{J} > \cdot)$. To justify this, we note, on the one hand, that by (2.2) $\mathcal{J}$ has all moments, so that $\mathbf{P}(\mathcal{J} > u) \leq c_N/u^N$ for any N, and therefore $f_j(u)$ is bounded and goes to 0 as u tends to ∞ for any j. On the other hand, $d^m g(z,u)/d^m u = e^{zu}$ with $d^k g(z,u)/d^k u = 0$ at $(z,0)$ for $k = 0, \ldots, m-1$ which controls the boundary terms at $u = 0$, and writing $g(z,u) = z^{-m}(e^{zu} - \sum_{n=0}^{m-1} z^n u^n/n!)$ and using the fact that $z < 0$ controls the boundary terms at $u = \infty$.

Since K is a convolution operator on $B(0,1)$ with locally $L^1\left(\mathbb{R}^d,\,dx\right)$ kernel, it follows easily as in [4, Corollary 12.3] that K is a (symmetric) compact operator. Moreover, the Fourier transform relation $\int e^{i(x\cdot p)}u^0(x)\,dx = |p|^{-\beta} > 0$ implies that K is strictly positive definite. By the standard theory for symmetric compact operators, K has discrete spectrum (except for a possible accumulation point at 0) with all eigenvalues positive, of finite multiplicity, and the corresponding eigenvectors of K, denoted $\{\phi_j\}$ form a complete orthonormal basis of $L^2\left(B(0,1),\,dx\right)$ (see [8, Theorems VI.15, VI.16]). Moreover, $(f, Kg)_{B(0,1)} > 0$ for any non-negative, non-zero, f, g, so by the generalized Perron-Frobenius Theorem, see [9, Theorem XIII.43], the eigenspace corresponding to $\Lambda = \lambda_1$ is one dimensional, and we may and shall choose ϕ_1 such that $\phi_1(y) > 0$ for all $y \in B(0,1)$. Noting that ϕ_j are also eigenvectors of $(\mathbf{I} - zK)^{-1}$ with corresponding eigenvalues $(1 - z\lambda_j)^{-1}$, we have by (2.3) and (2.4) that for $z \in [-3\theta^*/4, -\theta^*/4]$,

$$
\begin{aligned}
\int_0^\infty e^{zu} f_m(u)du = (1, (\mathbf{I} - zK)^{-1}v)_{B(0,1)} &= \sum_{j=1}^\infty \frac{c_j}{1 - z\lambda_j} \\
&= \int_0^\infty e^{zu}\left(\sum_{j=1}^\infty c_j \lambda_j^{-1} e^{-u/\lambda_j}\right) du\,, \qquad (2.5)
\end{aligned}
$$

where $c_j := (1, \phi_j)(v, \phi_j)$ is absolutely summable. Since both integrals in (2.5) are analytic in the strip Re $z \in [-3\theta^*/4, -\theta^*/4]$, and agree for z real inside the strip, they agree throughout this strip. Considering $z = -\theta^*/2 + it$, $t \in I\!\!R$ we have that

$$f_m(u) = \sum_{j=1}^{\infty} c_j \lambda_j^{-1} e^{-u/\lambda_j} \tag{2.6}$$

a.e. on $u \geq 0$ by Fourier inversion and hence for all $u > 0$ by right continuity. Considering the $(m-1)$-st derivative of (2.6), the uniform convergence of $\sum_j c_j \lambda_j^{-k} e^{-u/\lambda_j}$ for $k = 1, \ldots, m$ and u away from 0, shows that

$$\mathbf{P}\left(\mathcal{J} > u\right) = \sum_{j=1}^{\infty} c_j \lambda_j^{-m} e^{-u/\lambda_j}$$

for all $u > 0$. Recalling that v, the convolution kernel of K^m, is bounded and continuous, we have that $\lambda_j^m \phi_j = K^m \phi_j$ are continuous and bounded functions, and $K^m \phi_j(0) = (v, \phi_j)_{B(0,1)}$. Therefore $\phi_j(0) = \lambda_j^{-m}(v, \phi_j)_{B(0,1)}$ and thus $c_j \lambda_j^{-m} = (1, \phi_j)\phi_j(0) = \psi_j$ for all j, establishing (2.1). □

With the aid of (2.1) we next provide a localization result for the occupation measure of $\{X_t\}$.

Lemma 2.2 (The Localization Lemma) *Let $\{X_t\}$ be a symmetric stable process of index $\beta < d$ in $I\!\!R^d$. Then, with $\theta^* = \Lambda_{\beta,d}^{-1}$, for some $c_0, c_1 < \infty$, $t \geq c_0 u^{d/(d-\beta)}$, and all $u > 0$ sufficiently large*

$$c_1^{-1} e^{-\theta^* u} \leq \mathbf{P}\left(\mu_t^X\left(B(0,1)\right) \geq u\right) \leq \mathbf{P}\left(\mu_\infty^X\left(B(0,1)\right) \geq u\right) \leq c_1 e^{-\theta^* u} \tag{2.7}$$

With $h(\epsilon) = \epsilon^\beta |\log \epsilon|$ and any $\rho > d/(d-\beta)$, considering $u = a|\log \epsilon|$ in (2.7), by stable scaling we establish that for any $a > 0$ and $\epsilon > 0$ small enough

$$c_1^{-1} \epsilon^{a\theta^*} \leq \mathbf{P}\left(\mu_{\epsilon^\beta |\log \epsilon|^\rho}^X\left(B(0,\epsilon)\right) \geq ah(\epsilon)\right) \leq \mathbf{P}\left(\mu_\infty^X\left(B(0,\epsilon)\right) \geq ah(\epsilon)\right) \leq c_1 \epsilon^{a\theta^*} \tag{2.8}$$

Proof of Lemma 2.2: Let $\mathcal{J}_t := \mu_t^X(B(0,1))$. Recall that ϕ_1 is a strictly positive function on $B(0,1)$, hence in (2.1) we have $\psi_1 > 0$ and $\theta^* = \lambda_1^{-1} < \lambda_2^{-1}$, implying that

$$\lim_{u \to \infty} \mathbf{P}(\mathcal{J}_\infty > u) e^{\theta^* u} = \psi_1 \in (0, \infty) \tag{2.9}$$

out of which the upper bound of (2.7) immediately follows.

Turning to prove the corresponding lower bound, let $\tau_z := \inf\{s : |X_s| > z\}$, noting that by [1, Proposition VIII.3] for some $c > 0$ and any $u > 0$, $z > 1$, $t > 0$,

$$\mathbf{P}(\mathcal{J}_t > u) \geq \mathbf{P}(\mathcal{J}_{\tau_z} > u) - \mathbf{P}(\tau_z > t) \geq \mathbf{P}(\mathcal{J}_{\tau_z} > u) - c^{-1} \exp(-ctz^{-\beta}). \tag{2.10}$$

Let $\mathcal{J}$ and $\mathcal{J}'$ denote two independent copies of $\mathcal{J}_\infty$. Noting that $\mathbf{P}^y(\mathcal{J}' > u) \leq \mathbf{P}(\mathcal{J}' > u)$ for any $y \in I\!\!R^d$, $u > 0$, and using the strong Markov property, it is not hard to verify that

$$\mathbf{P}(\mathcal{J}_\infty > u) \leq \mathbf{P}(\mathcal{J}_{\tau_z} > u) + \mathbf{P}(\mathcal{J} + \mathcal{J}' > u) \sup_{|v| > z} \mathbf{P}^v(\inf_{s \geq 0} |X_s| < 1) \tag{2.11}$$

(c.f. [3, (3.6) and (3.7)] where this is obtained for the Brownian motion). Recall that

$$\mathbf{P}^{v}(\inf_{s\geq 0}|X_s|<1)\leq c(\beta,d)|v|^{-(d-\beta)}\wedge 1$$

(see [11, Lemma 4]). By (2.9) it follows that for some $C<\infty$ and all $u>0$,

$$\mathbf{P}(\mathcal{J}+\mathcal{J}'>u)\leq C(1+u)e^{-\theta^* u} \tag{2.12}$$

(c.f. [3, (3.8)] for the derivation of a similar result). Hence, for some C_1, C_2, c_0 large enough, taking $z=z_u:=C_1u^{1/(d-\beta)}$ and $t_u:=C_2uz_u^{\beta}=c_0u^{d/(d-\beta)}$ one gets from (2.10) and (2.11) that for some $c'>0$ and all $t\geq t_u$

$$\mathbf{P}(\mathcal{J}_t>u)\geq c'e^{-\theta^* u} \tag{2.13}$$

as needed to complete the proof of the lemma. □

3 Exponential integrability

Lemma 3.1 *Let $\{X_t\}$ denote the symmetric stable process of index β in $\mathbb{R}^d$ with $\beta<d$ and $\psi(x):=|x|^{-\beta}\mathbf{1}_{\{|x|\leq 1\}}$. Then, for any $\theta\in(0,d-\beta)$ and*

$$\lambda<\Lambda^{-1}_{\beta,d}(\theta):=2^{\beta}\frac{\Gamma(\frac{d-\theta}{2})\Gamma(\frac{\beta+\theta}{2})}{\Gamma(\frac{d-\theta-\beta}{2})\Gamma(\frac{\theta}{2})}, \tag{3.1}$$

there exists $\kappa_{\lambda,\theta}<\infty$ such that for all $|y|\leq 1$

$$\mathbb{E}^y\left(\exp(\lambda\int_0^\infty\psi(X_t)\,dt)\right)\leq\kappa_{\lambda,\theta}|y|^{-\theta}. \tag{3.2}$$

Proof of Lemma 3.1: As before, $u^0(x)=c_{\beta,d}|x|^{\beta-d}$ denotes the 0-potential density for $\{X_t\}$. Fixing $\theta\in(0,d-\beta)$ let $\Lambda_{\beta,d}(\theta)$ denote the norm of

$$Kf(y)=|y|^{\theta}\int_{B(0,1)}u^0(y-x)|x|^{-(\beta+\theta)}f(x)\,dx$$

considered as an operator from $L^\infty(B(0,1),\,dx)$ to itself. Recall the Fourier transform relation $\int e^{i(x\cdot p)}u^0(x)\,dx=|p|^{-\beta}$, implying that

$$\Lambda_{\beta,d}(\theta)=\sup_{y\in B(0,1)}(K1)(y)=\sup_y\int_{\mathbb{R}^d}\frac{c_{\beta,d}|y|^{\theta}dx}{|y-x|^{d-\beta}|x|^{\beta+\theta}}=\frac{c_{d-\theta,d}}{c_{d-\theta-\beta,d}}\,. \tag{3.3}$$

Since $c_{\alpha,d}=2^{-\alpha}\pi^{-d/2}\Gamma(\frac{d-\alpha}{2})/\Gamma(\frac{\alpha}{2})$ for any $\alpha\in(0,d)$, we obtain that

$$\Lambda_{\beta,d}(\theta)=2^{-\beta}\frac{\Gamma(\frac{d-\theta-\beta}{2})\Gamma(\frac{\theta}{2})}{\Gamma(\frac{d-\theta}{2})\Gamma(\frac{\beta+\theta}{2})}\,, \tag{3.4}$$

as stated in the lemma. For $g(y)=|y|^{\theta}$ and $\lambda\in(0,\Lambda_{\beta,d}(\theta)^{-1})$, the series

$$G(y)=\sum_{n=0}^{\infty}\lambda^n(Kg)^n(y)$$

converges uniformly in $L^\infty(B(0,1),\, dx)$. It is easy to check that for all n,

$$\begin{aligned}
I_n(y) &:= \frac{1}{n!}\mathbb{E}^y\left(\left\{\int_0^\infty \psi(X_t)\,dt\right\}^n\right) = \mathbb{E}^y\left(\int_{0\le t_1\le\ldots\le t_n<\infty}\prod_{j=1}^n \psi(X_{t_j})\,dt_j\right)\\
&= \int\cdots\int u^0(y-x_1)\psi(x_1)\prod_{j=2}^n u^0(x_{j-1}-x_j)\psi(x_j)\,dx_j\,dx_1\\
&= |y|^{-\theta}(Kg)^n(y)\ .
\end{aligned}$$

Therefore,

$$\mathbb{E}^y\left(\exp(\lambda\int_0^\infty \psi(X_t)\,dt)\right) = \sum_{n=0}^\infty \lambda^n I_n(y) = G(y)|y|^{-\theta}$$

resulting with the bound (3.2). □

4 Proofs

Throughout we write θ^* for $\Lambda_{\beta,d}^{-1}$, $h(\epsilon)$ for $\epsilon^\beta|\log\epsilon|$ and take $\rho(\epsilon) := |\log\epsilon|^\rho$ for some $\rho > d/(d-\beta)$ as in the localization bound of (2.8).

Proof of Theorem 1.2, lower bound: In proving the lower bound it suffices to assume that T is finite; by stable scaling, it is enough to consider $T=2$ or equivalently, $-1\le t\le 1$ in (1.6). Our proof follows closely the proof of [3, Corollary 4.1] to which the reader is referred for notation and details. Take $\epsilon_n = n^3 2^{-n/\beta}$, $n=1,2,\ldots$ and $b_{\epsilon_n} = 1-|\log\epsilon_n|^{-2}$. With $I=[t,t+2^{-n}]\in\mathcal{D}_n$, define $\tilde{I}=[t-n^v 2^{-n},t]$, $v=3\beta+\rho$. Let $Z_I=1$ if the following two (independent) conditions hold:

$$\int_{\tilde{I}} \mathbf{1}_{\{|X_t-X_s|<\epsilon_n b_{\epsilon_n}\}}ds \ge ah(\epsilon_n) \quad \text{and} \quad \sup_{t'\in I}|X_{t'}-X_t| \le \epsilon_n|\log\epsilon_n|^{-2}. \tag{4.1}$$

Therefore, if $I\in\mathcal{D}_n$ and $Z_I=1$, then $\int_{\tilde{I}}\mathbf{1}_{\{|X_s-X_{t'}|<\epsilon_n\}}ds \ge ah(\epsilon_n)$ for every $t'\in I$. Using stable scaling and [1, Proposition VIII.4] it is easily verified that the second condition in (4.1) has probability at least 1/2 for n sufficiently large. By stable scaling, the lower bound of (2.8) directly implies that for all $I\in\mathcal{D}_n$ and n sufficiently large

$$\mathbf{P}\left(\int_{\tilde{I}}\mathbf{1}_{\{|X_t-X_s|<\epsilon_n b_{\epsilon_n}\}}ds \ge ah(\epsilon_n)\right) \ge 2^{1-a\theta^* n/\beta}\ .$$

Noting that the two conditions in (4.1) are independent, we see that for $I\in\mathcal{D}_n$, and all n large enough,

$$p_n := \mathbf{P}(Z_I=1) \ge 2^{-a\theta^* n/\beta}\ . \tag{4.2}$$

We will now apply [3, Theorem 2.1] with gauge function

$$\varphi(r) = r^{1-a\theta^*/\beta}|\log_2 r|^{v+1}.$$

For intervals $I,J\in\mathcal{D}_n$ the variables Z_I and Z_J always satisfy $\mathrm{Cov}(Z_I,Z_J)\le\mathbb{E}(Z_I)=p_n$, and if $\mathrm{dist}(I,J)>n^v 2^{-n}$, then Z_I and Z_J are independent. Therefore, fixing $m<n$ and $D\in\mathcal{D}_m$, each $I\in\mathcal{D}_n$ satisfies $\mathrm{Cov}(Z_I,M_n(D))\le n^v p_n$. Consequently

$$\mathrm{Var}(M_n(D)) = \sum_{I\in\mathcal{D}_n,\, I\subset D}\mathrm{Cov}(Z_I,M_n(D)) \le 2^{n-m}n^v p_n\ .$$

The lower bound on the Hausdorff dimension in Theorem 1.2 now follows as in the proof of [3, Corollary 4.1], with the results about Packing dimension obtained by applying [3, Corollary 2.4]. The fact that $\mathsf{Thick}_a \neq \emptyset$ at the critical value $a = \beta\Lambda_{\beta,d}$ follows by the same argument as in [3, Section 4], using here the dense open sets

$$\mathsf{Thick}(a,h) := \bigcup_{\epsilon,\epsilon' \in (0,h)} \left\{ 0 < t < T \,\Big|\, \frac{\mu_T^X(B(X_t,\epsilon))}{\epsilon^\beta |\log \epsilon|} > a \text{ and } \frac{\mu_T^X(B(X_{t-},\epsilon'))}{\epsilon'^\beta |\log \epsilon'|} > a \right\},$$

where we used the fact that the process $\{X_t\}$ is right-continuous with left limits and has only a countable set of jumps.

Proof of Theorem 1.2, upper bound: This follows as in [3, Section 5] where the bound

$$\mathbf{P}\left(\mu_\infty^X(B(0,(1+\delta)\epsilon)) \geq (1-2\delta)ah(\epsilon)\right) \leq c\epsilon^{(1-4\delta)a\theta^*} \tag{4.3}$$

follows from (2.8), and for the standard estimate for stable hitting probabilities

$$\mathbf{P}(\sigma_\epsilon(x) < \infty) \leq c(\beta,d)(\frac{\epsilon}{|x|})^{d-\beta} \wedge 1, \tag{4.4}$$

with $\sigma_\epsilon(x) := \inf\{t \geq 0 : X_t \in B(x,\epsilon)\}$, see [11, Lemma 4].

Proof of Theorem 1.1: With the above estimates, as well as the lower bound (2.8) of the Localization Lemma 2.2, this follows directly as in the corresponding proof of [3, (1.7)], using now $\delta_\epsilon = \epsilon^\beta \rho(\epsilon)$.

Proof of Theorem 1.4: In the course of proving Lemma 2.1 we verified among other things that $\mathbb{E}(\exp(\theta\mu_\infty^X(B(0,1))) < \infty$ for all $\theta < \theta^*$ (see (2.3)). While $u^0 \notin L^2(B(0,1),dx)$ for $\beta \leq d/2$, we have that $K^i u^0 \in L^2(B(0,1),dx)$ for all i large enough, which is all that one needs when extending [3, Lemma 7.2] to the present context. Thus, adapting the proof of [3, Theorem 1.4] amounts to replacing each Brownian scaling in [3, Section 7] with stable scaling.

Proof of Theorem 1.3: Our proof follows closely the outline of [3, Section 8], to which the reader is referred for notation and details, where we take now $h(\epsilon) = \epsilon^\beta |\log \epsilon|$ and $\delta_n = \epsilon_n^\beta \rho(\epsilon_n)$ as needed for applying the lower bound of (2.8). For $\rho_n = \epsilon_n^\beta |\log \epsilon_n|^{-3\beta}$ and the i.i.d. random variables

$$Y_i^{(n)} := \frac{\mathbf{1}_{A(n,i)}}{h(\epsilon_n)} \int_{(2i-1)\delta_n}^{2i\delta_n} \mathbf{1}_{\{|X_{2i\delta_n} - X_s| < \epsilon_n b_{\epsilon_n}\}} ds\ ,$$

where

$$A(n,i) = \{ \sup_{t \in (0,\rho_n)} |X_{2i\delta_n + t} - X_{2i\delta_n}| \leq \epsilon_n |\log \epsilon_n|^{-2} \}$$

we combine [1, Proposition VIII.4] with (2.8) to provide the lower bound on $\mathbf{P}(Y^{(n)} \geq a/(1-\delta))$ leading to [3, (8.2)] (see the derivation of (4.2) for a similar application).

Proof of Theorem 1.5: We follow the outline of [3, Section 9], relying on Lemma 3.1 to control the exponential moments instead of Girsanov's theorem used in [3, Lemma 9.1].

To deal with the occupation measure of balls not centered at the origin take $0 < \alpha < 1$ and fix $b = 1 + \alpha > 1$. For $k \in (1,\infty)$, let $\Gamma_k = \{x : |x| \in [k^{-1},k]\}$ and for $a > 0$,

$$D_a := \{x \in \Gamma_k \,\Big|\, \liminf_{\epsilon \to 0} \frac{\mu_T^X(B(x,\epsilon))}{\epsilon^\beta} \geq a\}\ .$$

Set $\eta_n = 2^{-n}$ and $\delta_n = \eta_n^{1-b^{-1}}$ for $n = 1, 2, \ldots$. Let $\{x_j : j = 1, \ldots, K_n\}$, denote a maximal collection of points in Γ_k such that $\inf_{\ell \neq j} |x_\ell - x_j| \geq \alpha\eta_n$. Let $\mathcal{A}_n$ be the set of j ; $1 \leq j \leq K_n$ such that

$$\inf_{\epsilon \in [\eta_n, \delta_n]} \frac{\mu_T^X(B(x_j, b\epsilon))}{\epsilon^\beta} \geq \frac{a}{b}. \tag{4.5}$$

We will shortly prove that for some $C_{\beta,d} < \infty$ and all $a \geq C_{\beta,d}$

$$\mathbb{E}|\mathcal{A}_n| \leq c_2 \eta_n^\gamma \tag{4.6}$$

where $\gamma > 0$. Assuming this for the moment, let $\mathcal{V}_{n,j} = B(x_j, \alpha\eta_n)$. Then, for any $x \in \Gamma_k$ there exists $j \in \{1, \ldots, K_n\}$ such that $x \in \mathcal{V}_{n,j}$ and $B(x, \epsilon) \subseteq B(x_j, \epsilon + \alpha\eta_n) \subseteq B(x_j, b\epsilon)$ for all $\epsilon \geq \eta_n$. Fixing $a \geq C_{\beta,d}$, if $x \in D_a$ then a.s. for some $m_1(\omega, x, b) < \infty$ and all $n \geq m_1$,

$$\inf_{\epsilon \in [\eta_n, \delta_n]} \frac{\mu_T^X(B(x, \epsilon))}{\epsilon^\beta} \geq \frac{C_{\beta,d}}{b} \ .$$

Therefore, $D_a \subseteq \cup_{n \geq m} \cup_{j \in \mathcal{A}_n} \mathcal{V}_{n,j}$ for any $m \geq 1$. Therefore

$$\sum_{n=1}^\infty \mathbf{P}(|\mathcal{A}_n| \geq 1) \leq \sum_{n=1}^\infty \mathbb{E}|\mathcal{A}_n| \leq c_2 \sum_{n=1}^\infty \eta_n^\gamma < \infty.$$

Thus, by Borel-Cantelli, it follows that a.s. $\mathcal{A}_n$ is empty for all $n \geq m_2(\omega)$, implying that the sets D_a are a.s. empty for all $T < \infty$. By [13, Lemma 5], a.s.

$$\liminf_{\epsilon \to 0} \frac{\mu_\infty^X(B(0, \epsilon))}{\epsilon^\beta} = 0 \ .$$

Thus, taking $k \uparrow \infty$ completes the proof of the right side of (1.9), subject only to (4.6).

To prove (4.6) fix $T < \infty, a > 0, 1 < b < 2, \eta > 0, \delta = \eta^{1-b^{-1}}$ and $x \in \mathbb{R}^d$. Clearly, for $U_s := |X_s - x|$,

$$\{\mu_T^X(B(x, v)) > 0\} = \{\inf_{s \in [0,T]} U_s < v\} \ . \tag{4.7}$$

Setting $Z = \int_0^T U_s^{-\beta} ds$, also

$$\begin{aligned} b^\beta Z &= \int_0^T \int_{b^{-1}U_s}^\infty \frac{\beta d\epsilon}{\epsilon^{1+\beta}} \, ds = \int_0^T \int_0^\infty \mathbf{1}_{\{|X_s - x| \leq b\epsilon\}} \frac{\beta d\epsilon}{\epsilon^{1+\beta}} \, ds \\ &= \int_0^\infty \frac{\beta d\epsilon}{\epsilon^{1+\beta}} \mu_T^X(B(x, b\epsilon)) \geq \int_\eta^\delta \frac{\beta d\epsilon}{\epsilon^{1+\beta}} \mu_T^X(B(x, b\epsilon)) \ . \end{aligned} \tag{4.8}$$

If

$$\inf_{\epsilon \in [\eta, \delta]} \epsilon^{-\beta} \mu_T^X(B(x, b\epsilon)) \geq \frac{a}{b}$$

then $\mu_T^X(B(x, b\eta)) > 0$ and

$$\int_\eta^\delta \frac{\beta d\epsilon}{\epsilon^{1+\beta}} \mu_T^X(B(x, b\epsilon)) \geq \frac{a}{b} \int_\eta^\delta \frac{\beta d\epsilon}{\epsilon} = -\beta a b^{-2} \log \eta \ .$$

Thus, for $v = b\eta$, by (4.7), (4.8) and Chebycheff's inequality,

$$\begin{aligned}
\mathbf{P}(\inf_{\epsilon\in[\eta,\delta]} \frac{\mu_T^X(B(x,b\epsilon))}{\epsilon^\beta} \geq \frac{a}{b}) &\leq \mathbf{P}(Z \geq -\beta a b^{-2-\beta}\log\eta, \inf_{s\in[0,T]} U_s \leq v) \\
&\leq \eta^{\lambda\beta a b^{-2-\beta}} \mathbb{E}[e^{\lambda Z}\mathbf{1}_{\{\inf_{s\in[0,T]} U_s\leq v\}}] \\
&\leq \eta^{\lambda\beta a b^{-2-\beta}} \left(\mathbb{E}[e^{p\lambda Z}]\right)^{1/p} \left(\mathbf{P}[\inf_{s\in[0,T]} U_s \leq v]\right)^{1-1/p} \qquad (4.9)
\end{aligned}$$

for any $p > 1$. From Lemma 3.1 it follows that when $\lambda < \Lambda_{\beta,d}^{-1}(\theta)$,

$$\mathbb{E}\left(e^{\lambda Z}\right) = \mathbb{E}^x\left(\exp(\lambda\int_0^T |X_t|^{-\beta}\,dt)\right) \leq c_0 e^{\lambda T}|x|^{-\theta},$$

for some $c_0 = c_0(\lambda,\theta,k) < \infty$ and any x such that $|x| \in (0,k]$. Using this together with (4.4) and (4.9) we see that for $\lambda < p^{-1}\Lambda_{\beta,d}^{-1}(\theta)$ and some $c = c(\lambda,\theta,k,T) < \infty$,

$$\mathbf{P}(\inf_{\epsilon\in[\eta,\delta]} \frac{\mu_T^X(B(x,b\epsilon))}{\epsilon^\beta} \geq \frac{a}{b}) \leq c\eta^{\lambda\beta a b^{-2-\beta}+(d-\beta)(1-1/p)}|x|^{-(d-\beta)(1-1/p)-\theta/p}.$$

Choose $\theta = (d-\beta)/2$, $p > 1$, $\lambda < p^{-1}\Lambda_{\beta,d}^{-1}(\theta)$, and then $C_{\beta,d}$ so large that $f := \lambda\beta C_{\beta,d}b^{-2-\beta} + (d-\beta)(1-1/p) > d$. Note that $g := (d-\beta)(1-1/p) + \theta/p < d$ and we have

$$\mathbf{P}(\inf_{\epsilon\in[\eta,\delta]} \frac{\mu_T^X(B(x,b\epsilon))}{\epsilon^\beta} \geq \frac{C_{\beta,d}}{b}) \leq c\eta^f\,|x|^{-g}, \qquad (4.10)$$

where $g, f, C_{\beta,d}$ depend only on b, β, d and our free parameters p, λ.

Using (4.10), since $g < d$, for some $c, c_1, c_2 < \infty$ independent of n,

$$\begin{aligned}
\mathbb{E}|\mathcal{A}_n| &= \sum_{j=1}^{K_n} \mathbf{P}(\inf_{\epsilon\in[\eta_n,\delta_n]} \frac{\mu_T^X(B(x_j,b\epsilon))}{\epsilon^\beta} \geq \frac{C_{\beta,d}}{b}) \\
&\leq c\eta_n^f \sum_{j=1}^{K_n} |x_j|^{-g} \leq c_1\eta_n^{f-d}(1 + \int_{\{|x|\leq k\}} |x|^{-g}\,dx) \leq c_2\eta_n^\gamma
\end{aligned}$$

where $\gamma := f - d > 0$. This completes the proof of (4.6) and hence of our Theorem. $\square$

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