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PHASE TRANSITION FOR THE FROG MODEL[1]

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Abstract: We study a system of simple random walks on graphs, known as *frog model*. This model can be described as follows: There are active and sleeping particles living on some graph. Each active particle performs a simple random walk with discrete time and at each moment it may disappear with probability $1-p$. When an active particle hits a sleeping particle, the latter becomes active. Phase transition results and asymptotic values for critical parameters are presented for $\mathbb{Z}^d$ and regular trees.

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1 Introduction and results

The subject of this paper is the so-called frog model with death, which can be described as follows. Initially there is a random number of particles at each site of a graph $\mathcal{G}$. A site of $\mathcal{G}$ is singled out and called its root. All particles are sleeping at time zero, except for those that might be placed at the root, which are active. At each instant of time, each active particle may die with probability $(1-p)$. Once an active particle survives, it jumps on one of its nearest neighbors, chosen with uniform probability, performing a discrete time simple random walk (SRW) on $\mathcal{G}$. Up to the time it dies, it activates all sleeping particles it hits along its way. From the moment they are activated on, every such particle starts to walk, performing exactly the same dynamics, independent of everything else.

This model with $p = 1$ (i.e., no death) is a discrete-time version of the model for information spreading proposed by R. Durrett (1996, private communication), who also suggested the term "frog model". The first published result on this model is due to Telcs, Wormald [10], where it was referred to as the "egg model". They proved that, starting from the one-particle-per-site initial configuration, almost surely the origin will be visited infinitely often. Popov [8] proved that the same is true in dimension $d \geq 3$ for the initial configuration constructed as follows: A sleeping particle (or "egg") is added into each $x \neq 0$ with probability $\alpha/\|x\|^2$, where α is a large positive constant. In Alves et al. [1] for the frog model with no death it was proved that, starting from the one-particle-per-site initial configuration, the set of the original positions of all active particles, rescaled by the elapsed time, converges to a nonempty compact convex set. In Alves et al. [2] a similar result was obtained for the case of random initial configuration.

The authors have learned about this version (i.e., with death) of the frog model from I. Benjamini. The goal of the present work is to study the asymptotic dynamics of this particle system model, with regard to the parameter p, the graph where the random walks take place and the initial distribution of particles.

Let us define the model in a formal way. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an infinite connected non-oriented graph of locally bounded degree. Here $\mathcal{V} := \mathcal{V}(\mathcal{G})$ is the set of vertices (sites) of $\mathcal{G}$, and $\mathcal{E} := \mathcal{E}(\mathcal{G})$ is the set of edges of $\mathcal{G}$. Sites are said to be neighbors if they belong to a common edge. The *degree* of a site x is the number of edges which have x as an endpoint. A graph is *locally bounded* if all its sites have finite degree. Besides, a graph has *bounded degree* if its maximum degree is finite. The distance $\text{dist}(x,y)$ between sites x and y is the minimal amount of edges that one must pass in order to go from x to y. Fix a site $\mathbf{0} \in \mathcal{V}$ and call it the root of $\mathcal{G}$. With the usual abuse of notation, by $\mathbb{Z}^d$ we mean the graph with the vertex set $\mathbb{Z}^d$ and edge set $\{\langle (x^{(1)};\dots;x^{(d)}),(y^{(1)};\dots;y^{(d)})\rangle : |x^{(1)} - y^{(1)}| + \cdots + |x^{(1)} - y^{(d)}| = 1\}$. Also, $\mathbb{T}_d$, $d \geq 3$, denotes the degree d homogeneous tree.

Let η be a random variable taking values in $\mathbb{N} = \{0,1,2,\dots\}$ such that $\mathbf{P}[\eta \geq 1] > 0$, and define $\gamma_j = \mathbf{P}[\eta = j]$. Let $\{\eta(x); x \in \mathcal{V}\}, \{(S_n^x(i))_{n\in\mathbb{N}}; i \in \{1,2,3,\dots\}, x \in \mathcal{V}\}$ and $\{(\Xi_p^x(i)); i \in \{1,2,3,\dots\}, x \in \mathcal{V}\}$ be independent sets of i.i.d. random variables defined as follows. For each $x \in \mathcal{V}$, $\eta(x)$ has the same law as η, and gives the initial number of particles at site x. If $\eta(x) \geq 1$, then for each $0 < i \leq \eta(x)$, $(S_n^x(i))_{n\in\mathbb{N}}$ is a discrete time SRW on $\mathcal{G}$ starting from x (it describes the trajectory of i-th particle from x), and $\Xi_p^x(i)$, which stands for the lifetime of i-th particle from x, is a random variable whose law is given by $\mathbf{P}[\Xi_p^x(i) = k] = (1-p)p^{k-1}$, $k = 1,2,\dots$, where $p \in [0,1]$ is a fixed parameter. Thus, the i-th particle at site x follows the SRW $(S_n^x(i))_{n\in\mathbb{N}}$

and dies (disappears) $\Xi_p^x(i)$ units of time after being activated. For $x \neq y$ let

$$t(x,y) = \min_{1\le i\le \eta(x)} \min\{n < \Xi_p^x(i) : S_n^x(i) = y\}$$

(clearly, $t(x,y) = \infty$ with positive probability). The moment when all the particles in x are awakened is defined as

$$T(x) = \inf\{t(x_0,x_1) + \cdots + t(x_{m-1},x_m)\},$$

where the infimum is over all finite sequences $0 = x_0, x_1, \ldots, x_{m-1}, x_m = x$. Clearly, $T(x) = \infty$ means that the site x is never visited by active particles.

It is important to note that at the moment the particle disappears, it is not able to activate other particles (as first we decide whether the particle survives, and only after that the particle that survived is allowed to jump). Notice that there is no interaction between active particles, which means that each active particle moves independently of everything else. We denote by $\mathrm{FM}(\mathcal{G},p,\eta)$ the frog model on the graph $\mathcal{G}$ with survival parameter p and initial configuration ruled by η.

Let us consider the following definition.

Definition 1.1. *A particular realization of the frog model* survives *if for every instant of time there is at least one active particle. Otherwise, we say that it* dies out.

Now we observe that $\mathbf{P}[\mathrm{FM}(\mathcal{G},p,\eta)$ survives$]$ is nondecreasing in p and define

$$p_c(\mathcal{G},\eta) := \inf\{p : \mathbf{P}[\mathrm{FM}(\mathcal{G},p,\eta) \text{ survives}] > 0\}.$$

As usual, we say that $\mathrm{FM}(\mathcal{G},p,\eta)$ exhibits *phase transition* if

$$0 < p_c(\mathcal{G},\eta) < 1.$$

Now we present two lower bounds on $p_c(\mathcal{G},\eta)$ which can be obtained by a direct comparison with a Galton-Watson branching process. The next proposition shows that, provided that $\mathbf{E}\eta < \infty$, for small enough p (depending on η) the frog model dies out almost surely on any graph.

Proposition 1.1. *If $\mathbf{E}\eta < \infty$, then for arbitrary graph $\mathcal{G}$ it holds that $p_c(\mathcal{G},\eta) \geq (\mathbf{E}\eta + 1)^{-1}$.*

Proof. The set of active particles in the frog model is dominated by the population of the following Galton-Watson branching process. Each individual has a number of offspring distributed as $(\eta+1)\xi$, where the random variable ξ is independent of η, and $\mathbf{P}[\xi = 1] = p = 1 - \mathbf{P}[\xi = 0]$. Therefore, since the mean number of offspring by individual is $(1 + \mathbf{E}\eta)p$, the result follows by comparison with the Galton-Watson branching process. □

Next, again by comparison with Galton-Watson branching process, we give another lower bound to $p_c(\mathcal{G},\eta)$. This bound is better than the one presented in Proposition 1.1 for bounded degree graphs.

Proposition 1.2. *Suppose that $\mathcal{G}$ is a graph of maximum degree k, and $\mathbf{E}\eta < \infty$. Then*

$$p_c(\mathcal{G},\eta) \geq \frac{k}{1 + (k-1)(\mathbf{E}\eta + 1)}.$$

Proof. Consider a Galton-Watson branching process where particles produce no offspring with probability $1-p$, one offspring with probability p/k and the random number $\eta+1$ of offspring with probability $p(k-1)/k$. Observing that every site with at least one active particle at time $n>0$, has at least one neighbor site whose original particle(s) has been activated prior to time n, one gets that the frog model is dominated by the Galton-Watson process just defined. An elementary calculation shows that if $p < k(1+(k-1)(\mathbf{E}\eta+1))^{-1}$, the mean offspring in the Galton-Watson process defined above is less than one, therefore it dies out almost surely. So, the same happens to the frog model. □

Before going further, let us underline that in fact we are dealing with percolation. Indeed, let

$$\mathcal{R}_x^i = \{S_n^x(i) : 0 \le n < \Xi_p^x(i)\} \subset \mathcal{G}$$

be the "virtual" set of sites visited by the i-th particle placed originally at x. The set $\mathcal{R}_x^i$ becomes "real" in the case when x is actually visited (and thus all the sleeping particles from there are activated). We define the (virtual) range of site x by

$$\mathcal{R}_x := \begin{cases} \bigcup_{i=1}^{\eta(x)} \mathcal{R}_x^i, & \text{if } \eta(x) > 0, \\ \{x\}, & \text{if } \eta(x) = 0. \end{cases}$$

Notice that the frog model survives if and only if there exists an infinite sequence of distinct sites $\mathbf{0} = x_0, x_1, x_2, \dots$ such that, for all j,

$$x_{j+1} \in \mathcal{R}_{x_j}. \tag{1.1}$$

The last observation shows that the extinction of the frog model is equivalent to the finiteness of the cluster of $\mathbf{0}$ in the following oriented percolation model: from each site x the oriented edges are drawn to all the sites of the set $\mathcal{R}_x$. This approach is the key for the proof of most of the results of this paper.

Next we state the main results of this paper. The proofs are given in Section 2.

1.1 Extinction and survival of the process

We begin by showing that, under mild conditions on the initial number of particles, the process dies out a.s. (i.e., there is no percolation) in $\mathbb{Z}$ for every $p<1$. From now on, $a \vee b$ stands for $\max\{a,b\}$.

Theorem 1.1. *If* $\mathbf{E}\log(\eta \vee 1) < \infty$, *then* $p_c(\mathbb{Z}, \eta) = 1$.

Next, we find sufficient conditions to guarantee that the process becomes extinct for p small enough in $\mathbb{Z}^d$, $d \ge 2$, and in $\mathbb{T}_d$, $d \ge 3$.

Theorem 1.2. *Suppose that there exists* $\delta > 0$ *such that* $\mathbf{E}\eta^\delta < \infty$. *Then* $p_c(\mathbb{T}_d, \eta) > 0$, *i.e., the process on* $\mathbb{T}_d$ *dies out a.s. for* $p > 0$ *small enough.*

Theorem 1.3. *Suppose that* $\mathbf{E}(\log(\eta \vee 1))^d < \infty$. *Then* $p_c(\mathbb{Z}^d, \eta) > 0$.

Now, let us state the results related to the survival of the process. First, we show that for nontrivial η the frog model survives on $\mathbb{Z}^d$, $d \geq 2$, and on $\mathbb{T}_d$, when the parameter p is close enough to 1.

Theorem 1.4. *If* $\mathbf{P}[\eta \geq 1] > 0$*, then* $p_c(\mathbb{Z}^d, \eta) < 1$ *for all* $d \geq 2$*.*

Theorem 1.5. *If* $\mathbf{P}[\eta \geq 1] > 0$*, then* $p_c(\mathbb{T}_d, \eta) < 1$ *for all* $d \geq 3$*.*

Now we state the counterpart of Theorem 1.2. Note that Theorems 1.2 and 1.6 give the complete classification in η of the frog model on $\mathbb{T}_d$ from the point of view of positivity of $p_c(\mathbb{T}_d, \eta)$.

Theorem 1.6. *If* $\mathbf{E}\eta^\delta = \infty$ *for every* $\delta > 0$*, then* $p_c(\mathbb{T}_d, \eta) = 0$*.*

Besides, we are able to show that, for every fixed d, if η has a sufficiently heavy tail, then $\mathrm{FM}(\mathbb{Z}^d, p, \eta)$ survives with positive probability for all values of $p \in (0,1)$ (which would be the counterpart of Theorem 1.3). However, we do not state this result now, as in Section 1.3 we will give a stronger result (cf. Theorem 1.12).

1.2 Asymptotics for p_c

The following two theorems give asymptotic values for critical parameters (compare with Propositions 1.1 and 1.2) for the case of $\mathbb{Z}^d$ and regular trees.

Theorem 1.7. *We have, for the case* $\mathbf{E}\eta < \infty$*,*

$$\lim_{d\to\infty} p_c(\mathbb{T}_d, \eta) = \frac{1}{1 + \mathbf{E}\eta}.$$

Theorem 1.8. *We have, for the case* $\mathbf{E}\eta < \infty$*,*

$$\lim_{d\to\infty} p_c(\mathbb{Z}^d, \eta) = \frac{1}{1 + \mathbf{E}\eta}.$$

Remark. Observe that by truncating η and using a simple coupling argument one gets that if $\mathbf{E}\eta = \infty$, then

$$\lim_{d\to\infty} p_c(\mathbb{T}_d, \eta) = \lim_{d\to\infty} p_c(\mathbb{Z}^d, \eta) = 0.$$

Note that Theorems 1.7 and 1.8 suggest that there is some monotonicity of the critical probability in dimension. Then, a natural question to ask is the following: Is it true that $p_c(\mathbb{Z}^d, \eta) \geq p_c(\mathbb{Z}^{d+1}, \eta)$ for all d (and can one substitute "$\geq$" by "$>$")? In fact, there is a more general question: if $\mathcal{G}_1 \subset \mathcal{G}_2$, is it true that $p_c(\mathcal{G}_1, \eta) \geq p_c(\mathcal{G}_2, \eta)$? The last question has a trivial negative answer if we construct $\mathcal{G}_2$ from $\mathcal{G}_1$ by adding loops; if loops are not allowed, then this question is open. Note that for percolation that inequality is trivial; even the strict inequality can be proved in a rather general situation, cf. Menshikov [7].

1.3 Other types of phase transition and generalizations

There are other types of phase transitions for this model which may be of interest. For example, let p be such that $p_c(\mathcal{G},1) < p < 1$ and η_q be a 0-1 random variable with $\mathbf{P}[\eta_q = 1] = 1 - \mathbf{P}[\eta_q = 0] = q$. Then, the following result holds:

Proposition 1.3. *There is a phase transition in q, i.e., $\mathrm{FM}(\mathcal{G}, p, \eta_q)$ dies out when q is small and survives when q is large.*

Proof. First, note that $\mathrm{FM}(\mathcal{G}, p, \eta_q)$ is dominated by the following Galton-Watson branching process: An individual has 0 offspring with probability $1-p$, 1 offspring with probability $p(1-q)$, and 2 offspring with probability pq. The mean offspring of this branching process is $p(1+q)$, so $\mathrm{FM}(\mathcal{G}, p, \eta_q)$ dies out if $q < -1 + 1/p$.

Let us prove that $\mathrm{FM}(\mathcal{G}, p, \eta_q)$ survives when $p > p_c(\mathcal{G}, 1)$ and q is close enough to 1. Indeed, this model dominates a model described in the following way: The process starts from the one-particle-per-site initial configuration, and on each step active particles decide *twice* whether to disappear, the first time with probability $1-q$, and the second time with probability $1-p$. The latter model is in fact $\mathrm{FM}(\mathcal{G}, pq, 1)$, so the model $\mathrm{FM}(\mathcal{G}, p, \eta_q)$ survives if $q > p_c(\mathcal{G}, 1)/p$. $\square$

One may also be interested in the study of other types of critical behaviour with respect to the parameter p. Consider the following

Definition 1.2. *The model $\mathrm{FM}(\mathcal{G}, p, \eta)$ is called recurrent if*

$$\mathbf{P}[\mathbf{0} \textit{ is hit infinitely often in } \mathrm{FM}(\mathcal{G}, p, \eta)] > 0.$$

Otherwise, the model is called transient.

Note that, even in the case when a single SRW on $\mathcal{G}$ is transient, it is still reasonable to expect that the frog model with $p = 1$ is recurrent. For example, for the model $\mathrm{FM}(\mathbb{Z}^d, 1, 1)$ the recurrence was established in [10]. However, establishing the recurrence property for that model is nontrivial; it is still unclear to us whether $\mathrm{FM}(\mathbb{T}_d, 1, 1)$ is recurrent. Now, denote

$$p_u(\mathcal{G}, \eta) = \inf\{p : \mathbf{P}[\mathbf{0} \text{ is hit infinitely often in } \mathrm{FM}(\mathcal{G}, p, \eta)] > 0\}$$

(here, by definition, $\inf \emptyset = 1$); clearly, $p_u(\mathcal{G}, \eta) \geq p_c(\mathcal{G}, \eta)$ for every $\mathcal{G}$ and η. Now, we are interested in studying the existence of phase transition with respect to p_u.

First, we discuss some situations when the model is transient for every p, except possibly the case $p = 1$.

Theorem 1.9. *Suppose that $\mathbf{E}\eta^{\varepsilon} < \infty$ for every $0 < \varepsilon < 1$. Then $p_u(\mathbb{T}_d, \eta) = 1$.*

Theorem 1.10. *Suppose that $\mathbf{E}[\log(\eta \vee 1)]^d < \infty$. Then $p_u(\mathbb{Z}^d, \eta) = 1$.*

The next two theorems give sufficient conditions to have $p_u < 1$ on trees and on $\mathbb{Z}^d$.

Theorem 1.11. *Suppose that there exists $\beta < \frac{\log(d-1)}{2\log d}$ such that*

$$\mathbf{P}[\eta \geq n] \geq \frac{1}{n^{\beta}} \tag{1.2}$$

for all n large enough. Then $p_u(\mathbb{T}_d, \eta) < 1$.

Theorem 1.12. *Suppose that there exist $\beta < d$ such that*

$$\mathbf{P}[\eta \geq n] \geq \frac{1}{(\log n)^{\beta}} \tag{1.3}$$

for all n large enough. Then $p_u(\mathbb{Z}^d, \eta) = 0$.

Remark. It is possible to see that, if $\mathbf{P}[\eta \geq n] \geq (\log n)^{-\beta}$ for some $\beta < 1$ and all n large enough, then $\mathrm{FM}(\mathcal{G}, p, \eta)$ is recurrent on *any* infinite connected graph $\mathcal{G}$ of bounded degree. Indeed, for arbitrary graph $\mathcal{G}$ of bounded degree we do the following: First, fix a subgraph $\mathcal{G}_1$ of $\mathcal{G}$, which is isomorphic to $\mathbb{Z}_+$. If k_0 is the maximal degree of $\mathcal{G}$, it is easy to see that $\mathrm{FM}(\mathcal{G}, p, \eta)$ dominates $\mathrm{FM}(\mathcal{G}_1, p/k_0, \eta)$ (if a particle wants to leave $\mathcal{G}_1$, we just erase this particle). Then, we just apply Theorem 1.12 for the case of $\mathcal{G} = \mathbb{Z}$ (from the proof of Theorem 1.12 one gets that the argument for the case of $\mathbb{Z}$ also works for $\mathbb{Z}_+$).

Theorems 1.11 and 1.12 give sufficient conditions on the tail of the distribution of η for the process to be recurrent when $p < 1$. On the other hand, Theorems 1.9 and 1.10 show that for the one-particle-per-site initial configuration the process is not recurrent even if the parameter $p < 1$ is very close to 1. The model with one-particle-per-site initial configuration being the most natural example one has to hand, a natural question is raised: What can be done (i.e., how can one modify the model) to make the model recurrent without augmenting the initial configuration? Notice that, by definition, in our model the lifetime of active particles is geometrically distributed. In order to find answers to that question, we are going to change this and study the situation when the lifetime has another distribution, possibly more heavy-tailed one.

Let Ξ be a nonnegative integer-valued random variable. From this moment on we study the frog model on $\mathcal{G}$ with one-particle-per-site initial configuration, and the lifetimes of particles after activation are i.i.d. random variables $(\Xi^x, x \in \mathcal{G})$ having the same law as Ξ. This model will be called $\mathrm{FM}(\mathcal{G}, \Xi)$.

Theorem 1.13. *Suppose that one of the following alternatives holds:*

- $\mathcal{G} = \mathbb{Z}$ *and* $\mathbf{E}\sqrt{\Xi} < \infty$,
- $\mathcal{G} = \mathbb{Z}^2$ *and* $\mathbf{E}\frac{\Xi}{\log(\Xi \vee 2)} < \infty$,
- $\mathcal{G} = \mathbb{T}_d$ *or* $\mathbb{Z}^d, d \geq 3$ *and* $\mathbf{E}\,\Xi < \infty$.

Then $\mathrm{FM}(\mathcal{G}, \Xi)$ is transient.

Theorem 1.14. *For each dimension d there exists $\beta_d > 0$ such that if for all n large enough one of the following alternatives holds*

- $d = 1$ *and* $\mathbf{P}[\Xi \geq n^2] \geq \beta_1 n^{-1} \log\log n$,
- $d = 2$ *and* $\mathbf{P}[\Xi \geq n^2] \geq \beta_2 n^{-2} (\log n)^2$,
- $d \geq 3$ *and* $\mathbf{P}[\Xi \geq n^2] \geq \beta_d n^{-2} \log n$,

then $\mathrm{FM}(\mathbb{Z}^d, \Xi)$ is recurrent.

In fact, results of Popov [8] suggest that the following is true: For $d \geq 3$ there exist $\tilde{\alpha}_0(d), \tilde{\alpha}_1(d)$ such that if $\mathbf{P}[\Xi \geq n^2] \leq \tilde{\alpha}_0(d)n^{-2}$ for all n large enough, then $\mathrm{FM}(\mathbb{Z}^d, \Xi)$ is transient, and if $\mathbf{P}[\Xi \geq n^2] \geq \tilde{\alpha}_1(d)n^{-2}$ for all n large enough, then $\mathrm{FM}(\mathbb{Z}^d, \Xi)$ is recurrent. The heuristic explanation for this is as follows. The particle originally in x has a good chance (i.e., comparable with $\|x\|_2^{2-d}$, where $\|\cdot\|_2$ is the Euclidean norm) of ever getting to the origin only if it lives at least of order $\|x\|_2^2$ units of time (cf. Lemma 2.4 below), so one may expect that $\mathrm{FM}(\mathbb{Z}^d, \Xi)$ behaves roughly as the frog model with infinite lifetime of the particles and the initial configuration of sleeping particles constructed as follows: we add a sleeping particle into x with probability $h(x) := \mathbf{P}[\Xi \geq n^2]$, and add nothing with probability $1-h(x)$. For the case when $h(x) \simeq \alpha/\|x\|_2^2$ the latter model was studied in [8] and it was proved that it is recurrent when α is large and transient when α is small (note that the transience also can be proved by dominating the frog model by a branching random walk, cf. e.g. den Hollander et al. [3]). However, turning this heuristics into a rigorous proof is presently beyond our reach.

2 Proofs

2.1 Preliminaries

Here we state a few basic facts which will be necessary later in the Sections 2.2, 2.3, and 2.4.

For $0 \leq p \leq 1$ and integer numbers $k, i \geq 1$ denote $\Phi(i,k,p) = 1 - (1-p^k)^i$ and $\hat{k}(i,p) = \lfloor \log i / \log(1/p) \rfloor$, where $\lfloor x \rfloor$ stands for the largest integer which is less than or equal to x. The following fact can be easily obtained by using elementary calculus and is stated without proof.

Lemma 2.1. *There exist constants $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ such that for all i, p*

$$\hat{\beta}_1 \leq \Phi(i,k,p) \leq 1$$

for $k \leq \hat{k}(i,p)$ and

$$\hat{\beta}_2 p^{k-\hat{k}(i,p)} \leq \Phi(i,k,p) \leq \hat{\beta}_3 p^{k-\hat{k}(i,p)-1}$$

for $k \geq \hat{k}(i,p) + 1$.

In the sequel we will make use of the following large deviation result:

Lemma 2.2 (Shiryaev [9], p. 68.). *Let $\{X_i, i \geq 1\}$ be i.i.d. random variables with $\mathbf{P}[X_i = 1] = p$ and $\mathbf{P}[X_i = 0] = 1-p$. Then for all $0 < p < a < 1$ and for all $N \geq 1$ we have*

$$\mathbf{P}\Big[\frac{1}{N}\sum_{i=1}^{N} X_i \geq a\Big] \leq \exp\{-NH(a,p)\}, \tag{2.1}$$

where

$$H(a,p) = a \log \frac{a}{p} + (1-a) \log \frac{1-a}{1-p} > 0.$$

If $0 < a < p < 1$, then (2.1) holds with $\mathbf{P}[N^{-1}\sum_{i=1}^{N} X_i \leq a]$ in the left-hand side.

In order to prove Theorems 1.4 and 1.8 we need some auxiliary fact about projections of percolation models. Let $\mathcal{H}^d := \{H_x^d : x \in \mathbb{Z}^d\}$ be a collection of random sets such that $x \in H_x^d$ and the sets $H_x^d - x$, $x \in \mathbb{Z}^d$, are i.i.d. By using the sets of $\mathcal{H}^d$, define an oriented percolation process on $\mathbb{Z}^d$ analogously to what was done for the frog model (compare with (1.1)): If there is an infinite sequence of distinct sites, $\mathbf{0} = x_0, x_1, x_2, \dots$, such that $x_{j+1} \in H_{x_j}^d$ for all $j = 0, 1, 2, \dots$, we say that the cluster of the origin is infinite or, equivalently, that $\mathcal{H}^d$ survives.

Let $\Lambda := \{x \in \mathbb{Z}^d : x^{(k)} = 0 \text{ for } k \geq 3\} \subset \mathbb{Z}^d$ be a copy of $\mathbb{Z}^2$ immersed into $\mathbb{Z}^d$ and $\mathfrak{P} : \mathbb{Z}^d \to \Lambda$ be the projection on the first two coordinates. Let $\mathcal{H}^2 := \{H_x^2 : x \in \Lambda\}$ be a collection of random sets such that $x \in H_x^2$ and the sets $H_x^2 - x$, $x \in \Lambda$, are i.i.d. Analogously, one defines the percolation of the collection $\mathcal{H}^2$.

Lemma 2.3. *Suppose that there is a coupling of $H_{\mathbf{0}}^d$ and $H_{\mathbf{0}}^2$ such that*

$$\mathfrak{P}(H_{\mathbf{0}}^d) \supset H_{\mathbf{0}}^2, \tag{2.2}$$

i.e., the projection of $H_{\mathbf{0}}^d$ dominates $H_{\mathbf{0}}^2$. Then

$$\mathbf{P}[\mathcal{H}^d \text{ survives}] \geq \mathbf{P}[\mathcal{H}^2 \text{ survives}].$$

Proof. The proof of this fact is standard and can be done by carefully growing the cluster in Λ step by step, and comparing it with the corresponding process in $\mathbb{Z}^d$. See e.g. Menshikov [6] for details. ☐

Let $\hat{q}(n, x)$ be the probability that a SRW (starting from the origin) hits x until the moment n. The following fact about hitting probabilities of SRW is proved in [1], Theorem 2.2 (except for the case $d = 1$).

Lemma 2.4.

- *If $d = 1, x \neq 0$ and $n \geq \|x\|_2^2$, then there exists a number $w_1 > 0$ such that*

$$\hat{q}(n, x) \geq w_1. \tag{2.3}$$

- *If $d = 2, x \neq 0$ and $n \geq \|x\|_2^2$, then there exists a number $w_2 > 0$ such that*

$$\hat{q}(n, x) \geq \frac{w_2}{\log \|x\|_2}. \tag{2.4}$$

- *Suppose that $d \geq 3, x \neq 0$ and $n \geq \|x\|_2^2$. Then there exists a collection of positive numbers $w_d > 0$, $d \geq 3$, such that*

$$\hat{q}(n, x) \geq \frac{w_d}{\|x\|_2^{d-2}}. \tag{2.5}$$

Proof. To keep the paper self-contained, we give the proof of this fact. Let $\hat{p}_n(x)$ be the probability that the SRW is in x at time n, and τ_x be the moment of the first hitting of x. Also, denote by $G_n(x) = \sum_{k=0}^{n} \hat{p}_k(x)$ the mean number of visits to x until the moment n ($G_n(x)$ is usually called *Green's function*).

Suppose without loss of generality that $\|x\|_2^2 \le n < \|x\|_2^2 + 1$. Observe that

$$
\begin{aligned}
G_n(x) &= \sum_{j=0}^{n} \hat{p}_j(x) = \sum_{j=0}^{n}\sum_{k=0}^{j} \hat{p}_k(0)\mathbf{P}[\tau_x = j-k] \\
&= \sum_{k=0}^{n} \hat{p}_k(0)\hat{q}(n-k,x) \le \hat{q}(n,x)G_n(0).
\end{aligned}
$$

So

$$
\hat{q}(n,x) \ge \frac{G_n(x)}{G_n(0)} \ge \begin{cases} \dfrac{\sum_{j=\lfloor n/2\rfloor}^{n} \hat{p}_j(x)}{\sum_{j=0}^{n} \hat{p}_j(0)}, & d=1,2, \\ (G_\infty(0))^{-1}\sum_{j=\lfloor n/2\rfloor}^{n} \hat{p}_j(x), & d \ge 3. \end{cases}
$$

Using Theorem 1.2.1 of [5], after some elementary computations we finish the proof. □

2.2 Extinction and survival

Proof of Theorem 1.1. Notice that, for any graph $\mathcal{G}$ and all $x \neq y \in \mathcal{G}$, the following inequality holds:

$$
\mathbf{P}[y \in \mathcal{R}_x^1] \le p^{\operatorname{dist}(x,y)}. \tag{2.6}
$$

Clearly, for a fixed $y \in \mathbb{Z}$, we have

$$
\mathbf{P}[y \notin \mathcal{R}_x \text{ for all } x \neq y] = \prod_{x:x\neq y} (1 - \mathbf{P}[y \in \mathcal{R}_x]),
$$

and so the left-hand side of the above display is positive if and only if $\sum_{x:x\neq y} \mathbf{P}[y \in \mathcal{R}_x] < \infty$. Now, by using (2.6) and Lemma 2.1, for some $C_1, C_2 > 0$ (depending only on p) one gets

$$
\begin{aligned}
\sum_{x:x\neq y} \mathbf{P}[y \in \mathcal{R}_x] &= 2\sum_{x\ge 1} \mathbf{P}[\mathbf{0} \in \mathcal{R}_x] \\
&\le 2\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \gamma_i \Phi(i,k,p) \\
&= 2\sum_{i=1}^{\infty} \gamma_i \Big(\sum_{k \le \hat{k}(i,p)} \Phi(i,k,p) + \sum_{k \ge \hat{k}(i,p)+1} \Phi(i,k,p) \Big) \\
&\le 2\sum_{i=1}^{\infty} \gamma_i (C_1 \log i + C_2) < \infty.
\end{aligned}
$$

Thus, $\mathbf{P}[y \notin \mathcal{R}_x \text{ for all } x \neq y] > 0$, so, by the ergodic theorem there is an infinite sequence of sites $\cdots < y_{-1} < y_0 < y_1 < \cdots$ such that for all i, $y_i \notin \mathcal{R}_x$ for all $x \neq y_i$. Therefore, for almost every realization there is an infinite number of blocks of sites without "communication" with its exterior, which prevents the active particles to spread out. The result follows. □

Proof of Theorems 1.2 and 1.3. For $\mathcal{G} = \mathbb{Z}^d$ or $\mathbb{T}_d$ denote $s_k(\mathcal{G}) = |\{y \in \mathcal{G} : \operatorname{dist}(x,y) = k\}|$ (note that the right-hand side does not depend on the choice of the site x). Using (2.6) and

Lemma 2.1, one gets that for some positive constants C_1, C_2, C_3, C_4

$$\begin{aligned}
\mathbf{E}|\mathcal{R}_x \setminus \{x\}| &= \sum_{y:y\neq x} \mathbf{P}[y \in \mathcal{R}_x] \\
&= \sum_{y:y\neq x} \sum_{i=1}^{\infty} \gamma_i \mathbf{P}[y \in \mathcal{R}_x \mid \eta(x) = i] \\
&\leq \sum_{i=1}^{\infty} \gamma_i \sum_{k=1}^{\infty} s_k(\mathcal{G}) \Phi(i,k,p) \\
&\leq \sum_{i=1}^{\infty} \gamma_i \Big(\sum_{k=1}^{\hat{k}(i,p)} s_k(\mathcal{G}) + \sum_{k=1}^{\infty} \hat{\beta}_3 s_k(\mathcal{G}) p^{k-1} \Big) \\
&\leq \begin{cases} C_1 \sum_{i=1}^{\infty} \gamma_i i^{\frac{\log(d-1)}{\log(1/p)}} + C_2, & \mathcal{G} = \mathbb{T}_d, \\ C_3 \sum_{i=1}^{\infty} \gamma_i \Big(\dfrac{\log i}{\log(1/p)} \Big)^d + C_4, & \mathcal{G} = \mathbb{Z}^d, \end{cases}
\end{aligned}$$

which is finite for all $p < 1$ in the case $\mathcal{G} = \mathbb{Z}^d$ and for p small enough in the case of $\mathcal{G} = \mathbb{T}_d$. Now, as for some $p_0 > 0$ (which may depend on the graph $\mathcal{G}$) the convergence is uniform in $[0, p_0]$, there exists small enough p (which depends on $\mathcal{G}$) such that $\mathbf{E}|\mathcal{R}_x \setminus \{x\}| < 1$ for $\mathrm{FM}(\mathcal{G}, p, \eta)$, so one gets the proof by means of domination by a subcritical branching process. $\square$

In order to prove Theorems 1.4 and 1.5 it is enough to show that, for p large enough, the frog model survives with positive probability in $\mathbb{Z}^d$, $d \geq 2$, and in $\mathbb{T}_d$, $d \geq 3$. Let us define the modified initial configuration η' by

$$\eta'(x) = \mathbf{1}_{\{\eta(x) \geq 1\}}.$$

Since $\mathrm{FM}(\mathcal{G}, p, \eta)$ dominates $\mathrm{FM}(\mathcal{G}, p, \eta')$, without loss of generality we prove Theorems 1.4 and 1.5 assuming that the initial configuration is given by $\{\eta'(x) : x \in \mathcal{G}\}$.

Proof of Theorem 1.4. We start by considering the two dimensional frog model $\mathrm{FM}(\mathbb{Z}^2, p, \eta)$ which is equivalent to $\mathrm{FM}(\Lambda, p, \eta)$, since Λ is a copy of $\mathbb{Z}^2$ (recall the notation Λ from Section 2.1, it was introduced just before Lemma 2.3). It is a well-known fact that the two-dimensional SRW (no death) is recurrent. Then, given $N \in \mathbb{N}$ and assuming $\eta'(\mathbf{0}) = 1$, for sufficiently large $p = p(N)$, the probability that the first active particle hits all the sites in the square $[-2N, 2N]^2 \cap \mathbb{Z}^2$ before dying can be made arbitrarily large. Besides, the probability that there is a site $x \in [0, N)^2 \cap \mathbb{Z}^2$ such that $\eta'(x) = 1$, also can be made arbitrarily large by means of increasing N.

Let us define now a two-dimensional percolation process in the following way. Divide $\mathbb{Z}^2$ into disjoint squares of side N, i.e., write

$$\mathbb{Z}^2 = \bigcup_{(r;k)\in\mathbb{Z}^2} Q(r,k),$$

where $Q(r,k) = (rN;kN) + [0,N)^2 \cap \mathbb{Z}^2$. Declare $Q(r,k)$ open if the following happens (and closed otherwise)

$$\bigcup_{x\in Q(r,k)} (\{\eta'(x)=1\} \cap \{\mathcal{R}_x^1 \supseteq ([-2N,2N]^2 + x) \cap \mathbb{Z}^2\}) \neq \emptyset.$$

Observe that the events $\{Q(r,k) \text{ is open}\}$, $(r,k) \in \mathbb{Z}^2$, are independent. Notice that the frog model dominates this percolation process in the sense that if there is percolation then the frog model survives. It is not difficult to see that by suitably choosing N and p it is possible to make $\mathbf{P}[Q(r,k) \text{ is open}]$ arbitrarily close to 1, so the percolation process can be made supercritical, and thus the result follows for $\mathbb{Z}^2$.

Now, by using Lemma 2.3, we give the proof for dimensions $d \geq 3$. Let $\{\mathcal{R}_y(p,d) : y \in \mathbb{Z}^d\}$ be the collection of the ranges for the d dimensional frog model. For the moment we write $\mathcal{R}_y(p,d)$ instead of $\mathcal{R}_y$ to keep track of the dimension and the survival parameter. Analogously, let $\{\mathcal{R}_y(p,2) : y \in \Lambda\}$ be the collection of the ranges for the two dimensional frog model immersed in $\mathbb{Z}^d$. Notice that $\mathfrak{P}(\mathcal{R}_x(p,d)) \subset \Lambda$ is distributed as $\mathcal{R}_{\mathfrak{P}(x)}(p',2)$ for $p' = 2p/(d(1-p)+2p)$, where, as before, $\mathfrak{P} : \mathbb{Z}^d \to \Lambda$ is the projection on the first two coordinates. Since the fact $p' < 1$ implies that $p < 1$, and, as we just have proven, $p_c(\mathbb{Z}^2, \eta) < 1$ when $\mathbf{E}\eta < \infty$, by using Lemma 2.3 we finish the proof of Theorem 1.4. $\square$

Proof of Theorem 1.5. As in the previous theorem, we work with η' instead of η. In order to prove the result, we need some additional notation. Notice that for every $a \in \mathbb{T}_d$ there is a unique path connecting a to $\mathbf{0}$; we write $a \geq b$ if b belongs to that path. For $a \neq \mathbf{0}$ denote $\mathbb{T}_d^+(a) = \{b \in \mathbb{T}_d : b \geq a\}$. Fix an arbitrary site a_0 adjacent to the root and let $\mathbb{T}_d^+ = \mathbb{T}_d \backslash \mathbb{T}_d^+(a_0)$. For any $A \subset \mathbb{T}_d$ let us define the *external boundary* $\partial_e(A)$ in the following way:

$$\partial_e(A) = A \setminus \{a \in A : \text{ there exists } b \in A \text{ such that } b > a\}.$$

A useful fact is that if A, B are finite and $A \subset B$, then $|\partial_e(A)| \leq |\partial_e(B)|$. Now, denote by W_t the set of sites visited until time t by a SRW (no death) in $\mathbb{T}_d$ starting from $\mathbf{0}$. Note that

- as SRW on tree is transient, one gets that with positive probability $W_t \subset \mathbb{T}_d^+$ for all t;
- $|\partial_e(W_t)|$ is a nondecreasing sequence, and, moreover, it is not difficult to show that $|\partial_e(W_t)| \to \infty$ a.s. as $t \to \infty$.

The above facts show that for p large enough

$$\mathbf{E}|\partial_e(\mathcal{R}_\mathbf{0}) \cap \mathbb{T}_d^+| > \frac{1}{\mathbf{P}[\eta' > 0]}. \tag{2.7}$$

Now, all the initially sleeping particles in $\partial_e(\mathcal{R}_\mathbf{0}) \cap \mathbb{T}_d^+$ are viewed as the offspring of the first particle. By using (2.7) together with the fact that for all $x, y \in \partial_e(\mathcal{R}_\mathbf{0})$ such that $x \neq y$, we have $\mathbb{T}_d^+(x) \cap \mathbb{T}_d^+(y) = \emptyset$, one gets that the frog model dominates a Galton-Watson branching process with mean offspring greater than 1, thus concluding the proof of Theorem 1.5. $\square$

Proof of Theorem 1.6. First, note the following fact: For any graph $\mathcal{G}$ with maximal degree d, it is true that

$$\mathbf{P}[y \in \mathcal{R}_x^1] \geq \left(\frac{p}{d}\right)^{\mathrm{dist}(x,y)}. \tag{2.8}$$

Keeping the notation $\mathbb{T}_d^+(x)$ from the proof of Theorem 1.5, denote

$$L_k(x) = \{y \in \mathbb{T}_d^+(x) : \mathrm{dist}(x, y) = k\}.$$

Using (2.8) and Lemma 2.1, we have

$$\begin{aligned}\mathbf{E}|\mathcal{R}_x \cap L_{\hat{k}(i,p/d)}(x)| &\geq \sum_{i=1}^{\infty} \gamma_i (d-1)^{\hat{k}(i,p/d)} \Phi(i, \hat{k}(i,p/d), p/d) \\ &\geq \hat{\beta}_1 \sum_{i=1}^{\infty} \gamma_i i^{\frac{\log(d-1)}{\log(d/p)}} = \infty,\end{aligned}$$

so, by dominating a supercritical branching process by the frog model, one concludes the proof.
□

2.3 Asymptotics for the critical parameter

Proof of Theorem 1.7. By Proposition 1.1, $p_c(\mathbb{T}_d, \eta) \geq (1 + \mathbf{E}\eta)^{-1}$. So, it is enough to show that, fixing $p > (1 + \mathbf{E}\eta)^{-1}$, the model $\mathrm{FM}(\mathbb{T}_d, p, \eta)$ survives for d large enough.

Let us define

$$\eta^{(s)} = \begin{cases} \eta, & \text{if } \eta \leq s, \\ 0, & \text{if } \eta > s. \end{cases} \tag{2.9}$$

By the monotone convergence theorem it follows that $\mathbf{E}\eta^{(s)} \to \mathbf{E}\eta$ as $s \to \infty$, so, if $p > (1 + \mathbf{E}\eta)^{-1}$, then it is possible to choose s large enough so that $p > (1 + \mathbf{E}\eta^{(s)})^{-1}$. Fixed s and p, notice that $\mathrm{FM}(\mathbb{T}_d, p, \eta)$ dominates $\mathrm{FM}(\mathbb{T}_d, p, \eta^{(s)})$ in the sense that if the latter survives with positive probability, the same happens to the former. Therefore, it is enough to show that if $p > (1 + \mathbf{E}\eta^{(s)})^{-1}$, then $\mathrm{FM}(\mathbb{T}_d, p, \eta^{(s)})$ survives for d sufficiently large.

Let ξ_n be the set of active particles of $\mathrm{FM}(\mathbb{T}_d, p, \eta^{(s)})$, which are at level n (i.e., at distance n from the root) at time n. Next we prove that there exists a discrete time supercritical branching process, which is dominated by ξ_n. We do this by constructing an auxiliary process $\tilde{\xi}_n \subset \xi_n$. First of all, initially the particle(s) in $\mathbf{0}$ belong(s) to $\tilde{\xi}_0$. In general, the process $\tilde{\xi}_n$ is constructed by the following rules. If at time $n-1$ the set of particles $\tilde{\xi}_{n-1}$, which lives on the level $n-1$, is constructed, then at time n the set of particles $\tilde{\xi}_n$ (which all are at level n) is constructed in the following way. Introduce some ordering of the particles of $\tilde{\xi}_{n-1}$, they will be allowed to jump according to that order. Now, if the current particle survives, then

- if the particle jumps to some site of level n and does not encounter any particles that already belong to $\tilde{\xi}_n$ there, then this particle as well as all the particles possibly activated by it enter to $\tilde{\xi}_n$;
- otherwise it is deleted.

The particles of $\tilde{\xi}_{n+1}$ activated by some particle from $\tilde{\xi}_n$ are considered as the offspring of that particle; note that, due to the asynchronous construction of the process $\tilde{\xi}_n$, each particle has exactly one ancestor. Note also that the process $\tilde{\xi}_n$ was constructed in such a way that each site can be occupied by at most $s+1$ particles from $\tilde{\xi}_n$. So, it follows that process $\tilde{\xi}_n$ dominates a Galton-Watson process with mean offspring being greater than or equal to

$$\frac{d-1-s}{d}(\mathbf{E}\eta^{(s)}+1)p$$

(the "worst case" for a particle from $\tilde{\xi}_n$ is when it shares its site with another s particles from $\tilde{\xi}_n$, and all those particles have already jumped to the different sites of level $n+1$). Therefore, since $p > (\mathbf{E}\eta^{(s)}+1)^{-1}$, choosing d sufficiently large, one guarantees the survival of the process $\tilde{\xi}_n$. This concludes the proof of Theorem 1.7. □

Theorem 1.8 is a consequence of the following lemma.

Lemma 2.5. *Denote*

$$\mathcal{K} := \{x \in \mathbb{Z}^d : \max_{1\le i\le d} |x^{(i)}| \le 1\},$$

and consider $\mathrm{FM}(\mathbb{Z}^d, p, \eta)$*, where* $p > (1+\mathbf{E}\eta)^{-1}$*, restricted on* $\mathcal{K}$ *(this means that if a particle attempts to jump outside* $\mathcal{K}$*, then it disappears). There are constants* d_0*,* $a > 0$ *and* $\mu > 1$ *such that if* $d \ge d_0$*, then with probability greater than* a*, at time* $\sqrt{d}$ *there are more than* $\mu^{\sqrt{d}}$ *active particles in* $\mathcal{K}$*.*

Proof. First observe that it is enough to prove the lemma for $\mathrm{FM}(\mathbb{Z}^d, p, \eta^{(s)})$ with $\eta^{(s)}$ defined by (2.9), where s is such that $p > (1+\mathbf{E}\eta^{(s)})^{-1}$. Let us consider the sets

$$\mathcal{S}_k := \Big\{x \in \mathcal{K} : \sum_{i=1}^{d} |x^{(i)}| = k\Big\},$$

$k = 0, \dots, d$ and define ξ_k as the set of active particles which are in $\mathcal{S}_k$ at instant k. Similarly to the proof of Theorem 1.7, the idea is to show that up to time $\sqrt{d}$ the process ξ_k dominates a supercritical branching process to be defined later.

Let $x \in \mathcal{S}_k$ and $y \in \mathcal{S}_{k+1}$ be such that $\|x-y\| = 1$, where $\|x\| = \sum_{i=1}^{d} |x^{(i)}|$. Notice that if site x contains an active particle at instant k, then this particle can jump into y at the next instant of time. Keeping this in mind we define for $x \in \mathcal{S}_k$

$$\mathcal{E}_x := \Big\{z \in \mathcal{S}_k : \sum_{i=1}^{d} \mathbf{1}_{\{|x^{(i)}-z^{(i)}|\neq 0\}} = 2 \text{ and } \|x-z\| = 2\Big\}$$

called the set of the "enemies" of x. Observe that for $x \in \mathcal{S}_k$ and $z \in \mathcal{E}_x$ there exists $y \in \mathcal{S}_{k+1}$ such that $\|x-y\| = \|z-y\| = 1$ which in words means that if sites x and z have active particles at instant k, then these particles can jump into the same site next step. Moreover, for fixed x and z, the site y is the only one in $\mathcal{S}_{k+1}$ with this property and there are exactly $k+1$ sites in $\mathcal{S}_k$ whose particles might jump into y in one step. Notice also that $|\mathcal{E}_x| = 2k(d-k)$.

Let

$$\mathcal{D}_x := \{y \in \mathcal{S}_{k+1} : \|x-y\| = 1\}$$

be the set of "descendants" of $x \in \mathcal{S}_k$. It is a fact that $|\mathcal{D}_x| = 2(d-k)$. Finally, we define for $y \in \mathcal{S}_{k+1}$

$$\mathcal{A}_y := \{x \in \mathcal{S}_k : \|x-y\| = 1\},$$

called the set of "ancestors" of y, observing that for $x \in \mathcal{S}_k$

$$\mathcal{E}_x = \bigcup_{y \in \mathcal{D}_x} (\mathcal{A}_y \setminus \{x\}) \qquad \text{is a disjoint union,} \tag{2.10}$$

and $|\mathcal{A}_y| = k+1$ for all $y \in \mathcal{S}_{k+1}$.

Note that a single site $x \in \mathcal{S}_k$ can contain various particles from ξ_k. Now (as in the proof of Theorem 1.7) we define a process $\tilde{\xi}_k \subset \xi_k$ in the following way. First, initially the particle(s) in $\mathbf{0}$ belong(s) to $\tilde{\xi}_0$. If at time k the set of particles $\tilde{\xi}_k$ (which live in $\mathcal{S}_k$) was constructed, then at time $k+1$ the set of particles $\tilde{\xi}_{k+1}$ (which live in $\mathcal{S}_{k+1}$) is constructed in the following way. Introduce some ordering of the particles of $\tilde{\xi}_k$, they will be allowed to jump according to that order. Now, if the current particle survives, then

- if the particle jumps to some site of $\mathcal{S}_{k+1}$ and does not encounter any particles that already belong to $\tilde{\xi}_{k+1}$ there, then this particle as well as all the particles possibly activated by it enter to $\tilde{\xi}_{k+1}$;
- otherwise it is deleted.

For $x \in \mathcal{S}_k$ define $\mathcal{X}(x)$ to be the number of particles from $\tilde{\xi}_k$ in the site x. Note that, by construction, $0 \leq \mathcal{X}(x) \leq s+1$ for all x and k. For $x \in \mathcal{S}_k$ and $y \in \mathcal{D}_x$ we denote by $(x \to y)$ the event

$$\left\{ \begin{array}{c} \mathcal{X}(x) \geq 1 \text{ and at least one particle} \\ \text{from } \tilde{\xi}_k \text{ jumps from } x \text{ to } y \text{ at time } k+1 \end{array} \right\},$$

and let ζ_{xy}^k be the indicator function of the event

$$\{\text{there is } z \in \mathcal{E}_x \text{ such that } (z \to y)\}.$$

Picking $k \leq \sqrt{d}$, it is true that

$$\mathbf{P}[\zeta_{xy}^k = 1] \leq \mathbf{P}^*[\zeta_{xy}^k = 1] \leq \frac{C_1 k}{d} \leq C_1 (\sqrt{d})^{-1}$$

for some positive constant $C_1 = C_1(s)$, where

$$\mathbf{P}^*[\ \cdot\ ] = \mathbf{P}[\ \cdot \mid \mathcal{X}(z) = s+1 \text{ for all } z \in \mathcal{E}_x].$$

So, given an arbitrary $\sigma > 0$, it is possible to choose d so large that $\mathbf{P}^*[\zeta_{xy}^k = 1] < \sigma$ for $k \leq \sqrt{d}$. With this choice for d, if ζ_x^k is the indicator function of the event

$$\Big\{ |\mathcal{D}_x \cap \{y \in \mathcal{S}_{k+1} : \text{ there exists } z \in \mathcal{S}_k \setminus \{x\} \text{ such that } (z \to y)\}| > 2\sigma d \Big\},$$

then it follows that

$$\mathbf{P}[\zeta_x^k = 1] = \mathbf{P}\Big[\sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > 2\sigma d \Big].$$

Notice that by (2.10) the random variables $\{\zeta_{xy}^k : y \in \mathcal{D}_x\}$ are independent with respect to $\mathbf{P}^*$. Therefore, by Lemma 2.2, we get for $k \le \sqrt{d}$

$$\begin{aligned}
\mathbf{P}[\zeta_x^k = 1] &\le \mathbf{P}^*\Big[\sum_{y\in\mathcal{D}_x} \zeta_{xy}^k > 2\sigma d\Big] \\
&= \mathbf{P}^*\Big[\frac{\sum_{y\in\mathcal{D}_x} \zeta_{xy}^k}{2(d-k)} > \frac{2d\sigma}{2(d-k)}\Big] \\
&\le \mathbf{P}^*\Big[\frac{\sum_{y\in\mathcal{D}_x} \zeta_{xy}^k}{2(d-k)} > \sigma\Big] \\
&\le \exp\{-2C_2(d-k)\} \\
&\le \exp\{-C_3 d\},
\end{aligned} \tag{2.11}$$

with some positive constants C_2, C_3, which depend only on σ. Let us define the following event

$$B := \bigcup_{k=1}^{\sqrt{d}} \bigcup_{x\in\mathcal{S}_k} \{\zeta_x^k = 1\}.$$

Since $\eta^{(s)} \le s$ we have that $|\tilde{\xi}_k| \le (s+1)^{k+1}$. Therefore, from (2.11) it follows that

$$\mathbf{P}[B] \le \sqrt{d} \times (s+1)^{\sqrt{d}+1} \exp\{-C_3 d\},$$

and, as a consequence, $\mathbf{P}[B]$ can be made arbitrarily small for fixed σ and d large enough.

Suppose that the event B^c happens. In this case, since each site can be occupied by at most $s+1$ particles from $\tilde{\xi}_k$, for each $x \in \mathcal{S}_k$ there are at least $2(d-\sqrt{d}) - 2\sigma d - s$ *available sites* (i.e., sites which do not yet contain any particle from $\tilde{\xi}_{k+1}$) in $\mathcal{S}_{k+1}$ into which a particle from $\tilde{\xi}_k$ placed at site x could jump. So, it follows that up to time $\sqrt{d}$, the process $\tilde{\xi}_k$ dominates a Galton-Watson branching process with mean offspring being greater than or equal to

$$\frac{(2(d-\sqrt{d}) - 2\sigma d - s)(\mathbf{E}\eta^{(s)} + 1)p}{2d}. \tag{2.12}$$

Pick σ small enough and d large enough to make (2.12) greater than 1. The lemma follows since with positive probability a supercritical branching process grows exponentially in time. $\square$

Proof of Theorem 1.8. Let us first introduce some notation. Remember that

$$\Lambda := \{x \in \mathbb{Z}^d : x^{(i)} = 0 \text{ for } i \ge 3\}.$$

is a copy of $\mathbb{Z}^2$ immersed in $\mathbb{Z}^d$. For $M \in \mathbb{N}$ denote by

$$\Lambda_M = \{x \in \Lambda : \max(|x^{(1)}|, |x^{(2)}|) \le M\} \subset \Lambda$$

the square centered at the origin and with sides of size $2M$, parallel to the coordinate axes. For $x \in \Lambda_M$ let

$$\ell_x = \{y \in \mathbb{Z}^d : y^{(1)} = x^{(1)}, y^{(2)} = x^{(2)}\} \subset \mathbb{Z}^d$$

be the line orthogonal to Λ containing the site $x \in \Lambda_M$. By Lemma 2.5, for $d \ge d_0$, at instant $\sqrt{d}$ there are more than $\mu^{\sqrt{d}}$ active particles, where $\mu > 1$, in $\mathcal{K} = \{x \in \mathbb{Z}^d : \max_{1\le i\le d} |x^{(i)}| \le 1\}$ with probability larger than $a > 0$ for the process restricted on $\mathcal{K}$.

Fixed $M \in \mathbb{N}$, $y \in \mathcal{K}$ and $x \in \Lambda_M$, after at most $2M+2$ steps an active particle starting from y hits ℓ_x with probability at least $(p/2d)^{2M+2}$. So, for each fixed site x of Λ_M, the probability that at least one of those $\mu^{\sqrt{d}}$ particles enters ℓ_x after at most $2M+2$ steps is greater than

$$1 - \left(1 - \left(\frac{p}{2d}\right)^{2M+2}\right)^{\mu^{\sqrt{d}}}.$$

Consequently, defining

$$\begin{aligned} a' \;:=\; & \mathbf{P}[\ell_x \text{ is hit by some particle starting from } \mathcal{K}, \\ & \quad \text{for all } x \in \Lambda_M \mid \text{more than } \mu^{\sqrt{d}} \text{ particles start from } \mathcal{K}] \end{aligned}$$

one gets that, for fixed M,

$$a' \geq 1 - (2M+1)^2\left(1 - \left(\frac{p}{2d}\right)^{2M+2}\right)^{\mu^{\sqrt{d}}}$$

and so a' can be made arbitrarily close to 1 by choosing d large enough. So we see that with probability at least $aa'\mathbf{P}[\eta \geq 1]$ the projection of the trajectories of particles from $\mathcal{K}$ will fill up the square Λ_M (and, by choosing d large enough, M can be made as large as we want).

Note that we can repeat the above construction for all sites $x \in 3\Lambda$, and note also that if $x, y \in 3\Lambda$, $x \neq y$, then those constructions starting from x and y are independent (since $(\mathcal{K}+x) \cap (\mathcal{K}+y) = \emptyset$). Consider now the following percolation model: For $x \in 3\Lambda$, all the sites of the square $\Lambda_M + x$ are selected with probability $aa'\mathbf{P}[\eta \geq 1]$. Then, as in Theorem 1.4, one can prove that for M large enough this model percolates. Using Lemma 2.3, we obtain that the original frog model survives with positive probability, thus concluding the proof of Theorem 1.8. $\square$

2.4 Recurrence and transience

Proof of Theorems 1.9, 1.10, and 1.13. The idea of the proof of all the theorems about transience in this section is the following: *all* the particles are made active initially; clearly, if in such model with probability 1 the origin is hit only a finite number of times, then a coupling argument shows that the original frog model is transient.

To prove Theorem 1.9, we need an upper bound for $\mathbf{P}[y \in \mathcal{R}_x^1]$ which is better than (2.6). Note that on $\mathbb{T}_d$, the probability that a SRW (no death) starting from x will eventually hit y, is exactly $(d-1)^{-\operatorname{dist}(x,y)}$. This shows that, on $\mathbb{T}_d$,

$$\mathbf{P}[y \in \mathcal{R}_x^1] \leq \sum_{i=\operatorname{dist}(x,y)}^{\infty} p^i(1-p)\frac{1}{(d-1)^{\operatorname{dist}(x,y)}} = \left(\frac{p}{d-1}\right)^{\operatorname{dist}(x,y)}. \tag{2.13}$$

Now, using (2.13) and Lemma 2.1, one gets that for some $C > 0$

$$\begin{aligned} \sum_{x \neq \mathbf{0}} \mathbf{P}[\mathbf{0} \in \mathcal{R}_x] &= \sum_{i=1}^{\infty} \gamma_i \sum_{k=1}^{\infty} d(d-1)^{k-1}\Phi(i,k,p/(d-1)) \\ &\leq C\sum_{i=1}^{\infty} \gamma_i i^{\frac{\log(d-1)}{\log((d-1)/p)}} < \infty \end{aligned}$$

for every $p < 1$, so from Borel-Cantelli one gets that almost surely only a finite number of particles will ever enter $\mathbf{0}$, thus proving Theorem 1.9.

As for Theorem 1.10, we have, recalling the proof of Theorem 1.3, that when $\mathbf{E}(\log(\eta \vee 1))^d < \infty$,

$$\sum_{x \neq \mathbf{0}} \mathbf{P}[\mathbf{0} \in \mathcal{R}_x] = \sum_{x \neq \mathbf{0}} \mathbf{P}[x \in \mathcal{R}_{\mathbf{0}}] = \mathbf{E}|\mathcal{R}_{\mathbf{0}} \setminus \{\mathbf{0}\}| < \infty,$$

and Theorem 1.10 follows from Borel-Cantelli as well.

Let us turn to the proof of Theorem 1.13. Denote by $r_k(\mathcal{G})$ the expected size of the range of the SRW on $\mathcal{G}$ until the moment k. We have

$$\begin{aligned}
\sum_{x \neq \mathbf{0}} \mathbf{P}[\mathbf{0} \in \mathcal{R}_x] &= \sum_{x \neq \mathbf{0}} \mathbf{P}[x \in \mathcal{R}_{\mathbf{0}}] \\
&= \mathbf{E}|\mathcal{R}_{\mathbf{0}}| \\
&= \sum_{k=1}^{\infty} \mathbf{P}[\Xi = k] r_k(\mathcal{G}),
\end{aligned}$$

and using the fact that

$$r_k(\mathcal{G}) \simeq \begin{cases} \sqrt{k}, & \mathcal{G} = \mathbb{Z}, \\ \dfrac{k}{\log k}, & \mathcal{G} = \mathbb{Z}^2, \\ k, & \mathcal{G} = \mathbb{Z}^d \text{ or } \mathbb{T}_d, d \geq 3 \end{cases}$$

(see e.g. Hughes [4], p. 333, 338), one gets the result. □

Proof of Theorems 1.11, 1.12, and 1.14. In this section, theorems concerning the recurrence also are proved using a common approach. This approach can be roughly described as follows. We think of $\mathcal{G}$ as a disjoint union of sets $\mathfrak{J}_k, k = 1, 2, \ldots$, of increasing sizes, such that with large probability (increasing with k), the set $\mathfrak{J}_k$ contains a lot of particles in the initial configuration. Besides, given that $\mathfrak{J}_k$ contains many particles, also with large probability (increasing with k as well), the virtual paths of those particles will cover the whole set $\mathfrak{J}_{k+1}$ together with the origin, thus activating all particles placed originally in $\mathfrak{J}_{k+1}$. With a particular choice of that sequence of sets, the intersection of the events mentioned above occurs with strictly positive probability, which implies, consequently, that the process is recurrent (as in this case for each k there is a particle from $\mathfrak{J}_k$ which visits the origin, and so the total number of particles visiting the origin is infinite).

First, we give the proof of Theorem 1.11. Fix a number $\alpha > 1$ in such a way that $\frac{\log(d-1)}{2\log(\alpha d)} > \beta$, and fix the survival parameter p in such a way that $1/\alpha < p < 1$. Denote $\mathfrak{J}_n^d = \{y \in \mathbb{T}_d : \operatorname{dist}(\mathbf{0}, y) = n\}$, and define the events

$$\begin{aligned}
A_n^d &= \{\text{there exists } x \in \mathfrak{J}_n^d \text{ such that } \eta(x) \geq (\alpha d)^{2n}\}, \\
B_n^d &= \Big\{(\mathfrak{J}_n^d \cup \{\mathbf{0}\}) \subset \bigcup_{y \in \mathfrak{J}_{n-1}^d} \mathcal{R}_y\Big\}.
\end{aligned} \tag{2.14}$$

As $|\mathfrak{J}_n^d| > (d-1)^n$ and $\frac{d-1}{(\alpha d)^{2\beta}} > 1$, we get that

$$\begin{aligned}
\mathbf{P}[A_n^d] &\geq 1 - (1 - \mathbf{P}[\eta \geq (\alpha d)^{2n}])^{(d-1)^n} \\
&\geq 1 - \Big(1 - \frac{1}{(\alpha d)^{2\beta n}}\Big)^{(d-1)^n} \\
&\geq 1 - C_1 \exp\Big(-\Big(\frac{d-1}{(\alpha d)^{2\beta}}\Big)^n\Big)
\end{aligned} \tag{2.15}$$

for some $C_1 > 0$. Now, using the fact that

$$\max_{\substack{x\in\mathfrak{J}_n^d,\\ y\in\mathfrak{J}_{n+1}^d\cup\{\mathbf{0}\}}} \operatorname{dist}(x,y) = 2n+1$$

together with (2.8), one gets (note that $|\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}| \leq d^{n+1}$ for all n)

$$\begin{aligned}
\mathbf{P}[B_{n+1}^d \mid A_n^d, B_n^d] &\geq 1 - |\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}|\Big(1 - \Big(\frac{p}{d}\Big)^{2n+1}\Big)^{(\alpha d)^{2n}} \\
&\geq 1 - C_2 d^{n+1}\exp(-(\alpha p)^n)
\end{aligned}$$

for some $C_2 > 0$. The fact that $\alpha p > 1$ together with (2.15) imply that with strictly positive probability there exists a random number n_0 such that the events $B_n^d, n \geq n_0$, occur. Clearly, in this case $\mathbf{0}$ is hit infinitely often and so the process is recurrent.

Now, we start proving Theorem 1.12. Fix arbitrary $0 < p \leq 1$ and choose $\alpha > 1$ in such a way that $d - \alpha\beta > 0$. Let $\mathfrak{J}_n^d = \{x \in \mathbb{Z}^d : 2^{n-1} < \operatorname{dist}(\mathbf{0}, x) \leq 2^n\}$. Define the events B_n^d by means of (2.14) and

$$A_n^d = \{\text{there exists } x \in \mathfrak{J}_n^d \text{ such that } \eta(x) \geq \exp(2^{\alpha n})\}.$$

As $|\mathfrak{J}_n^d| \geq C_1 2^{dn}$ for some $C_1 > 0$ and all n, we have

$$\begin{aligned}
\mathbf{P}[A_n^d] &\geq 1 - (1 - \mathbf{P}[\eta \geq \exp(2^{\alpha n})])^{C_1 2^{dn}} \\
&\geq 1 - \Big(1 - \frac{1}{2^{\alpha\beta n}}\Big)^{C_1 2^{dn}} \\
&\geq 1 - C_2 \exp(-(C_1 2^{d-\alpha\beta})^n).
\end{aligned} \tag{2.16}$$

It is a fact that in this case

$$\max_{\substack{x\in\mathfrak{J}_n^d,\\ y\in\mathfrak{J}_{n+1}^d\cup\{\mathbf{0}\}}} \operatorname{dist}(x,y) \leq 2^{n+2},$$

and that $|\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}| \leq C_3 2^{d(n+1)}$, so using (2.8) we get, keeping in mind that $\alpha > 1$,

$$\begin{aligned}
\mathbf{P}[B_{n+1}^d \mid A_n^d, B_n^d] &\geq 1 - |\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}|\Big(1 - \Big(\frac{p}{2d}\Big)^{2^{n+2}}\Big)^{\exp(2^{\alpha n})} \\
&\geq 1 - C_4 2^{d(n+1)}\exp\Big(-\exp\Big(2^{\alpha n} - 2^{n+2}\log\frac{2d}{p}\Big)\Big).
\end{aligned} \tag{2.17}$$

As before, (2.16)–(2.17) imply that with positive probability infinite number of events B_n^d occur, so the process is recurrent.

Let us turn to the proof Theorem 1.14. The sets $\mathfrak{J}_n^d$ are now defined by $\mathfrak{J}_n^d = \{x \in \mathbb{Z}^d : 2^{n-1} < \|x\|_2 \leq 2^n\}$, and the sequence of events B_n^d is still defined by (2.14). Recall that Ξ^x is the lifetime of the particle originating from x. Now, the site $x \in \mathfrak{J}_n^d$ is called *good*, if $\Xi^x \geq 2^{2(n+2)}$ (intuitively, the site $x \in \mathfrak{J}_n^d$ is good if the corresponding particle lives long enough to be able to get to any fixed site of $\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}$). Define the events

$$A_n^d = \{\text{the number of good sites in } \mathfrak{J}_n^d \geq \varphi_{n,d}|\mathfrak{J}_n^d|\},$$

where

$$\varphi_{n,d} = \begin{cases} \beta_1 2^{-(n+3)} \log\log 2^{n+2}, & d = 1, \\ \beta_2 2^{-(2n+5)} (\log 2^{n+2})^2, & d = 2, \\ \beta_d 2^{-(2n+5)} \log 2^{n+2}, & d \geq 3 \end{cases}.$$

As, by the hypothesis, $\mathbf{P}[x \text{ is good}] \geq 2\varphi_{n,d}$ for every $x \in \mathfrak{J}_n^d$, by Lemma 2.2 we get (observe that $|\mathfrak{J}_n^d| \simeq 2^{dn}$)

$$\mathbf{P}[A_n^d] \geq \begin{cases} 1 - n^{-C_1\beta_1}, & d = 1, \\ 1 - \exp(-C_2\beta_2 n^2), & d = 2, \\ 1 - \exp(-C_3\beta_d n 2^{d(n-2)}), & d \geq 3, \end{cases}$$

so $\sum_{n=1}^{\infty}(1 - \mathbf{P}[A_n^d]) < \infty$ for arbitrary β_d, $d \geq 2$, and for $\beta_1 > 1/C_1$, $d = 1$. Using the inequality

$$\max_{\substack{x \in \mathfrak{J}_n^d, \\ y \in \mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}}} \|x - y\|_2 \leq 2^{n+2}$$

together with Lemma 2.4, one gets

$$\mathbf{P}[B_{n+1}^d \mid A_n^d, B_n^d] \geq \begin{cases} 1 - 3(1 - w_1)^{\varphi_{n,1}|\mathfrak{J}_n^1|}, & d = 1, \\ 1 - |\mathfrak{J}_{n+1}^2 \cup \{\mathbf{0}\}|\Big(1 - \dfrac{w_2}{\log 2^{n+2}}\Big)^{\varphi_{n,2}|\mathfrak{J}_n^2|}, & d = 2, \\ 1 - |\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}|\Big(1 - \dfrac{w_d}{2^{(d-2)(n+2)}}\Big)^{\varphi_{n,d}|\mathfrak{J}_n^d|}, & d \geq 3 \end{cases}$$

(the factor $|\mathfrak{J}_{n+1}^d \cup \{\mathbf{0}\}|$ was substituted by 3 in dimension 1, because in this case, to guarantee that all the sites of the set $\mathfrak{J}_{n+1}^1 \cup \{\mathbf{0}\}$ are hit, it is sufficient to visit the sites $\mathbf{0}$ and $\pm 2^{n+1}$). Then, elementary computations show that

$$\mathbf{P}[B_{n+1}^d \mid A_n^d, B_n^d] \geq \begin{cases} 1 - 3n^{-C_4\beta_1}, & d = 1, \\ 1 - C_5 2^{-(C_6 w_2 \beta_2 - 2)n}, & d = 2, \\ 1 - C_7 2^{-(C_8 w_d \beta_d - d)n}, & d \geq 3. \end{cases}$$

By choosing $\beta_1 > \max\{1/C_1, 1/C_4\}$, $\beta_2 > 2/C_6 w_2$, $\beta_d > d/C_8 w_d$, $d \geq 3$, once again one gets that with positive probability infinite number of events B_n^d occur, and so the frog model is recurrent. $\square$

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