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CONVERGENCE IN INCOMPLETE MARKET MODELS

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Abstract The problem of pricing and hedging of contingent claims in incomplete markets has led to the development of various valuation methodologies. This paper examines the mean-variance approach to risk-minimisation and shows that it is robust under the convergence from discrete- to continuous-time market models. This property yields new convergence results for option prices, trading strategies and value processes in incomplete market models.

Techniques from nonstandard analysis are used to develop new results for the lifting property of the minimal martingale density and risk-minimising strategies. These are applied to a number of incomplete market models:

It is shown that the convergence of the underlying models implies the convergence of strategies and value processes for multinomial models and approximations of the Black-Scholes model by direct discretisation of the price process. The concept of D^2-convergence is extended to these classes of models, including the construction of discretisation schemes. This yields new standard convergence results for these models.

For ease of reference a summary of the main results from nonstandard analysis in the context of stochastic analysis is given as well as a brief introduction to mean-variance hedging and pricing.

Keywords financial models, incomplete markets, D^2-convergence, nonstandard analysis

AMS subject classification 60J35, 60H05, 91B28, 28K05

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1. Introduction

Weak convergence results for option values and their associated trading strategies have so far been established in the context of *complete* market models, where unambiguous pricing of contingent claims is possible by means of hedging techniques. The results surveyed by Taqqu and Willinger [TW87] and the more recent work on stronger modes of convergence by Cutland, Kopp and Willinger [CKW93a] rely on market completeness.

For incomplete markets, convergence results for option values were obtained by Runggaldier and Schweizer [RS95], and in a more general framework in a recent work by Prigent [Pri97]. Both use the minimal martingale densities introduced by Föllmer and Schweizer [FS91] as their starting point and use the weak convergence of these densities to derive the convergence of option values. However, in neither case there is a discussion of hedging strategies associated with the various optimality criteria, although the formulation of the general mean-variance hedging problem for incomplete market models provides explicit expressions for such strategies under certain additional conditions on the price process. These conditions are satisfied in all the models discussed in the above papers, but it appears that the techniques developed there do not suffice to settle the question of convergence of trading strategies in this context.

In this paper we extend the techniques developed in [CKW93a] to show that the associated discrete-time strategies do indeed converge to their counterpart in the limiting continuous-time model, provided that the limiting model is complete. As in [CKW93a] we employ techniques from nonstandard analysis in proving our results, but are able to state these results in standard terms by means of the stronger concept of D^2-convergence developed in [CKW93a]. The necessary nonstandard tools are described briefly in the primer provided in [CKW91]; in this paper we outline new nonstandard results required for the extension of their results to incomplete market models.

2. Mean-Variance Hedging

2.1. Discrete Time.

The calculation of the minimal martingale measures and trading strategies we wish to use in our convergence theory is particularly simple in the discrete case. We review briefly the discrete-time setup, introduced by Schweizer in [Sch88], which will serve us throughout this paper.

Define a discrete time economy on the set $\mathbb{T} = \{0, 1, \ldots, T\}$ ($T \in \mathbb{N}$) of trading dates; the price of some risky asset is given as a stochastic process S on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$; assume that $\mathcal{F}_0$ is trivial (up to P-null sets) and that $\mathcal{F}_T = \mathcal{F}$. Without loss of generality we suppose that S already represents the *discounted* price process with respect to some numeraire (equivalently, the risk-free discount rate is zero).

We also assume that the price process S is $\mathbb{F}$-adapted and $S_t \in L^2(P)$ for $t \in \mathbb{T}$. Define $\Delta X_t = X_{t+1} - X_t$ for any process X on $(\Omega, \mathcal{F}, P)$. Note that, contrary to customary practice, we use the 'forward' increments.

A *trading strategy* in this economy is represented a as pair of processes $\phi = (\theta, \psi)$ satisfying the following conditions:

(i) (θ, ψ) is $\mathbb{F}$-adapted and $\theta_T = 0$;

(ii) $\displaystyle\sum_{u=0}^{t-1} \theta_u \Delta S_u \in L^2(P)$ for $t \in \mathbb{T}$;

(iii) $\theta_t S_t + \psi_t \in L^2(P)$ for $t \in \mathbb{T}$.

Here θ represents the number of units of the risky asset while ψ represents the amount held in a risk-free account. The usual definitions for the associated *value process* $V_t(\phi) = \theta_t S_t + \psi_t$, *gains process* $G_t(\phi) = \Sigma_{u=0}^{t-1} \theta_u \Delta S_u$ and *cost process* $C_t(\phi) = V_t(\phi) - G_t(\phi)$ apply. Recall that a strategy ϕ is *self-financing* if and only if $\Delta V(\phi) = \Delta G(\phi)$.

Let $H \in L^2(P)$ be a contingent claim; a strategy ϕ is called *H-admissible* if $V_T(\phi) = H$; we then say that ϕ *generates* H. A market model is *complete* if for any claim H there exists a self-financing strategy generating H. By contrast, in an incomplete model a claim H can in general not be generated by a self-financing strategy (note that there always exists an H-admissible strategy for any $H \in L^2$: take $\theta \equiv 0$, $\psi_t = H \cdot \mathbf{1}_{\{t=T\}}$). We can therefore only expect to find an H-admissible strategy which is optimal with respect to some optimality criterion. Here we use the notion of local risk-minimisation which was introduced in [Sch88]:

The local risk process $r(\phi)$ for a strategy ϕ is defined as

$$r_t(\phi) := E\left[(C_{t+1}(\phi) - C_t(\phi))^2 | \mathcal{F}_t\right]$$

for $t \in \mathbb{T}$. We are looking for a strategy ϕ which minimizes this local risk by an appropriate choice of θ_t and V_t (see [Sch88] for details). Note that $r \equiv 0$ for a self-financing strategy. It can be shown that under the nondegeneracy condition

$$\frac{\left(E\left[\Delta S_t | \mathcal{F}_t\right]\right)^2}{\operatorname{Var}\left[\Delta S_t | \mathcal{F}_t\right]} \leq L, \quad P\text{-a.s., } t \in \mathbb{T} \setminus \{0\} \quad \text{for some } L \in \mathbb{R} \tag{1}$$

there exists a unique H-admissible locally risk-minimising strategy ϕ^H. This strategy is given explicitly by backward sequential regression: setting $V_T^H := H$ and then defining recursively

$$\theta_t^H := \frac{\operatorname{Cov}\left[V_{t+1}^H, \Delta S_t | \mathcal{F}_t\right]}{\operatorname{Var}\left[\Delta S_t | \mathcal{F}_t\right]} \tag{2}$$

$$V_t := E\left[V_{t+1}^H - \theta_t^H \Delta S_t \big| \mathcal{F}_t\right] \tag{3}$$

$$\psi_t^H := V_t^H - \theta_t^H S_t \tag{4}$$

for $t \in \mathbb{T} \setminus \{T\}$. Finally, set $\theta_T^H := 0$ and $\psi_T^H := H$. As defined in (2) θ_t^H can be viewed as the best linear estimate for V_{t+1}^H based on the information at time t, while equations (3) and (4) ensure that ϕ^H is *mean-self-financing*, i.e. the cost process $C(\phi^H)$ is an $\mathbb{F}$-martingale.

The locally risk-minimising strategy given by equations (2)–(4) gives rise to a natural decomposition of the contingent claim H (cf. [Sch94a]): We define the Doob decomposition of the price process $S = S_0 + M + A$, where

$$\Delta A_t := E\left[\Delta S_t | \mathcal{F}_t\right], \quad \Delta M_t := \Delta S_t - \Delta A_t, \quad A_0 := 0, \; M_0 := 0.$$

Then M is a martingale and A is a predictable process; both M and A are square-integrable. We can then define the square-integrable martingale L by

$$\Delta L_t := V_{t+1}^H - E\left[V_{t+1}^H \big| \mathcal{F}_t\right] - \theta_t^H \Delta M_t, \quad L_0 := 0.$$

It is easy to see that the martingales M and L^H are orthogonal, i.e. their product is a martingale. Note that $\Delta L_t^H = \Delta C_t(\phi^H)$, so that L^H represents the extra cost required by the strategy ϕ^H. Using (3) we have

$$\begin{aligned} \Delta V_t^H &= \theta_t^H \Delta S_t + \Delta L_t^H, \quad \text{for } t \in \mathbb{T} \setminus \{T\} \\ \text{and} \quad V_0^H &= E\left[H - \sum_{s=0}^{T} \theta_s^H \Delta A_s\right], \end{aligned}$$

so we have obtained a decomposition

$$H = V_0^H + \sum_{u=0}^{T-1} \theta_u^H \Delta S_u + L_T^H, \tag{5}$$

where L^H is a square-integrable martingale orthogonal to the martingale part of S. Equation (5) is the discrete-time version of the Föllmer-Schweizer decomposition (see below); in the case where the price process S is already a martingale under P the decomposition (5) simplifies to the (discrete-time) Kunita-Watanabe decomposition of H.

2.2. **Continuous Time.** In continuous time the set of trading dates $\mathbb{T} := [0,T] \subset \mathbb{R}$ for $T > 0$. $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ is a filtration on some complete probability space $(\Omega, \mathcal{F}, P)$ which satisfies the usual conditions (right-continuity and completeness) with $\mathcal{F}_0$ trivial and $\mathcal{F}_T = \mathcal{F}$. The price process is given by a semimartingale $S \in \mathcal{S}^2(P)$ with Doob-Meyer decomposition $S = S_0 + A + M$ where M is a square-integrable martingale and A is a predictable process with square-integrable variation (see [Ell82] for definitions). We also assume that A is absolutely continuous with respect to $\langle M \rangle$ — this is a nondegeneracy condition akin to (1) above. A trading strategy ϕ is a pair (θ, ψ) with θ predictable and ψ adapted such that

(i) $\theta \in L^2(S)$, i.e.

$$E\left[\int_0^T \theta_s^2 d\langle M\rangle_s + \left(\int_0^T |\theta_s| d|A|_s\right)^2\right] < \infty,$$

(ii) the value process $V_t(\phi) := \theta_t S_t + \psi_t$ is square-integrable.

We can then define the gains process $G(\phi) := \int \theta dS$ and the cost process $C(\phi) := V(\phi) - G(\phi)$ analogous to the discrete-time situation. The notion of local risk-minimisation for the generating strategy of a contingent claim H as defined above can be transferred to this setting, although this involves sophisticated results on the differentiation of semimartingales (see [Sch90, Sch91]. It can be shown, however, that the existence of a locally risk-minimising strategy is equivalent to the existence of a Föllmer-Schweizer decomposition

$$H = V_0 + \int_0^T \theta_u^H dS_u + L_T^H, \tag{6}$$

where $V_0 \in \mathbb{R}$, $\theta^H \in L^2(S)$ and L^H is a square-integrable martingale orthogonal to M. The question of existence and uniqueness of the decomposition (6) for a general claim H was settled by Monat and Stricker [MS95].

Note that our market model is complete if $L^H \equiv 0$ for any $H \in L^2(P)$.

2.3. **Minimal Martingale Measures.** We wish to express the initial value V_0^H associated with the locally risk-minimising trading strategy ϕ^H as the expectation of the claim H under a signed martingale measure $\hat{P}$ for the price process S; the *minimal martingale measure* for S (cf. [Sch94b]). This measure will be described by its density relative to P. In the discrete-time setting the density is given as the final value (i.e. the value at time T) of the process

$$\hat{Z}_t = \prod_{s<t}(1 - \alpha_s \Delta M_s) \tag{7}$$

with

$$\alpha_t = \frac{E\left[\Delta S_t | \mathcal{A}_t\right]}{\mathrm{Var}\left[\Delta S_t | \mathcal{A}_t\right]} \quad \text{and} \quad \Delta M_t = \Delta S_t - E\left[\Delta S_t | \mathcal{A}_t\right] \tag{8}$$

The density $\frac{d\hat{P}}{dP} = \hat{Z}_T \in L^2(P)$ defines a signed measure $\hat{P}$ under which S is a martingale and $V_t^H = E_{\hat{P}}(H|\mathcal{F}_t)$ for all $t \in \mathbb{T}$.

In the continuous-time case we concentrate on conditions which ensure that $\hat{P}$ is a true measure equivalent to P, (thus coinciding with P on $\mathcal{F}_0$) such that any $L^2(P)-$martingale orthogonal to M remains a martingale under $\hat{P}$. In this case the density for the minimal martingale measure is defined via the stochastic exponential

$$\hat{z} = \mathcal{E}\left(-\int \alpha \, dm\right) \tag{9}$$

where α is the density of A with respect to $\langle M \rangle$ in the Doob-Meyer decomposition of S. It is then shown in [MR97, Chapter 26], that, under the assumption that $\alpha_t(M_t - M_{t-}) < 1$ for all $t \in \mathbb{T}$, the integrability condition $\hat{z}_t \in L^2(P)$ is equivalent to the existence of a unique minimal martingale measure $\hat{P}$, whose density relative to P is given by $\hat{z}_T$.

3. Nonstandard martingales

To compare the measures defined by these densities, we generalize the results of Section 3.1 of [CKW91], where techniques from nonstandard analysis were used to provide alternative derivations of Black-Scholes option pricing theory. We refer to Section 2 of [CKW91] for a primer in hyperfinite probability theory and the basic definitions needed for a nonstandard description of Brownian Motion and related concepts. However, we shall need to extend the setup somewhat in order to allow consideration of arbitrary cádlág processes rather than restrict the discussion to path-continuous processes as in [CKW91]. Two fundamental references for these developments are [AFHL86] and [HP83]; to set the scene for these extensions we recall (largely following [Cut00]) the principal ideas and definitions here.

3.1. **The Nonstandard Universe.** We shall assume given (as in [CKW91]) a fixed nonstandard extension ${}^*\mathbb{R}$ of the real line $\mathbb{R}$. The extension ${}^*\mathbb{R}$ includes elements defined as non-zero 'infinitesimals' ($x \in^* \mathbb{R}$ satisfying $|x| < \varepsilon$ for all $\varepsilon > 0$ in $\mathbb{R}$) and their 'infinite' multiplicative inverses. The extension itself is not unique: ${}^*\mathbb{R}$ can, for example, be defined as an *ultrapower* $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ of the reals by any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (i.e. a collection of subsets of $\mathbb{N}$ closed under intersections and supersets, containing no finite sets, and such that for any $A \subset \mathbb{N}$, either A or $\mathbb{N} \setminus A$ belongs to $\mathcal{U}$.) The existence of such ultrafilters is equivalent to the Axiom of Choice. What is important here is that the arithmetical and order operations valid in $\mathbb{R}$ extend to ${}^*\mathbb{R}$: the tuple $({}^*\mathbb{R}, +, \times, <)$ is an ordered field.

We may view all mathematical objects as sets. For any set S the *superstructure* $\mathbb{V}(S)$ *over* S is defined as $\mathbb{V}(S) = \bigcup_{n\in\mathbb{N}} V_n(S)$, where $V_0(S) = S$, and

$$V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S)) \quad (n \in \mathbb{N}).$$

This construction can be applied in turn to $\mathbb{R}$ and to $^*\mathbb{R}$. Their superstructures $\mathbb{V} = \mathbb{V}(\mathbb{R})$ and $\mathbb{V}(^*\mathbb{R})$ are then connected by a map $^* : \mathbb{V}(\mathbb{R}) \mapsto \mathbb{V}(^*\mathbb{R})$ which associates with each object $\mathcal{M}$ in $\mathbb{V}(\mathbb{R})$ its nonstandard extension $^*\mathcal{M}$ in $\mathbb{V}(^*\mathbb{R})$. The *nonstandard universe* (whose members are known as the *internal sets*) is then simply given by

$$^*\mathbb{V} = \{x : x \in^* \mathcal{M} \text{ for some } \mathcal{M} \in \mathbb{V}\}.$$

The *Transfer Principle* states that any bounded quantifier statement holds in $\mathbb{V}$ iff it holds in $^*\mathbb{V}$. (A bounded quantifier statement is a mathematical statement which can be written to ensure that all quantifiers range over a prescribed set. This includes most statements used in practice.) This result enables us to 'switch' from the 'standard world' $\mathbb{V}$ to *internal* objects (elements of $^*\mathbb{V}$) and back again: in proofs we can therefore 'translate' a statement into the language of internal sets, manipulate it within $^*\mathbb{V}$ and then (hope to) translate the results into the context of $\mathbb{V}$. For the *finite part* $\{x \in^* \mathbb{R} : |x| < r$ for some $r \in \mathbb{R}\}$ of the set $^*\mathbb{R}$ of *hyperreals* the important (topological) connection is made very simply via the Standard Part Theorem, which asserts that each finite $x \in^* \mathbb{R}$ can be written uniquely as $x = r + \delta$ for some $r \in \mathbb{R}$ and an infinitesimal δ. We write $x \approx y$ if $x - y$ is infinitesimal.

3.2. **Internal Probability Spaces and Loeb Measure.** We work principally with *hyperfinite* sets in $^*\mathbb{R}$, i.e. sets of the form $\Omega = \{1, 2, ..., N\}$ for some $N \in^* \mathbb{N} \setminus \mathbb{N}$. Thus we define a hyperfinite time line $(\mathbb{T}, \mathcal{F}, \Lambda)$, where $T \in \mathbb{R}$ is fixed throughout, N is a fixed hyperfinite integer, $\Delta t = T/N$,

$$\mathbb{T} = \{0, \Delta t, 2\Delta t, ..., (N-1)\Delta t, T\},$$

$\mathcal{F} = {}^*\mathcal{P}(\Omega)$, and $\Lambda(\{t\}) = \Delta t$. More generally, any hyperfinite set Ω and the algebra of its internal subsets $\mathcal{F}$, equipped with normalized counting measure P (i.e. $P(\{\omega\}) = 1/\mathrm{card}(\Omega)$) yields a *hyperfinite probability space*.

Associated with $(\Omega, \mathcal{F}, P)$ is a standard measure space, its *Loeb space* $(\Omega, \mathcal{F}_L, P_L)$, and Loeb-measurable functions $f : \Omega \mapsto \mathbb{R}$ are those which have *liftings* to internal $\mathcal{F}-$measurable functions $F : \Omega \mapsto {}^*\mathbb{R}$. The interplay between these (hyper-) discrete and (standard) continuous-time formulations enables us to derive new convergence results, generalizing those of [CKW93a].

Briefly, the Loeb measure construction proceeds as follows: the internal map $P : \mathcal{F} \mapsto {}^*[0,1]$ is finitely additive; and we may define the map $^\circ P : \mathcal{F} \mapsto [0,1]$ by setting $^\circ P(A) = {}^\circ(P(A))$. This makes $(\Omega, \mathcal{F}, {}^\circ P)$ into a (standard) finitely additive measure space. By the Caratheodory extension theorem $^\circ P$ will have a unique σ-additive extension to the σ-algebra $\sigma(\mathcal{F})$ provided that $^\circ P$ is σ-additive in $\mathcal{F}$. But this is a consequence of the $\aleph_1$-saturation of $^*\mathbb{V}$: this property, which is shared by any countable ultrapower, states that for any countable decreasing family $(A_m)_{m\in\mathbb{N}}$ of non-empty internal sets we have $\bigcap_{m\in\mathbb{N}} A_m \neq \emptyset$. Hence the completion P_L of $^\circ P$, defined on the completion $\mathcal{F}_L$ of $\sigma(\mathcal{F})$, defines the *Loeb space* $(\Omega, \mathcal{F}_L, P_L)$ as a standard probability space. A key property of $B \in \mathcal{F}_L$ is the existence of a set $A \in \mathcal{F}$ such that the symmetric difference $A\Delta B$ is P_L-null.

Example 3.1. Lebesgue measure on $[0,1]$ is the Loeb measure of counting measure Λ on $\mathbb{T}$ (with $T = 1$) as defined above. The Lebesgue σ-algebra consists of sets $B \subset [0,1]$ for which $\mathrm{st}_{\mathbb{T}}^{-1}(B) = \{t \in \mathbb{T} : {}^\circ t \in B\} \in \mathcal{F}_L$.

Example 3.2. Wiener measure W on $\mathcal{C} = C_0(0,T)$ can be defined similarly as the Loeb measure associated with counting probability W_N on the set $\mathcal{C}_N$ of all polygonal paths $(B_t)_{t\in\mathbb{T}}$ formed by joining the points of $\mathbb{T}$ linearly and satisfying $B_0 = 0$ and $\Delta B_t = B_{t+\Delta t} - B_t = \pm\sqrt{\Delta t}$. Wiener space is then the Loeb space associated with the internal probability space $(\mathcal{C}_N, {}^*\mathcal{P}(\mathcal{C}_N), W_N)$. More precisely: write $\mathcal{F}_N$ for the completion of $\sigma({}^*\mathcal{P}(\mathcal{C}_N)), P_N = (W_N)_L$. Then for P_N-a.a. $B \in \mathcal{C}_N$, $b = {}^\circ B$ is a continuous path in $\mathcal{C}$ and for Borel sets $D \subset \mathcal{C}$, the measure W defined by $W(D) = P_N(\mathrm{st}^{-1}(D))$ is Wiener measure. Thus Brownian motion can be regarded as the process $b : [0,T] \mapsto \mathbb{R}$ defined by $b_t(\omega) = {}^\circ\omega(t)$, where we have written ω instead of B for points of $\Omega = \mathcal{C}_N$, and $B_t(\omega) = \omega(t)$ is the canonical random walk on Ω.

A function $f : \Omega \mapsto \mathbb{R}$ on a Loeb space $(\Omega, \mathcal{F}_L, P_L)$ is *Loeb-measurable* if it is P_L-measurable, i.e. $f^{-1}([a,b]) \in \mathcal{F}_L$ for all $[a,b] \subset \mathbb{R}$. Similarly, $F : \Omega \mapsto {}^*\mathbb{R}$ is **-measurable* if $F^{-1}([\alpha,\beta]) \in \mathcal{F}$ for all hyperreal intervals $[\alpha,\beta]$. We say that F is a *lifting* of f if $f(\omega) \approx F(\omega)$ for P_L-a.a. $\omega \in \Omega$. The lifting is *$\mathcal{S}$-integrable* if $\int_\Omega |F|dP$ is finite and $P(A) \approx 0$ implies $\int_A |F|dP \approx 0$ for $A \in \mathcal{F}$. A Loeb-measurable function f has an $\mathcal{S}-$integrable lifting F iff f is P_L-integrable.

3.3. **Stochastic Processes.** If $f(t,\omega)$ is adapted to the Brownian filtration then we can find a *nonanticipating* function $F : \mathbb{T}\times\Omega \mapsto {}^*\mathbb{R}$ (i.e. $F(t,\omega)$ depends only on $\omega(s)$ for $s \le t$) such that $f({}^\circ t,\omega) \approx F(t,\omega)$ for almost all (t,ω) in $\mathbb{T}\times\Omega$. We call F a *nonanticipating lifting* of f. Such liftings exist for Itô integrals: given the pair (f,F), the hyperfinite sum $G(t,\omega) = \Sigma_{s<t}F(s,\omega)\Delta B(s,\omega)$ is a nonanticipating lifting of the Itô integral $I({}^\circ t,\omega) = \int_0^{{}^\circ t} f db$, that is, for almost all ω we have $I({}^\circ t,\omega) \approx G(t,\omega)$ for all t.

These ideas extend to the various functionals of special semimartingales used in continuous-time option pricing models, so that we can approximate these by appropriate 'discrete' hyperfinite counterparts in order to establish convergence results for option prices and the associated optimal trading strategies. The analogues in this context of the usual path-regularity properties of martingales are discussed exhaustively in [HP83], and we shall not repeat their technical results here. The concept of *SDJ-functions* is crucial: in essence this describes conditions under which an internal function $F : \mathbb{T} \mapsto^* \mathbb{R}$ has a cádlág standard part:

Definition 3.3. The *standard part* $\mathrm{st}(F) : [0,T] \mapsto \mathbb{R}$ of the SDJ-function F is defined by $\mathrm{st}(F)(u) = \lim_{{}^\circ t\downarrow u, t\in\mathbb{T}} {}^\circ F(t)$ for $u \in [0,T]$.

This definition is compatible with standard parts defined in the Skorohod topology on the space ${}^*\mathcal{D}[0,T]$ of *-cádlág functions. For this we need to demand that SDJ-functions should have at most one non-infinitesimal jump in each *monad* ('infinitesimal neighborhood') in $\mathbb{T}$ and be *$\mathcal{S}$-continuous* at 0, i.e. $t \approx 0$ should imply that $F(t) \approx 0$ (see [HP83] for details.)

For an internal stochastic process $X : \mathbb{T}\times\Omega \mapsto {}^*\mathbb{R}$ we say that X is SDJ (resp. $\mathcal{S}$-continuous) if P_L-a.a. paths $t \mapsto X_t(\omega)$ have this property. The internal probability space $(\Omega, \mathcal{F}, P)$ becomes an internal *filtered* space if it is endowed with an increasing internal sequence of algebras $\mathbb{A} = (\mathcal{A}_t)_{t\in\mathbb{T}}$ on Ω, and the process X is *nonanticipating* for $\mathbb{A}$ if $\omega \mapsto X_t(\omega)$ is $\mathcal{A}_t$-measurable for each $t \in \mathbb{T}$. Such a process M is an *SL^2-martingale* if $E(\Delta M_t|\mathcal{A}_t) = 0$ and M_t^2 is $\mathcal{S}$-integrable for each $t \in \mathbb{T}$. Defining the standard part $\mathrm{st}(M)$ as above we obtain a (standard) $L^2(P_L)$-martingale with respect to the filtration $\mathbb{B} = (\mathcal{B}_t)_{t\in\mathbb{T}}$, where $\mathcal{B}_u = (\bigcap_{u<{}^\circ t, t\in\mathbb{T}}(\mathcal{A}_t)_L) \cup \mathcal{N}$, with $\mathcal{N}$ denoting the P_L-null sets in $\mathcal{F}_L$.

The *(internal) quadratic variation* $[X] : \Omega\times\mathbb{T} \mapsto {}^*\mathbb{R}$ of an internal process X is given as $[X]_t = \Sigma_{s<t}(\Delta X_s)^2$. For an SL^2-martingale M, the quadratic variation $[M]$ is SDJ (resp. $\mathcal{S}$-continuous) whenever X is, and this operation commutes with taking standard parts, i.e. we have $\mathrm{st}([M]) = [\mathrm{st}(M)]$, where

the square-brackets process on the right is the usual (optional) quadratic variation of the L^2-martingale $\mathrm{st}(M)$ on the filtered space $(\Omega, \mathcal{F}_L, \mathbb{B}, P_L)$.

Stochastic integrals relative to M can again be defined as hyperfinite Stieltjes integrals by setting, for internal $\Theta : \Omega \times \mathbb{T} \mapsto {}^*\mathbb{R}$, $(\sum \Theta\Delta M)_t = \sum_{s<t} \Theta_s \Delta M_s$ for $t \in \mathbb{T}$. For this 'integral' to have a standard counterpart we need to introduce the *internal Doléans measure* ν_M on $\Omega \times \mathbb{T}$ endowed with the internal algebra $\mathcal{A}_{\Omega\times\mathbb{T}}$ generated by sets $\{A \times \{t\} : t \in \mathbb{T}, A \in \mathcal{A}_t\}$, by setting $\nu_M(A \times \{t\}) = E[\mathbf{1}_A(\Delta M_t)^2]$. The most useful class of integrands is the class $SL^2(\nu_M)$ of nonanticipating processes X for which X^2 is $\mathcal{S}$-integrable relative to ν_M on $\Omega \times \mathbb{T}$. The following results, taken largely from [HP83] relate the nonstandard and standard formulations to each other:

Theorem 3.4. *Let M be an internal martingale with $E(M_t^2)$ finite for all $t \in \mathbb{T}$.*

(i) M is $\mathcal{S}$-continuous if and only if its quadratic variation $[M]$ is $\mathcal{S}$-continuous.
(ii) If M is of class SDJ then $[M]$ is of class SDJ and $\mathrm{st}([M]) = [\mathrm{st}(M)]$, P_L-a.s., where the process on the right denotes the standard (optional) quadratic variation of $\mathrm{st}(M)$.

Theorem 3.5. *(i) If M is an $\mathcal{S}L^2$-martingale and $\Theta \in \mathcal{S}L^2(\nu_M)$ (in particular, Θ is nonanticipating) then $\sum \Theta\Delta M$ is an $\mathcal{S}L^2$-martingale.*
(ii) Furthermore, if M is of class SDJ, then so is $\sum \Theta\Delta M$.
(iii) If M is $\mathcal{S}$-continuous then $\sum \Theta\Delta M$ is $\mathcal{S}$-continuous.

Theorem 3.6. *Let M be an $\mathcal{S}L^2$-martingale of class SDJ. Let $m := \mathrm{st}(M)$ be its standard part and assume that $\theta \in L^2(\nu_m)$. Then θ has a 2-lifting Θ (i.e. an internal function $\Theta : \mathbb{T} \times \Omega \mapsto {}^*\mathbb{R}$ such that $\Theta(t,\omega) \approx \theta({}^\circ t, \omega)$ P_L-a.s. for all t) such that*

$$\int \theta dm = \mathrm{st}\left(\sum \Theta\Delta M\right), \qquad P_L\text{-a.s..}$$

Furthermore, $\int \theta dm = \mathrm{st}\left(\sum \bar{\Theta}\Delta M\right)$, P_L-a.s., for any other 2-lifting $\bar{\Theta}$ of θ.

3.4. **D^2-Convergence of Wiener Functionals.** Using nonstandard methods, the papers [CKW93a, CKW95] introduced a mode of convergence for functionals on Wiener space which is stable under stochastic differentiation and integration. This was motivated by the wish to consider convergence of a sequence of contingent claims in Cox-Ross-Rubinstein option pricing models (based upon simple random walks) together with convergence of their generating strategies and value process processes. Thus the desired mode of convergence would need to be stable under both stochastic differentiation and stochastic integration — which weak convergence is not.

The appropriate mode of convergence, called $D^2 - convergence$ in [CKW93a] (for reasons which will become clear below), can be formulated for arbitrary separable metric spaces and is stated very simply in terms of liftings:

Theorem 3.7. *Let $\mathcal{Y}$ be a separable metric space with a Borel probability μ, and suppose that (μ_n) is a family of probabilities on Borel sets $\mathcal{Y}_n \subset \mathcal{Y}$ converging weakly to μ.*

If $F_n : \mathcal{Y}_n \to \mathbb{R}$ is a family of measurable functions and $f : \mathcal{Y} \to \mathbb{R}$ is measurable then the following are equivalent:

(i) F_N is a lifting of f for all infinite N, i.e. for $L(\mu_N)$-a.a. $y \in \mathcal{Y}_N$, $F_N(y) \approx f({}^\circ y)$;

(ii) $\big(F_n(y), y\big) \to \big(f(y), y\big)$ *weakly as* $n \to \infty$. *This means that the distribution of* $\big(F_n(y), y\big) \in \mathbb{R} \times \mathcal{Y}_n$ *with* y *distributed according to* μ_n *converges to the distribution of* $\big(f(y), y\big) \in \mathbb{R}\times\mathcal{Y}$ *with* y *distributed under* μ.

Moreover, if (i) or (ii) holds, then for all $r > 0$, $F_N \in \mathcal{S}L^r(\mu_N)$ *for all infinite* N *if and only if* $E_{\mu_n}\left[|F_n|^r\right] \to E_\mu\left[|f|^r\right]$ *as* $n \to \infty$.

Example 3.8. This result is applied in [CKW95] to Wiener space: $\mathcal{C} = C[0,1] \cap \{x(0) = 0\}$, $W =$ Wiener measure, the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le 1}$ is generated by the canonical Brownian Motion $b(t,\omega) = \omega(t)$ for $\omega \in \mathcal{C}$, and $\mathcal{F} = \mathcal{F}_1$.

The associated discrete spaces are taken as $(\mathcal{C}_n, \mathcal{F}_n, \mathbb{A}_n, W_n)$, where $\mathcal{C}_n$ is the set of paths of a simple random walk based on $\mathbb{T}_n$ of step size $\pm\sqrt{\Delta_n}$ (with $\Delta_n = \frac{1}{n}$) and where we interpolate linearly between points of $\mathbb{T}_n$. Write $B_n(t,X) = X(t)$ for $X \in \mathcal{C}_n$, then $B_n(X)$ denotes the path $(B_n(t,X))_{t\in\mathbb{T}_n}$ and $\Delta B_n(t,X) = \Delta X(t)$. The filtration $\mathbb{A}_n$ is generated by $\{B_n(s,\cdot) : s \le t\}$, and W_n is counting probability on $\mathcal{C}_n$.

Thus, given a sequence of measurable functions $F_n : \mathcal{C}_n \mapsto \mathbb{R}$ and a measurable function $f : \mathcal{C} \mapsto \mathbb{R}$, we have: F_N is an $\mathcal{S}L^2$-lifting of f for all infinite N iff the sequence (F_n) converges to f 'weakly along the graphs' in the sense of Theorem 3.7.

Example 3.9. Convergence of stochastic processes can be handled similarly: discretise the time interval [0,1] as $\mathbb{T}_n = \{0, \Delta_n, 2\Delta_n, \ldots, 1\}$ and use the counting probability Λ_n on $\mathbb{T}_n$ and Lebesgue measure λ on [0,1]. Given $\phi \in L^2(\lambda \times W)$ and $\Phi_n : \mathbb{T}_n \times \mathcal{C}_n \to \mathbb{R}$ for $n \ge 1$, we then have: Φ_N is an $\mathcal{S}L^2$-lifting of ϕ for each infinite N iff $(\Phi_n(t, B_n), t, B_n) \to (\phi(t,b), t, b)$ weakly and $\mathbb{E}_{\Lambda_n\times W_n}(\Phi_n^2) \to \mathbb{E}_{\lambda\times W}(\phi^2)$.

Specializing to the convergence of claims and strategies based upon a Black-Scholes model, a second standard interpretation of the definition links it to L^2-convergence by factoring the functions through a discretisation of the path space — this explains the terminology (here Q denotes the equivalent martingale measure in the Black-Scholes model):

Definition 3.10. A family $(d_n)_{n\in\mathbb{N}}$ of mappings $d_n : \mathcal{C} \to \mathcal{C}_n$ is an *adapted* Q*-discretisation scheme* if

(i) d_n is $\mathbb{A}_n$-adapted; i.e. for each $t \in \mathbb{T}_n$ $(d_n(\cdot))(t)$ is $\mathcal{A}_{n,t}$-measurable.
(ii) d_n is Q-measure-preserving; i.e. $Q(d_n^{-1}(A)) = Q_n(A)$ for all $A \in \mathcal{F}_n$.
(iii) $d_n(\omega) \to \omega$ in Q-probability; i.e. for all $\varepsilon > 0$,

$$Q\big(\|d_n(\omega) - \omega\| < \varepsilon\big) \to 1 \quad \text{as } n \to \infty.$$

(here $\|\cdot\|$ denotes the supremum norm on $\mathcal{C}$.)

The existence of a adapted Q-discretisation scheme for Wiener space is shown in [CKW93a]. This leads to the following extension of Theorem 3.7 and Definition of D^2-convergence:

Theorem 3.11. *Let* $(H_n)_{n\in\mathbb{N}}$ *with* $H_n : \Omega_n \to \mathbb{R}$ *be a sequence of claims and let* $h \in L^2(Q)$. *Then the following are equivalent:*

(i) $H_n\big(d_n(\cdot)\big) \to h$ *in* $L^2(Q)$.
(ii) H_N *is an* $\mathcal{S}L^2(Q_N)$*-lifting of* h *for all infinite* N.
(iii) $\big(H_n(\omega), \omega\big) \to \big(h(\omega), \omega\big)$ *weakly and* $E_{Q_n}\left[H_n^2\right] \to E_Q\left[h^2\right]$.

Definition 3.12. Let $(H_n)_{n\in\mathbb{N}}$ and h be claims as in Theorem 3.11. We say that H_n D^2*-converges* to h if any of the equivalent conditions (i)–(iii) in Theorem 3.11 hold. We then write $H_n \overset{D^2}{\to} h$.

4. The Minimal Martingale Measure

We saw in 2.3 that for a semimartingale s with Doob-Meyer decomposition

$$s = s_0 + \int \alpha \, d\langle m \rangle + m \tag{10}$$

the density for the minimal martingale measure is defined via the stochastic exponential

$$\hat{z} = \mathcal{E}\Big(-\int \alpha \, dm\Big) \tag{11}$$

In discrete time the density is given by the process

$$\hat{Z}_t = \prod_{s<t} (1 - \alpha_s \Delta M_s) \tag{12}$$

with

$$\alpha_t = \frac{E[\Delta S_t | \mathcal{A}_t]}{\mathrm{Var}[\Delta S_t | \mathcal{A}_t]} \quad \text{and} \quad \Delta M_t = \Delta S_t - E[\Delta S_t | \mathcal{A}_t] \tag{13}$$

(cf. equation (8)). Proposition 4.1 below will help to establish the connection between (11) and (12) on the hyperfinite time line. Elements of $\mathbb{T}$ are usually denoted by s, t while u, v are used for elements of $[0, T]$.

Proposition 4.1. *Let $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an internal martingale of class SDJ. Let $x : \Omega \times [0, T] \to \mathbb{R}$ be its standard part, $x := \mathrm{st}(X)$. Define an internal martingale $Z : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ by*

$$Z_t := \prod_{s<t} (1 + \Delta X_s).$$

Then Z is of class SDJ and $z := \mathrm{st}(Z) = \mathcal{E}(x)$, P_L-a.s., where

$$\mathcal{E}(x)_v = \exp\Big(x_v - \frac{1}{2}[x]_v\Big) \prod_{u \le v} (1 + \Delta x_u) \exp\Big(-\Delta x_u + \frac{1}{2}(\Delta x_u)^2\Big)$$

denotes the stochastic exponential of x.

Proof. The proof of this proposition uses essentially the same techniques as the proof of Lemma 3.1 in [CKW91]. The only additional technical result that is required here is the approximation of the 'pure jump' part of a cádlág function by an internal function. This can be achieved pathwise by approximating the (at most countably many) jumps of $x(\omega)$ by points in $\mathbb{T}$ and employing Countable Comprehension to extend this to an internal subset; a complete proof is given in the Appendix. ☐

Corollary 4.2. *Let $M : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an $\mathcal{S}L^2$-martingale of class SDJ and $m := \mathrm{st}(M)$ its standard part. Let a be a predictable process in $L^2(\nu_m)$ and A be an $\mathcal{S}L^2(\nu_M)$-lifting of a. Then*

$$\mathrm{st}\Big(\prod (1 + A\Delta M)\Big) = \mathcal{E}\Big(-\int a \, dm\Big).$$

Proof. It follows from Theorems 3.5 and 3.6 that $\sum A\Delta M$ is a martingale of class SDJ and $\mathrm{st}(\sum A\Delta M) = \int a \, dm$. The result then follows from Proposition 4.1. ☐

Remark 4.3. It will be shown in Section 6 how Corollary 4.2 implies convergence results for option prices in a sequence of discrete-time models. Similar results are obtained in [Pri97]; however the above formulation was derived independently and makes use of different techniques. In [Pri97] the notion of *uniform tightness* of a sequence of martingales is used together with the results in [JMP89] to obtain the weak convergence of the sequence of associated stochastic integrals (see also [DP92]).

5. Trading Strategies and Value Processes

In this section we present a general result (extending Theorem 3.5 in [CKW95]) which will be used to relate the optimal trading strategies for discrete incomplete market models to those in *complete* continuous time models. We will illustrate these results in the next section when we consider (incomplete) discrete time approximations of the Black-Scholes model.

Suppose the process $S : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ is an internal $\mathcal{S}L^2(P)$-martingale of class SDJ. As usual we denote the internal Doléans measure of S by ν_S. Let $s := \mathrm{st}(S)$ be the standard part of S, so that $s : \Omega \times [0,T] \to \mathbb{R}$ is an $L^2(P_L)$-martingale.

We make the following assumptions (S1) and (S2) on the processes S and s. In the next section we will introduce a class of models for which these are satisfied. Furthermore, note that (S1) and (S2) are satisfied for the models considered in [CKW91, CKW93b].

(S1) For any nonanticipating process $\Theta : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ and any $\theta \in L^2(\nu_s)$ we have

$$\Theta \text{ is a } \mathcal{S}L^2(\nu_S)\text{-lifting of } \theta \quad \Leftrightarrow \quad \Theta S \text{ is a } \mathcal{S}L^2(P \times \Lambda)\text{-lifting of } \theta s.$$

(S2) Any $h \in L^2(P_L)$ can be represented as

$$h = v_0 + \int_0^T \theta_u ds_u \tag{14}$$

with a unique $\theta \in L^2(\nu_s)$ and $v_0 \in \mathbb{R}$.

Now let $h \in L^2(P_L)$ and v_0, θ be defined as in (14). Define processes $g, v, \psi : \Omega \times [0,T] \to \mathbb{R}$ by

$$g_t := \int_0^t \theta_u ds_u, \qquad v_t := v_0 + g_t, \qquad \psi_t := v_t - \theta_t s_t.$$

Let $H \in \mathcal{S}L^2(P)$ be a lifting of h. We want to obtain an internal decomposition of H of the form

$$H = V_0 + \sum_{u<t} \Theta_u \Delta S_u + L_t,$$

with $V_0 \in {}^*\mathbb{R}$ and L an internal martingale orthogonal to S. We can obtain this decomposition by the sequential regression described in Section 2: Let

$$V_t := E[H|\mathcal{A}_t], \qquad \Theta_t := \frac{E\left[V_{t+\Delta t}\Delta S_t \middle| \mathcal{A}_t\right]}{E\left[(\Delta S_t)^2 \middle| \mathcal{A}_t\right]}.$$

(with $\Theta_t = 0$ if $E[(\Delta S_t)^2|\mathcal{A}_t] = 0$) and set $\Theta_T := 0$. So V is a $\mathcal{S}L^2$-martingale. By transfer of Lemma A.3.3 in [Sch88] we see that

$$E\left[(\Theta_t \Delta S_t)^2\right] < \infty.$$

The internal martingale L defined by

$$L_0 := 0, \qquad \Delta L_t := \Delta V_t - \Theta_t \Delta S_t$$

is internally orthogonal to S, i.e.

$$\begin{aligned} E\left[\Delta L_t \Delta S_t \middle| \mathcal{A}_t\right] &= E\left[\Delta V_t \Delta S_t - \Theta_t (\Delta S_t)^2 | \mathcal{A}_t\right] \\ &= E\left[V_{t+\Delta t}\Delta S_t \middle| \mathcal{A}_t\right] - V_t E\left[\Delta S_t \middle| \mathcal{A}_t\right] - \Theta_t E\left[(\Delta S_t)^2 \middle| \mathcal{A}_t\right] \\ &= 0 \quad \text{for } t \in \mathbb{T} \setminus \{T\} \end{aligned}$$

(where the last equality uses the martingale property of S and the definition of Θ_t). Then

$$V_t = V_0 + \sum_{u<t} \Theta_u \Delta S_u + L_t \quad \text{and} \quad V_T = H.$$

Finally, we define internal processes $G, \Psi : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ by

$$G_t := \sum_{u<t} \Theta_u \Delta S_u, \qquad \Psi_t := V_t - \Theta_t S_t.$$

We need the following lemmata. The first characterizes orthogonality of martingales in terms of orthogonality with respect to the space generated by stochastic integrals. The following two give a simple criterion for an internal martingale to be infinitesimal.

Lemma 5.1. *Let $X, Y : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be internal martingales. Then X and Y are orthogonal, i.e.*

$$E\left[\Delta X_t \Delta Y_t | \mathcal{A}_t\right] = 0 \quad \text{for all } t \in \mathbb{T}, \tag{15}$$

if and only if for all nonanticipating $\Theta : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ the process Z defined by

$$Z_t = Y_t \sum_{s<t} \Theta_s \Delta X_s$$

is an internal martingale, i.e. Y and $\sum \Theta \Delta X$ are orthogonal.

Proof. Suppose X and Y are orthogonal. Calculating ΔZ_t yields

$$\Delta Z_t = Y_t \Theta_t \Delta X_t + \Delta Y_t \sum_{s<t} \Theta_s \Delta X_s + \Delta Y_t \Theta_t \Delta X_t$$

Since X and Y are martingales

$$E\left[\Delta Z_t | \mathcal{A}_t\right] = Y_t \Theta_t E\left[\Delta X_t | \mathcal{A}_t\right] + E\left[\Delta Y_t | \mathcal{A}_t\right] \sum_{s<t} \Theta_s \Delta X_s + \Theta_t\, E\left[\Delta Y_t \Delta X_t | \mathcal{A}_t\right] = 0.$$

For the converse note that the process

$$Z_t := X_t \sum_{s<t} \Delta Y_s = X_t Y_t$$

is a martingale. □

Lemma 5.2. *Let $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an $\mathcal{S}$-continuous process and $P_L(X_t \approx 0) = 1$ for each $t \in \mathbb{T}$. Then*

$$P_L(X_t \approx 0 \text{ for all } t \in \mathbb{T}) = 1.$$

Proof. This is a nonstandard version of the standard result that two continuous processes which are versions of each other are already indistinguishable.

Let $\tilde{\Omega} := \{\omega \in \Omega : X(\omega) \text{ is } \mathcal{S}\text{-continuous}\}$. Then $P_L(\tilde{\Omega}) = 1$. For $r \in [0, T]$ let $\bar{r} := \max\{t \in \mathbb{T} : t \leq r\}$, i.e. $\bar{r}$ is the element in $\mathbb{T}$ immediately to the left of r. Now let $\mathbb{S} := \{\bar{q} : q \in \mathbb{Q} \cap [0, T]\} \subset \mathbb{T}$. Since there is a one-to-one correspondence between the elements in $\mathbb{Q} \cap [0, T]$ and those in $\mathbb{S}$ we see that $\mathbb{S}$ is countable and $\mathcal{S}$-dense, i.e. $\mathrm{st}(\mathbb{S}) = [0, T]$.

For $s \in \mathbb{S}$ define

$$D_s := \{\omega \in \tilde{\Omega} : {}^\circ|X_s(\omega)| > 0\} = \{\omega \in \Omega : X_s(\omega) \not\approx 0\} \cap \tilde{\Omega}.$$

Then $P_L(D_s) = 0$ by assumption. Let

$$D := \bigcup_{s \in \mathbb{S}} D_s.$$

Then $P_L(D) = 0$ and $P_L(\tilde{\Omega} \setminus D) = 1$. For fixed $\omega \in \tilde{\Omega} \setminus D$ the path $X_\cdot(\omega)$ is $\mathcal{S}$-continuous and $X_t(\omega) \approx 0$ for all $t \in \mathbb{S}$. Let $x := \mathrm{st}(X(\omega))$, so x is a continuous function which is zero on $\mathbb{Q} \cap [0, T]$. Hence $X_t(\omega) \approx 0$ for all $t \in \mathbb{T}$. ☐

Lemma 5.3. *Let $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an internal martingale with $X_0 = 0$. If $E[X_T^2] \approx 0$ then $X_t \approx 0$ for all $t \in \mathbb{T}$, P_L-a.s., i.e. $\mathrm{st}(X) \equiv 0$.*

Proof. Let $[X]$ be the internal quadratic variation of X. We know that

$$E[X_t^2] = E[X_0^2] + E\big[[X]_t\big] = E\big[[X]_t\big] \quad \text{for } t \in \mathbb{T}.$$

It follows that $E\big[[X]_T\big] = E[X_T^2] \approx 0$. Since $[X]_T \geq 0$ this implies that $[X]_T(\omega) \approx 0$ for P_L-a.a. ω. But $[X]_0 = 0$ and $[X]$ is an increasing process, hence the path $[X]_\cdot(\omega)$ is $\mathcal{S}$-continuous for P_L-a.a. ω. Therefore X is $\mathcal{S}$-continuous.

Since $[X]$ is an increasing process we have

$$0 \leq E[X_t^2] = E\big[[X]_t\big] \leq E\big[[X]_T\big] \approx 0 \quad \text{for } t \in \mathbb{T}.$$

Hence, for fixed $t \in \mathbb{T}$, $X_t \approx 0$ P_L-a.s.. Lemma 5.2 then implies that $X_t \approx 0$ for all $t \in \mathbb{T}$, P_L-a.s.. ☐

We can now prove the main result of this section:

Theorem 5.4. *Under the assumptions (S1) and (S2) (cf. page 11) we have*

(i) Θ is an $\mathcal{S}L^2(\nu_S)$-lifting of θ;

(ii) Ψ is an $\mathcal{S}L^2(P \times \Lambda)$-lifting of ψ;

(iii) G and V are $\mathcal{S}L^2$-martingales of class SDJ and satisfy, P_L-a.s.,

$$\begin{aligned} \mathrm{st}(G) &= g, \\ \mathrm{st}(V) &= v, \\ \text{hence} \quad \mathrm{st}(V) &= \mathrm{st}(V_0 + G). \end{aligned} \tag{16}$$

Proof. First note that $v_0 = E_L[h] \approx E[H] = V_0$. Since S is an $\mathcal{S}L^2$-lifting of s and $\theta \in L^2(\nu_s)$ there exists a 2-lifting $\bar{\Theta}$ of θ such that

$$^\circ\Big(\sum_{t<T} \bar{\Theta}_t \Delta S_t\Big) = \int_0^T \theta_u ds_u, \quad P_L\text{-a.s.}$$

Then $\bar{H} := V_0 + \sum_{t<T} \bar{\Theta}_t \Delta S_t$ is also an $\mathcal{S}L^2$-lifting of h, hence

$$\begin{aligned} 0 \approx E\left[(H - \bar{H})^2\right] &= E\Big[\Big(\sum_{t<T} \Theta_t \Delta S_t + L_T - \sum_{t<T} \bar{\Theta}_t \Delta S_t\Big)^2\Big] \\ &= E\Big[\Big(\sum_{t<T} (\Theta_t - \bar{\Theta}_t) \Delta S_t\Big)^2\Big] + E\left[L_T^2\right] \\ &\quad +2E\Big[L_T \sum_{t<T} (\Theta_t - \bar{\Theta}_t) \Delta S_t\Big] \end{aligned} \tag{17}$$

The internal process

$$M_t := L_t \sum_{u<t} (\Theta_u - \bar{\Theta}_u) \Delta S_u$$

is a martingale by Lemma 5.1. Since $M_0 = 0$ we have

$$E\Big[L_T \sum_{t<T} (\Theta_t - \bar{\Theta}_t)\Delta S_t\Big] = 0.$$

Therefore the final term in (17) disappears and

$$\begin{aligned} \int_{\Omega\times\mathbb{T}} (\Theta - \bar{\Theta})^2 \, d\nu_S = E\Big[\Big(\sum_{t<T}(\Theta_t - \bar{\Theta}_t)\Delta S_t\Big)^2\Big] &\approx 0 \\ \text{and} \qquad E\left[L_T^2\right] &\approx 0. \end{aligned}$$

Hence Θ is also a 2-lifting of θ with respect to ν_S and $\Theta \in \mathcal{S}L^2(\nu_S)$. By assumption (S1) this is equivalent to ΘS being a $\mathcal{S}L^2(P \times \Lambda)$-lifting of θs. Furthermore, G is an $\mathcal{S}L^2$-martingale of class SDJ and hence

$$\mathrm{st}(G) = \mathrm{st}\Big(\sum \Theta\Delta S\Big) = \int \theta ds = g, \quad P_L\text{-a.s.}.$$

As $E[L_T^2] \approx 0$ it follows from Lemma 5.3 that the paths of $\mathrm{st}(L)$ are constant zero, P_L-a.s., hence

$$\mathrm{st}(V) = \mathrm{st}(H_0 + G + L) = {}^\circ H_0 + \mathrm{st}(G) = h_0 + g_t = v \qquad P_L\text{-a.s.},$$

Finally, this implies that $\Psi = V - \Theta S$ is an $\mathcal{S}L^2$-lifting of $\psi = v - \theta s$. □

Remark 5.5. In the language of mathematical finance Theorem 5.4 shows that the lifting property of a claim H implies the lifting property of the associated locally-risk-minimising (or variance-optimal[1]) strategy and its value and gains process.

Theorem 5.4 therefore includes the results of [CKW91, Thorem 3.5] and [CKW93b, Theorem 4.1] as special cases. It should be noted however that the models in [CKW91, CKW93b] are *internally complete*. It is therefore possible to obtain an internally self-financing trading strategy generating the claim H, so that equation (16) is automatically satisfied: for self-financing strategies we have $V = V_0 + G$.

The crucial point in Theorem 5.4 is that the additional cost process L is infinitesimal, so that the internal strategies here are self-financing 'in the limit'. This last statement will be made precise in the next section.

We summarize the results of this section in the following theorem which is a generalisation of the main result in [CKW93a]:

Theorem 5.6. *With the above notation and assumptions (S1) and (S2) the following are equivalent:*

(i) H *is an* $\mathcal{S}L^2(P)$*-lifting of* h.
(ii) Θ *is an* $\mathcal{S}L^2(\nu_S)$*-lifting of* θ*, and* Ψ *is an* $\mathcal{S}L^2(P \times \Lambda)$*-lifting of* ψ.
(iii) ΘS *and* Ψ *are* $\mathcal{S}L^2(P \times \Lambda)$*-liftings of* θs *and* ψ*, respectively.*
(iv) G *is an* $\mathcal{S}L^2$*-martingale of class SDJ and* $\mathrm{st}(V_0 + G) = v_0 + g$, P_L*-a.s..*
(v) V *is an* $\mathcal{S}L^2$*-martingale of class SDJ and* $\mathrm{st}(V) = v$, P_L*-a.s..*

[1] Since S is a martingale the locally-risk-minimising and variance-optimal strategies coincide in this case (cf. [Sch93]).

6. Applications

In this section we apply the results of Section 5 to two alternative approximations of the BS model. These models were introduced in [MV96] and [RS95] where convergence results for certain classes of contingent claims have been obtained. Using our results we can extend these to a wider class claims and — more importantly — to the associated trading strategies and value processes.

6.1. **The Multinomial Cox-Ross-Rubinstein Model.** Let $\beta \in \mathbb{N}$ be fixed and let $\mu, \sigma, s_0 \in \mathbb{R}$, $\sigma, s_0 > 0$. For $n \in \mathbb{N}$ we define the n-th $(\beta+1)$-*nomial CRR model* as follows: For $T \in \mathbb{R}^+$ let $\mathbb{T}_n := \{0, \Delta_n t, \dots, T\}$ with $\Delta_n t := T/n$. The price process S_n is obtained by sampling the price process of a complete binomial CRR model (on the finer time set $\mathbb{T}_n^\beta := \{0, \Delta_n t/\beta, 2\Delta_n t/\beta, \dots, T\}$) at the points $t \in \mathbb{T}_n$. Set $\Omega_n := \{0, 1, \dots, \beta\}^{\mathbb{T}_n \setminus \{T\}}$ and $\mathcal{F}_n := \mathcal{P}(\Omega_n)$. The counting measure on $\mathbb{T}_n$ is denoted by Λ_n.

Remark 6.1. It will be useful later to have some notation for the binomial model on $\mathbb{T}_n^\beta$: let $\Omega_n^\beta := \{-1, +1\}^{\mathbb{T}_n^\beta \setminus \{T\}}$ and Q_n^β be the measure on $(\Omega_n^\beta, \mathcal{P}(\Omega_n^\beta))$ given by the binomial probabilities q and $1-q$. Denote by W_n^β the binomial random walk on $\mathbb{T}_n^\beta$ with step size $\pm\sqrt{\Delta_n t/\beta}$ and the price process S_n^β is given by

$$S_{n,t}^\beta(\omega) = s_0 \prod_{s<t} \big(1 + \mu \frac{\Delta_n t}{\beta} + \sigma \Delta W_{n,s}^\beta\big), \quad s, t \in \mathbb{T}_n^\beta, \tag{18}$$

where $\Delta W_{n,t}^\beta := W_{n,t+\Delta_n t/\beta}^\beta - W_{n,t}^\beta$.

We want to specify a probability measure Q_n on Ω_n such that the price process

$$S_{n,t+\Delta} := S_{n,t} \cdot u^{\omega_t} d^{\beta-\omega_t}, \quad S_{n,0} := s_0 \tag{19}$$

with

$$u := 1 + \mu \frac{\Delta_n t}{\beta} + \sigma\sqrt{\Delta_n t/\beta} \quad \text{and} \quad d := 1 + \mu \frac{\Delta_n t}{\beta} - \sigma\sqrt{\Delta_n t/\beta}$$

is a martingale under Q_n. One possible choice would be the minimal martingale measure as defined in Section 2.3; however, we choose the unique martingale measure for the price process on $\mathbb{T}_n^\beta$ given by the probabilities

$$q := \frac{1-d}{u-d} = \frac{1}{2}\big(1 - \frac{\mu}{\sigma}\sqrt{\Delta_n t/\beta}\big) \quad \text{and} \quad 1-q = \frac{1}{2}\big(1 + \frac{\mu}{\sigma}\sqrt{\Delta_n t/\beta}\big)$$

for an 'up-' or 'down-movement' between times in $\mathbb{T}_n^\beta$. We then define the measure Q_n on $(\Omega_n, \mathcal{F}_n)$ by

$$Q_n(\omega_t = j) := \binom{\beta}{j} q^j (1-q)^{\beta-j}$$

for $\omega = (\omega_0, \dots, \omega_{T-\Delta_n t}) \in \Omega_n$ and $(\omega_t)_{t \in \mathbb{T} \setminus \{T\}}$ independent. We denote the expectation with respect to Q_n by $E_n[\,\cdot\,]$. A filtration $\mathbb{A}_n = (\mathcal{A}_{n,t})_{t \in \mathbb{T}_n}$ is again generated by the multinomial random walk $W_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ with

$$\Delta W_{n,t} := (-\beta + 2\omega_t)\sqrt{\Delta_n t/\beta}, \quad W_{n,0} := 0.$$

Since $E_n[\Delta W_{n,t} | \mathcal{A}_{n,t}] = -\frac{\mu}{\sigma}\Delta_n t$, W_n is not a martingale under Q_n. We therefore define an adjusted multinomial process $\tilde{W}_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ with

$$\Delta \tilde{W}_{n,t} := \Delta W_{n,t} + \frac{\mu}{\sigma}\Delta_n t, \quad \tilde{W}_{n,0} := 0,$$

so that

$$E_n\left[\Delta\tilde{W}_{n,t}|\mathcal{A}_{n,t}\right] = 0 \tag{20}$$

$$\mathrm{Var}_n\left[\Delta\tilde{W}_{n,t}|\mathcal{A}_{n,t}\right] = \Delta_n t\Big(1-(\frac{\mu}{\sigma})^2\Delta_n t/\beta\Big). \tag{21}$$

The n-th $(\beta+1)$-nomial CRR model satisfies the assumptions of Section 2.1, so that any claim $H:\Omega_n\to\mathbb{R}$ in this model can be replicated by a mean-self-financing strategy $\Phi^H=(\Theta^H,\Psi^H)$ which is risk-minimising since S_n is a martingale.

The Hyperfinite Version. For any infinite $N\in {}^*\mathbb{N}\setminus\mathbb{N}$ we have an internal $(\beta+1)$-nomial CRR model on the hyperfinite filtered probability space $(\Omega_N,\mathcal{F}_N,\mathbb{A}_N,Q_N)$ with associated Loeb space $(\Omega_N,L(\mathcal{F}_N),L(Q_N))$.

Lemma 6.2. *(i) The processes W_N and $\tilde{W}_N$ are $\mathcal{S}$-continuous.*
(ii) $\tilde{w}:=\mathrm{st}(\tilde{W}_N)$ is a standard Brownian motion on $(\Omega_N,L(\mathcal{F}_N),L(Q_N))$.
(iii) The price process S_N is $\mathcal{S}L^r(Q_N)$-integrable for each $r\in[1,\infty)$.

Proof. This is analogous to the proof of Theorem 3.3.5 in [AFHL86]. □

By Lemma 6.2(i) $W_{N,t}$ is finite, $L(Q_N)$-a.s.. It then follows from (19) and the proof of Lemma 3.1(a) in [CKW91] that, $L(Q_N)$-a.s.,

$$S_{N,t}\approx s_0\exp\big((\mu-\frac{1}{2}\sigma^2)t+\sigma W_{N,t}\big)=s_0\exp\big(-\frac{1}{2}\sigma^2 t+\sigma\tilde{W}_{N,t}\big)$$

for all $t\in\mathbb{T}_N$. Hence, S_N is $\mathcal{S}$-continuous and

$$s_u:=\mathrm{st}(S_N)_u=s_0\exp\big(-\frac{1}{2}\sigma^2 u+\sigma\tilde{w}_u\big)\quad\text{for } u\in[0,T],\quad L(Q_N)\text{-a.s..}$$

Since $\tilde{w}$ is a standard Brownian motion, s is indeed the price process in a Black-Scholes model under the unique martingale measure $Q:=L(Q_N)$. Hence each claim $h\in L^2(Q)$ can be represented as

$$h=v_0^h+\int_0^T\theta_u^h ds_u$$

for some $\theta^h\in L^2(\nu_s)$ and $v_0^h\in\mathbb{R}$. We now verify assumption (S1) on page 11 by employing the following lemma:

Lemma 6.3. *Suppose S_N satisfies*

$$E\left[(\Delta S_{N,t})^2\middle|\,\mathcal{A}_{N,t}\right]=(S_{N,t})^2K\Delta_N t \tag{22}$$

for some finite constant $K\in{}^*\mathbb{R}$ with ${}^\circ K>0$. For any nonanticipating process Θ and any $\theta\in L^2(\nu_s)$ we then have

$$\Theta \text{ is a } \mathcal{S}L^2(\nu_{S_N})\text{-lifting of } \theta \quad\Leftrightarrow\quad \Theta S_N \text{ is a } \mathcal{S}L^2(Q_N\times\Lambda_N)\text{-lifting of } \theta s.$$

Proof. The internal algebra $\mathcal{A}_{\Omega_N\times\Lambda_N}$ on $\Omega_N\times\mathbb{T}_N$ is generated by the sets $\{A\times\{t\}: t\in\mathbb{T}_N, A\in\mathcal{A}_{N,t}\}$ (cf. [AFHL86]). Using (22)

$$\nu_{S_N}(A\times\{t\})=E\left[\mathbf{1}_A(\Delta S_{N,t})^2\right]=E\left[\mathbf{1}_A S_{N,t}^2\right]K\Delta_N t$$

for $t\in\mathbb{T}_N, A\in\mathcal{A}_{N,t}$. Since $S_{N,t}$ is non-infinitesimal $L(Q_N)$-a.s., and square-$\mathcal{S}$-integrable we have

$$\nu_{S_N}(B)\approx 0\quad\Leftrightarrow\quad(Q_N\times\Lambda_N)(B)\approx 0$$

for $B \in \mathcal{A}_{\Omega_N \times \mathbb{T}_N}$. This implies that Θ lifts θ with respect to ν_{S_N} if and only if ΘS_N lifts θs with respect to $Q_N \times \Lambda_N$. Furthermore,

$$\begin{aligned}\int_{\Omega_N \times \mathbb{T}_N} \Theta^2 \, d\nu_{S_N} &= E\Big[\sum_{t<T} \Theta^2 (\Delta S_{N,t})^2\Big] = E\Big[\sum_{t<T} (\Theta S_{N,t})^2 K \Delta_N t\Big] \\ &= K \int_{\Omega_N \times \mathbb{T}_N} (\Theta S_N)^2 \, d(Q_N \times \Lambda_N)\end{aligned}$$

by (22) and the tower property for conditional expectations. Hence, $\Theta \in \mathcal{S}L^2(\nu_{S_N})$ if and only if $\Theta S_N \in \mathcal{S}L^2(Q_N \times \Lambda_N)$. ☐

In order to check (22) we calculate

$$E\left[(\Delta S_{N,t})^2 \middle| \mathcal{A}_{N,t}\right] = (S_{N,t})^2 \Big(E\left[\left(u^{\omega_t} d^{\beta-\omega_t}\right)^2\right] - 1\Big) \tag{23}$$

Now

$$\begin{aligned}E\left[\left(u^{\omega_t} d^{\beta-\omega_t}\right)^2\right] &= \sum_{j=0}^{\beta} \binom{\beta}{j} u^{2j} d^{2(\beta-j)} q^j (1-q)^{\beta-j} \\ &= \left(1 + \sigma^2 \frac{\Delta_N t}{\beta} - \mu^2 (\Delta_N t/\beta)^2\right)^{\beta} = 1 + (\sigma^2 + \varepsilon)\Delta_N t\end{aligned}$$

with $\varepsilon \approx 0$. Substituting into (23) we see that S_N satisfies (22) with $K = \sigma^2 + \varepsilon$.

We are now in a situation where we can apply Theorem 5.6, so that the valuation of claims and calculation of mean-self-financing trading strategies in the hyperfinite $(\beta+1)$-nomial CRR model are equivalent to the corresponding operations in the BS model.

Convergence Results. As in Section 3.4 we can consider the space of continuous paths $\mathcal{C} = \{x \in C[0,T] : x(0) = 0\}$ with the measure Q together with the finite subspaces $\mathcal{C}_n$ of polygonal paths of W_n and measure Q_n. Using Theorem 3.11 in the context of these spaces we have the following analogue of Theorem 4.7 in [CKW93a]:

Theorem 6.4. *Let $H_n : \Omega_n \to \mathbb{R}$ be a sequence of claims in the $(\beta+1)$-nomial CRR models and $h \in L^2(Q)$ a claim in the BS model. Then the following are equivalent:*

(i) $H_n \overset{D^2}{\to} h$.
(ii) $\Phi^{H_n} \overset{D^2}{\to} \phi^h$.
(iii) $V(\Phi^{H_n}) \overset{D^2}{\to} V(\phi^h)$.
(iv) $G(\Phi^{H_n}) \overset{D^2}{\to} G(\phi^h)$ *and* $V_0^{H_n} \to V_0^h$.

An important aspect of D^2-convergence is the existence of a discretisation scheme which maps paths in $\mathcal{C}$ back into $\mathcal{C}_n$ (see Definition 3.10).

Proposition 6.5. *There is an adapted Q-discretisation scheme for the $(\beta+1)$-nomial CRR model.*

Proof. Let $(d_n^\beta)_{n\in\mathbb{N}}$ be the adapted Q-discretisation scheme for the binomial CRR model on Ω_n^β (cf. [CKW93a]), so that $d_n^\beta : \mathcal{C} \to \mathcal{C}_n^\beta$ where $\mathcal{C}_n^\beta := \{W_n^\beta(\omega) : \omega \in \Omega_n^\beta\}$ denotes the path space for the binomial CRR model. Define a map $\tilde{d}_n : \mathcal{C}_n^\beta \to \mathcal{C}_n$ by

$$(\tilde{d}_n(\omega))(t) := \omega(t) \quad \text{for } \omega \in \mathcal{C}_n^\beta \text{ and } t \in \mathbb{T}_n$$

and filling in linearly between points in $\mathbb{T}_n$. So $\tilde{d}_n$ samples paths in $\mathcal{C}_n^\beta$ at points in $\mathbb{T}_n$ and 'forgets' what happens between these points. We now define $d_n : \mathcal{C} \to \mathcal{C}_n$ as

$$d_n := \tilde{d}_n \circ d_n^\beta.$$

To see that $(d_n)_{n\in\mathbb{N}}$ is an adapted Q-discretisation scheme we note that d_n is $\mathbb{A}_n$-adapted and Q-measure preserving since d_n^β and $\tilde{d}_n$ are. It only remains to show that $d_n(\omega) \to \omega$ in Q-probability. Fix $\varepsilon > 0$ and let $\delta > 0$. As $(d_n^\beta)_{n\in\mathbb{N}}$ is a discretisation scheme there exist $n_\delta \in \mathbb{N}$ such that

$$Q\big(\|d_n^\beta(\omega) - \omega\| < \varepsilon/2\big) > 1 - \delta \quad \text{for all } n \geq n_\delta. \tag{24}$$

Furthermore, $\|\tilde{d}_n(\omega) - \omega\| \leq 2(\beta+1)\sqrt{\Delta_n t/\beta}$ for all $\omega \in \mathcal{C}_n^\beta$. We therefore have $\tilde{n} \in \mathbb{N}$ such that $\|\tilde{d}_n(\omega) - \omega\| < \varepsilon/2$ for all $n \geq \tilde{n}$ and $\omega \in \mathcal{C}_n^\beta$. Hence, for $n \geq \max\{n_\delta, \tilde{n}\}$,

$$\begin{aligned}
Q\big(\|d_n(\omega) - \omega\| < \varepsilon\big) &= Q\big(\|\tilde{d}_n(d_n^\beta(\omega)) - d_n^\beta(\omega) + d_n^\beta(\omega) - \omega\| < \varepsilon\big) \\
&\geq Q\big(\|\tilde{d}_n(d_n^\beta(\omega)) - d_n^\beta(\omega)\| + \|d_n^\beta(\omega) - \omega\| < \varepsilon\big) \\
&\geq Q\big(\varepsilon/2 + \|d_n^\beta(\omega) - \omega\| < \varepsilon\big) \\
&> 1 - \delta.
\end{aligned}$$

□

Remark 6.6. The convergence result in Theorem 6.4 could also have been obtained under the minimal martingale measure for the multinomial CRR model. Therefore our choice of the measure Q_n might seem arbitrary; however, note that the minimal martingale measure depends on the specification of the 'physical' probability in the underlying model. We may therefore also use a martingale measure from the beginning. Furthermore, our choice of Q_n allows the construction of an adapted discretisation scheme from the already existing one for the binomial CRR model as in the proof of Proposition 6.5.

It was shown in [CKW93a, Theorem 4.4] that a sequence of D^2-convergent claims can always be obtained by means of an adapted discretisation scheme. However, when considering a specific claim h in the BS model a D^2-convergent sequence $(H_n)_{n\in\mathbb{N}}$ approximating h can often be found in a more direct and natural way:

Example 6.7. If the claim h only depends on the price of the risky asset at maturity, i.e. $h = f(s_T)$ for some function $f : \mathbb{R} \to \mathbb{R}$ then $H_n := f(S_{n,T})$ is a natural choice of an approximating sequence for h. Indeed, if f is piecewise continuous and satisfies a polynomial growth condition it is easy to see that $H_n \overset{D^2}{\to} h$.

Example 6.8 (Asian options). If $h = \big(\frac{1}{T}\int_0^T s_t dt - K\big)^+$ is an Asian call option with fixed strike price K then the claims $H_n := \big(\frac{1}{T}\sum_{t<T} S_{n,t}\Delta_n t - K\big)^+$ yield a D^2-convergent sequence approximating h. To see this note that, for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, S_N is $\mathcal{S}$-continuous, i.e. for $\omega \in \tilde{\Omega}_N \subset \Omega_N$ the path $S_{N,\cdot}(\omega)$ is $\mathcal{S}$-continuous where $L(Q_N)(\tilde{\Omega}_N) = 1$. In particular, $S_{N,\cdot}(\omega)$ is bounded, hence $\mathcal{S}$-integrable with respect to Λ_N. Therefore

$${}^\circ\Big(\sum_{t<T} S_{N,t}(\omega)\Delta_N t\Big) = \int_0^T s_t(\omega)\, dt \quad \text{for } \omega \in \tilde{\Omega}_N,$$

which shows that H_N is a lifting of h. By Lemma 6.2(iii) $S_{N,t} \in \mathcal{S}L^r(Q_N)$ for any $r \in [1,\infty)$. This implies that $\sum_{t<T} S_{N,t}\Delta_n t \in \mathcal{S}L^r(Q_N)$ for any $r \in [1,\infty)$. The same applies to an Asian option with average strike price, i.e. $h = \big(s_T - \frac{K}{T}\int_0^T s_t dt\big)^+$.

Summing up the results of this section we have shown that, for any $\beta \in \mathbb{N}$, the $(\beta+1)$-nomial CRR model has exactly the same convergence properties as the complete binomial CRR model, provided we are using mean-variance hedging for pricing and replicating claims in these models. When using finite models as approximations of the continuous time BS model the use of binomial models does therefore not offer any advantage over multinomial models. This gives a theoretical justification for the use of trinomial models in numerical methods for derivatives pricing and hedging; trinomial models are often preferred because they allow a more efficient calculation of prices (see [Wel98] for examples) and hedging parameters (the 'greeks'); it should also be noted that the use of trinomial models is equivalent to explicit finite difference methods for the numerical solutions of the pricing PDE (cf. for example [CS98]).

6.2. **Direct Discretisation of the Price Process.** In this section we consider a Black-Scholes model on a filtered probability space $(\Omega, \mathcal{F}, P)$ with time-dependent deterministic drift $(\mu_t)_{t\in[0,T]}$ and volatility $(\sigma_t)_{t\in[0,T]}$. We assume that the functions $\mu, \sigma : [0,T] \to \mathbb{R}$ are piecewise continuous and bounded. Furthermore, $\sigma_t > 0$ for all $t \in [0,T]$.

It is well-known that this model is complete (see e.g. [BK98, MR97]), so that there is a unique equivalent martingale measure Q for the price process s, and under this measure s is the solution to the stochastic differential equation

$$\frac{ds_t}{s_t} = \sigma_t \, d\tilde{w}_t$$

where $\tilde{w}$ is a standard Brownian motion under Q, i.e. s is given as

$$s_t = s_0 \exp\Big(-\frac{1}{2}\int_0^t \sigma_u^2 \, du + \int_0^t \sigma_u \, d\tilde{w}_u\Big). \tag{25}$$

From now on we will work with the measure Q; a filtration $\mathbb{F}$ on $(\Omega, \mathcal{F})$ is generated by $\tilde{w}$. For any $h \in L^2(Q)$ let ϕ^h be the unique self-financing strategy generating h.

To define a sequence of discrete time models approximating (25) we follow the 'direct discretisation' approach in [RS95]. We first approximate the volatility function σ by a sequence of piecewise constant functions: For $n \in \mathbb{N}$ and $\Delta_n t := T/n$ define $\sigma_n : [0,T] \to \mathbb{R}$ by

$$\sigma_n(t) := \sigma(0)\mathbf{1}_{\{0\}}(t) + \sum_{s<t} \sigma(s)\mathbf{1}_{(s,s+\Delta_n t]}(t) \tag{26}$$

(with the sum taken over $\mathbb{T}_n = \{0, \Delta_n t, \ldots, t\}$). This means that $\sigma_n(t) = \sigma(t)$ for $t \in \mathbb{T}_n$ and σ_n remains constant between points in $\mathbb{T}_n$. So σ_n is left-continuous with right limits, bounded and strictly positive.

We define another Q-martingale $s_n : \Omega \times [0,T] \to \mathbb{R}$ by

$$s_{n,t} := s_0 \exp\Big(-\frac{1}{2}\int_0^t \sigma_{n,u}^2 \, du + \int_0^t \sigma_{n,u} \, d\tilde{w}_u\Big)$$

We can then obtain a discrete time process $S_n : \Omega \times \mathbb{T}_n \to \mathbb{R}$ by evaluating s_n at the discretisation points $t \in \mathbb{T}_n$, so that S_n satisfies

$$\begin{aligned} S_{n,t+\Delta_n t} &= S_{n,t} \exp\Big(-\frac{1}{2}\sigma_{n,t}^2 \Delta_n t + \sigma_{n,t} \Delta \tilde{w}_t\Big), \quad t \in \mathbb{T}_n \setminus \{T\}, \\ S_{n,0} &= s_0, \end{aligned} \tag{27}$$

where $\Delta \tilde{w}_t := \tilde{w}_{t+\Delta_n t} - \tilde{w}_t \sim \mathcal{N}(0, \Delta_n t)$.

The Discrete Time Model and its Internal Version. We define the *n-th discretised BS model* as follows: Let $\Omega_n := \mathbb{R}^{\mathbb{T}_n \setminus \{T\}}$, $\mathcal{F}_n := \mathcal{B}(\Omega)$ and Q_n the probability defined by $\omega_t \sim \mathcal{N}(0, \Delta_n t)$ and $\omega_0, \ldots, \omega_{T-\Delta_n t}$ independent. A filtration $\mathbb{A}_n$ is generated by the process $W_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ with $\Delta W_{n,t} := \omega_t$, $W_{n,0} := 0$. The price process S_n is then defined as in (27), with $\Delta\tilde{w}_t$ replaced by $\Delta W_{n,t}$.

The discretised BS model satisfies the assumptions of Section 2.1 so that any claim $H \in L^2(Q_n)$ can be replicated by a risk-minimising mean-self-financing strategy $\Phi^H = (\Theta^H, \Psi^H)$. Again the variance-optimal strategy ξ^H coincides with Θ^H.

As before, for infinite N, this gives rise to an internal model on $(\Omega_N, \mathbb{A}_N, \mathcal{F}_N, Q_N)$ with associated Loeb space $(\Omega_N, L(\mathcal{F}_N), L(Q_N))$. By [Cut87, Theorem 2.2] the internal process W_N is $\mathcal{S}$-continuous and $w := \mathrm{st}(W_N)$ is a standard Brownian motion on the Loeb space.

Due to the piecewise continuity of σ we have

$$\sigma_N(t) = {}^*\sigma(t) \approx \sigma({}^\circ t) \quad \text{for } L(\Lambda_N)\text{-a.a. } t \in \mathbb{T}_N,$$

so that σ_N is an $\mathcal{S}$-bounded lifting of σ. Hence

$$S_{N,t} = s_0 \exp\Big(-\frac{1}{2}\sum_{s<t} \sigma_{N,s}^2 \Delta_N t + \sum_{s<t} \sigma_{N,s} \Delta W_{N,s}\Big)$$

is $\mathcal{S}$-continuous and

$$\mathrm{st}(S_N)_u = s_0 \exp\Big(-\frac{1}{2}\int_0^t \sigma_u^2 du + \int_0^t \sigma_u dw_u\Big).$$

Calculating

$$E\left[S_{N,t}^r\right] = s_0^r \exp\Big(-\frac{1}{2}(r - r^2)\sum_{s<t} \sigma_{N,s}^2 \Delta_N t\Big) \quad \text{for } r \in \mathbb{R}$$

shows that S_N is an $\mathcal{S}L^r$-martingale for any $r \in [1, \infty)$. Finally

$$E\left[(\Delta S_{N,t})^2 \,\middle|\, \mathcal{A}_{N,t}\right] = S_{N,t}^2\big(\exp(\sigma_{N,t}^2 \Delta_N t) - 1\big) = S_{N,t}^2(\sigma_{N,t}^2 + \varepsilon_t)\Delta_N t$$

with $\varepsilon_t \approx 0$ for all $t \in \mathbb{T}_N$, so that we can use the proof of Lemma 6.3 to show that assumption (S1) in Section 5 is satisfied. Hence Theorem 5.6 holds for this internal model and the BS model (25); we can now use it to obtain standard convergence results for the sequence of discretised BS models (27):

Convergence Results. As in Section 6.1 we assume that $\Omega = \mathcal{C}$ for the BS model and we consider the subspaces $\mathcal{C}_n \subset \mathcal{C}$ of polygonal paths of W_n. We therefore have the following analogue to Theorem 6.4:

Theorem 6.9. *Let $H_n : \Omega_n \to \mathbb{R}$ be a sequence of claims in the discretised BS models and $h \in L^2(Q)$ a claim in the BS model. Then the following are equivalent:*

(i) $H_n \overset{D^2}{\to} h$.
(ii) $\Phi^{H_n} \overset{D^2}{\to} \phi^h$.
(iii) $V(\Phi^{H_n}) \overset{D^2}{\to} V(\phi^h)$.
(iv) $G(\Phi^{H_n}) \overset{D^2}{\to} G(\phi^h)$ *and* $V_0^{H_n} \to V_0^h$.

We will see below how we can obtain an almost trivial adapted Q-discretisation scheme $(d_n)_{n\in\mathbb{N}}$ for these models, so that we also have the alternative characterization of D^2-convergence in terms of '$L^2(d_n(\cdot))$-convergence' given by Theorem 3.11:

Proposition 6.10. *The family of mappings $d_n : \mathcal{C} \to \mathcal{C}_n$ defined by*

$$(d_n(\omega))(t) := \omega(t) \quad \text{for } \omega \in \mathcal{C} \text{ and } t \in \mathbb{T}_n, \tag{28}$$

with $d_n(\omega)$ filled in linearly between points in $\mathbb{T}_n$, is an adapted Q-discretisation scheme.

Proof. Firstly, d_n is $\mathbb{A}_n$-adapted by definition. Furthermore, $Q(d_n^{-1}(A)) = Q_n(A)$ for all $A \in \mathcal{F}_n$ due to the fact that the finite-dimensional distributions of a Brownian motion are multivariate normal. It therefore only remains to show that $d_n(\omega) \to \omega$ in Q-probability.

Using [Cut87, Theorem 2.2] once again we see that, for infinite $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, the standard part of the process $d_N : {}^*\mathcal{C} \times \mathbb{T}_N \to {}^*\mathbb{R}$ defined by (28) is a standard Brownian motion on $({}^*\mathcal{C}, L({}^*\mathcal{F}), L({}^*Q))$, hence

$$L({}^*Q)\big(\|d_N({}^*\omega) - {}^*\omega\| \approx 0\big) = 1,$$

in particular, for any positive $\varepsilon \in \mathbb{R}$,

$$1 = L({}^*Q)\big(\|d_N({}^*\omega) - {}^*\omega\| < \varepsilon\big) \approx {}^*Q\big(\|d_N({}^*\omega) - {}^*\omega\| < \varepsilon\big).$$

By the nonstandard characterization of convergence of a sequence in $\mathbb{R}$ this means that

$$Q\big(\|d_n(\omega) - \omega\| < \varepsilon\big) \to 1 \quad \text{as } n \to \infty.$$

□

7. Conclusion

Although the proofs employed in this paper make extensive use of nonstandard stochastic analysis, as developed in [HP83] and [AFHL86], the main results themselves (including the formulation of D^2-convergence) are expressed entirely within the framework and terminology of standard stochastic analysis. The nonstandard tools used here, however, provide a degree of insight into the essential structure of the models and the relationship between them which is not easily obtained by purely standard methods.

The techniques developed in this paper enable us to extend to certain incomplete market models the convergence theory developed in [CKW93a] for approximations to the Black-Scholes model. By utilizing the stability properties of the minimal martingale measure approach and developing the appropriate liftings, we were able to demonstrate the equivalence of the D^2-convergence of contingent claims and that of their associated strategies, which distinguishes this mode of convergence from the more usual weak convergence methodology.

Weak convergence results for strategies do not, to our knowledge, exist in the literature for the various models dealt with in this paper. The D^2-convergence results obtained in the final section of this paper provide a theoretical justification for the multinomial approximations often used in practice and for direct sampling methods applied to continuous-time models.

However, our methodology still requires the limiting model to be complete, even though this restriction is not placed on the approximating discrete models. It would be interesting to see an extension of this approach to situations where the limit model is also allowed to be incomplete.

8. Appendix: Proof of Proposition 4.1

The following lemma is needed in the proof of Proposition 4.1. It shows how the "pure jump" part of a cadlag function can be approximated by an internal function.

Lemma 8.1. *Let $F : \mathbb{T} \to {}^*\mathbb{R}$ be an SDJ function such that*

$$\sum_{s<T} (\Delta F_s)^2 \quad \text{is finite.} \tag{29}$$

Let $f := \mathrm{st}(F)$ be the standard part of F. Then there exists an internal subset $\tilde{\mathbb{T}}$ of $\mathbb{T}$ such that the functions $J : \mathbb{T} \to {}^*\mathbb{R}$, $j : [0,T] \to \mathbb{R}$ defined as

$$J(t) := \sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} \Delta F_s + \frac{1}{2}(\Delta F_s)^2 \qquad \text{and} \qquad j(v) := \sum_{u\le v} \Delta f_u + \frac{1}{2}(\Delta f_u)^2$$

satisfy $j = \mathrm{st}(J)$. Furthermore, the functions $K : \mathbb{T} \to {}^*\mathbb{R}$, $k : [0,T] \to \mathbb{R}$ defined as

$$K(t) := \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1 + \Delta F_s) \qquad \text{and} \qquad k(v) := \prod_{u\le v} (1 + \Delta f_u)$$

satisfy $k = \mathrm{st}(K)$.

Proof. Let $B := \max\{|\Delta F_s| + 1 : s \in \mathbb{T}\}$. Then B is finite and $|{}^\circ\Delta F_s| \le B$ for all $s \in \mathbb{T}$. There are at most countably many points $u \in [0,T]$ such that $|\Delta f_u| > 0$. Since F is SDJ there exists a unique $s \in \mathbb{T}$ for every $u \in [0,T]$ with $|\Delta f_u| > 0$ such that $s \approx u$,

$${}^\circ F(s) = f(u-) \quad \text{and} \quad {}^\circ F(s+\Delta t) = f(u), \quad \text{hence} \quad {}^\circ\Delta F_s = \Delta f_u. \tag{30}$$

Now let $\mathbb{J} := \{u \in [0,T] : |\Delta f_u| > 0\} = \{u_1, u_2, \ldots\}$ be the (possibly) finite set of jump points of f.

Case (i): $\mathbb{J}$ is a finite — possibly empty — set. Let $\tilde{\mathbb{T}}$ be the set of corresponding jump points s of F as defined in (30). Then $\tilde{\mathbb{T}}$ is finite (or empty), hence internal and, for $t \in \mathbb{T}$,

$${}^\circ J(t) = \sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \qquad \text{and} \qquad {}^\circ K(t) = \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1 + {}^\circ\Delta F_s). \tag{31}$$

Case (ii): $\mathbb{J}$ is countably infinite. Due to assumption (29) the set of $u \in [0,T]$ such that $|\Delta f_u| > c$ is finite for any fixed $c > 0$. We may therefore assume (by applying Case (i) to a subset of $\mathbb{J}$ if necessary) that $|\Delta f_u| < \frac{1}{2}$ for all $u \in \mathbb{J}$. Let $(u_i)_{i\in\mathbb{N}}$ be an enumeration of the points in $\mathbb{J}$ and $(s_i)_{i\in\mathbb{N}}$ the sequence of corresponding non-infinitesimal jump points in $\mathbb{T}$ as defined in (30). By Countable Comprehension (see e.g. Section 1.2.3 in [Cut00]) this sequence can be extended to a sequence $(s_i)_{i\in{}^*\mathbb{N}}$ with $s_i \in \mathbb{T}$ for all $i \in {}^*\mathbb{N}$.

Now let $\bar{\varepsilon} := \max\{|F_s - {}^\circ F_s| : s \in \mathbb{T}\}$, so $\bar{\varepsilon} \approx 0$ and $\max\{|\Delta F_s - {}^\circ\Delta F_s| : s \in \mathbb{T}\} \le 2\bar{\varepsilon}$.

Choose $M \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $M\bar{\varepsilon} \approx 0$ and $M \le N$ (e.g. let $M = \min\{[1/\sqrt{\bar{\varepsilon}}], N\}$ where $[r]$ denotes the integer part of $r \in {}^*\mathbb{R}$).

Define $\tilde{\mathbb{T}} := \{s_i : i \le M\} \subset \mathbb{T}$. Then $\tilde{\mathbb{T}}$ is internal and

$${}^\circ J(t) = {}^\circ\Big(\sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} \Delta F_s + \frac{1}{2}(\Delta F_s)^2\Big) = \sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \tag{32}$$

as

$$
\begin{aligned}
&\Big|\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} \Delta F_s - {}^\circ\Delta F_s + \frac{1}{2}\big((\Delta F_s)^2 - ({}^\circ\Delta F_s)^2\big)\Big| \\
&\quad\leq \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} |\Delta F_s - {}^\circ\Delta F_s| + \frac{1}{2}|\Delta F_s - {}^\circ\Delta F_s|\cdot|\Delta F_s + {}^\circ\Delta F_s| \\
&\quad\leq M(2\bar{\varepsilon} + 2B\bar{\varepsilon}) \approx 0.
\end{aligned}
$$

Furthermore, writing $\log(1+x) = x + \varepsilon$,

$$
\begin{aligned}
\Big|\log\big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+\Delta F_s)\big) - \log\big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+{}^\circ\Delta F_s)\big)\Big| &\leq \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\log(1+\Delta F_s) - \log(1+{}^\circ\Delta F_s)\Big| \\
&= \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\log\Big(1+\frac{\Delta F_s - {}^\circ\Delta F_s}{1+{}^\circ\Delta F_s}\Big)\Big| \\
&\leq \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\frac{\Delta F_s - {}^\circ\Delta F_s}{1+{}^\circ\Delta F_s}\Big| + |\varepsilon_s|.
\end{aligned}
$$

Recall that $|\Delta f_u| < \frac{1}{2}$ for all $u \in \mathbb{J}$, so that, for $s \in \tilde{\mathbb{T}}$, $|{}^\circ\Delta F_s| \leq \frac{1}{2}$, which implies that $1 + {}^\circ\Delta F_s \geq \frac{1}{2}$. Since $|\log(1+x) - x| \leq |x|^2$ for $|x| \leq \frac{1}{2}$ we see that $|\varepsilon_s| \leq 4|\Delta F_s - {}^\circ\Delta F_s|^2$ and hence the final sum is bounded by

$$
\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} 2|\Delta F_s - {}^\circ\Delta F_s| + 4|\Delta F_s - {}^\circ\Delta F_s|^2 \leq M(4\bar{\varepsilon} + 16\bar{\varepsilon}^2) = 4M\bar{\varepsilon}(1+4\bar{\varepsilon}) \approx 0,
$$

thus, by the $\mathcal{S}$-continuity of the exponential function

$$
{}^\circ K(t) = {}^\circ\Big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+\Delta F_s)\Big) = \prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+{}^\circ\Delta F_s). \tag{33}
$$

We now show that $\mathrm{st}(J) = j$ and $\mathrm{st}(K) = k$ in either case: Let $v \in [0,T]$. Then there are points $v', v'' \in \mathbb{J}$ such that $v' \leq v < v''$ and $s \notin \mathbb{J}$ for all $v' < s < v''$ (i.e. v', v'' are the nearest jump points to the left and right of v — note that possibly $v' = v$). Let t', t'' be the corresponding points in $\tilde{\mathbb{T}}$ as defined in (30). Then ${}^\circ|v - t''| > 0$. Hence, using (31) and (32), and observing that j only changes values at points $u \in \mathbb{J}$ while ${}^\circ J(\cdot)$ changes values only at the corresponding points $s \in \tilde{\mathbb{T}}$,

$$
\begin{aligned}
\mathrm{st}(J)(v) = \lim_{{}^\circ t\downarrow v} {}^\circ J(t) &= \lim_{{}^\circ t\downarrow v} \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \\
&= \sum_{\substack{s<t'\\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \\
&= \sum_{u\leq v'} \Delta f_s + \frac{1}{2}(\Delta f_s)^2 = j(v).
\end{aligned}
$$

An analogous argument (using (31) and (33)) then shows that $\mathrm{st}(K) = k$. $\square$

Proof of Proposition 4.1. Let $\tilde{\Omega}$ be the subset of Ω such that $X(\omega)$ and $[X](\omega)$ are SDJ and $\mathrm{st}([X](\omega)) = [x](\omega)$ for all $\omega \in \tilde{\Omega}$. We know from Theorem 3.4(ii) that $P_L(\tilde{\Omega}) = 1$. Now fix $\omega \in \tilde{\Omega}$.

Let $\tilde{\mathbb{T}}$ be the internal subset of $\mathbb{T}$ obtained by applying Lemma 8.1 to the path $X(\omega)$ (i.e. $\tilde{\mathbb{T}}$ contains all $s \in \mathbb{T}$ such that ${}^\circ|\Delta X_s(\omega)| > 0$). Set $\mathbb{T}'' := \mathbb{T} \setminus \tilde{\mathbb{T}}$ and write $Z(\omega) = Z' \cdot Z''$ with

$$\begin{aligned} Z'_t &:= \prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s) \\ &= \exp\Big(\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} \Delta X_s - \frac{1}{2}(\Delta X_s)^2 - \Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big) \prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s), \qquad (34)\\ Z''_t &:= \prod_{\substack{s<t\\ s\in\mathbb{T}''}} (1+\Delta X_s). \qquad (35)\end{aligned}$$

Since $|\log(1+y) - y + \frac{1}{2}y^2| \le |y|^3$ for $|y| \le \frac{1}{2}$ and ${}^\circ|\Delta X_s| = 0$ for all $s \in \mathbb{T}''$ we have

$$\log(Z''_t) = \sum_{\substack{s<t\\ s\in\mathbb{T}''}} \log(1+\Delta X_s) = \sum_{\substack{s<t\\ s\in\mathbb{T}''}} \Delta X_s - \frac{1}{2}(\Delta X_s)^2 + \varepsilon_s \quad \text{with} \quad |\varepsilon_s| \le |\Delta X_s|^3.$$

Let $\bar{\varepsilon} := \max\{|\Delta X_s| : s \in \mathbb{T}''\}$. Then $\bar{\varepsilon} \approx 0$ and

$$\Big|\sum_{\substack{s<t\\ s\in\mathbb{T}''}} \varepsilon_s\Big| \le \bar{\varepsilon} \sum_{s<t} (\Delta X_s)^2 = \bar{\varepsilon} \cdot [X]_t \approx 0$$

as $[X]$ is SDJ; hence, by the $\mathcal{S}$-continuity of the exponential function,

$$Z''_t \approx \exp\Big(\sum_{\substack{s<t\\ s\in\mathbb{T}''}} \Delta X_s - \frac{1}{2}(\Delta X_s)^2\Big).$$

Therefore, by (34) and (35),

$$Z_t \approx \exp\Big(X_t - \frac{1}{2}[X]_t\Big) \prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s) \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} \exp\Big(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big).$$

Observing that $\mathrm{st}(X) = x$ and $\mathrm{st}([X]) = [x]$ it then follows from the definition of $\tilde{\mathbb{T}}$ in Lemma 8.1 and the $\mathcal{S}$-continuity of the exponential function that

$$\mathrm{st}(Z)_v = \exp\Big(x_v - \frac{1}{2}[x]_v\Big) \prod_{u\le v} (1+\Delta x_u) \exp\Big(-\Delta x_u + \frac{1}{2}(\Delta x_u)^2\Big).$$

□

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