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AN EXTENDED GENERATOR AND SCHRÖDINGER EQUATIONS

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Abstract The generator of a Borel right process is extended so that it maps functions to smooth measures. This extension may be defined either probabilistically using martingales or analytically in terms of certain kernels on the state space of the process. Then the associated Schrödinger equation with a (signed) measure serving as potential may be interpreted as an equation between measures. In this context general existence and uniqueness theorems for solutions are established. These are then specialized to obtain more concrete results in special situations.

Keywords Markov processes, Schrödinger equations, generators, smooth measures

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1 Introduction

During the past 20 years or so there has been considerable interest in equations of the form

$$(\Lambda + \mu)u = f. \tag{1.1}$$

Here Λ is a linear operator and μ a signed measure. Classically Λ was the Laplacian on a domain in $\mathbb{R}^n$ (possibly with boundary conditions) and μ was absolutely continuous with respect to Lebesgue measure, say with density p. Then (1.1) has the form $(\Delta + p)u = f$ and is often called the Schrödinger equation with potential p. Hence in the literature (1.1) is often called the (generalized) Schrödinger equation for Λ. The situation in which $\Lambda = \Delta$ was generalized first to Λ a reasonable second order elliptic partial differential operator, then when Λ is an operator derived from a Dirichlet form, a Markov process or a harmonic space. Of course even when $\Lambda = \Delta$ and μ is measure one must give a precise meaning to (1.1). For a sampling of the literature during this period see [BHH87], [FL88], [ABR89], [Ma91], [H93], [CZ95], [G99] and the references contained therein.

There seem to be (at least) three somewhat different but interrelated techniques for approaching (1.1) in the literature which might be described as:

(i) perturbation of Dirichlet forms,

(ii) perturbation of Markovian semigroups,

(iii) perturbation of harmonic spaces.

It seems to me that in all three approaches the basic idea is to define $\Lambda + \mu$ as an operator in some space of functions and then interpret (1.1) in some weak sense. For example in method (ii) one assumes the μ corresponds to a continuous additive functional A and then defines $\Lambda + \mu$ as the generator of the Feynman-Kac semigroup

$$Q_t g(x) := E^x[g(X_t)e^{A_t}]$$

acting in some reasonable function space, and then interprets (1.1) in an appropriate weak sense. Here $X = (X_t)$ is the underlying Markov process and E^x the expectation when $X_0 = x$. This approach is carried out in detail for very general Markov processes in my paper [G99].

In the present paper we introduce a rather different approach to (1.1). Namely we extend the domain of Λ so that it maps functions into measures and then interpret (1.1) as an equation between measures with the right side being the measure fm where m is a prescribed underlying measure—Lebesgue measure in the classical case. It then seems natural to replace the right side of (1.1) by a measure ν. So in fact we shall investigate the equation

$$(\Lambda + \mu)u = \nu. \tag{1.2}$$

This approach has several advantages over the perturbation approach—method(ii) above—used in [G99]. It seems more direct and natural (to me) and, more importantly, it is technically simpler and one obtains more general results under somewhat weaker hypotheses. Also the method lends itself to study (1.2) on suitable subsets of the state space of X. This allows a consideration of "boundary conditions" for (1.2) and leads to a notion of "harmonic functions" for $\Lambda + \mu$. This aspect will be explored in a subsequent paper. The relationship between u and the value of $\Lambda u = \lambda$ is that a certain process, Y, involving $u \circ X_t$ and the continuous additive functional A associated with the measure λ be a martingale. See Theorem 3.9 and Definition 4.1 for the precise statement. Of course the idea of using martingales to extend the generator goes back at least to Dynkin [D65]. This was extended further by Kunita [K69] to measures absolutely continuous with respect to a given measure. See also [CJPS80] for a discussion of various extended versions of the generator. However our point of view seems somewhat different from earlier work. It would be natural to extend the domain of our generator even further by requiring the process Y mentioned above to be a local martingale rather than a martingale. We have decided not to do this in the present paper for several reasons. Most importantly in order to write down the solution of (1.1) or (1.2) it is necessary to impose certain integrability conditions. Moreover the definition adopted in section 4 can be stated without reference to martingales. Finally the results in this paper would be needed for any localization of the definition and we decided not to complicate the basic idea with additional technicalities. In discussing harmonic functions—that is solutions of (1.1) when $f = 0$—in a subsequent paper it will be both natural and necessary to localize the current definition.

The remainder of the paper is organized as follows. Section 2 sets out the precise hypotheses under which we shall work and reviews some of the basic definitions that are needed. It also contains some preliminary results. Section 3 contains the equivalence of a martingale property and the analytic property that is used to define the generator. In section 4 the generator is defined and discussed. We proceed somewhat more generally than indicated so far. Namely we consider a finely open nearly Borel subset $D \subset E$ and define an operator Λ_D that we regard as an extension of the restriction of the generator of X to D. It maps functions on E to measures on D. In section 5 we study the equation (1.2). Again we are somewhat more general and consider

$$(q - \Lambda_D - \mu)u = \nu \tag{1.3}$$

on D. Here $q \geq 0$ is a parameter. We prove existence and uniqueness theorems for (1.3) under various hypotheses. Finally in section 6 we suppose that $\nu = fm$ where m is the distinguished underlying measure and $f \in L^p(m)$, $1 \leq p \leq \infty$. We specialize the results of section 5 to obtain existence and uniqueness theorems depending on p. If $1 < p < \infty$ our results are sharper than those obtain in [G99].

We close this introduction with some words on notation. If $(F, \mathcal{F}, \mu)$ is a measure space, then we also use $\mathcal{F}$ to denote the class of all $\bar{\mathbb{R}} = [-\infty, \infty]$ valued $\mathcal{F}$ measurable functions. If $\mathcal{M} \subset \mathcal{F}$, then $b\mathcal{M}$ (resp. $p\mathcal{M}$) denotes the class of bounded (resp. $[0,\infty]$-valued) functions in $\mathcal{M}$. For $f \in p\mathcal{F}$ we shall use $\mu(f)$ to denote the integral $\int f\,d\mu$; similarly, if $D \in \mathcal{F}$ then $\mu(f; D)$ denotes $\int_D f\,d\mu$. We write $\mathcal{F}^*$ for the universal completion of $\mathcal{F}$; that is, $\mathcal{F}^* = \cap_\nu \mathcal{F}^\nu$, where $\mathcal{F}^\nu$ is the ν-completion of $\mathcal{F}$ and the intersection is over all finite (equivalently σ-finite) measures ν on $(F, \mathcal{F})$. If $(E, \mathcal{E})$ is a second measurable space and $K = K(x, dy)$ is a kernel from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ (i.e., $F \ni x \mapsto K(x, A)$ is $\mathcal{F}$-measurable for each $A \in \mathcal{E}$ and $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$ for each $x \in F$), then we write μK for the measure $A \mapsto \int_F \mu(dx) K(x, A)$ and Kf for the function $x \mapsto \int_E K(x, dy) f(y)$. The symbol ":=" stands for "is defined to be." Finally $\mathbb{R}$ (resp. $\mathbb{R}^+$) denotes the real numbers (resp. $[0, \infty[$) and $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}(\mathbb{R}^+)$) the corresponding Borel σ-algebras, while $\mathbb{Q}$ denotes the rationals. A reference $(m.n)$ in the text refers to item $m.n$

in section m. Due to the vagaries of LaTeX this might be a numbered display or the theorem, proposition, etc. numbered $m.n$.

2 Preliminaries

Throughout the paper $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ will denote the canonical realization of a Borel right Markov process with state space $(E, \mathcal{E})$. We shall use the standard notation for Markov processes as found, for example, in [BG68], [G90], [DM87] and [Sh88]. Briefly, X is a strong Markov process with right continuous sample paths, the state space E (with Borel sets $\mathcal{E}$) is homeomorphic to a Borel subset of a compact metric space, and the transition semigroup $(P_t)_{t\geq 0}$ of X preserves the class $b\mathcal{E}$ of bounded $\mathcal{E}$-measurable functions. It follows that the resolvent operators $U^q := \int_0^\infty e^{-qt} P_t dt, q \geq 0$, also preserve Borel measurability. In the present situation q-excessive functions are nearly Borel and we let $\mathcal{E}^n$ denote the σ-algebra of nearly Borel subsets of E. *In the sequel, all named subsets of E are taken to be in $\mathcal{E}^n$ and all named functions are taken to be $\mathcal{E}^n$-measurable unless explicit mention is made to the contrary.*

We take Ω to be the canonical space of right continuous paths ω (with values in $E_\Delta := E \cup \{\Delta\}$) such that $\omega(t) = \Delta$ for all $t \geq \zeta(\omega) := \inf\{s : \omega(s) = \Delta\}$. The stopping time ζ is the *lifetime* of X and Δ is a cemetery state adjoined to E as an isolated point; Δ accounts for the possibility $P_t 1_E(x) < 1$ in that $P^x(\zeta < t) = 1 - P_t 1_E(x)$. The σ-algebras $\mathcal{F}_t$ and $\mathcal{F}$ are the usual completions of the σ-algebras $\mathcal{F}_t^\circ := \sigma\{X_s : 0 \leq s \leq t\}$ and $\mathcal{F}^\circ := \sigma\{X_s : s \geq 0\}$ generated by the coordinate maps $X_s : \omega \to \omega(s)$. The probability measure P^x is the law of X started at x, and for a measure μ on E, P^μ denotes $\int_E P^x(\cdot)\mu(dx)$. Finally, for $t \geq 0, \theta_t$ is the shift operator: $X_s \circ \theta_t = X_{s+t}$. We adhere to the convention that a function (resp. measure) on E (resp. $\mathcal{E}^*$) is extended to Δ by declaring its value at Δ (resp. $\{\Delta\}$) to be zero.

We fix once and for all an excessive measure m. Thus, m is a σ-finite measure on $(E, \mathcal{E}^*)$ and $mP_t \leq m$ for all $t > 0$. Since X is a right process, we then have $\lim_{t\to 0} mP_t = m$, setwise.

Recall that a set B is m-polar provided $P^m(T_B < \infty) = 0$, where $T_B := \inf\{t > 0 : X_t \in B\}$ denotes the hitting time of B. A property or statement $P(x)$ will be said to hold quasi-everywhere (q.e.), or for quasi-every $x \in E$, provided it holds for all x outside some m-polar subset of E. It would be more proper to use the term "m-quasi-everywhere," but since the measure m will remain fixed the abbreviation to "q.e." will cause no confusion. Similarly, the qualifier "a.e. m" will be abbreviated to "a.e." On the other hand, certain terms (e.g., polar) have a longstanding meaning without reference to a background measure, and so we shall use the more precise term "m-polar" to maintain the distinction. Notice that any finely open m-null set is m-polar. Consequently, any excessive function vanishing a.e. vanishes q.e. A set $B \subset E$ is m-semipolar provided it differs from a semipolar set by an m-polar set. It is known that B is m-semipolar if and only if

$$P^m(X_t \in B \text{ for uncountably many } t) = 0.$$

See [A73]. A set B is m-inessential provided it is m-polar and $E \setminus B$ is absorbing. According to [GS84–(6.12)] an m-polar set is contained in a Borel m-inessential set. Since m is excessive it follows that sets of potential zero are m-null. In particular m-polar and m-semipolar sets are m-null.

We now fix a finely open set D—remember this means D is finely open and nearly Borel. Let $\tau = \tau_D := \inf\{t > 0 : X_t \notin D\}$ be the exit time from D. Then τ is an exact terminal time. Let $D_p = \{x : P^x(\tau > 0) = 1\} = \{x : E^x(e^{-\tau}) < 1\}$ be the set of permanent points of τ. Then D_p is finely open (and nearly Borel). Let $D^c := E \setminus D$ be the complement of D and D^{cr} the set of regular points of D^c. Then $D_p = D \cup (D^c \setminus D^{cr})$ and $D^c \setminus D^{cr}$ is semipolar. Thus $D_p \setminus D$ is always semipolar. In many situations it is in fact m-polar and we shall say that D is *m-regular* when $D_p \setminus D$ is m-polar. However we shall state explicitly when we assume D is m-regular.

We shall make use of the process obtained by killing X at time τ which is denoted by (X, τ). The state space for (X, τ) is D_p and $(f \geq 0)$

$$Q_t f(x) := E^x[f(X_t); t < \tau], \qquad V^q f(x) := E^x \int_0^\tau e^{-qt} f(X_t)\, dt$$

denote the semigroup and resolvent of (X, τ). Then $Q_t f$ and $V^q f$ vanish on $D^{cr} = E \setminus D_p$ and, hence, a.e. on D^c.

Let $\mathcal{M}^+(D)$ denote the class of all (positive) σ-finite measures on D, and let $\mathcal{M}(D) = \mathcal{M}^+(D) - \mathcal{M}^+(D)$ denote the class of all formal differences of elements in $\mathcal{M}^+(D)$. Thus $\mu = (\mu_1, \mu_2) \in \mathcal{M}(D)$ is formally $\mu = \mu_1 - \mu_2$. Define equality in $\mathcal{M}(D)$ by $(\mu_1, \mu_2) = (\nu_1, \nu_2)$ provided $\mu_1 + \nu_2 = \mu_2 + \nu_1$ and introduce the obvious definitions of addition and scalar multiplication. Then $\mathcal{M}(D)$ becomes a real vector space. If $\mu \in \mathcal{M}(D)$ and $\mu = (\mu_1, \mu_2)$ for $\mu_1, \mu_2 \in \mathcal{M}^+(D)$, then we say that (μ_1, μ_2) represents μ. If $\mu = (\mu_1, \mu_2)$ then $-\mu = (\mu_2, \mu_1)$. It is not hard to see that there exists a unique representative (μ^+, μ^-) of $\mu \in \mathcal{M}(D)$ with $\mu^+ \perp \mu^-$. We then define $|\mu| = \mu^+ + \mu^-$. Of course $|\mu| \in \mathcal{M}^+(D)$. If f is defined on D and is finite a.e. $|\mu|$, then $f\mu := (f^+\mu^+ + f^-\mu^-, f^+\mu^- + f^-\mu^+) \in \mathcal{M}(D)$ where $f^+(x) = f(x)1_{\{f \geq 0\}}(x)$ and $f^-(x) = -f(x)1_{\{f<0\}}(x)$. Checking carriers one sees that, in fact, $(f\mu)^+ = f^+\mu^+ + f^-\mu^-$ and $(f\mu)^- = f^+\mu^- + f^-\mu^+$ so that $|f\mu| = |f||\mu|$.

The next definition is basic.

Definition 2.1 *A continuous additive functional, A, of (X, τ) is a real valued process $A = A_t(\omega)$ defined on $0 \leq t < \tau(\omega)$ if $\tau(\omega) > 0$ and for all $t \geq 0$ if $\tau(\omega) = 0$, for which there exists a defining set $\Lambda \in \mathcal{F}$ and an m-inessential set $N \subset D_p$—called an exceptional set for A—such that:*

(i) $A_t 1_{\{t<\tau\}} \in \mathcal{F}_t$ *for all* t.

(ii) $P^x(\Lambda) = 1$ *for* $x \notin N$.

(iii) If $\omega \in \Lambda$ *and* $t < \tau(\omega)$, *then* $\theta_t \omega \in \Lambda$.

(iv) For $\omega \in \Lambda$, $t \to A_t(\omega)$ *is continuous on* $[0, \tau(\omega)[$ *and of bounded variation on compact subintervals of* $[0, \tau(\omega)[$.

(v) For all $\omega \in \Lambda$; $s \geq 0$, $t \geq 0$, $s + t < \tau(\omega)$ *one has* $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$.

(vi) $A_t(\omega) = 0$ *for all* t *if* $\tau(\omega) = 0$.

Note that if $\omega \in \Lambda$ and $\tau(\omega) > 0$ it follows from (ν) that $A_0(\omega) = 0$. If A is increasing and we define for $\omega \in \Lambda$ and $t \geq \tau(\omega)$, $A_t(\omega) := \lim_{s \uparrow \tau(\omega)} A_s(\omega)$, then

$$A_{t+s}(\omega) = A_t(\omega) + 1_{[0,\tau(\omega)[}(t) A_s(\theta_t \omega) \tag{2.2}$$

for $\omega \in \Lambda$; $s, t \geq 0$. We denote the totality of all continuous additive functionals of (X, τ) by $\mathcal{A}(D)$ and by $\mathcal{A}^+(D)$ the increasing elements of $\mathcal{A}(D)$. If $A \in \mathcal{A}(D)$, $\omega \in \Lambda$ and $t < \tau(\omega)$ define $|A|_t(\omega)$ to be the total variation of $s \to A_s(\omega)$ on $[0, t]$. Then it is routine to check that $|A| \in \mathcal{A}^+(D)$ with the same defining and exceptional sets. Hence $A^+ := \frac{1}{2}[|A| + A]$ and $A^- := \frac{1}{2}[|A| - A]$ are in $\mathcal{A}^+(D)$ with the same defining and exceptional sets and $A = A^+ - A^-$. Two elements $A, B \in A(D)$ are equal provided they are m-equivalent; that is they have a common defining set Λ and a common exceptional set N such that $A_t(\omega) = B_t(\omega)$ for $\omega \in \Lambda$ and $0 \leq t < \tau(\omega)$. The argument below (3.1) in [FG96] may be adapted to show that $A = B$ if and only if $P^m(A_t \neq B_t; t < \tau) = 0$ for all $t > 0$. Note we assume that N is m-inessential for X and not just for (X, τ). If A is a PCAF of X as defined in [FG96], then the restriction of A to $[0, \tau[$ is in $\mathcal{A}^+(D)$. Also if $A, B \in \mathcal{A}^+(D)$, then $A - B \in \mathcal{A}(D)$. Finally note that if $A^1, A^2, B^1, B^2 \in \mathcal{A}^+(D)$ then $A = A^1 - A^2$ equals $B = B^1 - B^2$ if and only if $A^1 + B^2 = A^2 + B^1$. Of course we are using m-equivalence as our definition of equality in $\mathcal{A}(D)$.

Definition 2.3 *The Revuz measure associated with $A \in \mathcal{A}^+(D)$ is the measure ν_A defined by the formula*

$$\nu_A(f) :=\uparrow \lim_{t \downarrow 0} E^m \frac{1}{t} \int_0^t f(X_s)\, dA_s, \qquad f \geq 0. \tag{2.4}$$

Of course the integral in (2.4) extends only over the interval $[0, \tau(\omega)[$ since that is where A is defined, but may be considered over $[0, \infty[$ since by convention $A_s(\omega) = \lim_{t \uparrow \tau(\omega)} A_t(\omega)$ for $s \geq \tau$ when $A \in \mathcal{A}^+(D)$. See [FG88] for the fact that the limit in (2.4) exists in $[0, \infty]$. The limit in (2.4) coincides with $\uparrow \lim_{q \to \infty} qmV_A^q f$ where

$$V_A^q f(x) := E^x \int_0^\tau e^{-qt} f(X_t)\, dA_t; \qquad f \geq 0, q \geq 0, \tag{2.5}$$

is the q-potential operator associated with A. Since A is continuous a.s. P^m, it is clear that ν_A charges no m-semipolar set. It is evident that ν_A is carried by D and since $E^\cdot \int_0^t e^{-qs}\, dA_s$ and $V_A^q f$ vanish on D^{cr} one may replace m by m_D—its restriction to D—in (2.4) because m doesn't charge the semipolar set $D_p \setminus D$. Moreover it is known that ν_A is σ-finite [Re70, III.1]. It is also known that ν_A determines A (up to m-equivalence). See also [FG88] and [FG96] as well as [Re70] for further details on Revuz measures. Finally we have the classical uniqueness theorem. If $A, B \in \mathcal{A}^+(D)$ and for some $q \geq 0$, $V_A^q 1 = V_B^q 1 < \infty$ q.e., then $A = B$. See [DM, VI-(69)].

Definition 2.6 *A (positive) measure ν on D is smooth provided it is the Revuz measure of an $A \in \mathcal{A}^+(D)$. Let $\mathcal{S}^+(D)$ denote the class of smooth measures on D.*

One may characterize the elements of $\mathcal{S}^+(D)$ as measures not charging m-semipolars and subject to a finiteness condition, but we won't have specific need for this. See [FG96]. Obviously $\mathcal{S}^+(D) \subset \mathcal{M}^+(D)$. Define $\mathcal{S}(D) \subset \mathcal{M}(D)$ to be those elements $\mu = (\mu^+, \mu^-)$ in $\mathcal{M}(D)$ such that both μ^+ and μ^- are smooth; i.e. in $\mathcal{S}^+(D)$. If $A \in \mathcal{A}(D)$, then the Revuz "measure" of A is defined by $\nu_A := (\nu_{A^+}, \nu_{A^-}) \in \mathcal{S}(D)$ where $A = A^+ - A^-$. If one also has $A = B - C$ with $B, C \in \mathcal{A}^+(D)$, then (ν_B, ν_C) also represents $\nu_A \in \mathcal{S}(D)$. In particular A is uniquely determined by ν_A.

If $A \in \mathcal{A}(D)$ and $V^q_{|A|}|f| < \infty$ a.e., then, since $V^q_{|A|}|f|$ is q-excessive for (X, τ) restricted to $D_p \setminus N$ where N is an exceptional set for $|A|$, it follows that $V^q_{|A|}|f| < \infty$ q.e. Hence

$$V^q_A f(x) := E^x \int_0^\tau e^{-qt} f(X_t)\, dA_t$$

exists finite for q.e. x. Of course $V^q_A f$ vanishes on D^{cr}. One final piece of notation. If $A \in \mathcal{A}(D)$ and f is such that $t \to \int_0^t f(X_s)\, dA_s$ is finite a.s. on $[0, \tau[$, and is of bounded variation a.s. on compact subintervals of $[0, \tau[$, then we define $f * A \in \mathcal{A}(D)$ by

$$(f * A)_t = \int_0^t f(X_s)\, dA_s, \qquad t < \tau.$$

Clearly $\nu_{f*A} = f\nu_A$ and $V^q_A f = V^q_{f*A} 1$ provided either integral exists.

Definition 2.7 *A function h is q-invariant on D provided $|h|$ and $\sup_t Q_t|h|$ are finite q.e., and q.e. on D, $h = e^{-qt} Q_t h$ for each $t > 0$. It is convenient to let $Q^q_t := e^{-qt} Q_t$ and let $\mathcal{I}^q$ denote this class of functions.*

Remarks By convention $h = 0$ on D^c. One might require that q.e. on D_p, $h = Q^q_t h$ for $t > 0$ but since the measure $Q^q_t(x, \cdot)$ is carried by D when $t > 0$ even if $x \in D_p \setminus D$, the condition in (2.7) seems natural and turns out to be appropriate. Of course if D is m- regular so that $D_p \setminus D$ is m-polar, then the conditions are equivalent. We emphasize that the exceptional m-polar set off of which $h = Q^q_t h$ on D is independent of t.

There is another characterization of $\mathcal{I}^q$ in terms of the stopped process which helps explain some of the results in the next section. Let $\tau^* = \inf\{t \geq 0 : X_t \notin D\}$. Then the stopped process $X^a_t := X_{t \wedge \tau^*}$ has state space E. Note that $\tau^* = \tau$ a.s. P^x for $x \in D$. Define the stopped semigroups for $q \geq 0$

$$_aQ^q_t f = E^{\cdot}[e^{-q(t \wedge \tau^*)} f(X_{t \wedge \tau^*})]. \tag{2.8}$$

Note that if $x \notin D$, $_aQ^q_t f(x) = f(x)$, while if $x \in D$, $_aQ^q_t f(x) = Q^q_t f(x) + E^x[e^{-q\tau} f(X_\tau) : \tau \leq t]$, and that $_aQ^q_t \neq e^{-qt}\,_aQ_t$ where, as usual, we omit q when $q = 0$ so that $_aQ_t = {}_aQ^0_t$. It is easily verified that the $_aQ^q_t$ are indeed semigroups. We introduce the exit operators

$$P^q_\tau f := E^{\cdot}[e^{-q\tau} f(X_\tau); \tau < \infty] \tag{2.9}$$

and $P^q_{\tau^*} f$ defined similarly. Again we write P_τ and P_{τ^*} when $q = 0$. Of course $\tau = \tau^*$ a.s. P^x unless $x \in D^c \setminus D^{cr}$, in particular for a.e. x. If D is m-regular then $P^x(\tau = \tau^*) = 1$ q.e. A function u defined on E is q-invariant for X^a provided for q.e. x, $|u(x)| < \infty$, $\sup_t {}_aQ^q_t|u|(x) < \infty$ and $u(x) = {}_aQ^q_t u(x)$ for each $t > 0$. Again the exceptional set does not depend on t and since $_aQ^q_t u = u$ on D^c the critical condition is that $u = {}_aQ^q_t u$ q.e. on D. Let $\mathcal{I}^q_a$ denote this class of functions.

Proposition 2.10 *Let u be finite q.e. Then $u \in \mathcal{I}^q_a$ if and only if $P^q_{\tau^*}|u| < \infty$ q.e. and $u = P^q_\tau u + h$ q.e. on D where $h \in \mathcal{I}^q$.*

Remark Recall that $h = 0$ and $P^q_{\tau^*}u = u$ on D^c. Hence one may replace the equality in (2.10) by $u = P^q_{\tau^*}u + h$ q.e.

PROOF. Suppose $u \in \mathcal{I}^q_a$. Let N be m-inessential and such that for $x \in D \setminus N$ one has $|u(x)| < \infty$, $\sup_t {}_aQ^q_t|u|(x) < \infty$ and $u(x) = {}_aQ^q_t u(x)$. Fix $x \in D \setminus N$ for the moment. Then

$$u(x) = {}_aQ^q_t u(x) = Q^q_t u(x) + E^x[e^{-q\tau}u(X_\tau); \tau \le t]. \tag{2.11}$$

In addition the last equality in (2.11) also holds when u is replaced by $|u|$. Therefore

$$E^x[e^{-q\tau}|u|(X_\tau); \tau \le t] \le {}_aQ^q_t|u|(x) \le \sup_s {}_aQ^q_s|u|(x) < \infty.$$

Let $t \to \infty$ to obtain $P^q_\tau|u|(x) < \infty$. Now let $t \to \infty$ in (2.11) to obtain $u(x) = P^q_\tau u(x) + h(x)$ where $h(x) := \lim_{t\to\infty} Q^q_t u(x)$ exists and $|h(x)| \le |u(x)| + P^q_\tau|u|(x) < \infty$. Moreover $Q^q_t|u| \le {}_aQ^q_t|u|$ and

$$\begin{aligned} Q^q_t P^q_\tau|u|(x) &= E^x[e^{-q(t+\tau\circ\theta_t)}|u|(X_\tau)\circ\theta_t; \tau\circ\theta_t < \infty, t < \tau] \\ &= E^x[e^{-q\tau}|u|(X_\tau); t < \tau < \infty] \le P^q_\tau|u|(x). \end{aligned}$$

Therefore h exists and the preceding relations hold on $D \setminus N$ and because N^c is absorbing, if $x \in D \setminus N$, $X_t \in D \setminus N$ on $0 \le t < \tau$ a.s. P^x. Hence $\sup_t Q^q_t|h| < \infty$ on $D \setminus N$. Also from (2.11), $|Q^q_t u| \le |u| + P^q_\tau|u|$ and from the above $Q^q_s[|u| + P^q_\tau|u|] < \infty$. Consequently using the dominated convergence theorem $Q^q_t h = Q^q_t \lim_{s\to\infty} Q^q_s u = \lim_{s\to\infty} Q^q_{t+s}u = h$ where the equalities and inequalities in this and preceding sentence hold identically on $D \setminus N$. Hence $h \in \mathcal{I}^q$ and one half of (2.10) is established.

For the converse since $h = 0$ on D^c, ${}_aQ^q_t h = Q^q_t h$ and ${}_aQ^q_t|h| = Q^q_t|h|$. Hence $\sup_t {}_aQ^q_t|h| < \infty$ q.e. and $h = {}_aQ^q_t h$ q.e. Also recalling that $P^q_\tau u = P^q_{\tau^*}u$ on D and writing $u^* = |u|$ for notational simplicity

$$\begin{aligned} {}_aQ^q_t P^q_{\tau^*}u^* &= E^\cdot[e^{-qt}P^q_{\tau^*}u^*(X_t); t < \tau] + E^\cdot[e^{-q\tau}P^q_{\tau^*}u^*(X_\tau); \tau \le t] \\ &= E^\cdot[e^{-qt}P^q_\tau u^*(X_t); t < \tau] + E^\cdot[e^{-q\tau}u^*(X_\tau); \tau \le t] \\ &= E^\cdot[e^{-q\tau}u^*(X_\tau); t < \tau < \infty] + E^\cdot[e^{-q\tau}u^*(X_\tau); \tau \le t] \\ &= P^q_\tau u^*. \end{aligned}$$

Hence $\sup_t {}_aQ^q_t P^q_{\tau^*}|u| < \infty$ q.e. and the same computation now shows ${}_aQ^q_t P^q_{\tau^*}u = P^q_{\tau^*}u$ q.e. Therefore $P^q_{\tau^*}u + h$ is in $\mathcal{I}^q_a$. But $P^q_{\tau^*}u = P^q_\tau u$ on D and $P^q_{\tau^*}u = u$ on D^c. Hence $u = P^q_{\tau^*}u + h$ q.e. and so $u \in \mathcal{I}^q_a$. □

3 The Basic Machinery

In this section we shall develop the necessary machinery which will enable us to define the extended generator in the next section. The notation is that of the preceding section. The next result is basic.

Lemma 3.1 *Let $q \geq 0$. Suppose that $|u|$ and $P_\tau^q|u|$ are finite q.e., that $A \in \mathcal{A}(D)$ with $V_{|A|}^q 1 < \infty$ q.e. and h is in $\mathcal{I}^q$. If $u = P_\tau^q u + V_A^q 1 + h$ q.e. on D, then for $p > q$, $u = P_\tau^p u + V_B^p 1$ q.e. on D, where $B \in \mathcal{A}(D)$ is given by*

$$B_t = A_t + (p-q)\int_0^t u(X_s)\, ds, \qquad t < \tau.$$

Moreover q.e., $P_\tau^p|u| < \infty$ and $V_{|B|}^p 1 < \infty$.

Proof. Since $p > q$, it is clear that $P_\tau^p|u| < \infty$ and $V_{|A|}^p 1 < \infty$ q.e. Then one readily checks that for q.e. x

$$P_\tau^q u(x) = P_\tau^p u(x) + (p-q)V^p P_\tau^q u(x), \tag{3.2}$$

$$V_A^q 1(x) = V_A^p 1(x) + (p-q)V^p V_A^q 1(x). \tag{3.3}$$

See [GSt87, (6.4)] for (3.2) and [BG68, IV-(2.3)] for (3.3). Next q.e. on D

$$V^p|h| = \int_0^\infty e^{-pt} Q_t|h|\, dt \leq p^{-1} \sup_t Q_t|h| < \infty,$$

and then

$$V^p h = \int_0^\infty e^{-(p-q)t} Q_t^q h\, dt = (p-q)^{-1} h.$$

Combining these results and the facts that for q.e. x, the measure $V^p(x,\cdot)$ does not charge the m-polar set $\{u \neq P_\tau^q u + V_A^q 1 + h\}$ and that $V^p(x,\cdot)$ is carried by D one finds

$$u = P_\tau^p u + V_A^p 1 + (p-q)V^p u \qquad \text{q.e. on } D.$$

But (3.2) and (3.3) hold everywhere if u is replaced by $|u|$ and A by $|A|$. Hence $V^p|u| < \infty$ q.e. Now define $B_t = A_t + (p-q)\int_0^t u(X_s)\, ds$ for $t < \tau$. Then $V_{|B|}^p 1 \leq V_{|A|}^p 1 + (p-q)V^p|u| < \infty$ q.e., completing the proof of 3.1. □

Remark 3.4 In the course of the proof of (3.1) it was shown that $V^p|u| = E^\cdot \int_0^\tau e^{-pt}|u|(X_t)\, dt < \infty$ q.e. It follows that $t \to \int_0^t e^{-ps} u(X_s)\, ds$ on $[0,\tau[$ is in $\mathcal{A}(D)$ and has Revuz measure um_D where m_D is the restriction of m to D. In particular $um_D \in \mathcal{S}(D)$. Also as remarked in section 2, if $V_{|A|}^q 1 < \infty$ a.e., then it is finite q.e. and so it would suffice to suppose that $V_{|A|}^q 1 < \infty$ a.e. in (3.1).

Definition 3.5 *A function f is quasi-finely continuous on D provided there exists an m-inessential set N_f such that f is finite and finely continuous on the finely open set $D \setminus N_f$. We abbreviate this by saying that f is q-f-continuous on D.*

Proposition 3.6 *Let u satisfy the hypotheses of (3.1). Then u and h are q-f-continuous on D and*

$$Y_t := e^{-q(t\wedge\tau)} u(X_{t\wedge\tau}) + \int_0^{t\wedge\tau} e^{-qs}\, dA_s - e^{-qt} h(X_t) 1_{\{t<\tau\}} \tag{3.7}$$

is a P^x uniformly integrable right continuous martingale for q.e. $x \in D$.

PROOF. Let N be a Borel m-inessential set which contains the union of the sets $\{|u| = \infty\}$, $\{|h| = \infty\}$, $\{P^q_\tau|u| = \infty\}$, $\{V^q_{|A|}1 = \infty\}$ and $\{u \neq P^q_\tau u + V^q_A 1 + h\}$. We shall first show that (Y_t) is a uniformly integrable strong martingale for each $x \in D \setminus N$. Fix such an x and let $Y_\tau := e^{-q\tau}u(X_\tau)1_{\{\tau<\infty\}} + \int_0^\tau e^{-qs}\, dA_s$. Given a bounded stopping time T we shall show that $Y_T = E^x[Y_\tau \mid \mathcal{F}_T]$ which will establish the assertion that Y is a uniformly integrable strong martingale. Now

$$E^x[e^{-q\tau}u(X_\tau)1_{\{\tau<\infty\}} \mid \mathcal{F}_T] = e^{-q\tau}u(X_\tau)1_{\{\tau\le T\}} + E^x[e^{-q\tau}u(X_\tau)1_{\{T<\tau<\infty\}} \mid \mathcal{F}_T].$$

Since τ is a terminal time the last term equals

$$\begin{aligned} &E^x[e^{-qT}e^{-q\tau\circ\theta_T}u(X_\tau)\circ\theta_T; \tau\circ\theta_T < \infty; T < \tau \mid \mathcal{F}_T] \\ &= e^{-qT}1_{\{T<\tau\}}P^q_\tau u(X_T). \end{aligned}$$

Also

$$E^x\Big[\int_0^\tau e^{-qt}\, dA_t \mid \mathcal{F}_T\Big] = \int_0^\tau e^{-qt}\, dA_t 1_{\{\tau\le T\}} + E^x\Big[\int_0^\tau e^{-qt}\, dA_t; T < \tau \mid \mathcal{F}_T\Big].$$

This time the last term becomes

$$\begin{aligned} &\int_0^T e^{-qt}\, dA_t 1_{\{T<\tau\}} + E^x\Big[\int_T^\tau e^{-qt}\, dA_t; T < \tau \mid \mathcal{F}_T\Big] \\ &= \int_0^T e^{-qt}\, dA_t 1_{\{T<\tau\}} + e^{-qT}1_{\{T<\tau\}}V^q_A 1(X_T). \end{aligned}$$

Combining these calculations gives

$$\begin{aligned} E^x(Y_\tau \mid \mathcal{F}_T) = &\int_0^{T\wedge\tau} e^{-qt}\, dA_t + e^{-q\tau}u(X_\tau)1_{\{\tau\le T\}} \\ &+ e^{-qT}1_{\{T<\tau\}}[P^q_\tau u(X_T) + V^q_A 1(X_T)]. \end{aligned} \tag{3.8}$$

But $u - h = P^q_\tau u + V^q_A 1$ on $D \setminus N$ and since $x \in D \setminus N$, $X_T \in D \setminus N$ a.s. P^x on $\{T < \tau\}$ because $E \setminus N$ is absorbing. Consequently $Y_T = E^x(Y_\tau \mid \mathcal{F}_T)$ proving that (Y_t) is a uniformly integrable strong martingale.

If $p > q$ then what we have shown so far and (3.1) yield the fact that $Z_t := e^{-p(t\wedge\tau)}u(X_{t\wedge\tau}) + \int_0^{t\wedge\tau} e^{-ps}\, dB_s$ is a P^x uniformly integrable strong martingale for $x \in D \setminus L$ where L is an m-inessential set and B is defined in (3.1). Fix such an x. Since u is nearly Borel, Z is optional. Then the optional section theorem [DM, IV-(8.6)] implies that Z is indistinguishable from its right continuous modification. But $t \to \int_0^{t\wedge\tau} e^{-ps}\, dB_s$ is continuous a.s. P^x and hence $t \to u(X_t)$ is right continuous P^x a.s. on $[0, \tau[$. It now follows that u is finely continuous on $D \setminus L$.

Finally the same argument shows that Y defined in (3.7) is right continuous a.s. P^x for q.e. $x \in D$. From this and the right continuity of $t \to u(X_{t\wedge\tau})$ we conclude that h is q-f-continuous on D. □

Theorem 3.9 *Let $q \geq 0$. Given a function u, $A \in \mathcal{A}(D)$ and $h \in \mathcal{I}^q$ such that $|u|$, $P^q_\tau|u|$ and $V^q_{|A|}1$ are finite q.e., then $u = P^q_\tau u + V^q_A 1 + h$ q.e. on D if and only if $h = \lim_{t\to\infty} Q^q_t u$ q.e. on D and Y_t defined in (3.7) is a P^x uniformly integrable right continuous martingale for q.e. $x \in D$.*

PROOF. Fix an m-inessential set N as in the first sentence of the proof of (3.6). Then for $x \in D \setminus N$

$$Q^q_t P^q_\tau |u|(x) = E^x[e^{-q\tau}|u(X_\tau)|; t < \tau < \infty] \to 0$$

as $t \to \infty$ by dominated convergence, while

$$Q^q_t V^q_{|A|}1(x) = E^x \int_t^\tau e^{-qs}\, d|A|_s \to 0$$

as $t \to \infty$ for the same reason. Thus if $u = P^q_\tau u + V^q_A 1 + h$ q.e. on D, then $Q^q_t u \to h$ as $t \to \infty$ q.e. on D. In view of (3.6) this proves one half of (3.9). Conversely suppose Y is a P^x uniformly integrable right continuous martingale. Then $Y_\infty := \lim_{t\to\infty} Y_t$ exists a.s. and in L^1 relative to P^x. But a.s. P^x,

$$Y_\infty = e^{-q\tau}u(X_\tau)1_{\{\tau<\infty\}} + \int_0^\tau e^{-qs}\, dA_s + Z \tag{3.10}$$

where for definiteness we define

$$Z = \limsup_{s\to\infty, s\in\mathbb{Q}} e^{-qs}[u(X_s) - h(X_s)]1_{\{\tau=\infty\}}.$$

Then (3.10) holds a.s. P^y for all $y \in D \setminus N$. Let $k(y) = E^y(Z)$. Then $k(y)$ exists finite for $y \in D \setminus N$ and one easily checks that $k = Q^q_t k$ on $D \setminus N$. Taking expectations in (3.10), $u - h = P^q_\tau u + V^q_A 1 + k$ on $D \setminus N$. But $Q^q_t u \to h$ as $t \to \infty$ q.e. on D and as in the first part of the proof $Q^q_t P^q_\tau u \to 0$ and $Q^q_t V^q_A 1 \to 0$ q.e. on D as $t \to \infty$. Therefore $k = 0$ q.e. on D completing the proof of (3.9). □

4 The Extended Generator

In this section we shall define an operator Λ_D which we regard as an extension of the generator of X restricted to D. It will map (equivalence classes modulo m of) functions defined on E into $\mathcal{S}(D)$. However with the usual abuse of notation we shall regard it as a map from functions to $\mathcal{S}(D)$. If $\tilde{u} = u$ a.e., then we say that $\tilde{u}$ is a *version* of u.

Definition 4.1 *The domain $\mathcal{D}(\Lambda_D)$ of Λ_D consists of functions u on E which have a version $\tilde{u}$ which is finite q.e. and such that there exist $q \geq 0$, $A \in \mathcal{A}(D)$ and $h \in \mathcal{I}^q$ with $P^q_\tau|\tilde{u}|$ and $V^q_{|A|}1$ finite q.e. and satisfying q.e. on D*

$$\tilde{u} = P^q_\tau \tilde{u} + V^q_A 1 + h, \qquad h = \lim_{t\to\infty} Q^q_t \tilde{u}. \tag{4.2}$$

If $u \in \mathcal{D}(\Lambda_D)$, $\Lambda_D u := qum_D - \nu_A \in \mathcal{S}(D)$. Here m_D is the restriction of m to D and ν_A is the Revuz "measure" of A as defined in section 2.

Remark Since only um_D appears in the expression for $\Lambda_D u$ it is clear that $\Lambda_D u$ depends only on the equivalence class $\mathrm{mod}\, m$ containing u. See (3.4) for the fact that $um_D \in \mathcal{S}(D)$. It is often convenient to write $\Lambda_D u = qu|_D - \nu_A$ rather than $qum_D - \nu_A$.

In order to simplify the notation in what follows we shall suppose that we have chosen $\tilde{u}$ as a version of u; that is u itself satisfies the conditions imposed on $\tilde{u}$ and we shall drop the notation $\tilde{u}$. The next result justifies the definition of Λ_D.

Theorem 4.3 *Λ_D is a well-defined linear map from $\mathcal{D}(\Lambda_D)$ to $\mathcal{S}(D)$.*

Proof. We shall first show that $\mathcal{D}(\Lambda_D)$ is a vector space. We often omit the qualifying phrase "q.e. on D" where it is clearly required. Obviously if $u \in \mathcal{D}(\Lambda_D)$ and $\alpha \in \mathbb{R}$, then $\alpha u \in \mathcal{D}(\Lambda_D)$. Suppose $u, v \in \mathcal{D}(\Lambda_D)$ with $u = P^q_\tau u + V^q_A 1 + h$, $v = P^p_\tau v + V^p_C 1 + k$.

We may suppose $p \geq q$. Clearly $u + v \in \mathcal{D}(\Lambda_D)$ when $p = q$. If $p > q$, then according to Lemma 3.1, $u = P^p_\tau u + V^p_B 1$ where B is defined in Lemma 3.1. Hence $u+v = P^p_\tau(u+v) + V^p_{C+B}1 + k$ and the appropriate finiteness conditions are satisfied. Consequently $u + v \in \mathcal{D}(\Lambda_D)$.

Next we show that Λ_D is well-defined. Once again we omit the "q.e. on D" in places where it is obviously required. Suppose $u = P^q_\tau u + V^q_A 1 + h = P^p_\tau u + V^p_C 1 + k$ and $p \geq q$. If $p = q$, $h = \lim_{t\to\infty} Q^q_t u = k$. Hence $V^q_A 1 = V^q_C 1$. This implies that the PCAF's $A^+ + C^-$ and $A^- + C^+$ have the same finite q-potential relative to (X, τ). Hence $A = C$ and then $\nu_A = \nu_C$, so $qum_D - \nu_A = qum_D - \nu_C$. If $p > q$, then $u = P^p_\tau u + V^p_B 1$ where $B_t = A_t + (p - q)\int_0^t u(X_s)\, ds$. As remarked in (3.4), $V^p|u| < \infty$ q.e. Hence B_t is finite P^x a.s. on $[0, \tau[$ and $\nu_B = \nu_A + (p - q)um_D$. Therefore $pum_D - \nu_B = pum_D - \nu_A - (p - q)um_D = qum_D - \nu_A$. These manipulations in $\mathcal{S}(D)$ are easily justified and so Λ_D is well-defined. Finally if $u, v \in \mathcal{D}(\Lambda_D)$ we may suppose $u = P^p_\tau u + V^p_A 1$ and $v = P^p_\tau v + V^p_B 1$ for the same p and with $h = 0$. Then $\Lambda_D(u + v) = \Lambda_D u + \Lambda_D v$ and $\Lambda_D(\alpha u) = \alpha\Lambda_D u$. Thus Λ_D is linear. □

Remark 4.4 Because of Proposition 3.6, u—that is the version $\tilde{u}$ in (4.1)—is q-f-continuous on D. Thus we may suppose that u is q-f-continuous on D when $u \in \mathcal{D}(\Lambda_D)$. We stress that elements in $\mathcal{D}(\Lambda_D)$ are defined on E although they may vanish off D.

Here are some examples. Let $u = V^q f$ where $V^q|f| < \infty$ a.e. and hence q.e. Then $u = 0$ on D^{cr} and so $1_D u = u$ a.e. and $P^q_\tau 1_D u = 0$. Therefore $1_D u = P^q_\tau 1_D u + V^q f$ on D. Hence $u \in \mathcal{D}(\Lambda_D)$ and $\Lambda_D u = qum_D - fm_D = (qu - f)m_D$. Note that $P^q_\tau u$ itself need not vanish a.e. on D, let alone q.e. on D, unless D is m-regular. If $u = U^q f$ with $U^q|f| < \infty$ a.e., then $u = P^q_\tau u + V^q f$ so that $u \in \mathcal{D}(\Lambda_D)$ and $\Lambda_D u = (qu - f)m_D$. If, for example, $q > 0$ and f bounded, then $u = U^q f$ is in the domain of the "generator" Λ of X and $\Lambda u = qu - f$. In this case $\Lambda_D u$ is the restriction of Λu to D. As a final example if $A \in \mathcal{A}(D)$ and for some $q \geq 0$ $V^q_{|A|}1 < \infty$, then $u = V^q_A 1 \in \mathcal{D}(\Lambda_D)$ and $\Lambda_D u = qum_D - \nu_A$.

When, as in the first two examples, $\Lambda_D u << m$ it is often convenient to regard $\Lambda_D u$ as the function $d\Lambda_D u/dm_D$. Thus if $u = U^q f$ one would write $\Lambda_D u = (qu - f)|_D = \Lambda u|_D$.

5 The Schrödinger Equation

The assumptions and notation are as in the previous sections. If $q \geq 0$ and $\mu \in \mathcal{S}(D)$ are fixed, we consider the equation

$$(q - \Lambda_D - \mu)u = \nu \tag{5.1}$$

where $\nu \in \mathcal{S}(D)$. A solution u of (5.1) is an element $u \in \mathcal{D}(\Lambda_D)$ such that

$$qum_D - \Lambda_D u - u\mu = \nu \text{ in } \mathcal{S}(D). \tag{5.2}$$

One could absorb the parameter q into μ by replacing μ by $\mu - qm_D$. But the basic data are μ and ν and one is often interested in the dependence of the solution on q and so it is preferable to keep q explicitly in (5.1)

We need to introduce some notation and prepare several lemmas before discussing existence and uniqueness results for (5.1).

Let $A, B \in \mathcal{A}(D)$ and $q \geq 0$. Define the following operations on functions whenever the integrals make sense:

$$V_B^{q,A} f = E^{\cdot} \int_0^{\tau} e^{-qt} e^{A_t} f(X_t)\, dB_t \tag{5.3}$$

$$V^{q,A} f = E^{\cdot} \int_0^{\tau} e^{-qt} e^{A_t} f(X_t)\, dt. \tag{5.4}$$

For example if $A \in \mathcal{A}(D)$, $B \in \mathcal{A}^+(D)$ and $f \geq 0$ then these integrals exist although they might be identically infinite. Note that in the notation of the previous sections $V_B^{q,0} f = V_B^q f$ and $V^{q,0} f = V^q f$.

Lemma 5.5 *Let $A, B \in \mathcal{A}^+(D)$, $q \geq 0$ and $f \geq 0$. Then*

$$V_B^{q,A} f = V_B^q f + V_A^{q,A} V_B^q f = V_B^q f + V_A^q V_B^{q,A} f, \quad q.e. \tag{5.6}$$

If $A, B \in \mathcal{A}(D)$ and f is arbitrary and if $V_{|B|}^{q,|A|}|f| < \infty$ q.e., then (5.6) holds for A, B and f.

Proof. Let $A, B \in \mathcal{A}^+(D)$ and let N_A and N_B be the exceptional sets for A and B respectively. Fix $x \notin N_A \cup N_B$. Then $e^{A_t} = 1 + \int_0^t e^{A_s}\, dA_s$ on $0 \leq t < \tau$ a.s. P^x and so for $f \geq 0$

$$V_B^{q,A} f(x) = V_B^q f(x) + E^x \int_0^{\tau} e^{-qt} \int_0^t e^{A_s}\, dA_s f(X_t)\, dB_t.$$

The last term in the display equals

$$\begin{aligned} E^x & \int_0^{\tau} e^{A_s} \int_s^{\tau} e^{-qt} f(X_t)\, dB_t\, dA_s \\ &= E^x \int_0^{\tau} e^{A_s} e^{-qs} E^{X(s)} \int_0^{\tau} e^{-qt} f(X_t)\, dB_t\, dA_s = V_A^{q,A} V_B^q f(x) \end{aligned}$$

proving the first equality in (5.6). A similar argument using the identity $e^{A_t} = 1 + e^{A_t} \int_0^t e^{-A_s}\, dA_s$ completes the proof of (5.6). Suppose $A, B \in \mathcal{A}(D)$. If $V_{|B|}^{q,|A|}|f|(x) < \infty$ and (5.6) holds for this x with $|A|$, $|B|$ and $|f|$, then the previous manipulations are valid and the assertion in the second sentence of (5.5) holds. $\square$

Lemma 5.7 *Let $A \in \mathcal{A}(D)$, $B \in \mathcal{A}^+(D)$ and $A = A^+ - A^-$ be the decomposition defined below (2.2). If $f \geq 0$, then q.e.*

$$V_B^q f + V_{A^+}^q V_B^{q,A} f = V_B^{q,A} f + V_{A^-}^q V_B^{q,A} f \leq V_B^{q,A^+} f. \tag{5.8}$$

Also $V_B^{q,A^+} f$ is q-excessive for (X, τ). If $V_B^{q,A^+} f < \infty$ a.e., then $V_B^{q,A^+} f$, $V_B^{q,A} f$ and $V_A^q V_B^{q,A} f$ are finite q.e. and q-f-continuous on D.

Proof. Let $N = N_A \cup N_B$ where N_A and N_B are the exceptional sets for A and B. On $E \setminus N$ one has

$$\begin{aligned} V_B^q f + V_{A^+}^q V_B^{q,A} f = E^{\cdot} &\int_0^\tau e^{-qt} f(X_t)\, dB_t \\ &+ E^{\cdot} \int_0^\tau e^{-qt} \int_0^{\tau\circ\theta_t} e^{-qs} e^{A_s\circ\theta_t} f(X_{s+t})\, dB_s \circ \theta_t\, dA_t^+. \end{aligned}$$

The last term equals

$$\begin{aligned} E^{\cdot} &\int_0^\tau \int_t^\tau e^{-qs} e^{A_s} e^{-A_t} f(X_s)\, dB_s\, dA_t^+ \\ &= E^{\cdot} \int_0^\tau e^{-qs} e^{A_s} f(X_s) \int_0^s e^{-A_t}\, dA_t^+\, dB_s. \end{aligned}$$

Therefore

$$V_B^q f + V_{A^+}^q V_B^{q,A} f = E^{\cdot} \int_0^\tau e^{-qs} f(X_s) \Big[1 + e^{A_s} \int_0^s e^{-A_t}\, dA_t^+\Big]\, dB_s.$$

A similar computation shows that

$$V_B^{q.A} f + V_{A^-}^q V_B^{q,A} f = E^{\cdot} \int_0^\tau e^{-qs} f(X_s) \Big[e^{A_s} + e^{A_s} \int_0^s e^{-A_t}\, dA_t^-\Big]\, dB_s.$$

A simple integration by parts shows that the expressions in square brackets in the last two displays are equal proving the equality in (5.8). But $1 + \int_0^s e^{-A_t}\, dA_t^- \leq 1 + \int_0^s e^{A_t^-}\, dA_t^- = e^{A_s^-}$ if $s < \tau$. Combining this with the last displayed expression yields the inequality in (5.8).

Now

$$Q_t^q V_B^{q,A^+} f = E^{\cdot}\Big[e^{-A_t^+} \int_t^\tau e^{-qs} e^{A_s^+} f(X_s)\, dB_s; t < \tau\Big] \uparrow V_B^{q,A^+} f$$

as $t \downarrow 0$ and so $V_B^{q,A^+} f$ is q-excessive for (X, τ). Therefore if it is finite a.e., it is finite q.e.—of course it vanishes off D_p. Consequently $V_B^q f$, $V_{A^+}^q V_B^{q,A} f$, $V_{A^-}^q V_B^{q,A} f$ and $V_B^{q,A} f$ are finite q.e. But the first three functions in the last sentence are q-excessive for (X, τ) and, hence, finely continuous on D_p. It follows that $V_A^q V_B^{q,A} f$ is q-f-continuous on D, and then so is $V_B^{q,A} f$ since by (5.8) it equals $V_B^q f + V_A^q V_B^{q,A} f$ q.e. □

Remark 5.9 In fact part of the last assertion may be improved. Namely it is readily checked that $V_B^{q,A}f$ is q-excessive for the subprocess of (X,τ) corresponding to the multiplicative function $M_t = e^{-A_t^-}$. See [BG68, III-5.7]. Consequently if $V_B^{q,A}f < \infty$ a.e., it is finite q.e. and q-f-continuous on D. Therefore if $A, B \in \mathcal{A}(D)$ and f arbitrary with $V_{|B|}^{q,A}|f| < \infty$ a.e., it follows that $V_B^{q,A}f$ is q-f-continuous on D, while a direct application of (5.7) would require $V_{|B|}^{q,A^+}|f| < \infty$ a.e.

We next formulate a general existence and uniqueness theorem for solutions of (5.1). Subsequently we shall investigate conditions which guarantee that its hypotheses hold.

Theorem 5.10 *Let $\mu, \nu \in \mathcal{S}(D)$ and set $A = A^\mu$, $B = A^\nu$. Fix $q \geq 0$. If $V_{|B|}^{q,A^+}1$ is finite a.e., then $u = V_B^{q,A}1$ is a solution of (5.1) and u satisfies*

$$u = 0 \qquad \text{a.e. on } D^c \tag{5.11}$$

$$\lim_{t\to\infty} Q_t^q u = 0 \qquad \text{a.e.} \tag{5.12}$$

If, in addition,

$$V_{|A|}^{q,A}|u| < \infty \qquad \text{a.e.}, \tag{5.13}$$

then u is the unique solution of (5.1) satisfying (5.11), (5.12) and (5.13).

Proof. By (5.7), $V_{|B|}^{q,A^+}1$ is, in fact, finite q.e. whenever it is finite a.e. Also if w is a function with $V_{|A|}^{q,A}|w| < \infty$ a.e., then in light of (5.9) it is finite q.e. Thus in using (5.13) one may replace a.e. by q.e.

Since (5.1) is linear and the hypotheses involve $|B|$ only, in showing that u is a solution we may, and shall, suppose $\nu \geq 0$ so that $B \in \mathcal{A}^+(D)$. We shall omit the qualifier "q.e. on D" in those places where it is clearly required. Now $0 \leq u = V_B^{q,A}1 \leq V_B^{q,A^+}1 < \infty$ and $V_B^q 1 \leq V_B^{q,A^+}1 < \infty$. Consequently (5.7) implies that $V_{A^+}^q u$ and $V_{A^-}^q u$ are dominated by $V_B^{q,A^+}1$ and hence $V_{|A|}^q u$ is finite. Thus from (5.5),

$$u = V_B^q 1 + V_A^q u = V_{B+u*A}^q 1$$

and $V_{|B+u*A|}^q 1 \leq V_B^q 1 + V_{|A|}^q u < \infty$. Therefore $u \in \mathcal{D}(\Lambda_D)$ and $\Lambda_D u = qum_D - \nu - u\mu$ or $(q - \Lambda_D - \mu)u = \nu$ in $\mathcal{S}(D)$. Clearly $u = 0$ on $(D^c)^r$ and hence a.e. on D^c. In fact $\{u \neq 0\} \cap D^c$ is semipolar. Finally $Q_t^q u \leq Q_t^q[V_B^q 1 + V_{|A|}^q u] \to 0$ as $t \to \infty$ by dominated convergence because $V_B^q 1$ and $V_{|A|}^q u$ are finite.

In the proving the uniqueness assertion we can no longer assume $\nu \geq 0$. By hypothesis $u = V_B^{q,A}1$ satisfies (5.13). The following lemma is the key step in proving uniqueness.

Lemma 5.14 *Suppose $\mu \in \mathcal{S}(D)$ and v satisfies $(q - \Lambda_D - \mu)v = 0$, $v = 0$ a.e. on D^c and $\lim_{t\to\infty} Q_t^q v = 0$ a.e. Then $v = V_A^q v$ a.e. and $V_{|A|}^q|v| < \infty$ a.e.*

We shall use (5.14) to complete the proof of Theorem 5.10 before giving the proof of (5.14). Let u_1 and u_2 be solutions of (5.1) satisfying (5.11), (5.12) and (5.13). We may suppose that u_1 and u_2 are q-f-continuous on D. Then so is $v := u_1 - u_2$. By (5.14), $v = V^q_A v$ and $V^q_{|A|}|v| < \infty$ a.e. and then q.e. on D since $V^q_{|A|}|v|$ is q-excessive for (X, τ). By hypothesis q.e. on D, $V^{q,A}_{|A|}|v| < \infty$. Now using (5.5) with $A = B$ results in $V^{q,A}_A v = v + V^{q,A}_A v$ q.e. on D forcing $v = 0$ since $V^{q,A}_A v$ is finite.

We shall now prove (5.14) which will complete the proof of (5.10). Since $v \in \mathcal{D}(\Lambda_D)$ we may choose a version of v which is q-f-continuous of D and vanishes on D^c. Then $P^q_\tau v = 0$ and so $v = V^q_C 1 + h$ where $C \in \mathcal{A}(D)$ satisfies $V^q_{|C|}1 < \infty$ and $h \in \mathcal{I}^q$. Again we omit "q.e. on D". But $Q^q_t V^q_{|C|}1 \to 0$ and hence $h = 0$ a.e. and then being q-f-continuous on D, q.e. on D. Now $\Lambda_D v = qvm_D - \nu_C$ and by hypothesis $(q - \Lambda_D)v = v\mu$. Hence $\nu_C = v\mu$ and so $C = v * A$ and then $v = V^q_C 1 = V^q_A v$. Since $\nu_{|C|} = |v||\mu|$, $V^q_{|A|}|v| = V^q_{|C|}1 < \infty$. □

Remarks 5.15 (i) One can also establish the existence of a solution under slightly different conditions. Throughout these remarks we omit the phrase "q.e. on D". For example suppose $V^{q,A}_{|B|}1$, $V^q_{|B|}1$ and $V^{q,A}_{|A|}V^{q,A}_{|B|}1$ are finite, then $u = V^{q,A}_B 1$ is the unique solution of (5.1) satisfying (5.11), (5.12) and (5.13). To prove that $u = V^{q,A}_B 1$ is a solution we may suppose $\nu \geq 0$. Clearly u and $V^{q,A}_{|A|}|u|$ are finite. Our finiteness assumptions are strong enough to justify the following ($B \in \mathcal{A}^+(D)$)

$$\begin{aligned} V^q_A V^{q,A}_B 1 &= E^\cdot \int_0^\tau \int_t^\tau e^{-qs} e^{A_s} \, dB_s e^{-A_t} \, dA_t \\ &= E^\cdot \int_0^\tau e^{-qs} e^{A_s} \int_0^s e^{-A_t} \, dA_t \, dB_s = V^{q,A}_B 1 - V^q_B 1, \end{aligned}$$

whence $u = V^q_B 1 + V^q_A u$. The remainder of the argument is the same as before.

(ii) The hypotheses in (5.10) that imply that $u = V^{q,A}_B 1$ is a solution of (5.1) involve only A^+; A^- could be any element of $\mathcal{A}^+(D)$. However the uniqueness hypotheses involve A^- through (5.13). The conditions for a solution in (i) above involve both A^+ and A^-, although A^+ in a rather different manner.

(iii) A straightforward argument using the Fubini theorem shows that if $A \in \mathcal{A}(D)$, B and C in $\mathcal{A}^+(D)$ and $f \geq 0$, then

$$V^{q,A}_C V^{q,A}_B f = E^\cdot \int_0^\tau e^{-qs} e^{A_s} C_s f(X_s) \, dB_s.$$

Taking $C = |A|$ it follows that a sufficient condition that (5.13) holds is that

$$E^\cdot \int_0^\tau e^{-qs} e^{A_s} |A|_s \, d|B|_s < \infty.$$

We shall specialize (5.10) and (5.15–i)in several directions. The next lemma contains relationships that are needed to establish the results to follow. It complements (5.7).

Lemma 5.16 *Let $A \in \mathcal{A}(D)$, $f \geq 0$ and $q \geq 0$. Then*

$$V^q f + V^{q,A}_{A^+} V^q f = V^{q,A} f + V^{q,A}_{A^-} V^q f \quad q.e. \tag{5.17}$$

and

$$1+V^{q,A}_{A^+}1\geq qV^{q,A}1+V^{q,A}_{A^-}1 \quad q.e. \tag{5.18}$$

PROOF. If $t<\tau$ an integration by parts yields

$$1+\int_0^t e^{A_s}\,dA^+_s=e^{A_t}+\int_0^t e^{A_s}\,dA^-_s.$$

Multiply this by $e^{-qt}f(X_t)$, integrate from 0 to τ and then take expectations with respect to P^x for $x\notin N_A$—the exceptional set for A—to obtain (5.17). For (5.18), integrating by parts one has for $t<\tau$

$$\int_0^t e^{-qs}e^{A_s}dA^+_s=\int_0^t e^{-qs}e^{-A^-_s}\,de^{A^+_s}=e^{-qt}e^{A_t}-1+\int_0^t e^{-qs}e^{A_s}[qds+dA^-_s].$$

Drop the positive term $e^{-qt}e^{A_t}$, let $t\uparrow\tau$ and then take P^x expectations for $x\notin N_A$ to obtain (5.18). □

Remarks It follows from (5.18) that q.e. $V^{q,A}_{A^-}1\leq 1+V^{q,A}_{A^+}1$ and so

$$V^{q,A}_{|A|}1\leq 1+2V^{q,A}_{A^+}1 \qquad \text{q.e.} \tag{5.19}$$

Also q.e. using (5.18) for $q>0$ and (5.17) for $q=0$

$$\begin{cases} V^{q,A}1\leq q^{-1}[1+V^{q,A}_{A^+}1], & q>0\\ V^A1\leq V1+V^A_{A^+}V1. \end{cases} \tag{5.20}$$

In what follows we often omit the "q.e. on D" in proofs where it is obviously needed, but we include it in our hypotheses.

Corollary 5.21 *Suppose that q.e.* $V^{q,A^+}_{|B|}1<\infty$ *and* $u=V^{q,A}_B1$ *is bounded. If* $V^{q,A}_{A^+}1$ *is finite q.e., then* u *is the unique bounded solution of (5.1) satisfying (5.11), (5.12) and (5.13).*

PROOF. ¿From (5.10), u is a bounded solution satisfying (5.11) and (5.12). But if M is a bound for $|u|$, then $V^{q,A}_{|A|}|u|\leq MV^{q,A}_{|A|}1$ and so by (5.19) u satisfies (5.13). Similarly any bounded solution satisfies (5.13). Whence (5.21) is a consequence of (5.10). □

Remarks The proof actually shows that u is the unique solution in the class of all solutions satisfying (5.11), (5.12) and (5.13). Since a solution is only determined a.e., here bounded really means in $L^\infty(m)$. Also if $q>0$, then $Q^q_tu\to 0$ as $t\to\infty$ is automatic for u bounded.

We next give a somewhat different criterion for existence and uniqueness in the spirit of section 3.3 of [CZ95]. Given $\mu\in\mathcal{S}(D)$ we define, following Chung and Zhao, $\mathbb{F}=\mathbb{F}(D,\mu)$ to consist of those $\nu\in\mathcal{S}(D)$ for which there exist constants α and β—depending on ν—such that

$$|\nu|\leq\alpha m+\beta|\mu|. \tag{5.22}$$

Let A correspond to μ.

Theorem 5.23 *Let $\nu \in \mathbb{F}(D,\mu)$. (i) If $q > 0$ and q.e. on D, $V^q_{|A|}1 < \infty$ and $V^{q,A}_{A^+}1$ is bounded, then $u = V^{q,A}_B 1$ is the unique bounded solution of (5.1) vanishing a.e. on D^c. (ii) If $q = 0$ and, in addition, q.e. on D either $V1 = E^{\cdot}(\tau)$ is bounded or $V^A 1$ is bounded and $V1 < \infty$, then the same conclusion holds. If $\beta = 0$ in (5.22), then the condition $V^q_{|A|}1 < \infty$ is not needed in either (i) or (ii).*

Proof. ¿From (5.19) if $q \geq 0$, then $V^{q,A}_{|A|}1$ is bounded. ¿From (5.20) if $q > 0$, $V^{q,A}1$ is bounded, and if $q = 0$ and $V1$ is bounded then so is $V^A 1$. Thus in all cases both $V^{q,A}_{A^+}1$ and $V^{q,A}1$ are bounded. The inequality (5.22) implies that $d|B|_t \leq \alpha dt + \beta d|A|_t$. Therefore $V^{q,A}_{|B|}1 \leq \alpha V^{q,A}1 + \beta V^{q,A}_{|A|}1$ is bounded and hence so is $u = V^{q,A}_B 1$. Moreover $V^q_{|B|}1 \leq \alpha V^q 1 + \beta V^q_{|A|}1$ is finite. Clearly when $\beta = 0$, the hypothesis on $V^q_{|A|}1$ is not needed. Using (5.19) again $V^{q,A}_{|A|}V^{q,A}_{|B|}1$ is bounded. Therefore according to (5.15–i) $u = V^{q,A}_B 1$ is the unique solution of (5.1) vanishing a.e. on D^c and satisfying (5.12) and (5.13). But under the present hypotheses any bounded function f satisfies (5.12) and (5.13) since when $q = 0$, $E^{\cdot}(\tau) = V1 < \infty$ so that $Q_t f(x) = E^x[f(X_t); t < \tau] \to 0$ as $t \to \infty$. □

We next give some, perhaps more familiarly, conditions guaranteeing the hypotheses of (5.23).

Definition 5.24 *$A \in \mathcal{A}^+(D)$ satisfies the "Kato" condition provided there exist $r > 0$ and $\beta < 1$ with $E^{\cdot}(A_r) \leq \beta$ q.e. on D.*

The usual Kato condition, for example as in [CZ95], assumes $\limsup_{r \downarrow 0} \sup_{x \in D} E^x(A_r) = 0$. It is well-known that if A satisfies (5.24) then there exist constants M and ω, depending only on β and r, such that

$$E^{\cdot}(e^{A_t}) \leq M e^{\omega t} \qquad \text{q.e. on } D. \tag{5.25}$$

Proposition 5.26 *Let $A \in \mathcal{A}^+(D)$ satisfy (5.24). (i) If $q > \omega$, $V^{q,A}1$ and $V^{q,A}_A 1$ are bounded q.e. on D where ω is the constant in (5.25). (ii) If $q = 0$, $\mu << m$ and $V^A 1$ is bounded q.e., then $V^A_A 1$ is bounded q.e. on D.*

Proof. (i) By (5.25) if $q > \omega$, then q.e. on D

$$V^{q,A}1 = \int_0^\infty e^{-qt} E^{\cdot}(e^{A_t}; t < \tau)\, dt \leq \frac{M}{q-\omega} < \infty.$$

For $t > 0$, integrating by parts gives

$$\int_0^t e^{-qs} e^{A_s}\, dA_s = e^{-qt} e^{A_t} - 1 + q \int_0^t e^{-qs} e^{A_s}\, ds.$$

Take expectations and let $t \to \infty$ to obtain for $q > \omega$, $V^{q,A}_A 1 \leq \frac{Mq}{q-\omega} - 1 \leq \frac{Mq}{q-\omega}$, q.e. on D.
(ii) Here the argument comes form Chung and Zhao [CZ95, Th. 3.18]. Suppose $\mu = fm$. Then $dA_t = f(X_t)\, dt$ and $V^A_A 1 = V^A f$. By (5.25) the semigroup $Q^A_t q = E^{\cdot}[e^{A_t} g(X_t); t < \tau]$ is well-defined on functions which are bounded q.e. as well as positive functions. Also $Q^A_t 1 = E^{\cdot}(e^{A_t}; t <$

$\tau) \leq Me^{\omega t}$ and

$$\int_0^t Q_s^A f\, ds = E^{\cdot} \int_0^{t\wedge\tau} e^{A_s} f(X_s)\, ds \leq E^{\cdot}(e^{A_t}) - 1 \leq Me^{\omega t}.$$

Thus if $t > 0$,

$$V^A f = \int_0^t Q_s^A f\, ds + \int_t^\infty Q_s^A f ds \leq Me^{\omega t} + V^A Q_t^A f \leq Me^{\omega t}[1 + V^A 1]. \quad \square$$

Remarks If A^+ satisfies the Kato condition, then for $q > \omega$, $V_{A^+}^{q,A} 1 \leq V_{A^+}^{q,A^+} 1$ and $V^{q,A} 1 \leq V^{q,A^+} 1$ are bounded q.e. Thus if $V_{A^-}^q 1$ is finite q.e. the hypotheses in (5.23–i) hold. If $\mu^+ << m$ and A^+ satisfies the Kato condition and if $V^{A^+} 1$ is bounded q.e., then the hypotheses in the (5.23–ii) hold provided $E^{\cdot}(A_\tau^-) < \infty$ q.e.

In [G99] we gave a condition that implies (5.24) that involves μ more directly. Namely let $(\hat{V}^q)$ be the resolvent of the moderate Markov dual relative to m of (X, τ). Then condition (5.24) holds for $A \in \mathcal{A}^+(D)$ provided that for some $q < \infty$ one has

$$\sup \left\{ \mu \hat{V}^q(f); f \geq 0, \int f dm \leq 1 \right\} < 1, \tag{5.27}$$

or in terms of the moderate Markov, dual semigroup $(\hat{Q}_t)$ for some $s > 0$

$$\sup \left\{ \int_0^s \mu \hat{Q}_t(f)\, dt; f \geq 0, \int f dm \leq 1 \right\} < 1,$$

where, of course, μ is the Revuz measure of A. See, for example, section 3 of [G99].

6 L^p Theory

In this section we shall investigate the situation where $\nu << m$ and under the assumption that $f = \frac{d\nu}{dm} \in L^p(m, D)$, $1 \leq p \leq \infty$. Let $L^p = L^p(m, D)$ be the *real* L^p space over (D, m). We shall need the moderate Markov dual of (X, τ) which we denote by $(\hat{X}, \hat{\tau})$. If $A \in \mathcal{A}^+(D)$ then $\hat{A}$ is its dual as defined in section 4 of [G99]. In particular A and $\hat{A}$ have the same Revuz measure. If $A \in \mathcal{A}(D)$, then $\hat{A} := \hat{A}^+ - \hat{A}^-$. Also $(\hat{V}^q)$ and $(\hat{Q}_t)$ denote the resolvent and semigroup of $(\hat{X}, \hat{\tau})$. For notational simplicity we write $\hat{V}^{q,A}$ and $\hat{V}_B^{q,A}$ in place of $\hat{V}^{q,\hat{A}}$ and $\hat{V}_{\hat{B}}^{q,\hat{A}}$ and so forth. The following duality relations are crucial. Let $(f, g) := \int fg dm$ whenever the integral makes sense, not necessarily finite. In the next two propositions, $A \in \mathcal{A}^+(D)$ with Revuz measure μ and $f, g \geq 0$.

Proposition 6.1 *(i)* $(f, V^{q,A} g) = (\hat{V}^{q,A} f, g)$. *(ii)* $(f, V_A^q g) = \int g \hat{V}^q f d\mu$ *and* $(f, \hat{V}_A^q g) = \int g V^q f d\mu$.

See [G99] Proposition 4.8 for the first assertion and Proposition 4.6 for the second. We shall say that a kernel W on D is *m-proper* provided there exists $h > 0$ on D with $Wh < \infty$ a.e. The next result is (5.6) in [G99]. See also (5.8) in [G99].

Proposition 6.2 *Let $A \in \mathcal{A}^+(D)$ and $f, g \geq 0$. If $q > 0$ and $V^{q,A}$ is m-proper, then $(f, V_A^{q,A} g) = \int g \hat{V}^{q,A} f d\mu$. If V and $\hat{V}$ are m-proper this also holds when $q = 0$. Dually if $\hat{V}^{q,A}$ is m-proper, then $(f, \hat{V}_A^{q,A} g) = \int g V^{q,A} f d\mu$.*

We refer the reader to [G99] for information about the dual process but warn him that the notation is slightly different there.

We are now going to investigate the equation (5.1) when $\nu = fm$ with $f \in L^p$. This means that $\nu = f^+ m - f^- m \in \mathcal{M}(D)$. Then $|\nu| = |f| m$ and $|\nu|$, hence ν^+ and ν^-, are in $\mathcal{S}(D)$ provided $\int_0^t |f|(X_s)\, ds < \infty$ on $0 \leq t < \tau$ a.s. $P^\times$ for q.e. $x \in D$. This certainly is the case in $V^q |f| < \infty$ q.e. for some $q \geq 0$. But m is excessive and so if $q > 0$, $V^q : L^p \to L^p$, $1 \leq p \leq \infty$. Therefore $\nu = fm \in \mathcal{S}(D)$ whenever $f \in L^p$. Then equation (5.1) is written

$$(q - \Lambda_D - \mu) u = f, \tag{6.3}$$

and u is a solution provided $u \in \mathcal{D}(\Lambda_D)$ and $qum_D - \Lambda_D u - u\mu = fm_D$ in $\mathcal{M}(D)$. In particular a solution depends only on the equivalence class $\bmod\ m$ to which f belongs, so (6.3) is well-defined for $f \in L^p$.

The next lemma is necessary in order to show that $V^{q,A}$ is well-defined on L^p.

Lemma 6.4 *Let $A \in \mathcal{A}(D)$ and $J \subset D$ with $m(J) = 0$. Then $V^{q,A} 1_J = 0$ a.e. If $V^{q,A}|f| < \infty$ a.e. and $g = f$ a.e., then $V^{q,A} f = V^{q,A} g$ q.e.*

Proof. Using (6.1–i), if $f \geq 0$ then $(f, V^{q,|A|} 1_J) = (\hat{V}^{q,|A|} f, I_J) = 0$. Therefore $V^{q,|A|} 1_J$ and $V^{q,A} 1_J$ vanish a.e. The second assertion is an immediate consequence of the first and (5.9). $\square$

In what follows $\mu \in \mathcal{S}(D)$ and A corresponds to μ. Any additional hypotheses on μ or A will be explicitly stated. The case $p = \infty$ is a simple corollary of (5.23). If $\nu = fm$ with $f \in L^\infty$, then $|\nu| = |f| m \leq \|f\|_\infty m$ and so ν satisfies (5.22) with $\beta = 0$. In this situation $dB_t = f(X_t)\, dt$ and $V_B^{q,A} 1 = V^{q,A} f$. Thus the following proposition is an immediately consequence of (5.23) and (6.4).

Proposition 6.5 *Let $\nu = fm$ with $f \in L^\infty$. Suppose that $V_{A^+}^{q,A} 1$ is bounded q.e. on D and when $q = 0$ suppose, in addition, that q.e. on D either $E^\cdot(\tau)$ is bounded or $V^A 1$ is bounded and $E^\cdot(\tau) < \infty$. Then $u = V^{q,A} f$ is the unique solution of (6.3) in L^∞ that vanishes on D^c.*

When $1 \leq p < \infty$ the following space of functions turns out to be the space in which solutions are unique. Define

$$W^p(\mu) = \{u \in L^p : u \text{ has a } q\text{-}f\text{-continuous version } \tilde{u} \text{ with } \int |\tilde{u}|^p d|\mu| < \infty\}. \tag{6.6}$$

We next consider the case $p = 1$.

Theorem 6.7 *Let $\nu = fm$ with $f \in L^1$. Suppose that $\hat{V}_{|A|}^{q,|A|} 1$ is bounded q.e. and in addition, when $q = 0$ that $\hat{V}^{|A|} 1$ is bounded q.e. Then $u = V^{q,A} f$ is the unique solution of (6.3) in $W^1(\mu)$ vanishing a.e. on D^c and with $\lim_{t\to\infty} e^{-qt} Q_t u = 0$ a.e.*

PROOF. By (the dual of) (5.20) applied to $|A|$, $\hat{V}^{q,|A|}1 \leq q^{-1}[1+\hat{V}^{q,|A|}_{|A|}1]$ so $\hat{V}^{q,|A|}1$ is bounded q.e. for all $q \geq 0$. If $f \in L^1$ by (6.1–i), $\int V^{q,|A|}|f|\,dm = (|f|, \hat{V}^{q,|A|}1) < \infty$, hence because of (6.4), $V^{q,|A|}$ and $V^{q,A}$ map L^1 into L^1; in particular they are m-proper. Fix $f \in L^1$. Then $u := V^{q,A}f$ is a solution of (6.3) by Theorem 5.10. Moreover $V^{q,|A|}$ is m-proper. Now from (5.7) one sees that $V^{q,|A|}|f|$ and $V^{q,A}f$ are q-f-continuous. But $\hat{V}^{|A|}$ and $V^{|A|}$ being m-proper imply that $\hat{V}$ and V are m-proper. Therefore (6.2) implies

$$\int V^{q,|A|}|f|\,d|\mu| = \int |f|\hat{V}^{q,|A|}_{|A|}1\,dm < \infty.$$

Consequently $V^{q,A} : L^1 \to W^1(\mu)$. Thus $u = V^{q,A}f$ is a solution in $W^1(\mu)$. On the other hand if $w \in W^1(\mu)$ one may suppose that w itself is q-f-continuous, then using (6.2) again

$$\int V^{q,|A|}_{|A|}|w|\,dm = \int |w|\hat{V}^{q,|A|}1\,d|\mu| < \infty$$

because $\hat{V}^{q,|A|}1$ is bounded q.e. and hence a.e. $|\mu|$ since $|\mu|$ does not charge m-semipolars. It now follows from (5.10) that $u = V^{q,A}f$ is the unique solution of (6.3) in $W^1(\mu)$ which vanishes on D^c and satisfies $\lim_{t\to\infty} e^{-qt}Q_t u = 0$ a.e. $\square$

Remarks 6.8 (i) The proof shows that u is the unique solution subject to (5.11) and (5.12) in the possibly larger class of solutions w having a q-f-continuous version $\tilde{w}$ satisfying $\int |\tilde{w}|\hat{V}^{q,|A|}1\,d|\mu| < \infty$. (ii) Using (2.9), (2.6) and (2.3) of [G99] one can show that if $\hat{V}^{q,|A|}_{|A|}1$ and $\hat{V}^{|A|}1$ are bounded a.e., then they are bounded q.e. Hence the hypotheses in (6.7) may be relaxed to this extent.

When $1 < p < \infty$ the situation is similar but we require the dual assumptions as well as those of (6.7).

Theorem 6.9 *Let $\nu = fm$ with $f \in L^p$, $1 < p < \infty$. Suppose that $V^{q,|A|}_{|A|}1$ and $\hat{V}^{q,|A|}_{|A|}1$ are bounded q.e. and, in addition, when $q = 0$, that $V^{|A|}1$ and $\hat{V}^{|A|}1$ are bounded q.e. Then $u = V^{q,A}f$ is the unique solution of (6.3) in $W^p(\mu)$ vanishing a.e. on D^c and with $\lim_{t\to\infty} e^{-qt}Q_t u = 0$ a.e.*

PROOF. As in the proof of (6.7), the hypotheses and (5.20) imply that $V^{q,|A|}1$ and $\hat{V}^{q,|A|}1$ are bounded q.e. for all $q \geq 0$. Let $p' = p/(p-1)$ be the conjugate exponent for p. Then

$$(V^{q,|A|}|f|)^p \leq (V^{q,|A|}1)^{\frac{p}{p'}} V^{q,|A|}|f|^p \leq cV^{q,|A|}|f|^p, \text{q.e.}$$

and so if $f \in L^p$ by (6.1–i)

$$\int \left(V^{q,|A|}|f|\right)^p dm \leq c\int V^{q,|A|}|f|^p\,dm = c\int |f|^p\hat{V}^{q,|A|}1\,dm < \infty.$$

Thus by (6.4), $V^{q,|A|}$ and $V^{q,A}$ map L^p into L^p and for $f \in L^p$, $V^{q,|A|}|f|$ and $V^{q,|A|}|f|^p$ are determined q.e. Fix $f \in L^p$. Then $u = V^{q,A}f$ is a solution of (6.3) and $u \in L^p$. From (5.7), u is q-f-continuous and using (6.2)

$$\int \left(V^{q,|A|}|f|\right)^p d|\mu| \leq c\int V^{q,|A|}|f|^p\,d|\mu| = c\int |f|^p\hat{V}^{q,|A|}_{|A|}1\,dm < \infty.$$

Hence $V^{q,|A|}$ and $V^{q,A}$ map L^p into $W^p(\mu)$. If $w \in W^p(\mu)$ and w is q-f-continuous,

$$\big(V^{q,|A|}_{|A|}|w|\big)^p \le \big(V^{q,|A|}_{|A|}1\big)^{\frac{p}{p'}} V^{q,|A|}_{|A|}|w|^p \le c' V^{q,|A|}_{|A|}|w|^p, \quad \text{q.e.}$$

Therefore as in the proof of (6.7)

$$\int \big(V^{q,|A|}_{|A|}|w|\big)^p \, dm \le c' \int |w|^p \hat{V}^{q,|A|} 1 \, d|\mu| < \infty$$

and $V^{q,|A|}_{|A|}|w| < \infty$ a.e. Now an appeal to (5.10) completes the proof of (6.9). □

As remarked in (6.8) it would suffice to suppose that $\hat{V}^{q,|A|}_{|A|}1$ and $\hat{V}^{|A|}1$ are bounded a.e. Of course (5.7) implies that if $V^{q,|A|}_{|A|}1$ and $V^{|A|}1$ are bounded a.e. then they are bounded q.e.

Final Remark Let $\mu = (\mu^+, \mu^-)$ with μ^+ satisfying (5.27) relative to X (that is with $D = E$) and μ^- smooth. Let (Q_t) and $(\hat{Q}_t)$ be the dual semigroups corresponding to $A = A^\mu$ and $\hat{A}$. See [G99] for the precise definitions. Then the arguments in section 5 of [SV96] are readily modified to show that if (P_t) and $(\hat{P}_t)$—the semigroups of X and $\hat{X}$—are continuous from $L^1(m)$ to $L^\infty(m)$, then (Q_t) and $(\hat{Q}_t)$ are continuous from $L^p(m)$ to $L^q(m)$ for $1 \le p \le q \le \infty$. This was supposed to appear as an added note in [G99], but somehow was omitted by the printer.

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