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Convex concentration inequalities and forward-backward stochastic calculus

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Abstract

Given $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ respectively a forward and a backward martingale with jumps and continuous parts, we prove that $E[\phi(M_t + M_t^*)]$ is non-increasing in t when ϕ is a convex function, provided the local characteristics of $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ satisfy some comparison inequalities. We deduce convex concentration inequalities and deviation bounds for random variables admitting a predictable representation in terms of a Brownian motion and a non-necessarily independent jump component

Key words: Convex concentration inequalities, forward-backward stochastic calculus, deviation inequalities, Clark formula, Brownian motion, jump processes

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1 Introduction

Two random variables F and G satisfy a convex concentration inequality if

$$\mathbb{E}[\phi(F)] \leq \mathbb{E}[\phi(G)] \tag{1.1}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$. By a classical argument, the application of (1.1) to $\phi(x) = \exp(\lambda x)$, $\lambda > 0$, entails the deviation bound

$$P(F \geq x) \leq \inf_{\lambda>0} \mathbb{E}[e^{\lambda(F-x)} 1_{\{F\geq x\}}] \leq \inf_{\lambda>0} \mathbb{E}[e^{\lambda(F-x)}] \leq \inf_{\lambda>0} \mathbb{E}[e^{\lambda(G-x)}], \tag{1.2}$$

$x > 0$, hence the deviation probabilities for F can be estimated via the Laplace transform of G, see [2], [3], [15] for more results on this topic. In particular, if G is Gaussian then Theorem 3.11 of [15] shows moreover that

$$P(F \geq x) \leq \frac{e^2}{2} P(G \geq x), \qquad x > 0.$$

On the other hand, if F is more convex concentrated than G then $E[F] = E[G]$ as follows from taking successively $\phi(x) = x$ and $\phi(x) = -x$, and applying the convex concentration inequality to $\phi(x) = x \log x$ we get

$$\begin{aligned}
\mathrm{Ent}[F] &= \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F] \\
&= \mathbb{E}[F \log F] - \mathbb{E}[G] \log \mathbb{E}[G] \\
&\leq \mathbb{E}[G \log G] - \mathbb{E}[G] \log \mathbb{E}[G] \\
&= \mathrm{Ent}[G],
\end{aligned}$$

hence a logarithmic Sobolev inequality of the form $\mathrm{Ent}[G] \leq \mathcal{E}(G, G)$ implies

$$\mathrm{Ent}[F] \leq \mathcal{E}(G, G).$$

In this paper we obtain convex concentration inequalities for the sum $M_t + M_t^*$, $t \in \mathbb{R}_+$, of a forward and a backward martingale with jumps and continuous parts. Namely we prove that $M_t + M_t^*$ is more concentrated than $M_s + M_s^*$ if $t \geq s \geq 0$, i.e.

$$\mathbb{E}[\phi(M_t + M_t^*)] \leq \mathbb{E}[\phi(M_s + M_s^*)], \quad 0 \leq s \leq t,$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the local characteristics of $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ satisfy the comparison inequalities assumed in Theorem 3.2 below. If further $\mathbb{E}[M_t^* | \mathcal{F}_t^M] = 0$, $t \in \mathbb{R}_+$, where $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$ denotes the filtration generated by $(M_t)_{t\in\mathbb{R}_+}$, then Jensen's inequality yields

$$\mathbb{E}[\phi(M_t)] \leq \mathbb{E}[\phi(M_s + M_s^*)], \qquad 0 \leq s \leq t,$$

and if in addition we have $M_0 = 0$, then

$$\mathbb{E}[\phi(M_T)] \leq \mathbb{E}[\phi(M_0^*)], \qquad T \geq 0. \tag{1.3}$$

In other terms, we will show that a random variable F is more concentrated than M_0^*:

$$\mathbb{E}[\phi(F - E[F])] \leq \mathbb{E}[\phi(M_0^*)],$$

provided certain assumptions are made on the processes appearing in the predictable representation of $F - E[F] = M_T$ in terms of a point process and a Brownian motion.

Consider for example a random variable F represented as

$$F = \mathbb{E}[F] + \int_0^{+\infty} H_t dW_t + \int_0^{+\infty} J_t (dZ_t - \lambda_t dt),$$

where $(Z_t)_{t\in\mathbb{R}_+}$ is a point process with compensator $(\lambda_t)_{t\in\mathbb{R}_+}$, $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and $(H_t)_{t\in\mathbb{R}_+}$, $(J_t)_{t\in\mathbb{R}_+}$ are predictable square-integrable processes satisfying $J_t \le k$, $dPdt$-a.e., and

$$\int_0^{+\infty} |H_t|^2 dt \le \beta^2, \quad \text{and} \quad \int_0^{+\infty} |J_t|^2 \lambda_t dt \le \alpha^2, \quad P-a.s.$$

By applying (1.3) or Theorem 4.1$-ii)$ below to forward and backward martingales of the form

$$M_t = \mathbb{E}[F] + \int_0^t H_u dW_u + \int_0^t J_u (dZ_u - \lambda_u du), \quad t \in \mathbb{R}_+,$$

and

$$M_t^* = \hat{W}_{\beta^2} - \hat{W}_{V^2(t)} + k(\hat{N}_{\alpha^2/k^2} - \hat{N}_{U^2(t)/k^2}) - (\alpha^2 - U^2(t))/k, \quad t \in \mathbb{R}_+,$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$, $(\hat{N}_t)_{t\in\mathbb{R}_+}$, are a Brownian motion and a left-continuous standard Poisson process, $\beta \ge 0$, $\alpha \ge 0$, $k > 0$, and $(V(t))_{t\in\mathbb{R}_+}$, $(U(t))_{t\in\mathbb{R}_+}$ are suitable random time changes, it will follow in particular that F is more concentrated than

$$M_0^* = \hat{W}_{\beta^2} + k\hat{N}_{\alpha^2/k^2} - \alpha^2/k,$$

i.e.

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \le \mathbb{E}\left[\phi(\hat{W}_{\beta^2} + k\hat{N}_{\alpha^2/k^2} - \alpha^2/k)\right] \tag{1.4}$$

for all convex functions ϕ such that ϕ' is convex.

From (1.2) and (1.4) we get

$$P(F - \mathbb{E}[F] \ge x) \le \inf_{\lambda>0} \exp\left(\frac{\alpha^2}{k^2}(e^{\lambda k} - \lambda k - 1) + \frac{\beta^2\lambda^2}{2} - \lambda x\right),$$

i.e.

$$P(F - \mathbb{E}[F] \ge x) \le \exp\left(\frac{x}{k} - \frac{\beta^2\lambda_0(x)}{2k}(2 - k\lambda_0(x)) - (x + \alpha^2/k)\lambda_0(x)\right),$$

where $\lambda_0(x) > 0$ is the unique solution of

$$e^{k\lambda_0(x)} + \frac{k\lambda_0(x)\beta^2}{\alpha^2} - 1 = \frac{kx}{\alpha^2}.$$

When $H_t = 0$, $t \in \mathbb{R}_+$, we can take $\beta = 0$, then $\lambda_0(x) = k^{-1}\log(1 + xk/\alpha^2)$ and this implies the Poisson tail estimate

$$P(F - \mathbb{E}[F] \ge y) \le \exp\left(\frac{y}{k} - \left(\frac{y}{k} + \frac{\alpha^2}{k^2}\right)\log\left(1 + \frac{ky}{\alpha^2}\right)\right), \qquad y > 0. \tag{1.5}$$

Such an inequality has been proved in [1], [19], using (modified) logarithmic Sobolev inequalities and the Herbst method when $Z_t = N_t$, $t \in \mathbb{R}_+$, is a Poisson process, under different hypotheses on the predictable representation of F via the Clark formula, cf. Section (6). When $J_t = \lambda_t = 0$, $t \in \mathbb{R}_+$, we recover classical Gaussian estimates which can be independently obtained from the expression of continuous martingales as time-changed Brownian motions.

We proceed as follows. In Section 3 we present convex concentration inequalities for martingales. In Sections 4 and 5 these results are applied to derive convex concentration inequalities with respect to Gaussian and Poisson distributions. In Section 6 we consider the case of predictable representations obtained from the Clark formula. The proofs of the main results are formulated using forward/backward stochastic calculus and arguments of [10]. Section 7 deals with an application to normal martingales, and in the appendix (Section 8) we prove the forward-backward Itô type change of variable formula which is used in the proof of our convex concentration inequalities. See [4] for a reference where forward Itô calculus with respect to Brownian motion has been used for the proof of logarithmic Sobolev inequalities on path spaces.

2 Notation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with an increasing filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and a decreasing filtration $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$. Consider $(M_t)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t$-forward martingale and $(M_t^*)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t^*$-backward martingale. We assume that $(M_t)_{t\in\mathbb{R}_+}$ has right-continuous paths with left limits, and that $(M_t^*)_{t\in\mathbb{R}_+}$ has left-continuous paths with right limits. Denote respectively by $(M_t^c)_{t\in\mathbb{R}_+}$ and $(M_t^{*c})_{t\in\mathbb{R}_+}$ the continuous parts of $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$, and by

$$\Delta M_t = M_t - M_{t^-}, \qquad \Delta^* M_t^* = M_t^* - M_{t^+}^*,$$

their forward and backward jumps. The processes $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ have jump measures

$$\mu(dt, dx) = \sum_{s>0} 1_{\{\Delta M_s \neq 0\}} \delta_{(s,\Delta M_s)}(dt, dx),$$

and

$$\mu^*(dt, dx) = \sum_{s>0} 1_{\{\Delta^* M_s^* \neq 0\}} \delta_{(s,\Delta^* M_s^*)}(dt, dx),$$

where $\delta_{(s,x)}$ denotes the Dirac measure at $(s, x) \in \mathbb{R}_+ \times \mathbb{R}$. Denote by $\nu(dt, dx)$ and $\nu^*(dt, dx)$ the $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$-dual predictable projections of $\mu(dt, dx)$ and $\mu^*(dt, dx)$, i.e.

$$\int_0^t \int_{-\infty}^{\infty} f(s, x)(\mu(ds, dx) - \nu(ds, dx)) \quad \text{and} \quad \int_t^{\infty} \int_{-\infty}^{\infty} g(s, x)(\mu^*(ds, dx) - \nu^*(ds, dx))$$

are respectively $\mathcal{F}_t$-forward and $\mathcal{F}_t^*$-backward local martingales for all sufficiently integrable $\mathcal{F}_t$-predictable, resp. $\mathcal{F}_t^*$-predictable, process f, resp. g. The quadratic variations $([M, M])_{t\in\mathbb{R}_+}$, $([M^*, M^*])_{t\in\mathbb{R}_+}$ are defined as the limits in uniform convergence in probability

$$[M, M]_t = \lim_{n\to\infty} \sum_{i=1}^{n} |M_{t_i^n} - M_{t_{i-1}^n}|^2,$$

and

$$[M^*, M^*]_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} |M^*_{t_i^n} - M^*_{t_{i+1}^n}|^2,$$

for all refining sequences $\{0 = t_0^n \leq t_1^n \leq \cdots \leq t_{k_n}^n = t\}$, $n \geq 1$, of partitions of $[0, t]$ tending to the identity. We then let $M_t^d = M_t - M_t^c$, $M_t^{*d} = M_t^* - M_t^{*c}$,

$$[M^d, M^d]_t = \sum_{0<s\leq t} |\Delta M_s|^2, \qquad [M^{*d}, M^{*d}]_t = \sum_{0\leq s<t} |\Delta^* M_s^*|^2,$$

and

$$\langle M^c, M^c\rangle_t = [M, M]_t - [M^d, M^d]_t, \qquad \langle M^{*c}, M^{*c}\rangle_t = [M^*, M^*]_t - [M^{*d}, M^{*d}]_t,$$

$t \in \mathbb{R}_+$. Note that $([M, M]_t)_{t\in\mathbb{R}_+}$, $(\langle M, M\rangle_t)_{t\in\mathbb{R}_+}$, $([M^*, M^*]_t)_{t\in\mathbb{R}_+}$ and $(\langle M^*, M^*\rangle_t)_{t\in\mathbb{R}_+}$ are $\mathcal{F}_t$-adapted, but $([M^*, M^*]_t)_{t\in\mathbb{R}_+}$ and $(\langle M^*, M^*\rangle_t)_{t\in\mathbb{R}_+}$ are not $\mathcal{F}_t^*$-adapted. The pairs

$$(\nu(dt, dx), \langle M^c, M^c\rangle) \qquad \text{and} \qquad (\nu^*(dt, dx), \langle M^{*c}, M^{*c}\rangle)$$

are called the local characteristics of $(M_t)_{t\in\mathbb{R}_+}$, cf. [8] in the forward case. Denote by $(\langle M^d, M^d\rangle_t)_{t\in\mathbb{R}_+}$, $(\langle M^{*d}, M^{*d}\rangle_t)_{t\in\mathbb{R}_+}$ the conditional quadratic variations of $(M_t^d)_{t\in\mathbb{R}_+}$, $(M_t^{*d})_{t\in\mathbb{R}_+}$, with

$$d\langle M^d, M^d\rangle_t = \int_{\mathbb{R}} |x|^2 \nu(dt, dx) \quad \text{and} \quad d\langle M^{*d}, M^{*d}\rangle_t = \int_{\mathbb{R}} |x|^2 \nu^*(dt, dx).$$

The conditional quadratic variations $(\langle M, M\rangle_t)_{t\in\mathbb{R}_+}$, $(\langle M^*, M^*\rangle_t)_{t\in\mathbb{R}_+}$ of $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ satisfy

$$\langle M, M\rangle_t = \langle M^c, M^c\rangle_t + \langle M^d, M^d\rangle_t, \quad \text{and} \quad \langle M^*, M^*\rangle_t = \langle M^{*c}, M^{*c}\rangle_t + \langle M^{*d}, M^{d*}\rangle_t,$$

$t \in \mathbb{R}_+$. In the sequel, given η, resp. η^*, a forward, resp. backward, adapted and sufficiently integrable process, the notation $\int_0^t \eta_u dM_u$, resp. $\int_t^\infty \eta_u^* dM_u$, will respectively denote the right, resp. left, continuous version of the indefinite stochastic integral, i.e. we have

$$\int_0^t \eta_u dM_u = \int_0^{t^+} \eta_u dM_u \quad \text{and} \quad \int_t^\infty \eta_u^* dM_u = \int_{t^-}^\infty \eta_u^* dM_u, \qquad t \in \mathbb{R}_+,\ dP - a.e.$$

3 Convex concentration inequalities for martingales

In the sequel we assume that

$$(M_t)_{t\in\mathbb{R}_+} \text{ is an } \mathcal{F}_t^*\text{-adapted, } \mathcal{F}_t\text{-forward martingale,} \tag{3.1}$$

and

$$(M_t^*)_{t\in\mathbb{R}_+} \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{F}_t^*\text{-backward martingale,} \tag{3.2}$$

whose characteristics have the form

$$\nu(du, dx) = \nu_u(dx)du \qquad \text{and} \qquad \nu^*(du, dx) = \nu_u^*(dx)du, \tag{3.3}$$

and

$$d\langle M^c, M^c\rangle_t = |H_t|^2 dt, \quad \text{and} \quad d\langle M^{*c}, M^{*c}\rangle_t = |H_t^*|^2 dt, \tag{3.4}$$

where $(H_t)_{t\in\mathbb{R}_+}$, $(H_t^*)_{t\in\mathbb{R}_+}$, are respectively predictable with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$.

Hypotheses (3.1) and (3.2) may seem artificial but they are actually crucial to the proofs of our main results. Indeed, Theorem 3.2 and Theorem 3.3 are based on a forward/backward Itô type change of variable formula (Theorem 8.1 below) for $(M_t, M_t^*)_{t\in\mathbb{R}_+}$, in which (3.1) and (3.2) are needed in order to make sense of the integrals

$$\int_{s^+}^{t} \phi'(M_{u^-} + M_u^*) dM_u$$

and

$$\int_{s}^{t^-} \phi'(M_u + M_{u^+}^*) d^* M_u^*.$$

Note that in our main applications (see Sections 4, 5, 6 and 7), these hypotheses are fulfilled by construction of $\mathcal{F}_t$ and $\mathcal{F}_t^*$.

Recall the following Lemma.

Lemma 3.1. *Let m_1, m_2 be two measures on $\mathbb{R}$ such that $m_1([x,\infty)) \leq m_2([x,\infty)) < \infty$, $x \in \mathbb{R}$. Then for all non-decreasing and m_1, m_2-integrable function f on $\mathbb{R}$ we have*

$$\int_{-\infty}^{\infty} f(x) m_1(dx) \leq \int_{-\infty}^{\infty} f(x) m_2(dx).$$

If m_1, m_2 are probability measures then the above property corresponds to stochastic domination for random variables of respective laws m_1, m_2.

Theorem 3.2. *Let*

$$\bar{\nu}_u(dx) = x\nu_u(dx), \qquad \bar{\nu}_u^*(dx) = x\nu_u^*(dx), \quad u \in \mathbb{R}_+,$$

and assume that:

i) $\bar{\nu}_u([x,\infty)) \leq \bar{\nu}_u^*([x,\infty)) < \infty$, $x, u \in \mathbb{R}$, *and*

ii) $|H_u| \leq |H_u^*|, \quad dPdu - a.e.$

Then we have:

$$\mathbb{E}[\phi(M_t + M_t^*)] \leq \mathbb{E}[\phi(M_s + M_s^*)], \qquad 0 \leq s \leq t, \tag{3.5}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

Next is a different version of the same result, under L^2 hypotheses.

Theorem 3.3. *Let*

$$\tilde{\nu}_u(dx) = |x|^2\nu_u(dx) + |H_u|^2\delta_0(dx), \quad \tilde{\nu}^*_u(dx) = |x|^2\nu^*_u(dx) + |H^*_u|^2\delta_0(dx),$$

$u \in \mathbb{R}_+$*, and assume that:*

$$\tilde{\nu}_u([x,\infty)) \leq \tilde{\nu}^*_u([x,\infty)) < \infty, \qquad x \in \mathbb{R}, \quad u \in \mathbb{R}_+. \tag{3.6}$$

Then we have:

$$\mathbb{E}[\phi(M_t + M^*_t)] \leq \mathbb{E}[\phi(M_s + M^*_s)], \qquad 0 \leq s \leq t, \tag{3.7}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ' is convex.

Remark 3.4. *Note that in both theorems, $(M_t)_{t\geq 0}$ and $(M^*_t)_{t\geq 0}$ do not have to be independent.*

In the proof we may assume that ϕ is $\mathcal{C}^2$ since a convex ϕ can be approximated by an increasing sequence of $\mathcal{C}^2$ convex Lipschitz functions, and the results can then be extended to the general case by an application of the monotone convergence theorem. In order to prove Theorem 3.2 and Theorem 3.3, we apply Itô's formula for forward/backward martingales (Theorem 8.1 in the Appendix Section 8), to $f(x_1, x_2) = \phi(x_1 + x_2)$:

$$\begin{aligned}
\phi(M_t + M^*_t) &= \phi(M_s + M^*_s) \\
&+ \int_{s^+}^t \phi'(M_{u^-} + M^*_u)dM_u + \frac{1}{2}\int_s^t \phi''(M_u + M^*_u)d\langle M^c, M^c\rangle_u \\
&+ \sum_{s<u\leq t} \left(\phi(M_{u^-} + M^*_u + \Delta M_u) - \phi(M_{u^-} + M^*_u) - \Delta M_u \phi'(M_{u^-} + M^*_u)\right) \\
&- \int_s^{t^-} \phi'(M_u + M^*_{u^+})d^*M^*_u - \frac{1}{2}\int_s^t \phi''(M_u + M^*_u)d\langle M^{*c}, M^{*c}\rangle_u \\
&- \sum_{s\leq u<t} \left(\phi(M_u + M^*_{u^+} + \Delta^* M^*_u) - \phi(M_u + M^*_{u^+}) - \Delta^* M^*_u \phi'(M_u + M^*_{u^+})\right),
\end{aligned}$$

$0 \leq s \leq t$, where d and d^* denote the forward and backward Itô differential, respectively defined as the limits of the Riemann sums

$$\sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})\phi'(M_{t_{i-1}^n} + M^*_{t_{i-1}^n})$$

and

$$\sum_{i=0}^{n-1} (M^*_{t_i^n} - M^*_{t_{i+1}^n})\phi'(M_{t_{i+1}^n} + M^*_{t_{i+1}^n})$$

for all refining sequences $\{s = t_0^n \leq t_1^n \leq \cdots \leq t_{k_n}^n = t\}$, $n \geq 1$, of partitions of $[s,t]$ tending to the identity.

Proof of Theorem 3.2. Taking expectations on both sides of the above Itô formula we get

$$\mathbb{E}[\phi(M_t + M^*_t)] = \mathbb{E}[\phi(M_s + M^*_s)] + \frac{1}{2}\mathbb{E}\left[\int_s^t \phi''(M_u + M^*_u)d(\langle M^c, M^c\rangle_u - \langle M^{*c}, M^{*c}\rangle_u)\right]$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_s^t\int_{-\infty}^{+\infty}(\phi(M_u+M_u^*+x)-\phi(M_u+M_u^*)-x\phi'(M_u+M_u^*))\nu_u(dx)du\right]\\
&-\mathbb{E}\left[\int_s^t\int_{-\infty}^{+\infty}(\phi(M_u+M_u^*+x)-\phi(M_u+M_u^*)-x\phi'(M_u+M_u^*))\nu_u^*(dx)du\right]\\
=\ &\mathbb{E}[\phi(M_s+M_s^*)]+\frac{1}{2}\mathbb{E}\left[\int_s^t\phi''(M_u+M_u^*)(|H_u|^2-|H_u^*|^2)du\right]\\
&+\mathbb{E}\left[\int_s^t\int_{-\infty}^{+\infty}\varphi(x,M_u+M_u^*)(\bar{\nu}_u(dx)-\bar{\nu}_u^*(dx))du\right],
\end{aligned}
$$

where

$$
\varphi(x,y)=\frac{\phi(x+y)-\phi(y)-x\phi'(y)}{x},\qquad x,y\in\mathbb{R}.
$$

The conclusion follows from the hypotheses and the fact that since ϕ is convex, the function $x\mapsto\varphi(x,y)$ is increasing in $x\in\mathbb{R}$ for all $y\in\mathbb{R}$. $\square$

Proof of Theorem 3.3. Using the following version of Taylor's formula

$$
\phi(y+x)=\phi(y)+x\phi'(y)+|x|^2\int_0^1(1-\tau)\phi''(y+\tau x)d\tau,\qquad x,y\in\mathbb{R},
$$

which is valid for all $\mathcal{C}^2$ functions ϕ, we get

$$
\begin{aligned}
&\mathbb{E}[\phi(M_t+M_t^*)]=\mathbb{E}[\phi(M_s+M_s^*)]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\int_s^t\phi''(M_u+M_u^*)(|H_u|^2-|H_u^*|^2)du\right]\\
&\quad+\mathbb{E}\left[\int_s^t\int_{-\infty}^{+\infty}|x|^2\int_0^1(1-\tau)\phi''(M_u+M_u^*+\tau x)d\tau\nu_u(dx)du\right]\\
&\quad-\mathbb{E}\left[\int_s^t\int_{-\infty}^{+\infty}|x|^2\int_0^1(1-\tau)\phi''(M_u+M_u^*+\tau x)d\tau\nu_u^*(dx)du\right]\\
&=\ \mathbb{E}[\phi(M_s+M_s^*)]\\
&\quad+\mathbb{E}\left[\int_0^1(1-\tau)\int_s^t\int_{-\infty}^{+\infty}\phi''(M_u+M_u^*+\tau x)(\tilde{\nu}_u(dx)-\tilde{\nu}_u^*(dx))dud\tau\right],
\end{aligned}
$$

and the conclusion follows from the hypothesis and the fact that ϕ is convex implies that ϕ'' is non-decreasing. $\square$

Note that if ϕ is $\mathcal{C}^2$ and ϕ'' is also convex, then it suffices to assume that $\tilde{\nu}_u$ is more convex concentrated than $\tilde{\nu}_u^*$ instead of hypothesis (3.6) in Theorem 3.3.

Remark 3.5. *In case $|H_t|=|H_t^*|$ and $\nu_t=\nu_t^*$, $dPdt$-a.e., from the proof of Theorem 3.2 and Theorem 3.3 we get the identity*

$$
\mathbb{E}[\phi(M_t+M_t^*)]=\mathbb{E}[\phi(M_s+M_s^*)],\qquad 0\le s\le t,\tag{3.8}
$$

for all sufficiently integrable functions $\phi:\mathbb{R}\to\mathbb{R}$.

In particular, Relation (3.8) extends its natural counterpart in the independent increment case: given $(Z_s)_{s\in[0,t]}$, $(\tilde{Z}_s)_{s\in[0,t]}$ two independent copies of a Lévy process without drift, define the backward martingale $(Z^*_s)_{s\in[0,t]}$ as $Z^*_s = \tilde{Z}_{t-s}$, $s \in [0,t]$, then by convolution $\mathbb{E}[\phi(Z_s + Z^*_s)] = \mathbb{E}[\phi(Z_t)]$ does clearly not depend on $s \in [0,t]$.

Remark 3.6. *If ϕ is non-decreasing, the proofs and statements of Theorem 3.2, Theorem 3.3, Corollary 3.9 and Corollary 3.8 extend to semi-martingales $(\hat{M}_t)_{t\in\mathbb{R}_+}$, $(\hat{M}^*_t)_{t\in\mathbb{R}_+}$ represented as*

$$\hat{M}_t = M_t + \int_0^t \alpha_s ds \quad \textit{and} \quad \hat{M}^*_t = M^*_t + \int_t^{+\infty} \beta_s ds, \tag{3.9}$$

provided $(\alpha_t)_{t\in\mathbb{R}_+}$, $(\beta_t)_{t\in\mathbb{R}_+}$, are respectively $\mathcal{F}_t$ and $\mathcal{F}^*_t$-adapted with $\alpha_t \leq \beta_t$, $dPdt$-a.e.

Let now $(\mathcal{F}^M_t)_{t\in\mathbb{R}_+}$, resp. $(\mathcal{F}^{M^*}_t)_{t\in\mathbb{R}_+}$, denote the forward, resp. backward, filtration generated by $(M_t)_{t\in\mathbb{R}_+}$, resp. $(M^*_t)_{t\in\mathbb{R}_+}$.

Corollary 3.7. *Under the hypothesis of Theorem 3.2, if further $\mathbb{E}[M^*_t|\mathcal{F}^M_t] = 0$, $t \in \mathbb{R}_+$, then*

$$\mathbb{E}[\phi(M_t)] \leq \mathbb{E}[\phi(M_s + M^*_s)], \qquad 0 \leq s \leq t. \tag{3.10}$$

Proof. From (3.19) we get

$$\begin{aligned}
\mathbb{E}\left[\phi\left(M_s + M^*_s\right)\right] &\geq \mathbb{E}\left[\phi(M_t + M^*_t)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\phi(M_t + M^*_t)|\mathcal{F}^M_t\right]\right] \\
&\geq \mathbb{E}\left[\phi\left(M_t + \mathbb{E}[M^*_t|\mathcal{F}^M_t]\right)\right] \\
&= \mathbb{E}\left[\phi(M_t)\right],
\end{aligned}$$

$0 \leq s \leq t$, where we used Jensen's inequality. □

In particular, if $M_0 = E[M_t]$ is deterministic (or $\mathcal{F}^M_0$ is the trivial σ-field), Corollary 3.7 shows that $M_t - E[M_t]$ is more concentrated than M^*_0:

$$\mathbb{E}[\phi(M_t - E[M_t])] \leq \mathbb{E}[\phi(M^*_0)], \qquad t \geq 0.$$

The filtrations $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}^*_t)_{t\in\mathbb{R}_+}$ considered in Theorem 3.2 can be taken as $\mathcal{F}_t = \mathcal{F}^{M^*}_0 \vee \mathcal{F}^M_t$, $\mathcal{F}^*_t = \mathcal{F}^M_\infty \vee \mathcal{F}^{M^*}_t$, $t \in \mathbb{R}_+$, provided $(M_t)_{t\in\mathbb{R}_+}$ and $(M^*_t)_{t\in\mathbb{R}_+}$ are independent. In this case, if additionally we have $M^*_T = 0$, then $\mathbb{E}[M^*_t|\mathcal{F}^M_t] = \mathbb{E}[M^*_t] = \mathbb{E}[M^*_T] = 0$, $0 \leq t \leq T$, hence the hypothesis of Corollary 3.7 is also satisfied. However the independence of $\langle M, M\rangle_t$ with $\langle M^*, M^*\rangle_t$, $t \in \mathbb{R}_+$, is not compatible (except in particular situations) with the assumptions imposed in Theorem 3.2.

In applications to convex concentration inequalities between random variables (admitting a predictable representation) and Poisson or Gaussian random variables, the independence of $(M_t)_{t\in\mathbb{R}_+}$ with $(M^*_t)_{t\in\mathbb{R}_+}$ will not be required, see Sections 4 and 5.

The case of bounded jumps

Assume now that $\nu^*(dt, dx)$ has the form

$$\nu^*(dt, dx) = \lambda_t^* \delta_k dt, \tag{3.11}$$

where $k \in \mathbb{R}_+$ and $(\lambda_t^*)_{t\in\mathbb{R}_+}$ is a positive $\mathcal{F}_t^*$-predictable process. Let

$$\lambda_{1,t} = \int_{-\infty}^{+\infty} x \nu_t(dx), \qquad \lambda_{2,t}^2 = \int_{-\infty}^{+\infty} |x|^2 \nu_t(dx), \qquad t \in \mathbb{R}_+,$$

denote respectively the compensator and quadratic variation of the jump part of $(M_t)_{t\in\mathbb{R}_+}$, under the respective assumptions

$$\int_{-\infty}^{+\infty} |x| \nu_t(dx) < \infty, \qquad \text{and} \qquad \int_{-\infty}^{+\infty} |x|^2 \nu_t(dx) < \infty, \tag{3.12}$$

$t \in \mathbb{R}_+$, P-a.s.

Corollary 3.8. *Assume that $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ have jump characteristics satisfying* (3.11) *and* (3.12)*, that $(M_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t^*$-adapted, and that $(M_t^*)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted. Then we have:*

$$\mathbb{E}[\phi(M_t + M_t^*)] \le \mathbb{E}[\phi(M_s + M_s^*)], \qquad 0 \le s \le t, \tag{3.13}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided any of the three following conditions is satisfied:

i) $0 \le \Delta M_t \le k$, $dPdt - a.e.$, *and*

$$|H_t| \le |H_t^*|, \qquad \lambda_{1,t} \le k\lambda_t^*, \quad dPdt - a.e.,$$

ii) $\Delta M_t \le k$, $dPdt - a.e.$, *and*

$$|H_t| \le |H_t^*|, \qquad \lambda_{2,t}^2 \le k^2\lambda_t^*, \quad dPdt - a.e.,$$

iii) $\Delta M_t \le 0$, $dPdt - a.e.$, *and*

$$|H_t|^2 + \lambda_{2,t}^2 \le |H_t^*|^2 + k^2\lambda_t^*, \quad dPdt - a.e.,$$

with moreover ϕ' convex in cases ii) and iii).

Proof. The conditions $0 \le \Delta M_t \le k$, $\Delta M_t \le k$, $\Delta M_t \le 0$, are respectively equivalent to $\nu_t([0, k]^c) = 0$, $\nu_t((k, \infty)) = 0$, $\nu_t((0, \infty)) = 0$, hence under condition (i), the result follows from Theorem 3.2-i), and under conditions $(ii) - (iii)$ it is an application of Theorem 3.2-ii). □

For example we may take $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ of the form

$$M_t = M_0 + \int_0^t H_s dW_s + \int_0^t \int_{-\infty}^{+\infty} x(\mu(ds, dx) - \nu_s(dx)ds), \qquad t \ge 0, \tag{3.14}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and

$$M_t^* = \int_t^{+\infty} H_s^* d^* W_s^* + k\left(Z_t^* - \int_t^{+\infty} \lambda_s^* ds\right), \tag{3.15}$$

where $(W_t^*)_{t\in\mathbb{R}_+}$ is a backward Brownian motion and $(Z_t^*)_{t\in\mathbb{R}_+}$ is a backward point process with intensity $(\lambda_t^*)_{t\in\mathbb{R}_+}$. However in Section 5 we will consider an example for which the decomposition (3.15) does not hold.

The case of point processes

In particular, $(M_t)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ can be taken as

$$M_t = M_0 + \int_0^t H_s dW_s + \int_0^t J_s(dZ_s - \lambda_s ds), \quad t \in \mathbb{R}_+, \tag{3.16}$$

and

$$M_t^* = \int_t^{+\infty} H_s^* d^* W_s^* + \int_t^{+\infty} J_s^*(d^* Z_s^* - \lambda_s^* ds), \quad t \in \mathbb{R}_+, \tag{3.17}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, $(Z_t)_{t\in\mathbb{R}_+}$ is a point process with intensity $(\lambda_t)_{t\in\mathbb{R}_+}$, $(W_t^*)_{t\in\mathbb{R}_+}$ is a backward standard Brownian motion, and $(Z_t^*)_{t\in\mathbb{R}_+}$ is a backward point process with intensity $(\lambda_t^*)_{t\in\mathbb{R}_+}$, and $(H_t)_{t\in\mathbb{R}_+}$, $(J_t)_{t\in\mathbb{R}_+}$, resp. $(H_t^*)_{t\in\mathbb{R}_+}$, $(J_t^*)_{t\in\mathbb{R}_+}$ are predictable with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, resp. $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$.

In this case, taking

$$\nu(dt, dx) = \nu_t(dx) = \lambda_t \delta_{J_t}(dx) dt \quad \text{and} \quad \nu^*(dt, dx) = \nu_t^*(dx) = \lambda_t^* \delta_{J_t^*}(dx) dt \tag{3.18}$$

in Theorem 3.3 yields the following corollary.

Corollary 3.9. *Let $(M_t)_{t\in\mathbb{R}_+}$, $(M_t^*)_{t\in\mathbb{R}_+}$ have the jump characteristics* (3.18) *and assume that $(M_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t^*$-adapted and $(M_t^*)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted. Then we have:*

$$\mathbb{E}[\phi(M_t + M_t^*)] \leq \mathbb{E}[\phi(M_s + M_s^*)], \qquad 0 \leq s \leq t, \tag{3.19}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided any of the three following conditions are satisfied:

i) $0 \leq J_t \leq J_t^*$, $\lambda_t dPdt - a.e.$ *and*

$$|H_t| \leq |H_t^*|, \qquad \lambda_t J_t \leq \lambda_t^* J_t^*, \quad dPdt - a.e.,$$

ii) $J_t \leq J_t^*$, $\lambda_t dPdt - a.e.$, *and*

$$|H_t| \leq |H_t^*|, \qquad \lambda_t |J_t|^2 \leq \lambda_t^* |J_t^*|^2, \quad dPdt - a.e..$$

iii) $J_t \leq 0 \leq J_t^*$, $\lambda_t dPdt - a.e.$, *and*

$$|H_t|^2 + \lambda_t |J_t|^2 \leq |H_t^*|^2 + \lambda_t^* |J_t^*|^2, \quad dPdt - a.e..$$

with moreover ϕ' convex in cases ii) and iii).

Note that condition *i)* in Corollary 3.9 can be replaced with the stronger condition:

i') $0 \leq J_t \leq J_t^*$, $\lambda_t dPdt - a.e.$ and

$$|H_t| \leq |H_t^*|, \qquad \lambda_t \leq \lambda_t^*, \quad dPdt - a.e.$$

4 Application to point processes

Let $(W_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion and a point process, generating a filtration $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$. We will assume that $(W_t)_{t\in\mathbb{R}_+}$ is also an $\mathcal{F}_t^M$-Brownian motion and that $(Z_t)_{t\in\mathbb{R}_+}$ has compensator $(\lambda_t)_{t\in\mathbb{R}_+}$ with respect to $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$, which does not in general require the independence of $(W_t)_{t\in\mathbb{R}_+}$ from $(Z_t)_{t\in\mathbb{R}_+}$. Consider F a random variable with the representation

$$F = \mathbb{E}[F] + \int_0^{+\infty} H_t dW_t + \int_0^{+\infty} J_t(dZ_t - \lambda_t dt), \tag{4.1}$$

where $(H_u)_{u\in\mathbb{R}_+}$ is a square-integrable $\mathcal{F}_t^M$-predictable process and $(J_t)_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t^M$-predictable process which is either square-integrable or positive and integrable. Theorem 4.1 is a consequence of Corollary 3.9 above, and shows that the possible dependence of $(W_t)_{t\in\mathbb{R}_+}$ from $(Z_t)_{t\in\mathbb{R}_+}$ can be decoupled in terms of independent Gaussian and Poisson random variables. Note that inequality (4.2) below is weaker than (4.3) but it holds for a wider class of functions, i.e. for all convex functions instead of all convex functions having a convex derivative.

Theorem 4.1. *Let F have the representation* (4.1)*:*

$$F = \mathbb{E}[F] + \int_0^{+\infty} H_t dW_t + \int_0^{+\infty} J_t(dZ_t - \lambda_t dt),$$

and let $\tilde{N}(c)$, $W(\beta^2)$ be independent random variables with compensated Poisson law of intensity $c > 0$ and centered Gaussian law with variance $\beta^2 \geq 0$, respectively.

i) Assume that $0 \leq J_t \leq k$, $dPdt$-a.e., for some $k > 0$, and let

$$\beta_1^2 = \left\| \int_0^{+\infty} |H_t|^2 dt \right\|_\infty \quad \text{and} \quad \alpha_1 = \left\| \int_0^{+\infty} J_t \lambda_t dt \right\|_\infty .$$

Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi\left(W(\beta_1^2) + k\tilde{N}(\alpha_1/k)\right)\right], \tag{4.2}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

ii) Assume that $J_t \leq k$, $dPdt$-a.e., for some $k > 0$, and let

$$\beta_2^2 = \left\| \int_0^{+\infty} |H_t|^2 dt \right\|_\infty \quad \text{and} \quad \alpha_2^2 = \left\| \int_0^{+\infty} |J_t|^2 \lambda_t dt \right\|_\infty .$$

Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi\left(W(\beta_2^2) + k\tilde{N}(\alpha_2^2/k^2)\right)\right], \tag{4.3}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ' is convex.

iii) Assume that $J_t \leq 0$, $dPdt$-a.e., and let

$$\beta_3^2 = \left\| \int_0^{+\infty} |H_t|^2 dt + \int_0^{+\infty} |J_t|^2 \lambda_t dt \right\|_\infty .$$

Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi(W(\beta_3^2))\right], \tag{4.4}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ' is convex.

Proof. Consider the $\mathcal{F}_t^M$-martingale

$$M_t = \mathbb{E}[F|\mathcal{F}_t^M] - \mathbb{E}[F] = \int_0^t H_s dW_s + \int_0^t J_s(dZ_s - \lambda_s ds), \qquad t \geq 0,$$

and let $(\hat{N}_s)_{s\in\mathbb{R}_+}$, $(\hat{W}_s)_{s\in\mathbb{R}_+}$ respectively denote a left-continuous standard Poisson process and a standard Brownian motion which are assumed to be mutually independent, and also independent of $(\mathcal{F}_s^M)_{s\in\mathbb{R}_+}$.

i) $-$ *ii)* For $p = 1, 2$, let the filtrations $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$ be defined by

$$\mathcal{F}_t^* = \mathcal{F}_\infty^M \vee \sigma(\hat{W}_{\beta_p^2} - \hat{W}_{V_p^2(s)}, \hat{N}_{\alpha_p^p/k^p} - \hat{N}_{U_p^p(s)/k^p} \ : \ s \geq t\},$$

and $\mathcal{F}_t = \sigma(\hat{W}_s,\ \hat{N}_s \ : \ s \geq 0) \vee \mathcal{F}_t^M$, $t \in \mathbb{R}_+$, and let

$$M_t^* = \hat{W}_{\beta_p^2} - \hat{W}_{V_p^2(t)} + k(\hat{N}_{\alpha_p^p/k^p} - \hat{N}_{U_p^p(t)/k^p}) - (\alpha_p^p - U_p^p(t))/k^{p-1}, \tag{4.5}$$

where

$$V_p^2(t) = \int_0^t |H_s|^2 ds \quad \text{and} \quad U_p^p(t) = \int_0^t J_s^p \lambda_s ds, \qquad P-a.s., \quad s \geq 0.$$

Then $(M_t^*)_{t\in\mathbb{R}_+}$ satisfies the hypothesis of Corollary 3.9$-i) - ii)$, as well as the condition $E[M_t^*|\mathcal{F}_t^M] = 0$, $t \in \mathbb{R}_+$, with $H_s^* = H_s$, $J_s^* = k$, $\lambda_s^* = J_s^p\lambda_s/k^p$, $dPds$-a.e., hence

$$\mathbb{E}[\phi(M_t)] \leq \mathbb{E}[\phi(M_0^*)],$$

and letting t go to infinity we obtain (4.2) and (4.3), respectively for $p = 1$ and $p = 2$.

iii) Let

$$M_s^* = W_{\beta_3^2} - W_{U_3^2(s)}, \tag{4.6}$$

where

$$U_3^2(s) = \int_0^s |H_u|^2 du + \int_0^s |J_u|^2 \lambda_u du, \qquad P-a.s.$$

Then $(M_t^*)_{t\in\mathbb{R}_+}$ satisfies the hypothesis of Corollary 3.9$-iii)$ with $|H_s^*|^2 = |H_s|^2 + |J_s|^2\lambda_s$ and $\lambda_s^* = J_s^* = 0$, $dPds$-a.e., hence

$$\mathbb{E}[\phi(M_t)] \leq \mathbb{E}[\phi(M_0^*)],$$

and letting t go to infinity we obtain (4.4). □

Remark 4.2. *The proof of Theorem 4.1 can also be obtained from Corollary 3.8.*

Proof. Let

$$\mu(dt, dx) = \sum_{\Delta Z_s \neq 0} \delta_{(s,J_s)}(dt, dx), \qquad \nu_t(dx) = \lambda_t \delta_{J_t}(dx).$$

i) $-$ *ii)* In both cases $p = 1, 2$, let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$ and $(M_t^*)_{t\in\mathbb{R}_+}$ be defined in (4.5), with $V_p^2(t) = \int_0^t |H_s|^2 ds$ and $U_p^p(t) = \int_0^t |J_s|^p ds$, P-a.s., $t \geq 0$. Then $(M_t^*)_{t\in\mathbb{R}_+}$ satisfies the hypothesis of Corollary 3.8$-i) - ii)$, with $H_s^* = H_s$, $\nu_s^* = |J_s|^p/k^p$, $dPds$-a.e.

iii) Let $(M_s^*)_{s\in\mathbb{R}_+}$ be defined as in (4.6), and let $U_3^2(s) = \int_0^s |H_u|^2 du + \int_0^s |J_u|^2 du$, $|H_s^*|^2 = |H_s|^2 + |J_s|^2$ and $\nu_s^* = 0$, $dPds$-a.e. □

In the pure jump case, Theorem 4.1-ii) yields

$$P(M_T \geq x) \leq \exp\left(\frac{y}{k} - \left(\frac{y}{k} + \frac{\alpha_2^2}{k^2}\right)\log\left(1 + \frac{ky}{\alpha_2^2}\right)\right) \leq \exp\left(-\frac{y}{2k}\log\left(1 + \frac{ky}{\alpha_2^2}\right)\right),$$

$y > 0$, with $\alpha_2^2 = \|\langle M, M\rangle_T\|_\infty$, cf. Theorem 23.17 of [9], although some differences in the hypotheses make the results not directly comparable: here no lower bound is assumed on jump sizes, and the presence of a continuous component is treated in a different way.

The results of this section and the next one apply directly to solutions of stochastic differential equations such as

$$dX_t = a(t, X_t)dW_t + b(t, X_t)(dZ_t - \lambda_t dt),$$

with $H_t = a(t, X_t)$, $J_t = b(t, X_t)$, $t \in \mathbb{R}_+$, for which the hypotheses can be formulated directly on the coefficients $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ without explicit knowledge of the solution.

5 Application to Poisson random measures

Since a large family of point processes can be represented as stochastic integrals with respect to Poisson random measures (see e.g. [7], Section 4, Ch. XIV), it is natural to investigate the consequences of Theorem 3.2 in the setting of Poisson random measures. Let σ be a Radon measure on $\mathbb{R}^d$, diffuse on $\mathbb{R}^d \setminus \{0\}$, such that $\sigma(\{0\}) = 1$, and

$$\int_{\mathbb{R}^d\setminus\{0\}} (|x|^2 \wedge 1)\sigma(dx) < \infty,$$

and consider a random measure $\omega(dt, dx)$ of the form

$$\omega(dt, dx) = \sum_{i\in\mathbb{N}} \delta_{(t_i, x_i)}(dt, dx)$$

identified to its (locally finite) support $\{(t_i, x_i)\}_{i\in\mathbb{N}}$. We assume that $\omega(dt, dx)$ is Poisson distributed with intensity $dt\sigma(dx)$ on $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$, and consider a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, independent of $\omega(dt, dx)$, under a probability P on Ω. Let

$$\mathcal{F}_t = \sigma(W_s,\ \omega([0, s] \times A)\ :\ 0 \leq s \leq t,\ A \in \mathcal{B}_b(\mathbb{R}^d \setminus \{0\})), \qquad t \in \mathbb{R}_+,$$

where $\mathcal{B}_b(\mathbb{R}^d \setminus \{0\}) = \{A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})\ :\ \sigma(A) < \infty\}$. The stochastic integral of a square-integrable $\mathcal{F}_t$-predictable process $u \in L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d, dP \times dt \times d\sigma)$ is written as

$$\int_0^{+\infty} u(t, 0)dW_t + \int_{\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}} u(t, x)(\omega(dt, dx) - \sigma(dx)dt), \tag{5.1}$$

and satisfies the Itô isometry

$$E\left[\left(\int_0^{+\infty} u(t, 0)dW_t + \int_0^{+\infty}\int_{\mathbb{R}^d\setminus\{0\}} u(t, x)(\omega(dt, dx) - \sigma(dx)dt)\right)^2\right]$$

$$
\begin{aligned}
&= E\left[\int_0^{+\infty} u^2(t,0)dt\right] + E\left[\int_{\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}} u^2(t,x)\sigma(dx)dt\right] \\
&= E\left[\int_{\mathbb{R}_+\times\mathbb{R}^d} u^2(t,x)\sigma(dx)dt\right]. \qquad (5.2)
\end{aligned}
$$

Recall that due to the Itô isometry, the predictable and adapted version of u can be used indifferently in the stochastic integral (5.1), cf. p. 199 of [5] for details. When $u \in L^2(\mathbb{R}_+ \times \mathbb{R}^d, dt \times d\sigma)$, the characteristic function of

$$
I_1(u) := \int_0^{+\infty} u(t,0)dW_t + \int_{\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}} u(t,x)(\omega(dt,dx) - \sigma(dx)dt),
$$

is given by the Lévy-Khintchine formula

$$
E\left[e^{iI_1(u)}\right] = \exp\left(-\frac{1}{2}\int_0^{+\infty} u^2(t,0)dt + \int_{\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}} (e^{iu(t,x)} - 1 - iu(t,x))\sigma(dx)dt\right).
$$

Theorem 5.1. *Let F with the representation*

$$
F = E[F] + \int_0^{+\infty} H_s dW_s + \int_0^{+\infty}\int_{\mathbb{R}^d\setminus\{0\}} J_{u,x}(\omega(du,dx) - \sigma(dx)du),
$$

where $(H_t)_{t\in\mathbb{R}_+} \in L^2(\Omega \times \mathbb{R}_+)$, and $(J_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ are $\mathcal{F}_t$-predictable with $(J_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d} \in L^1(\Omega\times\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}, dP\times dt\times d\sigma)$ and $(J_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d} \in L^2(\Omega\times\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}, dP\times dt\times d\sigma)$ respectively in (i) and in $(ii-iii)$ below.

i) Assume that $0 \le J_{u,x} \le k$, $dP\sigma(dx)du$-a.e., for some $k > 0$, and let

$$
\beta_1^2 = \left\| \int_0^{+\infty} |H_u|^2 du \right\|_\infty, \quad \text{and} \quad \alpha_1(x) = \left\| \int_0^{+\infty} J_{u,x} du \right\|_\infty, \quad \sigma(dx) - a.e.
$$

Then we have

$$
\mathbb{E}[\phi(F - \mathbb{E}[F])] \le \mathbb{E}\left[\phi\left(W(\beta_1^2) + k\tilde{N}\left(\int_{\mathbb{R}^d\setminus\{0\}} \frac{\alpha_1(x)}{k}\sigma(dx)\right)\right)\right],
$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

ii) Assume that $J_{u,x} \le k$, $dP\sigma(dx)du$-a.e., for some $k > 0$, and let

$$
\beta_2^2 = \left\| \int_0^{+\infty} |H_u|^2 du \right\|_\infty, \quad \text{and} \quad \alpha_2^2(x) = \left\| \int_0^{+\infty} |J_{u,x}|^2 du \right\|_\infty, \quad \sigma(dx) - a.e.
$$

Then we have

$$
\mathbb{E}[\phi(F - \mathbb{E}[F])] \le \mathbb{E}\left[\phi\left(W(\beta_2^2) + k\tilde{N}\left(\int_{\mathbb{R}^d\setminus\{0\}} \frac{\alpha_2^2(x)}{k^2}\sigma(dx)\right)\right)\right],
$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ' is convex.

iii) Assume that $J_{u,x} \leq 0$, $dP\sigma(dx)du$-a.e., and let

$$\beta_3^2 = \left\| \int_0^{+\infty} |H_u|^2 du + \int_0^{+\infty} \int_{\mathbb{R}^d\setminus\{0\}} |J_{u,x}|^2 du\sigma(dx) \right\|_\infty .$$

Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi(W(\beta_3^2))\right],$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ' is convex.

Proof. The proof is similar to that of Theorem 4.1, replacing the use of Corollary 3.9 by that of Corollary 3.8. Let

$$M_t = M_0 + \int_0^t H_u dW_u + \int_{\mathbb{R}^d\setminus\{0\}} \int_0^t J_{u,x}(\omega(du, dx) - \sigma(dx)du),$$

generating the filtration $(\mathcal{F}_t^M)_{t\in\mathbb{R}_+}$. Here, $\nu_t(dx)$ denotes the image measure of $\sigma(dx)$ by the mapping $x \mapsto J_{t,x}$, $t \geq 0$, and $\mu(dt, dx)$ denotes the image measure of $\omega(dt, dx)$ by $(s, y) \mapsto (s, J_{s,y})$, i.e.

$$\mu(dt, dx) = \sum_{\omega(\{(s,y)\})=1} \delta_{(s,J_{s,y})}(dt, dx).$$

$i) - ii)$ For $p = 1, 2$, let the filtrations $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{F}_t^*)_{t\in\mathbb{R}_+}$ be defined by

$$\mathcal{F}_t^* = \mathcal{F}_\infty^M \vee \sigma(\hat{W}_{\beta_p^2} - \hat{W}_{V_p^2(s)} + \hat{N}_{\alpha_p^p/k^p} - \hat{N}_{U_p^p(s)/k^p} \ : \ s \geq t\}),$$

and

$$\mathcal{F}_t = \mathcal{F}_t^M \vee \sigma(\hat{W}_s, \ \hat{N}_s \ : \ s \geq 0), \quad t \in \mathbb{R}_+,$$

and let

$$M_t^* = \hat{W}_{\beta_p^2} - \hat{W}_{V_p^2(t)} + k(\hat{N}_{\alpha_p^p/k^p} - \hat{N}_{U_p^p(t)/k^p}) - (\alpha_p^p - U_p^p(t))/k^{p-1},$$

where

$$V_p^2(t) = \int_0^t |H_s|^2 ds \quad \text{and} \quad U_p^p(t) = \int_0^t \int_{-\infty}^{+\infty} x^p \nu_s(dx) ds, \qquad P - a.s., \quad t \geq 0.$$

Then $(M_t^*)_{t\in\mathbb{R}_+}$ satisfies the hypothesis of Theorem 3.2$-i) - ii)$, and also the condition $E[M_t^*|\mathcal{F}_t^M] = 0$, $t \in \mathbb{R}_+$, with $H_s^* = H_s$, $\nu_s^* = \int_{-\infty}^{+\infty} x^p \nu_s(dx)$, $dPds$-a.e., hence

$$\mathbb{E}[\phi(M_t)] \leq \mathbb{E}[\phi(M_0^*)].$$

Letting t go to infinity we obtain (4.2) and (4.3), respectively for $p = 1, 2$.

$iii)$ Let

$$M_s^* = W_{\beta_3^2} - W_{U_3^2(s)},$$

where

$$U_3^2(s) = \int_0^s |H_u|^2 du + \int_0^s \int_{-\infty}^{+\infty} |x|^2 \nu_u(dx) du, \qquad P - a.s., \quad s \geq 0.$$

Then $(M^*_t)_{t\in\mathbb{R}_+}$ satisfies the hypotheses of Theorem 3.2$-iii)$ with

$$|H^*_s|^2 = |H_s|^2 + \int_{-\infty}^{+\infty} |x|^2 \nu_s(dx)$$

and $\nu^*_s = 0$, $dPds$-a.e., hence

$$\mathbb{E}[\phi(M_t)] \le \mathbb{E}[\phi(M^*_0)],$$

and letting t go to infinity we obtain (4.4). □

In Theorem 4.1, $(Z_t)_{t\in\mathbb{R}_+}$ can be taken equal to the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, which also satisfies the hypotheses of Theorem 5.1 since it can be defined with $d = 1$ and $\sigma(dx) = 1_{[0,1]}(x)dx$ as

$$N_t = \omega([0,t] \times [0,1]), \qquad t \ge 0.$$

In other terms, being a point process, $(N_t)_{t\in\mathbb{R}_+}$ is at the intersection of Corollary 3.8 and Corollary 3.9, as already noted in Remark 4.2.

6 Clark formula

In this section we examine the consequence of results of Section 5 when the predictable representation of random variables is obtained via the Clark formula. We work on a product

$$(\Omega, P) = (\Omega_W \times \Omega_X, P_W \otimes P_X),$$

where (Ω_W, P_W) is the classical Wiener space on which is defined a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ and

$$\Omega_X = \left\{ \omega_X(dt,dx) = \sum_{i\in\mathbb{N}} \delta_{(t_i,x_i)}(dt,dx) \ : \ (t_i,x_i) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\}), \ i \in \mathbb{N} \right\}.$$

The elements of Ω_X are identified to their (by assumption locally finite) support $\{(t_i,x_i)\}_{i\in\mathbb{N}}$, and $\omega_X \mapsto \omega_X(dt,dx)$ is Poisson distributed under P_X with intensity $dt\sigma(dx)$ on $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$.

The multiple stochastic integral $I_n(h_n)$ of $h_n \in L^2(\mathbb{R}_+ \times \mathbb{R}^d, dtd\sigma)^{\circ n}$ can be defined by induction with

$$I_n(h_n) = n\int_0^\infty I_{n-1}(\pi^n_{t,0}h_n)dW_t + n\int_{\mathbb{R}_+\times\mathbb{R}^d} I_{n-1}(\pi^n_{t,x}h_n)(\omega_X(dt,dx) - \sigma(dx)dt),$$

where

$$(\pi^n_{t,x}h_n)(t_1,x_1,\ldots,t_{n-1},x_{n-1}) := h_n(t_1,x_1,\ldots,t_{n-1},x_{n-1},t,x)1_{[0,t]}(t_1)\cdots 1_{[0,t]}(t_{n-1}),$$

$t_1,\ldots,t_{n-1},t \in \mathbb{R}_+$, $x_1,\ldots,x_{n-1},x \in \mathbb{R}^d$. The isometry property

$$E\left[I_n(h_n)^2\right] = n!\|h_n\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d, dt\otimes\sigma)^{\otimes n}}$$

follows by induction from (5.2). Let the linear, closable, finite difference operator

$$D : L^2(\Omega, P) \longrightarrow L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d, dP \times dt \times d\sigma)$$

be defined as

$$D_{t,x} I_n(f_n) = n I_{n-1}(f_n(*, t, x)), \qquad \sigma(dx)dtdP - a.e.,$$

cf. e.g. [12], [17], with in particular

$$D_{t,0} I_n(f_n) = n I_{n-1}(f_n(*, t, 0)), \qquad dtdP - a.e.,$$

Recall that the closure of D is also linear, and given $F \in \mathrm{Dom}\,(\mathrm{D})$, for $\sigma(dx)dt$-a.e. every $(t, x) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$ we have

$$D_{t,x} F(\omega_W, \omega_X) = F(\omega_W, \omega_X \cup \{(t, x)\}) - F(\omega_W, \omega_X), \qquad P(d\omega) - a.s.,$$

cf. e.g. [12], [14], while $D_{t,0}$ has the derivation property, and

$$D_{t,0} f(I_1(f_1^{(1)}), \ldots, I_1(f_1^{(n)})) = \sum_{k=1}^{n} f_1^{(i)}(t, 0) \partial_k f(I_1(f_1^{(1)}), \ldots, I_1(f_1^{(n)})),$$

$dtdP$-a.e., $f_1^{(1)}, \ldots, f_1^{(d)} \in L^2(\mathbb{R}_+ \times \mathbb{R}^d, dtd\sigma)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, cf. e.g. [16].

The Clark formula for Lévy processes, cf. [13], [16], states that every $F \in L^2(\Omega)$ has the representation

$$F = \mathbb{E}[F] + \int_0^{+\infty} \mathbb{E}[D_{s,0} F | \mathcal{F}_s] dW_s + \int_0^{+\infty} \int_{\mathbb{R}^d \setminus \{0\}} \mathbb{E}[D_{s,x} F | \mathcal{F}_s](\omega_X(ds, dx) - \sigma(dx)ds). \quad (6.1)$$

(The formula originally holds for F in the domain of D but its extension to $L^2(\Omega)$ is straightforward, cf. [16], Proposition 12). Theorem 5.1 immediately yields the following corollary when applied to any $F \in L^2(\Omega)$ represented as in (6.1).

Corollary 6.1. *Let $F \in L^2(\Omega)$ have the representation* (6.1)*, and assume additionally that $\int_0^{+\infty} \int_{\mathbb{R}^d \setminus \{0\}} |\mathbb{E}[D_{s,x} F | \mathcal{F}_s]| \sigma(dx) ds < \infty$ a.s. in (i) below.*

i) Assume that $0 \leq \mathbb{E}[D_{u,x} F | \mathcal{F}_u] \leq k$, $dP\sigma(dx)du$-a.e., for some $k > 0$, and let

$$\beta_1^2 = \left\| \int_0^{+\infty} (\mathbb{E}[D_{u,0} F | \mathcal{F}_u])^2 du \right\|_\infty, \quad \textit{and} \quad \alpha_1(x) = \left\| \int_0^{+\infty} \mathbb{E}[D_{u,x} F | \mathcal{F}_u] du \right\|_\infty,$$

$\sigma(dx)$-a.e. Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi\left(W(\beta_1^2) + k\tilde{N}\left(\int_{\mathbb{R}^d \setminus \{0\}} \frac{\alpha_1(x)}{k} \sigma(dx)\right)\right)\right], \qquad (6.2)$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

ii) Assume that $\mathbb{E}[D_{u,x}F|\mathcal{F}_u] \leq k$, $dP\sigma(dx)du$*-a.e., for some* $k > 0$*, and let*

$$\beta_2^2 = \left\| \int_0^{+\infty} (\mathbb{E}[D_{u,0}F|\mathcal{F}_u])^2 du \right\|_\infty, \quad \textit{and} \quad \alpha_2^2(x) = \left\| \int_0^{+\infty} (\mathbb{E}[D_{u,x}F|\mathcal{F}_u])^2 du \right\|_\infty,$$

$\sigma(dx)$*-a.e. Then we have*

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi\left(W(\beta_2^2) + k\tilde{N}\left(\int_{\mathbb{R}^d\setminus\{0\}} \frac{\alpha_2^2(x)}{k^2}\sigma(dx)\right)\right)\right], \tag{6.3}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ *such that* ϕ' *is convex.*

iii) Assume that $\mathbb{E}[D_{u,x}F|\mathcal{F}_u] \leq 0$, $dP\sigma(dx)du$*-a.e., and let*

$$\beta_3^2 = \left\| \int_0^{+\infty} (\mathbb{E}[D_{u,0}F|\mathcal{F}_u])^2 du + \int_0^{+\infty} \int_{\mathbb{R}^d\setminus\{0\}} (\mathbb{E}[D_{u,x}F|\mathcal{F}_u])^2 du\sigma(dx) \right\|_\infty.$$

Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi(W(\beta_3^2))\right], \tag{6.4}$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$ *such that* ϕ' *is convex.*

As mentioned in the introduction, from (6.4) we deduce the deviation inequality

$$P(F - \mathbb{E}[F] \geq y) \leq \frac{e^2}{2} P(W(\beta_3^2) > y) \leq \frac{e^2}{2} \exp\left(-\frac{y^2}{2\beta_3^2}\right), \qquad y > 0,$$

provided $\mathbb{E}[D_{u,x}F|\mathcal{F}_u] \leq 0$, $dP\sigma(dx)du$-a.e., and

$$\int_0^{+\infty} (\mathbb{E}[D_{u,0}F|\mathcal{F}_u])^2 du + \int_0^{+\infty} \int_{\mathbb{R}^d\setminus\{0\}} (\mathbb{E}[D_{u,x}F|\mathcal{F}_u])^2 du\sigma(dx) \leq \beta_3^2, \quad P-a.s.$$

Similarly from (6.3) we get

$$P(E - \mathbb{E}[F] \geq y) \leq \exp\left(\frac{y}{k} - \left(\frac{y}{k} + \frac{\alpha_2^2}{k^2}\right) \log\left(1 + \frac{ky}{\alpha_2^2}\right)\right), \qquad y > 0, \tag{6.5}$$

provided

$$E[D_{t,x}F|\mathcal{F}_t] \leq k, \quad dP\sigma(dx)dt - a.e., \tag{6.6}$$

and

$$\int_{\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}} (E[D_{t,x}F|\mathcal{F}_t])^2 \sigma(dx)dt \leq \alpha_2^2, \quad P-a.s.,$$

for some $k > 0$ and $\alpha_2^2 > 0$. In [1] this latter estimate has been proved using (modified) logarithmic Sobolev inequalities and the Herbst method under the stronger condition

$$|D_{t,x}F| \leq k, \quad dP\sigma(dx)dt\text{-a.e.}, \tag{6.7}$$

and

$$\int_{\mathbb{R}_+\times\mathbb{R}^d\setminus\{0\}} |D_{t,x}F|^2 \sigma(dx)dt \leq \alpha_2^2, \quad P-a.s., \tag{6.8}$$

for some $k > 0$ and $\alpha_2^2 > 0$. In [19] it has been shown, using sharp logarithmic Sobolev inequalities, that the condition $|D_{t,x}F| \le k$ can be relaxed to

$$D_{t,x}F \le k, \qquad dP\sigma(dx)dt\text{-a.e.}, \tag{6.9}$$

which is nevertheless stronger than (6.6).

In the next result, which however imposes uniform almost sure bounds on DF, we consider Poisson random measures on $\mathbb{R}^d \setminus \{0\}$ instead of $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$.

Corollary 6.2. *i) Assume that $0 \le D_xF \le \beta(x) \le k$, $dP\sigma(dx)-a.e.$, where $\beta(\cdot) : \mathbb{R}^d \setminus \{0\} \to [0,k]$ is deterministic and $k > 0$. Then for all convex functions ϕ we have*

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \le \mathbb{E}\left[\phi\left(k\tilde{N}\left(\int_{\mathbb{R}^d\setminus\{0\}} \frac{\beta(x)}{k}\sigma(dx)\right)\right)\right].$$

ii) Assume that $|D_xF| \le \beta(x) \le k$, $dP\sigma(dx)$-a.e., where $\beta(\cdot) : \mathbb{R} \to [0,k]$ and $k > 0$ are deterministic. Then for all convex functions ϕ with a convex derivative ϕ' we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \le \mathbb{E}\left[\phi\left(k\tilde{N}\left(\int_{\mathbb{R}^d\setminus\{0\}} \frac{\beta^2(x)}{k^2}\sigma(dx)\right)\right)\right].$$

iii) Assume that $-\beta(x) \le D_xF \le 0$, $dP\sigma(dx)$-a.e., where $\beta(\cdot) : \mathbb{R} \to [0,\infty)$ is deterministic. Then for all convex functions ϕ with a convex derivative ϕ' we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \le \mathbb{E}\left[\phi\left(W\left(\int_{\mathbb{R}^d\setminus\{0\}} \beta^2(x)\sigma(dx)\right)\right)\right].$$

Proof. Assume that $\omega_X(dt,dx)$ has intensity $1_{[0,1]}(s)\sigma(dx)ds$ on $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$, we define the random measure $\hat{\omega}$ on $\mathbb{R}^d \setminus \{0\}$ with intensity $\sigma(dx)$ as

$$\hat{\omega}_X(A) = \omega_X([0,1] \times A), \quad A \in \mathcal{B}_b(\mathbb{R}^d \setminus \{0\}).$$

Then it remains to apply Corollary 6.1 to $\hat{F}(\omega_W, \omega_X) := F(\omega_W, \hat{\omega}_X)$. $\square$

In Corollary 6.2, $\mathbb{R}^d \setminus \{0\}$ can be replaced by $\mathbb{R}^d$ without additional difficulty.

7 Normal martingales

In this section we interpret the above results in the framework of normal martingales. Let $(Z_t)_{t\in\mathbb{R}^+}$ be a normal martingale, i.e. $(Z_t)_{t\in\mathbb{R}^+}$ is a martingale such that $d\langle Z, Z\rangle_t = dt$. If $(Z_t)_{t\in\mathbb{R}^+}$ is in L^4 and has the chaotic representation property it satisfies the structure equation

$$d[Z,Z]_t = dt + \gamma_t dZ_t, \quad t \in \mathbb{R}^+,$$

where $(\gamma_t)_{t\in\mathbb{R}_+}$ is a predictable square-integrable process, cf. [6]. Recall that the cases $\gamma_s = 0$, $\gamma_s = c \in \mathbb{R} \setminus \{0\}$, $\gamma_s = \beta Z_s$, $\beta \in (-2, 0)$, correspond respectively to Brownian motion, the compensated Poisson process with jump size c and intensity $1/c^2$, and to the Azéma martingales. Consider the martingale

$$M_t = M_0 + \int_0^t R_u dZ_u, \tag{7.1}$$

where $(R_u)_{u\in\mathbb{R}_+} \in L^2(\Omega \times \mathbb{R}_+)$ is predictable. We have

$$d\langle M^c, M^c\rangle_t = 1_{\{\gamma_t=0\}}|R_t|^2 dt$$

and

$$\mu(dt, dx) = \sum_{\Delta Z_s\neq 0} \delta_{(s,R_s\gamma_s)}(dt, dx), \qquad \nu(dt, dx) = \frac{1}{\gamma_t^2} \sum_{\Delta Z_s\neq 0} \delta_{R_s\gamma_s}(dx)dt,$$

and the Itô formula, cf. [6]:

$$\begin{aligned}\phi(M_t) = \phi(M_s) &+ \int_s^t 1_{\{\gamma_u=0\}} R_u\phi'(M_u)dZ_u + \int_s^t 1_{\{\gamma_u\neq 0\}} \frac{\phi(M_{u^-} + \gamma_u R_u) - \phi(M_{u^-})}{\gamma_u} dZ_u \\ &+\frac{1}{2}\int_s^t 1_{\{\gamma_u=0\}}|R_u|^2\phi''(M_u)du + \int_s^t 1_{\{\gamma_u\neq 0\}} \frac{\phi(M_u + \gamma_u R_u) - \phi(M_u) - \gamma_u R_u\phi'(M_u)}{|\gamma_u|^2} du,\end{aligned}$$

$\phi \in \mathcal{C}^2(\mathbb{R})$. The multiple stochastic integrals with respect to $(M_t)_{t\in\mathbb{R}_+}$ are defined as

$$I_n(f_n) = n! \int_0^{+\infty} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n)dM_{t_1} \cdots dM_{t_n},$$

for f_n a symmetric function in $L^2(\mathbb{R}_+^n)$. As an application of Corollary 3.9 we have the following result.

Theorem 7.1. *Let $(M_t)_{t\in\mathbb{R}_+}$ have the representation (7.1), let $(M_t^*)_{t\in\mathbb{R}_+}$ be represented as*

$$M_t^* = \int_t^{+\infty} H_s^* d^*W_s^* + \int_t^{+\infty} J_s^*(d^*Z_s^* - \lambda_s^* ds),$$

assume that $(M_t)_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t^*$-adapted $\mathcal{F}_t$-martingale and that $(M_t^*)_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t$-adapted $\mathcal{F}_t^*$-martingale. Then we have

$$\mathbb{E}[\phi(M_t + M_t^*)] \leq \mathbb{E}[\phi(M_s + M_s^*)], \qquad 0 \leq s \leq t,$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided any of the following three conditions is satisfied:

i) $0 \leq \gamma_t R_t \leq J_t^*$, $1_{\{\gamma_t=0\}}|R_t|^2 \leq |H_t^*|^2$, *and*

$$1_{\{\gamma_t\neq 0\}}\frac{R_t}{\gamma_t} \leq \lambda_t^* J_t^*, \quad dPdt - a.e.,$$

ii) $\gamma_t R_t \leq J_t^*$, $1_{\{\gamma_t=0\}}|R_t|^2 \leq |H_t^*|^2$, *and*

$$1_{\{\gamma_t\neq 0\}}|R_t|^2 \leq \lambda_t^* |J_t^*|^2, \quad dPdt - a.e.,$$

and ϕ' is convex,

iii) $\gamma_t R_t \leq 0$, $|R_t|^2 \leq |H_t^*|^2$, $J_t^* = 0$, *dPdt - a.e., and* ϕ' *is convex.*

As above, if further $E[M_t^* | \mathcal{F}_t^M] = 0$, $t \in \mathbb{R}_+$, we obtain

$$\mathbb{E}[\phi(M_t)] \leq \mathbb{E}[\phi(M_s + M_s^*)], \qquad 0 \leq s \leq t.$$

As a consequence we have the following result which admits the same proof as Theorem 4.1.

Theorem 7.2. *Let* $F \in L^2(\Omega, \mathcal{F}, P)$ *have the predictable representation*

$$F = \mathbb{E}[F] + \int_0^{+\infty} R_t dZ_t.$$

i) Assume that $0 \leq \gamma_t R_t \leq k$, *dPdt-a.e., for some* $k > 0$, *and let*

$$\beta_1^2 = \left\| \int_0^{+\infty} 1_{\{\gamma_s = 0\}} |R_s|^2 ds \right\|_\infty \quad \text{and} \quad \alpha_1 = \left\| \int_0^{+\infty} 1_{\{\gamma_s \neq 0\}} \frac{R_s}{\gamma_s} ds \right\|_\infty.$$

Then we have

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi\left(W(\beta_1^2) + k\tilde{N}(\alpha_1/k)\right)\right],$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

ii) Assume that $\gamma_u R_u \leq k$, *dPdt-a.e., for some* $k > 0$ *and*

$$\beta_2^2 = \left\| \int_0^{+\infty} 1_{\{\gamma_s = 0\}} |R_s|^2 ds \right\|_\infty \quad \text{and} \quad \alpha_2^2 = \left\| \int_0^{+\infty} 1_{\{\gamma_s \neq 0\}} |R_s|^2 ds \right\|_\infty.$$

Then for all convex functions ϕ *with a convex derivative* ϕ', *we have*

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}\left[\phi\left(W(\beta_2^2) + k\tilde{N}(\alpha_2^2/k^2)\right)\right].$$

iii) Assume that $\gamma_u R_u \leq 0$ *and let*

$$\beta_3^2 = \left\| \int_0^{+\infty} |R_s|^2 ds \right\|_\infty.$$

Then for all convex functions ϕ *with a convex derivative* ϕ', *we have*

$$\mathbb{E}[\phi(F - \mathbb{E}[F])] \leq \mathbb{E}[\phi(\hat{W}(\beta_3^2))].$$

Let now

$$D : L^2(\Omega, \mathcal{F}, P) \mapsto L^2(\Omega \times [0, T], dP \times dt)$$

denote the annihilation operator on multiple stochastic integrals defined as $D_t = I_n(f_n) = nI_n(f_n(*, t))$, $t \in \mathbb{R}_+$. The Clark formula for normal martingales [11] provides a predictable representation for $F \in \mathrm{Dom(D)} \subset \mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P})$, which can be used in Theorem 7.2:

$$F = \mathbb{E}[F] + \int_0^{+\infty} \mathbb{E}[D_t F | \mathcal{F}_t] dZ_t,$$

where $\mathcal{F}_t = \sigma(Z_s, 0 \leq s \leq t)$.

8 Appendix

In this section we prove the Itô type change of variable formula for forward/backward martingales which has been used in the proofs of Theorem 3.2 and Theorem 3.3. Assume that $(\Omega, \mathcal{F}, P)$ is equipped with an increasing filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and a decreasing filtration $(\mathcal{F}^*_t)_{t\in\mathbb{R}_+}$.

Theorem 8.1. *Consider $(M_t)_{t\in\mathbb{R}_+}$ an $\mathcal{F}^*_t$-adapted, $\mathcal{F}_t$-forward martingale with right-continuous paths and left limits, and $(M^*_t)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t$-adapted, $\mathcal{F}^*_t$-backward martingale with left-continuous paths and right limits, whose characteristics have the form* (3.3) *and* (3.4). *For all $f \in C^2(\mathbb{R}^2, \mathbb{R})$ we have*

$$\begin{aligned}
& f(M_t, M^*_t) - f(M_0, M^*_0) \\
&= \int_{0^+}^t \frac{\partial f}{\partial x_1}(M_{u^-}, M^*_u)dM_u + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_1^2}(M_u, M^*_u)d\langle M^c, M^c\rangle_u \\
&\quad + \sum_{0<u\leq t}\left(f(M_u, M^*_u) - f(M_{u^-}, M^*_u) - \Delta M_u \frac{\partial f}{\partial x_1}(M_{u^-}, M^*_u)\right) \\
&\quad - \int_0^{t^-} \frac{\partial f}{\partial x_2}(M_u, M^*_{u^+})d^*M^*_u - \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_2^2}(M_u, M^*_u)d\langle M^{*c}, M^{*c}\rangle_u \\
&\quad - \sum_{0\leq u<t}\left(f(M_u, M^*_u) - f(M_u, M^*_{u^+}) - \Delta M^*_u \frac{\partial f}{\partial x_2}(M_u, M^*_{u^+})\right),
\end{aligned}$$

where d^* denotes the backward Itô differential and $(M^c_t)_{t\in\mathbb{R}_+}$, $(M^{*c}_t)_{t\in\mathbb{R}_+}$ respectively denote the continuous parts of $(M_t)_{t\in\mathbb{R}_+}$, $(M^*_t)_{t\in\mathbb{R}_+}$.

Proof. We adapt the arguments of Theorem 32 of Chapter II in [18], using here the following version of Taylor's formula:

$$\begin{aligned}
f(y_1, y_2) - f(x_1, x_2) &= f(y_1, y_2) - f(y_1, x_2) + f(y_1, x_2) - f(x_1, x_2) \qquad (8.1)\\
&= (y_1 - x_1)\frac{\partial f}{\partial x_1}(x_1, x_2) + \frac{1}{2}(y_1 - x_1)^2\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) \\
&\quad +(y_2 - x_2)\frac{\partial f}{\partial x_2}(y_1, y_2) - \frac{1}{2}(y_2 - x_2)^2\frac{\partial^2 f}{\partial x_2^2}(y_1, y_2) \\
&\quad +R(x, y),
\end{aligned}$$

where $R(x, y) \leq o(|y - x|^2)$. Assume first that $(M_s)_{s\in[0,t]}$ and $(M^*_s)_{s\in[0,t]}$ take their values in a bounded interval, and let $\{0 = t^n_0 \leq t^n_1 \leq \cdots \leq t^n_{k_n} = t\}$, $n \geq 1$, be a refining sequence of partitions of $[0, t]$ tending to the identity. As in [18], for any $\varepsilon > 0$, consider $A_{\varepsilon,t}$, $B_{\varepsilon,t}$ two random subsets of $[0, t]$ such that

i) $A_{\varepsilon,t}$ is finite, P-a.s.,

ii) $A_{\varepsilon,t} \cup B_{\varepsilon,t}$ exhausts the jumps of $(M_s)_{s\in[0,t]}$ and $(M^*_s)_{s\in[0,t]}$,

iii) $\sum_{s\in B_{\varepsilon,t}} |\Delta M_s|^2 + |\Delta^* M^*_s|^2 \leq \varepsilon^2$,

iv) for each $1 \leq i \leq n$, exactly one of the two sets $A_{\varepsilon,t}\cap(t^n_{i-1}, t^n_i]$ or $B_{\varepsilon,t}\cap(t^n_{i-1}, t^n_i]$ is non-empty, P-a.s.

We have

$$\begin{aligned} f(M_t, M_t^*) - f(M_0, M_0^*) &= \sum_{A_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} f(M_{t_i^n}, M_{t_i^n}^*) - f(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad + \sum_{B_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} f(M_{t_i^n}, M_{t_i^n}^*) - f(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*), \end{aligned}$$

and from Taylor's formula (8.1) we get

$$\begin{aligned} & f(M_t, M_t^*) - f(M_0, M_0^*) \\ &= \sum_{A_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} f(M_{t_i^n}, M_{t_i^n}^*) - f(M_{t_i^n}, M_{t_{i-1}^n}^*) + f(M_{t_i^n}, M_{t_{i-1}^n}^*) - f(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad + \sum_{B_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} f(M_{t_i^n}, M_{t_i^n}^*) - f(M_{t_i^n}, M_{t_{i-1}^n}^*) + f(M_{t_i^n}, M_{t_{i-1}^n}^*) - f(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &= \sum_{A_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} f(M_{t_i^n}, M_{t_i^n}^*) - f(M_{t_i^n}, M_{t_{i-1}^n}^*) + f(M_{t_i^n}, M_{t_{i-1}^n}^*) - f(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad + \sum_{B_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} (M_{t_i^n} - M_{t_{i-1}^n}) \frac{\partial f}{\partial x_1}(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) + \frac{1}{2} |M_{t_i^n} - M_{t_{i-1}^n}|^2 \frac{\partial^2 f}{\partial x_1^2}(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad + \sum_{B_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} (M_{t_i^n}^* - M_{t_{i-1}^n}^*) \frac{\partial f}{\partial x_2}(M_{t_i^n}, M_{t_i^n}^*) - \frac{1}{2} |M_{t_i^n}^* - M_{t_{i-1}^n}^*|^2 \frac{\partial^2 f}{\partial x_2^2}(M_{t_i^n}, M_{t_i^n}^*) \\ &\quad + \sum_{B_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} R(M_{t_i^n}, M_{t_i^n}^*, M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &= \sum_{A_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} f(M_{t_i^n}, M_{t_i^n}^*) - f(M_{t_i^n}, M_{t_{i-1}^n}^*) + f(M_{t_i^n}, M_{t_{i-1}^n}^*) - f(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad + \sum_{i=1}^{n} (M_{t_i^n} - M_{t_{i-1}^n}) \frac{\partial f}{\partial x_1}(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) + \frac{1}{2} |M_{t_i^n} - M_{t_{i-1}^n}|^2 \frac{\partial^2 f}{\partial x_1^2}(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad - \sum_{A_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} (M_{t_i^n} - M_{t_{i-1}^n}) \frac{\partial f}{\partial x_1}(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) + \frac{1}{2} |M_{t_i^n} - M_{t_{i-1}^n}|^2 \frac{\partial^2 f}{\partial x_1^2}(M_{t_{i-1}^n}, M_{t_{i-1}^n}^*) \\ &\quad + \sum_{i=1}^{n} (M_{t_i^n}^* - M_{t_{i-1}^n}^*) \frac{\partial f}{\partial x_2}(M_{t_i^n}, M_{t_i^n}^*) - \frac{1}{2} |M_{t_i^n}^* - M_{t_{i-1}^n}^*|^2 \frac{\partial^2 f}{\partial x_2^2}(M_{t_i^n}, M_{t_i^n}^*) \\ &\quad - \sum_{A_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} (M_{t_i^n}^* - M_{t_{i-1}^n}^*) \frac{\partial f}{\partial x_2}(M_{t_i^n}, M_{t_i^n}^*) - \frac{1}{2} |M_{t_i^n}^* - M_{t_{i-1}^n}^*|^2 \frac{\partial^2 f}{\partial x_2^2}(M_{t_i^n}, M_{t_i^n}^*) \\ &\quad + \sum_{B_{\varepsilon,t} \cap (t_{i-1}^n, t_i^n] \neq \emptyset} R(M_{t_i^n}, M_{t_i^n}^*, M_{t_{i-1}^n}, M_{t_{i-1}^n}^*). \end{aligned}$$

By the same arguments as in [18] and from conditions (3.1) and (3.2), letting n tend to infinity we get

$f(M_t, M_t^*) - f(M_0, M_0^*)$

$$
\begin{aligned}
= & \sum_{u\in A_{\varepsilon,t}} \left(f(M_u,M_u^*) - f(M_{u^-},M_u^*) - \Delta M_u \frac{\partial f}{\partial x_1}(M_{u^-},M_u^*) - \frac{1}{2}|\Delta M_u|^2 \frac{\partial^2 f}{\partial x_1^2}(M_{u^-},M_u^*) \right) \\
& - \sum_{u\in A_{\varepsilon,t}} \left(f(M_u,M_{u^+}^*) - f(M_u,M_u^*) - \Delta^* M_u^* \frac{\partial f}{\partial x_2}(M_u,M_{u^+}^*) + \frac{1}{2}|\Delta^* M_u^*|^2 \frac{\partial^2 f}{\partial x_2^2}(M_u,M_{u^+}^*) \right) \\
& + \int_0^t \frac{\partial f}{\partial x_1}(M_{u^-},M_u^*) dM_u + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_1^2}(M_{u^-},M_u^*) d[M,M]_u \\
& - \int_0^t \frac{\partial f}{\partial x_2}(M_u,M_{u^+}^*) d^* M_u^* - \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_2^2}(M_u,M_{u^+}^*) d[M^*,M^*]_u .
\end{aligned}
$$

Then letting ε tend to 0, the above sum converges to

$$
\begin{aligned}
& f(M_t,M_t^*) - f(M_0,M_0^*) \\
& = \int_{0^+}^t \frac{\partial f}{\partial x_1}(M_{u^-},M_u^*) dM_u + \frac{1}{2}\int_{0^+}^t \frac{\partial^2 f}{\partial x_1^2}(M_{u^-},M_u^*) d[M,M]_u \\
& + \sum_{0<u\le t} \left(f(M_u,M_u^*) - f(M_{u^-},M_u^*) - \Delta M_u \frac{\partial f}{\partial x_1}(M_{u^-},M_u^*) - \frac{1}{2}|\Delta M_u|^2 \frac{\partial^2 f}{\partial x_1^2}(M_{u^-},M_u^*) \right) \\
& - \int_0^{t^-} \frac{\partial f}{\partial x_2}(M_u,M_{u^+}^*) d^* M_u^* - \frac{1}{2}\int_0^{t^-} \frac{\partial^2 f}{\partial x_2^2}(M_u,M_{u^+}^*) d[M^*,M^*]_u \\
& - \sum_{0\le u<t} \left(f(M_u,M_u^*) - f(M_u,M_{u^+}^*) - \Delta^* M_u^* \frac{\partial f}{\partial x_2}(M_u,M_{u^+}^*) + \frac{1}{2}|\Delta^* M_u^*|^2 \frac{\partial^2 f}{\partial x_2^2}(M_u,M_{u^+}^*) \right),
\end{aligned}
$$

which yields

$$
\begin{aligned}
f(M_t,M_t^*) - f(M_0,M_0^*) = & \int_{0^+}^t \frac{\partial f}{\partial x_1}(M_{u^-},M_u^*) dM_u + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_1^2}(M_u,M_u^*) d\langle M^c,M^c\rangle_u \\
& + \sum_{0<u\le t} \left(f(M_u,M_u^*) - f(M_{u^-},M_u^*) - \Delta M_u \frac{\partial f}{\partial x_1}(M_{u^-},M_u^*) \right) \\
& - \int_0^{t^-} \frac{\partial f}{\partial x_2}(M_u,M_{u^+}^*) d^* M_u^* - \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_2^2}(M_u,M_u^*) d\langle M^{*c},M^{*c}\rangle_u \\
& - \sum_{0\le u<t} \left(f(M_u,M_u^*) - f(M_u,M_{u^+}^*) - \Delta^* M_u^* \frac{\partial f}{\partial x_2}(M_u,M_{u^+}^*) \right),
\end{aligned}
$$

where the integral with respect to $(\langle M^*,M^*\rangle_t)_{t\in\mathbb{R}_+}$ is defined as a Stieltjes integral with respect to a (not necessarily $\mathcal{F}_t^*$-adapted) increasing process. In the general case, define the stopping times

$$
R_m = \inf\{u\in[0,t] \ : \ |M_u|\ge m\}, \quad \text{and} \quad R_m^* = \sup\{u\in[0,t] \ : \ |M_u^*|\ge m\}.
$$

The stopped process $(M_{u\wedge R_m}, M^*_{u\vee R_m^*})_{u\in[0,t]}$ is bounded by $2m$ and since Itô's formula is valid for $(X_u^{R_m})_{u\in[0,t]}$ for each m, it is also valid for $(X_u)_{u\in\mathbb{R}_+}$. $\square$

Note that the cross partial derivative $\dfrac{\partial^2 f}{\partial x_1 \partial x_2}(M_u,M_u^*)$ does not appear in the formula and there is no need to consider or define a bracket of the form $d\langle M,M^*\rangle_t$.

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