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Large deviations and isoperimetry over convex probability measures with heavy tails *

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Abstract

Large deviations and isoperimetric inequalities are considered for probability distributions, satisfying convexity conditions of the Brunn-Minkowski-type.

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1 Introduction

A Radon probability measure μ on a locally convex space L is called κ-concave, where $-\infty \leq \kappa \leq 1$, if for all Borel subsets A, B of L with positive measure and all $t \in (0,1)$,

$$\mu_*(tA + (1-t)B) \;\geq\; (t\mu(A)^\kappa + (1-t)\mu(B)^\kappa)^{1/\kappa}, \tag{1.1}$$

where $tA + (1-t)B = \{ta + (1-t)b : a \in A,\, b \in B\}$ denotes the Minkowski average, and where μ_* stands for the inner measure. The mean power function

$$M_\kappa(u,v) = M_\kappa^{(t)}(u,v) = (tu^\kappa + (1-t)v^\kappa)^{1/\kappa}, \qquad u,v \geq 0,$$

appearing on the right hand side of (1.1), is understood as $u^t v^{1-t}$ when $\kappa = 0$, and as $\min\{u,v\}$ when $\kappa = -\infty$. Inequality (1.1) is getting stronger, as κ increases, so the case $\kappa = -\infty$ is the weakest one, describing the largest class in the hierarchy of the so-called convex probability measures (according to C. Borell's terminology, or hyperbolic measures, according to V. D. Milman).

In view of the homogeneity of M, the definition (1.1) also makes sense without the normalizing assumption $\mu(L) = 1$. For example, the Lebesgue measure on $L = \mathbf{R}^n$ or its restriction to an arbitrary convex set of positive Lebesgue measure is $\frac{1}{n}$- concave (by the Brunn-Minkowski inequality). The case $\kappa = 0$ corresponds to the family of log-concave measures. The n-dimensional Cauchy distribution is κ-concave for $\kappa = -1$. Actually, many interesting multidimensional (or infinite dimensional) distributions are κ-concave with $\kappa < 0$, cf. (1) for more examples.

The dimension-free parameter κ may be viewed as the one characterizing the strength of convexity. A full description and comprehensive study of basic properties of κ-concave probability distributions was performed by C. Borell (1; 2); cf. also H. J. Brascamp and E. H. Lieb (12). As it turns out, such measures inherit a number of interesting properties from Gaussian measures, such as 0-1 law, integrability of (functions of) norms, absolute continuity of distributions of norms, etc. In this note we consider some geometric properties of κ-concave probability measures, which allow one to study large deviations of functionals from a wide class including arbitrary norms and polynomials. As one of the purposes, we try to unify a number of results, essentially known in the log-concave case, and to explore the role of κ in various inequalities.

For simplicity, we always assume L is finite-dimensional, although many dimension-free properties of such measures can easily be extended to infinite dimensional spaces (perhaps, under further mild assumptions about the space L). With every Borel measurable function f on L we associate its "modulus of regularity"

$$\delta_f(\varepsilon) = \sup_{x,y \in L} \mathrm{mes}\{t \in (0,1) : |\, f(tx + (1-t)y)\,| \leq \varepsilon\, |f(x)|\}, \quad 0 \leq \varepsilon \leq 1, \tag{1.2}$$

where mes stands for the Lebesgue measure. The behaviour of δ_f near zero is connected with probabilities of large and small deviations of f. Moreover, the corresponding inequalities can be made independent of μ. This may be seen from the following:

Theorem 1.1. *Let f be a Borel measurable function on L, and let m be a median for $|f|$ with respect to a κ-concave probability measure μ on L, $\kappa < 0$. For all $h \geq 1$,*

$$\mu\{|f| > mh\} \;\leq\; C_\kappa\, \delta_f(1/h)^{-1/\kappa}, \tag{1.3}$$

where the constant C_k depends on k, only.

This inequality may further be refined to reflect a correct behaviour as $\kappa \to 0$, cf. Theorem 5.2 below. In the limit log-concave case $\kappa = 0$, the refined form yields an exponential bound

$$\mu\{|f| > mh\} \le \exp\left\{-\frac{c}{\delta_f(1/h)}\right\} \tag{1.4}$$

with some universal $c > 0$.

For example, $\delta_f(\varepsilon) \le 2\varepsilon$ for any norm $f(x) = \|x\|$, and then (1.3)-(1.4) correspond to a well-known result of C. Borell (1). In the case of polynomials of degree at most d, one has $\delta_f(\varepsilon) \le 2d\,\varepsilon^{1/d}$, and then (1.4) represents a slightly improved version of a theorem of J. Bourgain (11), who studied Khinchin-type inequalities over high-dimensional convex bodies. In the general log-concave case, the bound (1.4) was recently obtained by different methods by F. Nazarov, M. Sodin and A. Volberg (25) and by the author in (9). While the approach of (25) is based on the bisection technique, here we follow (9) to extend the transportation argument, going back to the works of H. Knothe (17) and J. Bourgain (11).

In the second part of this note, we consider isoperimetric and analytic inequalities and derive, in particular:

Theorem 1.2. *Let μ be a non-degenerate κ-concave probability measure μ on $\mathbf{R}^n$, $-\infty < \kappa \le 1$. For any Borel set A in $\mathbf{R}^n$,*

$$\mu^+(A) \ge \frac{c(\kappa)}{m}\,(\min\{\mu(A), 1-\mu(A)\})^{1-\kappa}, \tag{1.5}$$

where m is the μ-median of the Euclidean norm $x \to |x|$, and where $c = c(\kappa)$ is a positive continuous function in the range $(-\infty, 1]$.

Here $\mu^+(A)$ stands for the μ-perimeter of A, defined by

$$\mu^+(A) = \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\,\mu\{x \in \mathbf{R}^n \setminus A : |x-a| < \varepsilon, \text{ for some } a \in A\}.$$

Up to the factor $\frac{c}{m}$, the right hand side of (1.5) describes the asymptotically worst possible behaviour with respect to $\mu(A)$.

In the log-concave case, the median m is equivalent to the mean $\int |x|\,d\mu(x)$, and (1.5) turns into the Cheeger-type isoperimetric inequality, first obtained by R. Kannan, L. Lovász and M. Simonovits for the uniform distribution μ in an arbitrary convex body K (up to a universal multiplicative factor, cf. (16)). In that case, $m = m(K)$, the so-called volume radius of K, may considerably be smaller than the diameter of the body. Actually, when $\kappa = \frac{1}{n}$, the inequality (1.5) is stronger than the Cheeger-type and resembles the usual isoperimetric inequality for the Lebesgue measure. The general log-concave case in (1.5) was treated in (8) with a different functional argument, which was afterwards modified and pushed forward by F. Barthe (4).

Theorem 1.2 may equivalently be formulated in terms of large deviations of Lipschitz functions under the measure μ. It may also be related to various analytic inequalities. For example, in case $\kappa < 0$ it follows from (1.5) that

$$\left(\int |f|^q \, d\mu \right)^{1/q} \leq C_{q,\kappa} \, m \int |\nabla f| \, d\mu, \qquad 0 < q < \frac{1}{1-\kappa}, \tag{1.6}$$

for all locally Lipschitz functions f on $\mathbf{R}^n$ with μ-median zero. Of independent interest are also weak Poincaré inequalities, which perfectly reflect important properties of probability measures with heavy tails.

The paper is organized as follows. In section 2 we recall Borell's characterization of the κ-concavity and consider separately the one-dimensional case. Some useful results about triangular maps are collected in section 3. They are used in section 4 to derive a geometric inequality of dilation-type for κ-concave probability measures. Theorem 1.1 is proved in section 5, together with some refinements. Section 6 deals with Khinchin-type inequalities and the problem on small deviations. In section 7, we derive Theorem 1.2. Related analytic inequalities are discussed in section 8. Finally, in section 9 general convex measures with a compact support are shown to share a Poincaré-type inequality of L. E. Payne and H. F. Weinberger.

2 Characterizations of convex measures

As was shown by C. Borell (1; 2), any convex probability measure on $\mathbf{R}^n$ has an affine supporting subspace L, where it is absolutely continuous with respect to Lebesgue measure on L. For any κ-concave measure μ, it is necessary that $\kappa \leq \frac{1}{\dim(L)}$, unless μ is a delta-measure. More precisely, when $L = \mathbf{R}^n$ (this may be assumed in further considerations), the following characterization holds.

Lemma 2.1. *Let $-\infty \leq \kappa \leq \frac{1}{n}$. An absolutely continuous probability measure μ on $\mathbf{R}^n$ is κ-concave if and only if it is concentrated on an open convex set K in $\mathbf{R}^n$ and has there a positive density p, which is $\kappa(n)$-concave in the sense that, for all $t \in (0,1)$ and $x, y \in K$,*

$$p(tx + (1-t)y) \geq M_{\kappa(n)}(p(x), p(y)), \quad \textit{where} \quad \kappa(n) = \frac{\kappa}{1 - \kappa n}. \tag{2.1}$$

In particular, p must be continuous on the supporting set K.

The family of all full-dimensional convex probability measures μ on $\mathbf{R}^n$ is described by (2.1) with $\kappa(n) = -\frac{1}{n}$.

The case $\kappa = \frac{1}{n}$ is only possible when p is constant, i.e., when K is bounded and μ is the uniform distribution in K.

If $\kappa > 0$, μ has to be compactly supported. For $\kappa = 0$, the density function must decay exponentially fast at infinity, that is, for some positive C and c, we have $p(x) \leq Ce^{-c|x|}$, for all $x \in K$. If $\kappa < 0$, the density admits the bound

$$p(x) \leq \frac{C}{1 + |x|^{\alpha+n}}, \qquad x \in K, \quad \alpha = -\frac{1}{\kappa}. \tag{2.2}$$

To see this, define the function p to be zero outside K. By Lemma 2.1, the sets of the form $K(\lambda) = \{x \in K : p(x) > \lambda\}$ are convex. Since $1 \geq \int_{K(\lambda)} p \geq \lambda\,\mathrm{mes}(K(\lambda))$, they are bounded for all $\lambda > 0$. Hence, $p(x) \to 0$, as $|x| \to +\infty$.

Secondly, p is bounded. For, in the other case, pick a sequence $x_\ell \in K$, such that $p(x_\ell) \uparrow +\infty$, as $\ell \to \infty$. Since $K(\lambda)$ are bounded, we may assume x_ℓ has a limit point $x_0 \in \mathrm{clos}(K)$. Let $x_0 = 0$ (without loss of generality). By (2.1) with $x = x_\ell$ and $s = 1 - t$, we have $p(tx_\ell + sy) \geq M_{-1/n}(p(x_\ell), p(y)) \uparrow s^{-n} p(y)$, so $p(sy) \geq s^{-n} p(y)$, for any $y \in K$ and $s \in (0,1)$. But $\int p(sy)\,dy = s^{-n} \int p(y)\,dy$, which is only possible when $p(sy) = s^{-n} p(y)$. The latter would imply that p is not integrable.

Finally, put $A = \sup_x p(x)$ with the assumption that the sup is asymptotically attained at $x_0 = 0$. By the first step, choose $r > 0$ large enough so that $p(x) \leq \frac{1}{2} A$, whenever $|x| \geq r$. By Lemma 2.1, the function $g(x) = (\frac{p(x)}{A})^{\kappa(n)}$ is convex, $g(x) \to 1$, as $x \to 0$, $x \in K$, and $g(x) \geq 2^{-\kappa(n)}$ for $|x| \geq r$. Hence, if $|x| = r$ and $\lambda x \in K$ with $\lambda \geq 1$, we have $g(\lambda x) - 1 \geq \lambda\,(2^{-\kappa(n)} - 1)$. This yields (2.2).

This argument also shows that p is compactly supported in the case $\kappa > 0$. Then the function g is concave, so, for $\lambda \geq 1$, it satisfies $g(\lambda x) \leq 1 - \lambda(1 - g(x)) \leq 1 - \lambda\,(1 - 2^{-\kappa(n)})$, where $|x| = r$ and $\lambda x \in K$, as before. Since $g \geq 0$, necessarily $\lambda \leq \lambda_0 = \frac{1}{1 - 2^{-\kappa(n)}}$, and this means that K is contained in the Euclidean ball of radius at most $\lambda_0 r$.

In dimension 1, one can complement Lemma 2.1 with another characterization of the κ-concavity, which may be useful for the study of isoperimetric and large deviations inequalities. Let a probability measure μ be concentrated on some finite or infinite interval of the real line, say (a,b), and have there a positive, continuous density p. With it, we associate the function $I(t) = p(F^{-1}(t))$, defined in $0 < t < 1$, where $F^{-1} : (0,1) \to (a,b)$ denotes the inverse of the distribution function $F(x) = \mu((-\infty, x])$, $a < x < b$. Up to shifts, the correspondence $\mu \to I$ is one-to-one between the family of all such measures μ and the family of all positive, continuous functions I on $(0,1)$. If the median of μ is at the origin, then the measure may uniquely be determined via the associated function by virtue of the relation $F^{-1}(t) = \int_{1/2}^t \frac{ds}{I(s)}$.

Lemma 2.2. *A non-degenerate probability measure μ on the real line is κ-concave, $-\infty < \kappa < 1$, if and only if its associated function I is such that $I^{1/(1-\kappa)}$ is concave. The measure is convex, if and only if the function* $\log(I)$ *is concave on* $(0,1)$.

When $\kappa = 0$, we obtain a well-known characterization of log-concave probability measures on the line ((7), Proposition A.1). The general case may easily be treated with the help of Lemma 2.1 and the identity

$$\kappa\,(I^{1/(1-\kappa)})'(F(x)) = (p(x)^{\kappa/(1-\kappa)})', \quad \kappa \neq 0.$$

For an example, let us start with $\kappa \leq 1$ and define a symmetric probability measure μ_κ on the real line by requiring that its associated function is $I_\kappa(t) = (\min\{t, 1-t\})^{1-\kappa}$. By Lemma 2.2, μ_κ is κ-concave. Its distribution function is given by

$$1 - F_\kappa(x) = \frac{1}{(2^{-\kappa} - \kappa x)^{-1/\kappa}}, \quad x \geq 0. \tag{2.3}$$

When $0 < \kappa \leq 1$, μ_k is supported on the finite interval $[-\frac{2^{-\kappa}}{k}, \frac{2^{-\kappa}}{k}]$. If $\kappa = 1$, we obtain a uniform distribution on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

When $-\infty < \kappa \leq 0$, μ_k is not supported on a finite interval. If $\kappa = 0$, we obtain the two-sided exponential distribution with density $p(x) = \frac{1}{2}\, e^{-|x|}$. If $\kappa < 0$, the tails $1 - F_\kappa(x)$ behave at infinity like $\frac{1}{x^\alpha}$, $\alpha = -\frac{1}{\kappa}$.

Let us note that, for any two probability measures μ and ν on the real line, having positive, continuous densities on their supporting intervals, the corresponding associated functions I and J satisfy $I \geq cJ$ $(c > 0)$, if and only if the (unique) increasing map T, which pushes forward ν into μ, has a Lipschitz semi-norm $\|T\|_{\rm Lip} \leq \frac{1}{c}$. This observation can be used when μ is κ-concave and $\nu = \mu_k$. By Lemma 2.2, $I^{1/(1-\kappa)}$ is concave, so $I^{1/(1-\kappa)}(t) \geq 2\, I^{1/(1-\kappa)}(\frac{1}{2})\, \min\{t, 1-t\}$, for all $t \in (0,1)$. Equivalently, $I(t) \geq 2^{1-\kappa} p(m)\, I_\kappa(t)$, where p is density of μ and m is its median. Hence:

Corollary 2.3. *Let μ be a κ-concave probability measure μ on the real line with density p and median m $(-\infty < \kappa \leq 1)$. Then μ represents the image of the measure μ_k under a Lipschitz map T with $\|T\|_{\rm Lip} \leq \frac{1}{2^{1-\kappa} p(m)}$.*

A similar observation will be made in the n-dimensional case, cf. Corollary 8.1.

3 Triangular maps

Here we recall definitions and sketch a few facts about triangular maps that will be needed for the proof of Theorem 1.1. In the convex body case such maps were used by H. Knothe (17) to reach certain generalizations of the Brunn-Minkowski inequality.

A map $T = (T_1, \dots, T_n) : A \to \mathbf{R}^n$ defined on an open non-empty set A in $\mathbf{R}^n$ is called triangular, if its components are of the form

$$T_i = T_i(x_1, \dots, x_i), \qquad x = (x_1, \dots, x_n) \in A.$$

It is called increasing, if every component T_i is a (strictly) increasing function with respect to the x_i-coordinate while the other coordinates are fixed. It is easy to see that, if the triangular map T is continuous and increasing on the open set A, then the image $B = T(A)$ is open, and T represents a homeomorphism between A and B.

Lemma 3.1. *Let $T = (T_1, \dots, T_n)$ be a continuous, increasing triangular map, defined on an open set A, such that every component T_i has a continuous positive partial derivative $\frac{\partial T_i}{\partial x_i}$. Then, for every integrable function f on $\mathbf{R}^n$,*

$$\int_A f(T(x)) J(x)\, dx = \int_{T(A)} f(y)\, dy, \tag{3.1}$$

where $J(x) = \prod_{i=1}^n \frac{\partial T_i(x)}{\partial x_i}$.

Equality (3.1) is a slight generalization of the usual change of the variable formula. The point of the generalization is that T is not required to be C^1-smooth. Nevertheless, one may still use $J(x)$ as a "generalized" Jacobian.

As a consequence, assume we have two absolutely continuous probability measures P and Q concentrated on A and $B = T(A)$ and having there densities p and q, respectively. If they are related by the equality

$$p(x) = q(T(x))J(x), \tag{3.2}$$

holding almost everywhere on A, then T must push forward P to Q. Indeed, applying (3.1) to $f(y) = g(y)q(y)$ with Q-integrable g and using (3.2), we would get that

$$\int g\,dQ = \int_A g(T(x))\,q(T(x))J(x)\,dx = \int_A g(T(x))p(x)\,dx = \int g(T(x))\,dP(x).$$

That is, $Q = PT^{-1}$ or $Q = T^{\#}(P)$.

Conversely, starting from P and Q, one may ask whether Q may be obtained from P as the image of T as above. The existence of a triangular map T satisfying (3.2) and with properties as in Lemma 3.1 requires, however, certain properties of P and Q. We say that a probability measure P concentrated on an open set A in $\mathbf{R}^n$ is regular, if it has a density p which is positive and continuous on A, and for each $i \leq n-1$, the integrals

$$p_i(x) = \int_{\mathbf{R}^{n-i}} p(x_1, \dots, x_i, u_{i+1}, \dots, u_n)\,du_{i+1} \dots du_n,$$

$$\int_{-\infty}^{x_i} \int_{\mathbf{R}^{n-i}} p(x_1, \dots, x_{i-1}, u_i, \dots, u_n)\,du_i \dots du_n$$

represent continuous functions on the projection $A_i = \{x \in \mathbf{R}^i : \exists u \in \mathbf{R}^{n-i},\ (x,u) \in A\}$.

Lemma 3.2. *Let P and Q be regular probability measures supported on an open set A and on an open convex set B in $\mathbf{R}^n$ with densities p and q, respectively. There exists a unique increasing, continuous, triangular map $T : A \to B$, which pushes forward P to Q. Moreover:*

a) the partial derivatives $\frac{\partial T_i}{\partial x_i}$ are positive and continuous on A_i;

b) for all $x \in A$, we have $p(x) = q(T(x))J(x)$, where $J(x) = \prod_{i=1}^n \frac{\partial T_i(x)}{\partial x_i}$.

If A is convex, then $T(A) = B$, so T is bijective. In the general case, $B \setminus T(A)$ may be non-empty, but has Lebesgue measure zero.

The components $T_i = T_i(x_1, \dots, x_i)$, $1 \leq i \leq n$, of the map T can be constructed recursively via the relation for the conditional probabilities

$$\mathbf{P}\{X_i \leq x_i \,|\, X_1 = x_1, \dots, X_{i-1} = x_{i-1}\} = \mathbf{P}\{Y_i \leq T_i \,|\, Y_1 = T_1, \dots, Y_{i-1} = T_{i-1}\}, \tag{3.3}$$

where $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ are random vectors in $\mathbf{R}^n$ with the distributions P and Q, respectively (when $i = 1$ these probabilities are unconditional). Thus, the (unique, increasing, continuous) function $x_i \to T_i(x_1, \dots, x_i)$ is defined as the one, transporting the conditional distribution of X_i given fixed $x_1, \dots, x_{i-1}$ to the conditional distribution of Y_i given

$y_1 = T_1, \dots, y_{i-1} = T_{i-1}$, like in dimension one. This is where convexity of B and the regularity assumption are needed to ensure continuity of T and to justify derivation of the properties $a) - b)$ in Lemma 3.2, cf. (9) for more details. Differentiating the equality (3.3) with respect to x_i leads to

$$\frac{p_i(x)}{p_{i-1}(x)} = \frac{q_i(T(x))}{q_{i-1}(T(x))}\,\frac{\partial T_i(x)}{\partial x_i}, \qquad i = 1, \dots, n,$$

with the convention that $p_0 = q_0 = 1$. Multiplying these relations by each other, we arrive eventually at $p(x) = q(T(x))J(x)$.

To give some examples of regular measures in the above sense, for every set A in $\mathbf{R}^n$, together with the projections A_i consider its sections $A_{x_1,\dots,x_i} = \{u \in \mathbf{R}^{n-i} : (x_1, \dots, x_i, u) \in A\}$. Let us say that A is regular, if for all $i \leq n-1$ and for all $(x_1, \dots, x_i) \in A_i$, the section $(\partial A)_{x_1,\dots,x_i}$ of the boundary of A has the $(n-i)$-dimensional Lebesgue measure zero. An arbitrary open convex set or the union of finitely many open Euclidean balls in $\mathbf{R}^n$ represent regular sets. A straightforward application of the Lebesgue dominated convergence theorem gives the following sufficient condition.

Lemma 3.3. *If the probability measure P is concentrated on a open regular set A in $\mathbf{R}^n$ and has there a continuous density p, such that $\int_{\mathbf{R}^{n-i}} \sup_{x \in A_i} p(x,u)\,du < \infty$, $i = 1, \dots, n-1$, then it is regular.*

Without regularity assumption on the supporting set, we might be lead to certain singularity problems, so that part of the conclusions in Lemma 3.2 may fail. For example, the uniform distribution P on $A = (0,2) \times (0,1) \cup (0,1) \times (0,2) \subset \mathbf{R}^2$ is not regular. In this case, the density p_1 of the first coordinate is discontinuous at $x_1 = 1$. But continuity is necessary for the property $a)$ in Lemma 3.2.

In the next section we apply Lemmas 3.1-3.3 to κ-concave measures μ restricted to regular subsets of $\mathbf{R}^n$.

4 Dilation

Given a Borel subset F of a (Borel) convex set K in $\mathbf{R}^n$ and a number $\delta \in [0,1]$, define

$$F_\delta = \left\{ x \in K : \frac{\mathrm{mes}(F \cap \Delta)}{\mathrm{mes}(\Delta)} \geq 1 - \delta, \text{ for any interval } \Delta, \text{ such that } x \in \Delta \subset K \right\}.$$

We use mes to denote the one-dimensional Lebesgue measure on the (non-degenarate, closed) interval $\Delta \subset \mathbf{R}^n$. In the definition the requirement that $x \in \Delta$ may be replaced with "x is an endpoint of Δ". Hence, the complement of F_δ in K may be represented as

$$K \setminus F_\delta = \{x \in K : \varphi(x,y) < 1 - \delta, \text{ for some } y \in K,\ y \neq x\},$$

where $\varphi(x,y) = \int_0^1 1_F(tx + (1-t)y)\,dt$. The latter function is Borel measurable on $K \times K$, so $K \setminus F_\delta$ may be viewed as the x-projection of a Borel set in $\mathbf{R}^n \times \mathbf{R}^n$. Therefore, the set F_δ is universally measurable, and we may freely speak about its measure. Note also, if F is closed, then F_δ is a closed subset of F $(0 < \delta < 1)$.

To illustrate the δ-operation, let us take a centrally symmetric, open, convex set $B \subset K$ and put $F = K \setminus B$. Then $F_\delta = K \setminus (\frac{2}{\delta} - 1)B$, for any $\delta \in (0,1)$, which is the complement to the dilated set B. So, relations between $\mu(F)$ and $\mu(F_\delta)$ belong to the family of inequalities of dilation-type.

As a basic step, we prove:

Theorem 4.1. *Let μ be a κ-concave probability measure supported in a convex set $K \subset \mathbf{R}^n$, where $-\infty < \kappa \le 1$. Let $0 \le \delta \le 1$, $0 \le c \le 1 - \delta$, and $t = \frac{\delta}{1-c}$. For any Borel subset F of K, such that $\mu(F_\delta) > 0$,*

$$\mu(F) \,\ge\, c\,(t\,\mu(F_\delta)^\kappa + (1-t))^{1/\kappa}\,. \tag{4.1}$$

In the case $\kappa \le 0$, the right-hand side of (4.1) is vanishing when $\mu(F_\delta) = 0$, so the requirement $\mu(F_\delta) > 0$ may be ignored. When $\kappa = 0$, (4.1) reads as

$$\mu(F) \ge c\mu(F_\delta)^{\delta/(1-c)}, \quad 0 \le c \le 1-\delta. \tag{4.2}$$

In an equivalent functional form, which we discuss in the next section, the above inequality was obtained in (9). Actually, (4.2) can be improved to

$$\mu(F) \ge \mu(F_\delta)^\delta. \tag{4.3}$$

This is a correct relation obtained by F. Nazarov, M. Sodin and A. Volberg (25) in the spirit of localization results due to R. Kannan, L. Lovász and M. Simonovits (23), (16), with technique, going back to the bisection method of L. E. Payne and H. F. Weinberger (26) (cf. also (14)). This approach may be used to get many other sharp geometric inequalities for log-concave probability distributions. Although (4.2) is somewhat weaker, we do not know whether the more general inequality (4.1) can be sharpened in a similar manner as

$$\mu(F) \,\ge\, (\delta\,\mu(F_\delta)^\kappa + (1-\delta))^{1/\kappa}\,.$$

The refinement may improve absolute constants in some applications, e.g., the constant C_κ in Theorem 1.1, which does not seem crucial. In some others, the refinement would be desirable. For example, $\sqrt{\delta_f(\varepsilon)}$ could be then replaced with $\delta_f(\varepsilon)$ in Theorem 6.1.

Let us also mention that in the log-concave case (4.2) implies (4.3) for convex F. Indeed, one may automatically sharpen (4.2) by applying it to the product measure μ^N on $\mathbf{R}^{nN}$ and the product set $F^N = F \times \dots \times F$. Then with respect to K^N, we have $(F^N)_\delta = (F_\delta)^N$, so that $\mu(F)^N \ge c\mu(F_\delta)^{N\delta/(1-c)}$, by (4.2). Letting first $N \to \infty$ and then $c \to 0$ yields (4.3).

In the non-convex case, we only have the inclusion $(F^N)_\delta \subset (F_\delta)^N$, and we do not know whether the above argument still works. However, it is to be emphasized that necessarily $\delta < 1$ in the refined form (4.3). For, when $\delta = 1$, we have $F_\delta = K$, which is larger than F. Hence, the factor c may not be removed in (4.1)-(4.2) with our definition of the δ-operation.

In the proof of Theorem 4.1, we use the following elementary lemma, known as (a partial case of) the generalized Hölder inequality.

Lemma 4.2. *$M_{\kappa(n)}(u_1, v_1)\, M_{1/n}(u_2, v_2) \ge M_k(u_1 u_2, v_1 v_2)$, whenever $-\infty \le \kappa \le \frac{1}{n}$, $0 < t < 1$, and $u_1, u_2, v_1, v_2 \ge 0$.*

Proof of Theorem 4.1. We may assume μ is full-dimensional, that is, absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^n$. In particular, $\kappa \leq \frac{1}{n}$. Moreover, the supporting convex set K may be assumed to be open, and that μ has there a positive, $\kappa(n)$-concave density p.

When $\delta = 1$, necessarily $c = 0$, and (4.1) is immediate. When $\delta = 0$, since F_δ is non-empty, necessarily $\mu(F) = 1$, and (4.2) is obvious. Indeed, taking a point $x \in F_\delta$, we get $\int_0^1 1_F(tx + (1-t)y)\,dt = 1$, for all $y \in K$. Integrating this equality over y with respect to some (any) absolutely continuous probability measure on K, having a positive density, we obtain that $\nu(F) = 1$, for some absolutely continuous probability measure ν on K. This implies $\mu(K \setminus F) = 0$.

Now, let $0 < \delta < 1$. Take an open neighborhood G of F, containing in K, and take a regular subset A of K, such that $\mu(A \cap F_\delta) > 0$ (e.g., a finite union of open balls). Denote by μ_A the normalized restriction of μ to the set A. By Lemmas 3.2–3.3 and by the boundedness of p (cf. (2.2)), there is a continuous triangular map $T : A \to K$, which pushes forward $P = \mu_A$ to the measure $Q = \mu$. Moreover, the components $T_i = T_i(x_1, \dots, x_i)$ of T are C^1-smooth with respect to the x_i-coordinates and satisfy $\frac{\partial T_i}{\partial x_i} > 0$, so that the Jacobian $J(x) = \prod_{i=1}^n \frac{\partial T_i(x)}{\partial x_i}$ is continuous and positive on A. Since μ_A has the density $p_A(x) = \frac{p(x)}{\mu(A)}$, the property $b)$ of Lemma 3.2 becomes

$$\frac{p(x)}{\mu(A)} = p(T(x))\,J(x), \quad x \in A. \tag{4.4}$$

Now, for each $t \in (0,1)$, introduce the map $T_t(x) = tx + (1-t)T(x)$, $x \in A$, which is also continuous, triangular, with components that are C^1-smooth with respect to x_i-coordinates. Moreover, for all $x \in A$, its "generalized" Jacobian $J_t(x)$ satisfies

$$J_t(x) = \prod_{i=1}^n \left(t + (1-t)\,\frac{\partial T_i(x)}{\partial x_i}\right) \geq \left(t + (1-t)J(x)^{1/n}\right)^n = M_{1/n}(1, J(x)), \tag{4.5}$$

where we applied an elementary inequality

$$\prod_{i=1}^n (ta_i + (1-t)b_i)^{1/n} \geq t\prod_{i=1}^n a_i^{1/n} + (1-t)\prod_{i=1}^n b_i^{1/n}, \quad a_i, b_i \geq 0.$$

Consider an open set $B(t) = \{x \in A : T_t(x) \in G\}$ and its image $D(t) = T_t(B(t))$. By the definition, $D(t) \subset G$, so $\mu(D(t)) \leq \mu(G)$. On the other hand, by Lemma 3.1 applied to T_t,

$$\mu(D(t)) = \int_{D(t)} p(y)\,dy = \int_{B(t)} p(T_t(x))\,J_t(x)\,dx. \tag{4.6}$$

Since p is $\kappa(n)$-concave (Lemma 2.1), $p(T_t(x)) \geq M_{\kappa(n)}(p(x), p(T(x)))$. Moreover, combining (4.5) with the inequality of Lemma 4.2, we get that

$$\begin{aligned} p(T_t(x))\,J_t(x) \ &\geq\ M_{\kappa(n)}(p(x), p(T(x)))\,M_{1/n}(1, J(x)) \\ &\geq\ M_\kappa(p(x), p(T(x))J(x)) = M_\kappa\bigg(p(x), \frac{p(x)}{\mu(A)}\bigg) = p(x)M_\kappa\bigg(1, \frac{1}{\mu(A)}\bigg), \end{aligned}$$

where we made use of (4.4) and of the homogeneity of the function M_κ. Plugging this bound into (4.6), we get $\mu(D(t)) \geq M_\kappa(1, \frac{1}{\mu(A)})\,\mu(B(t))$, so,

$$\mu(G)\ \geq\ M_\kappa\bigg(1, \frac{1}{\mu(A)}\bigg)\,\mu(B(t)). \tag{4.7}$$

Now, we need a lower bound on the last term in (4.7). By the definition of the δ-operation, for all $x \in G_\delta$ and $y \in K$, $\mathrm{mes}\{t \in (0,1) : tx + (1-t)y \in G\} \geq 1 - \delta$. Hence, with $y = T(x)$, for any $x \in A \cap G_\delta$,

$$\mathrm{mes}\{t \in (0,1) : T_t(x) \in G\} \geq 1 - \delta,$$

or equivalently $\int_0^1 1_{\{T_t(x) \in G\}}\, dt \geq 1 - \delta$. Integrating this inequality over the set $A \cap G_\delta$ with respect to the normalized restriction ν of μ and interchanging the integrals gives

$$\int_0^1 \nu(B(t) \cap G_\delta)\, dt \geq 1 - \delta. \tag{4.8}$$

Therefore, the function $\psi(t) = \nu(B(t) \cap G_\delta)$ satisfies $\int_0^1 \psi(t)\, dt \geq 1 - \delta$. On the other hand, it is bounded by 1. This actually implies that $\psi(t) \geq c$, for some $t \in (0, t_0]$, where $t_0 = \frac{\delta}{1-c} \in (0,1]$. Indeed, assuming that $\psi(t) < c$ in $0 < t \leq t_0$, we would get that

$$\int_0^1 \psi(t)\, dt = \int_0^{t_0} \psi(t)\, dt + \int_{t_0}^1 \psi(t)\, dt < ct_0 + (1 - t_0) = 1 - \delta.$$

But this contradicts to (4.8). We may therefore conclude that

$$\frac{\mu(B(t) \cap G_\delta)}{\mu(A \cap G_\delta)} = \nu(B(t) \cap G_\delta) \geq c, \quad \text{for some} \quad t \in (0, t_0],$$

and thus $\mu(B(t)) \geq \mu(B(t) \cap G_\delta) \geq c\mu(A \cap G_\delta) \geq c\mu(A \cap F_\delta)$. Recalling (4.7), we arrive at the bound

$$\mu(G) \,\geq\, cM_\kappa^{(t)}\Big(\mu(A \cap F_\delta), \frac{\mu(A \cap F_\delta)}{\mu(A)}\Big). \tag{4.9}$$

Note that the function $t \to M_\kappa^{(t)}(u,v)$ is non-increasing for $u \leq v$, so (4.9) also holds with $t = t_0$. Finally, letting $G \downarrow F$ and approximating F_δ with regular A's, so that $\mu(A \cap F_\delta) > 0$ and $\mu((A \setminus F_\delta) \cup (F_\delta \setminus A)) \to 0$, we obtain (4.1).

Theorem 4.1 is proved.

5 Theorem 1.1 and refinements

Let f be a Borel measurable function on $\mathbf{R}^n$. By the very definition of the δ_f-function, $\{x \in \mathbf{R}^n : |f(x)| \geq \lambda\} \subset F_\delta$ with $\delta = \delta_f(\varepsilon)$, $0 \leq \varepsilon \leq 1$, $\lambda \geq 0$, $K = \mathbf{R}^n$, where

$$F = \{x \in \mathbf{R}^n : |f(x)| > \lambda\varepsilon\}.$$

Indeed, given that $|f(x)| \geq \lambda$, $|f(tx + (1-t)y)| \leq \lambda\varepsilon$ implies $|f(tx + (1-t)y)| \leq \varepsilon|f(x)|$, so

$$\mathrm{mes}\{t \in (0,1) : |f(tx + (1-t)y)| \leq \lambda\varepsilon\} \;\leq$$

$$\mathrm{mes}\{t \in (0,1) : |f(tx + (1-t)y)| \leq \varepsilon|f(x)|\} \;\leq\; \delta.$$

Hence, $\mathrm{mes}\{t \in (0,1) : |f(tx + (1-t)y)| > \lambda\varepsilon\} \geq 1 - \delta$, which means that $x \in F_\delta$.

As a result, Theorem 4.1 gives:

Theorem 5.1. *Let μ be a κ-concave probability measure on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$. Let f be a Borel measurable function on $\mathbf{R}^n$, and $0 \leq \lambda < \text{ess sup}\,|f|$. Then, for all $\varepsilon \in [0,1]$ and $c \in [0, 1-\delta_f(\varepsilon)]$,*

$$\mu\{|f| > \lambda\varepsilon\} \ \geq\ c\,(t\,\mu\{|f| \geq \lambda\}^\kappa + (1-t))^{1/\kappa}, \tag{5.1}$$

where $t = \frac{\delta_f(\varepsilon)}{1-c}$.

If $\kappa \leq 0$, the assumption $\lambda < \operatorname{ess\,sup}|f|$ may be removed.

If μ is supported on a convex set K in $\mathbf{R}^n$, one may also apply (5.1) to functions defined on K (rather than on the whole space). Then in the definition (1.2) of δ_f the supremum should be taken over all points x, y in K.

In fact, (5.1) represents an equivalent functional form for (4.1). To see this, let F be a closed subset of the supporting convex set K of μ and let $\mu(F_\delta) > 0$ $(0 < \delta < 1)$. We have $F_\delta \subset F \subset K$. Define

$$f(x) = \begin{cases} 1, & \text{if} \quad x \in F_\delta, \\ \frac{1}{2}, & \text{if} \quad x \in F \setminus F_\delta, \\ \frac{1}{4}, & \text{if} \quad x \in K \setminus F. \end{cases}$$

Put $\Delta_\varepsilon(x,y) = \{t \in (0,1) : f(tx + (1-t)y) > \varepsilon f(x)\}$. Then, if $\frac{1}{4} \leq \varepsilon < \frac{1}{2}$ and $y \in K$,

$$\Delta_\varepsilon(x,y) = \begin{cases} t \in (0,1) : tx + (1-t)y \in F\}, & \text{if} \quad x \in F_\delta, \\ (0,1), & \text{if} \quad x \in K \setminus F_\delta. \end{cases}$$

By the definition of F_δ, $1-\delta_f(\varepsilon) = \inf_x \inf_y \operatorname{mes}(\Delta_\varepsilon(x,y)) \geq 1-\delta$, or equivalently, $\delta_f(\varepsilon) \leq \delta$. Applying (5.1) with $\lambda = 1$ and with an arbitrary value $\varepsilon \in [\frac{1}{4}, \frac{1}{2})$, we get $\mu(F) \geq cM_\kappa^{(t(\varepsilon))}(\mu(F_\delta), 1)$ with $t(\varepsilon) = \frac{\delta_f(\varepsilon)}{1-c}$, $0 < c \leq 1 - \delta_f(\varepsilon)$. Since $\delta_f(\varepsilon) \leq \delta$ and since the function $t \to M_\kappa^{(t)}(u,v)$ is non-increasing for $u \leq v$, we arrive at the desired inequality $\mu(F) \geq cM_\kappa^{(t)}(\mu(F_\delta), 1)$ with $t = \frac{\delta}{1-c}$, $0 < c \leq 1 - \delta$, i.e., (4.1).

An attempt to choose an optimal c in (5.1) complicates this inequality and in essense does not give an improvement. For applications, at the expense of some loss in constants, one may use Theorem 5.1 with $c = \frac{3}{4}$, for example.

Theorem 5.2. *Let f be a Borel measurable function on $\mathbf{R}^n$, and let m be a median for $|f|$ with respect to a κ-concave probability measure μ on $\mathbf{R}^n$, $\kappa < 0$. Then, for all $h \geq 1$, such that $4\delta_f(\frac{1}{h}) \leq 1$, up to some numerical constant $C > 0$,*

$$\mu\{|f| > mh\} \ \leq\ \left[1 + \frac{C}{\delta_f(\frac{1}{h})\,\alpha}\right]^{-\alpha}, \qquad \alpha = -\frac{1}{\kappa}. \tag{5.2}$$

If $\kappa \to 0$, in the limit we obtain an exponential bound

$$\mu\{|f| > mh\} \ \leq\ e^{-C/\delta_f(\frac{1}{h})}, \tag{5.3}$$

holding true for all log-concave probability measures μ.

Note the right-hand side of (5.1) does not exceed $(\frac{\alpha}{C})^\alpha \delta_f(\frac{1}{h})^\alpha$, and we arrive at Theorem 1.1 with $C_\kappa = \max\{4^\alpha, (\frac{\alpha}{C})^\alpha\}$. Indeed, in case $4\delta_f(\frac{1}{h}) \geq 1$, inequality (1.3) holds true with constant 4^α.

Proof of Theorem 5.2. Define the tail function $u(\lambda) = \mu\{|f| > \lambda\}$, $\lambda \geq 0$. Given $\varepsilon \in (0,1]$, let $\delta = \delta_f(\varepsilon)$. If $\lambda\varepsilon = m$, then $u(\lambda\varepsilon) \leq \frac{1}{2}$, and Theorem 4.1 yields, for any $c \in (0, 1-\delta]$,

$$\frac{1}{2} \geq c\,(tu(\lambda)^\kappa + (1-t))^{1/\kappa}, \quad t = \frac{\delta}{1-c}.$$

Equivalently, raising to the negative power k, we have $(2c)^{-\kappa} \leq tu(\lambda)^\kappa + (1-t)$, or

$$u(\lambda)^\kappa \geq \frac{(2c)^{-\kappa} - (1-t)}{t} = 1 + \frac{\psi(c)}{\delta},$$

where $\psi(c) = (1-c)((2c)^{-\kappa} - 1)$. We apply this with $\lambda = mh$ and $\varepsilon = \frac{1}{h}$. Since $\delta \leq \frac{1}{4}$, we may take $c = \frac{3}{4}$. In this case, $\psi(c) = \frac{1}{4}\,(e^{-\kappa \log(3/2)} - 1) \geq -\kappa\,\frac{\log(3/2)}{4}$. Hence, $u(\lambda)^\kappa \geq 1 - \kappa\,\frac{\log 1.5}{4\delta} > 1 - \frac{\kappa}{10\,\delta}$ and $u(\lambda) \leq (1 - \frac{\kappa}{10\,\delta})^{1/\kappa}$. This is the desired estimate (5.2) with $C = \frac{1}{10}$.

Note the inequality (1.3) of Theorem 1.1 is useless, when $\delta_f(\varepsilon)$ is not getting small, as ε approaches zero. In fact, a bound similar to (1.3) can also be obtained in case $\lim_{\varepsilon \downarrow 0} \delta_f(\varepsilon) > 0$.

Theorem 5.3. *For each $\kappa < 0$, there is a constant $\delta(\kappa) \in (0,1]$ with the following property. Let f be a Borel measurable function on $\mathbf{R}^n$ such that $\delta_f(\varepsilon_0) \leq \delta_0 < \delta(\kappa)$, for some $\varepsilon_0 \in (0,1)$. Then, with respect to any κ-concave probability measure μ on $\mathbf{R}^n$,*

$$\mu\{|f| > mh\} \leq Ch^{-\beta}, \qquad h \geq 1, \tag{5.4}$$

where $m > 0$ is a μ-median for $|f|$, and where C and β are positive constants depending on κ, ε_0, and δ_0, only.

Proof. With notations as in proof of Theorem 5.2, let $v = u^\kappa$. Then by (5.1), for all $\lambda > 0$ and $c \in (0, 1-\delta_0)$, we have $v(\lambda\varepsilon_0) \leq av(\lambda) + b$ with $a = c^\kappa t_0$, $b = c^\kappa(1-t_0)$, and $t_0 = \frac{\delta_0}{1-c}$. The repeated use of this inequality leads to

$$v(\lambda\varepsilon_0^i) \leq a^i v(\lambda) + \frac{1-a^i}{1-a}\,b, \quad i \geq 1 \text{ (integer)}. \tag{5.5}$$

If $\delta(\kappa) > 0$ is small enough and $0 < \delta_0 \leq \delta(\kappa)$, the value $c = 1 - 2\delta_0$ satisfies $a = b = \frac{c^\kappa}{2} < 1$ and $\frac{b}{1-a} < 2^{-\kappa}$. Choosing $\lambda = \frac{m}{\varepsilon_0^i}$, we have $v(\lambda\varepsilon_0^i) = u(m)^\kappa \geq 2^{-\kappa}$, so (5.5) gives $C \equiv 2^{-\kappa} - \frac{b}{1-a} \leq a^i v(\lambda)$, that is, $u(m\varepsilon_0^{-i}) \leq C^{1/\kappa} a^{-i/\kappa}$. Since the integer $i \geq 1$ is arbitrary, the latter easily yields (5.4) with a different C.

Remark. In the log-concave case, a similar argument, based on the limiting inequality (5.3), yields $\delta(0) = 0$. More precisely (cf. (8)), if $\delta_f(\varepsilon_0) \leq \delta_0$, for some $\varepsilon_0 \in (0,1)$ and $\delta_0 \in (0,1)$, then

$$\mu\{|f| > mh\} \leq C\,e^{-ch^r}, \quad h \geq 1, \tag{5.6}$$

with positive numbers C, c, r, depending on $(\varepsilon_0, \delta_0)$, only. The power r appearing in (5.6) can be chosen as close to the number $r_0 = \frac{\log(1/\delta_0)}{\log(1/\varepsilon_0)}$, as we wish. In particular, f has finite μ-moments $\mathbf{E}_\mu |f|^q = \int |f|^q \, d\mu$ of any order $q > 0$, and, moreover, for all $0 < r < r_0$,

$$\int \exp\{|f|^r\} \, d\mu < +\infty.$$

For polynomials f with respect to the uniform distribution μ over an arbitrary convex body K in $\mathbf{R}^n$, such results were first obtained by J. Bourgain (11). In this case $\kappa = \frac{1}{n}$ is positive, and one may hope to get an additional information about the distribution of f in terms of κ and δ_f (where the dependence on the dimension might be hidden in κ). Indeed, by Theorem 5.1 with $\kappa > 0$ and $c = \frac{1-\delta}{2}$, $\delta = \delta_f(\varepsilon)$,

$$\mu\{|f| > \lambda\varepsilon\} \, \geq \, c\,(1-t)^{1/\kappa} = \frac{(1-\delta)^{1+1/\kappa}}{2\,(1+\delta)^{1/\kappa}},$$

holding true whenever $0 \leq \varepsilon \leq 1$ and $0 \leq \lambda < \operatorname{ess\,sup}|f|$. By Chebyshev's inequality, the left-hand side is bounded by $\mathbf{E}_\mu|f|/(\lambda\varepsilon)$. Letting $\lambda \to \operatorname{ess\,sup}|f|$, we arrive at

$$\operatorname{ess\,sup}|f| \leq \; C\,\mathbf{E}_\mu|f| \quad \text{with} \quad C \, = \, 2 \inf_{0 \leq \varepsilon \leq 1} \frac{(1+\delta_f(\varepsilon))^{1/\kappa}}{\varepsilon\,(1-\delta_f(\varepsilon))^{1+1/\kappa}}.$$

In particular:

Corollary 5.4. *Let μ be a κ-concave probability measure on $\mathbf{R}^n$ with $\kappa > 0$. Then, for any μ-integrable function f on $\mathbf{R}^n$,*

$$\operatorname{ess\,sup}|f| \, \leq \, \frac{C}{\varepsilon_k} \int |f| \, d\mu, \tag{5.7}$$

where $\varepsilon_k \in (0,1]$ is chosen so that $\delta_f(\varepsilon_k) \leq \frac{\kappa}{2}$, and where C is universal.

For example, for any norm $f(x) = \|x\|$, we have $\sup_{x \in K} \|x\| \, \leq \, \frac{C}{\kappa} \int_K \|x\| \, d\mu(x)$, where K is the supporting convex set of μ (which has to be bounded in case $\kappa > 0$). At the expense of the constant on the right, the expectation may be replaced with the μ-median m of f. Precise relations have been studied by O. Guédon (15), who showed that, for any κ-concave probability measure μ on $\mathbf{R}^n$, $0 < \kappa \leq 1$, and any convex symmetric set B,

$$(1-\mu(hB))^\kappa \, \leq \, \max\left\{1 - \frac{h+1}{2}\,(1-(1-\mu(B))^\kappa), \, 0\right\}, \qquad h \geq 1. \tag{5.8}$$

For $\kappa = 0$ the above turns into

$$1 - \mu(hB) \leq (1-\mu(B))^{(h+1)/2}, \tag{5.9}$$

which is due to L. Lovász and M. Simonovits (23) in case of Euclidean balls. If $\kappa > 0$ and $B = B_m(0)$ is the Euclidean ball with center at the origin and radius m, that is, $\mu(B) = \frac{1}{2}$, (5.8) implies that

$$\sup_{x \in K} |x| \, \leq \, \frac{1+2^{-\kappa}}{1-2^{-\kappa}}\, m \, \leq \, \frac{4}{\kappa}\, m. \tag{5.10}$$

6 Small deviations. Khinchin-type inequalities

If $f(x) = \|x\|$ is a non-degenerate finite semi-norm on $\mathbf{R}^n$, it is easy to check that $\delta_f(\varepsilon) = \frac{2\varepsilon}{1+\varepsilon}$. Therefore, for any κ-concave probability measure μ on $\mathbf{R}^n$, $\kappa < 0$, by Theorem 1.1,

$$\mu\{f > mh\} \, \leq \, C_\alpha \, h^{-\alpha}, \qquad \alpha = -\frac{1}{\kappa}, \quad h \geq 1, \tag{6.1}$$

where m is a median for f with respect to μ, and the constant C_α depends on α, only. By a different argument, this inequality was already obtained by C. Borell in (1). A more precise bound, useful when κ is close to zero, is given in Theorem 5.2.

The estimate (6.1) implies that the moments $\|f\|_q = (\int |f|^q \, d\mu)^{1/q}$ are finite for $q < \alpha$ and are equivalent to each other in the sense that

$$\|f\|_q \leq C \|f\|_{q_0}, \quad 0 < q_0 < q < \alpha, \tag{6.2}$$

with constants C depending on q, q_0, and κ. For various applications, it is however crucial to know whether or not it is possible to make C independent of q_0 and thus to involve the case $q_0 = 0$ in (6.2). Note that in general

$$\|f\|_0 = \lim_{q \downarrow 0} \|f\|_q = \exp \int \log |f| \, d\mu$$

represents the geometic mean of $|f|$ (provided that $\|f\|_q < +\infty$, for some $q > 0$). Hence, in order to sharpen (6.2) by replacing $\|f\|_{q_0}$ with $\|f\|_0$ one needs to derive also bounds on small deviations of f. For log-concave probability measures, this question was settled by R. Latala in (18) by showing that $\mu\{\|x\| \leq m\varepsilon\} \leq C\varepsilon$, $0 \leq \varepsilon \leq 1$, for some absolute constant C. Actually, this estimate implies (6.2) for a larger range $q_0 > -1$. A different argument was suggested in (7). Further refinements are due to O. Guédon (15), who also considered the behaviour of constants for κ-concave probablity measures with $\kappa > 0$. While the argument of (15) is based on the localization lemma of L. Lovász and M. Simonovits (23) one can still use the transportation argument to derive (6.2) for a large class of functionals f, including arbitrary norms and polynomials. Namely, from Theorem 5.1 we have the following simple corollary:

Theorem 6.1. *Let f be a Borel measurable function on $\mathbf{R}^n$, and let m be a median for $|f|$ with respect to a κ-concave probability measure μ on $\mathbf{R}^n$, $\kappa \leq 0$. Then,*

$$\mu\{|f| \leq m\varepsilon\} \, \leq \, C_\kappa \sqrt{\delta_f(\varepsilon)}, \qquad 0 < \varepsilon < 1, \tag{6.3}$$

where the constant C_κ depends on κ, only.

For the proof, assume $\kappa < 0$. Using the previous notations $\delta = \delta_f(\varepsilon)$, $t = \frac{\delta}{1-c}$, $0 < c < 1 - \delta$, $\alpha = -\frac{1}{\kappa}$, we obtain from (5.1) with $\lambda = m$ that $\mu\{|f| \leq m\varepsilon\} \leq \varphi(x)$, where $\varphi(x) = 1 - c\,(1+x)^{-\alpha}$ and $x = (2^{-\kappa} - 1)\,t$. Since this function is concave in $x > -1$,

$$\varphi(x) \, \leq \, \varphi(0) + \varphi'(0)x \, \leq \, (1-c) + \alpha x = (1-c) + \frac{\alpha\delta \cdot (2^{-\kappa} - 1)}{1-c}.$$

Optimizing over $c \in (0, 1-\delta]$, we get $\varphi(x) \leq 2\sqrt{\alpha\delta \cdot (2^{-\kappa} - 1)}$, provided that $\delta \leq \alpha\,(2^{-\kappa} - 1)$. In the other case, we may use

$$\mu\{|f| \leq m\varepsilon\} \leq \mu\{|f| < m\} \leq \frac{1}{2} < \alpha\,(2^{-\kappa} - 1) < \sqrt{\alpha\,(2^{-\kappa} - 1)}\ \sqrt{\delta}.$$

Thus, (6.3) follows with $C_\kappa = 2\sqrt{\frac{2^{-\kappa}-1}{-\kappa}}$, $\kappa < 0$, and $C_0 = \lim_{\kappa \to 0} C_\kappa = 2\sqrt{\log 2}$.

Corollary 6.2. *Let f be a Borel measurable function on $\mathbf{R}^n$, such that $\delta_f(\varepsilon) \leq R\,\varepsilon^r$ in $0 < \varepsilon < 1$, for some $R, r > 0$. Then with respect to any κ-concave probability measure μ on $\mathbf{R}^n$, $\kappa \leq 0$,*

$$\|f\|_q \leq C\,\|f\|_0, \quad 0 \leq q < -\frac{r}{\kappa}, \tag{6.4}$$

where the constant C depends on R, r, q, and κ.

Proof. Let m be a μ-median of $|f|$. By Theorem 6.1, since $\delta_f(\varepsilon) \to 0$, as $\varepsilon \to 0$, m must be positive. Let $\alpha = -\frac{1}{\kappa}$, $\kappa < 0$. Introduce the distribution function $F(h) = \mu\{|f| \leq mh\}$. If $0 < q < \alpha r$, we have, by Theorem 1.1 and the assumption on the growth of $\delta_f(\varepsilon)$,

$$\begin{aligned}
\int \left|\frac{f}{m}\right|^q d\mu \ &= \ q\int_0^{+\infty} h^{q-1}(1 - F(h))\,dh \ \leq \ 1 + q\int_1^{+\infty} h^{q-1}(1 - F(h))\,dh \\
&\leq \ 1 + qC_\kappa \int_1^{+\infty} h^{q-1}\delta_f(1/h)^\alpha\,dh \ \leq \ 1 + qC_\kappa R^\alpha \int_1^{+\infty} h^{q-\alpha r-1}\,dh,
\end{aligned}$$

so, $\|f\|_q \leq m\,(1 + C_\kappa R^\alpha\,\frac{q}{\alpha r - q})^{1/q}$. On the other hand, by (6.3),

$$\int \log\left|\frac{f}{m}\right| d\mu \ \geq \ -\int_0^1 \frac{F(\varepsilon)}{\varepsilon}\,d\varepsilon \ \geq \ -C_\kappa\sqrt{R}\int_0^1 \varepsilon^{(r-2)/2}\,d\varepsilon \ = \ -\frac{2C_\kappa\sqrt{R}}{r},$$

so, $m \leq \exp\{\frac{2C_\kappa\sqrt{R}}{r}\}\,\|f\|_0$. The two estimates imply (6.4).

As an example, one may apply Corollary 6.2 to an arbitrary norm or polynomial $f(x_1, \dots, x_n)$ in n real variables of degree at most $d \geq 1$. In the second case, f represents a polynomial on every line of degree at most d, so the maximal possible value of $\delta_f(\varepsilon)$ is determined in dimension one. Writing $f(t) = \prod_{i=1}^d (t - z_i)$ with $z_i \in \mathbf{C}$, $1 \leq i \leq d$, we have with respect to the Lebesgue measure on (0,1) that

$$\mathrm{mes}\{|f(t)| \leq \varepsilon\,|f(0)|\} \ \leq \ \mathrm{mes}\bigcup_{i=1}^d \{|t - z_i| \leq \varepsilon^{1/d}\,|z_i|\} \ \leq \ \sum_{i=1}^d \mathrm{mes}\{|t - z_i| \leq \varepsilon^{1/d}\,|z_i|\}.$$

Since $|t - z_i| \geq |t - |z_i|\,|$, the roots z_i may be assumed to be real non-negative numbers. But, for any $c \in (0,1)$ and $z > 0$, the quantity $\mathrm{mes}\{|t - z| \leq cz\}$ is maximized at $z = \frac{1}{1+c}$ and is equal to $\frac{2c}{1+c}$. This gives $\delta_f(\varepsilon) \leq 2d\,\varepsilon^{1/d}$, that is, the hypothesis of Corollary 6.2 with $R = 2d$ and $r = \frac{1}{d}$. Hence,

$$\|f\|_q \leq C\,\|f\|_0, \quad 0 \leq q < -\frac{1}{d\kappa},$$

with C depending on d, q and κ. For norms, the above estimate holds with $d = 1$.

7 Isoperimetry. Proof of Theorem 1.2

Now, we turn to geometric inequalities, relating μ-perimeter of sets to their μ-measures.

Lemma 7.1. (4) *If μ is a non-degenerate κ-concave probability measure on $\mathbf{R}^n$, for any Borel set A in $\mathbf{R}^n$, any point $x_0 \in \mathbf{R}^n$ and $r > 0$,*

$$2r\,\mu^+(A) \,\geq\, \frac{1 - \big(\mu(A)^{1-\kappa} + (1-\mu(A))^{1-\kappa}\big)\,\mu(B_r(x_0))^{\kappa}}{-\kappa}, \tag{7.1}$$

where $B_r(x_0)$ is the Euclidean ball of radius r with center at x_0.

In the log-concave case ($\kappa = 0$), inequality (7.1) should read as

$$2r\,\mu^+(A) \,\geq\, \mu(A)\log\frac{1}{\mu(A)} + (1-\mu(A))\log\frac{1}{1-\mu(A)} + \log\mu(B_r(x_0)). \tag{7.2}$$

By virtue of Prékopa-Leindler's functional form of the Brunn-Minkowski inequality, (7.2) was derived in (7). The arbitrary κ-concave case was treated by F. Barthe (4), who applied an extension of Prékopa-Leindler's theorem in the form of Borell and Brascamp-Lieb. Inequality (7.1) was used in (4) to study the concept of the isoperimetric dimension for κ-concave measures with $\kappa > 0$.

Without loss of generality, one may state Lemma 7.1 with $x_0 = 0$. To make the proof of Theorem 1.2 to be self-contained, let us give a direct argument for (7.1), not appealing to any functional form. It is based on the following representation for the μ-perimeter, explicitely relating it to measure convexity properties. Namely, let a given probability measure μ on $\mathbf{R}^n$ be absolutely continuous and have a continuous density $p(x)$ on an open supporting convex set, say K. Then, for any sufficiently "nice" set A, e.g., a finite union of closed balls in K or the complement in $\mathbf{R}^n$ to the finite union of such balls,

$$\mu^+(A) \,=\, \lim_{\varepsilon\downarrow 0}\frac{\mu((1-\varepsilon)A + \varepsilon B_r) + \mu((1-\varepsilon)\bar{A} + \varepsilon B_r) - 1}{2r\varepsilon}\,, \tag{7.3}$$

where B_r is the Euclidean ball of radius r with center at $x_0 = 0$. Indeed, the μ-perimeter is described as the $(n-1)$-dimensional integral

$$\mu^+(A) \,=\, \lim_{\varepsilon\downarrow 0}\frac{\mu(A + \varepsilon B_1) - \mu(A)}{\varepsilon} \,=\, \int_{\partial A} p(x)\,d\mathcal{H}_{n-1}(x)$$

with respect to Lebesgue measure $\mathcal{H}_{n-1}$ on the boundary ∂A of the set A. Moreover, for $\mathcal{H}_{n-1}$-almost all points x in ∂A, the outer normal unit vector $n_A(x)$ at x is well-defined, and for any $r > 0$,

$$\lim_{\varepsilon\downarrow 0}\frac{\mu((1-\varepsilon)A + \varepsilon B_r) - \mu(A)}{\varepsilon} \,=\, r\mu^+(A) - \int_{\partial A}\langle n_A(x), x\rangle\,p(x)\,d\mathcal{H}_{n-1}(x).$$

The complement $\bar{A} = \mathbf{R}^n \setminus A$ has the same boundary and the same μ-perimeter $\mu^+(\bar{A})$, and we have $n_{\bar{A}}(x) = -n_A(x)$. Hence, for $\bar{A}$ the above relation reads as

$$\lim_{\varepsilon\downarrow 0}\frac{\mu((1-\varepsilon)\bar{A} + \varepsilon B_r) - \mu(\bar{A})}{\varepsilon} \,=\, r\mu^+(A) + \int_{\partial A}\langle n_A(x), x\rangle\,p(x)\,d\mathcal{H}_{n-1}(x).$$

Summing the two relations, we arrive at (7.3).

In the case of a κ-concave μ, it remains to apply in (7.3) the original convexity property (1.1), and then we obtain (7.1).

Remark. More generally, one could start from a Brunn-Minkowski-type inequality of the form

$$\mu((1-\varepsilon)A+\varepsilon B) \,\geq\, F((1-\varepsilon)F^{-1}(\mu(A))+\varepsilon F^{-1}(\mu(B))), \qquad 0<\varepsilon<1,$$

where $F:(\alpha,\beta)\to(0,1)$ is a C^1-smooth monotone function with inverse F^{-1} and where we assume $0<\mu(A),\mu(B)<1$. Then, a similar argument leads to

$$\begin{aligned} 2r\,\mu^+(A) \;\geq\; & I(t)\,(F^{-1}(\mu(B_r))-F^{-1}(t)) \,+ \\ & I(1-t)\,(F^{-1}(\mu(B_r))-F^{-1}(1-t)) \end{aligned} \tag{7.4}$$

with $t=\mu(A)$ and $I(t)=F'(F^{-1}(t))$. The case of a κ-concave μ corresponds to $F(u)=u^{1/\kappa}$, and then (7.4) reduces to (7.1). When F represents the distribution function of some symmetric probability measure on the line, (7.4) simplifies to

$$\mu^+(A) \,\geq\, I(t)\,\frac{F^{-1}(\mu(B_r))}{r}. \tag{7.5}$$

For example, the famous Ehrhard's inequality states that the standard Gaussian measure μ on $\mathbf{R}^n$ satisfies the Brunn-Minkowski-type inequality with the standard normal distribution function $F=\Phi$. In this case, as easy to check, $\frac{F^{-1}(\mu(B_r))}{r}\to 1$, as $r\to+\infty$, for any fixed n, and (7.5) becomes the Gaussian isoperimetric inequality

$$\mu^+(A)\geq\Phi'(\Phi^{-1}(\mu(A))). \tag{7.6}$$

That the Ehrhard inequality may directly be used to derive (7.6) in its "integral" form, $\mu(A+B_r)\geq\Phi(\Phi^{-1}(\mu(A))+r)$, was noted by M. Ledoux (19).

Proof of Theorem 1.2.

In terms of the isoperimetric function $I_\mu(t)=\inf_{\mu(A)=t}\mu^+(A)$, associated to the given κ-concave probability measure μ, inequality (1.5) may be written as

$$I_\mu(t) \,\geq\, \frac{c(\kappa)}{m}\,(\min\{\,t,1-t\})^{1-\kappa}, \qquad 0<t<1,$$

where $c=c(\kappa)$ is a positive, continuous function, defined on the half-axis $\kappa\leq 1$. Recall that m is defined to be a real number, such that $\mu\{x:|x|\leq m\}=\frac{1}{2}$.

By symmetry, let $0<t\leq\frac{1}{2}$. Introduce the tail function $u(h)=\mu\{x:|x|>mh\}$, $h\geq 1$. Choosing $x_0=0$ and $r=mh$ in Lemma 7.1, one may write inequality (7.1) as

$$2m\,I_\mu(t) \,\geq\, \psi(t,h)\equiv\frac{1-\left(t^{1-\kappa}+(1-t)^{1-\kappa}\right)(1-u(h))^\kappa}{-\kappa h}. \tag{7.7}$$

Thus, our task is to properly estimate the right hand side of (7.7) by choosing somewhat optimal h. It will be conventient to consider separately the two cases.

Case $\kappa<0$. Let $R=(1-u(h))^\kappa$ so that $R\geq 1$. For the enumerator we have

$$\begin{aligned}
1-(t^{1-\kappa}+(1-t)^{1-\kappa})R &= \big[1-(t^{1-\kappa}+(1-t)^{1-\kappa})\big]-(t^{1-\kappa}+(1-t)^{1-\kappa})(R-1)\\
&\geq \big[1-(t^{1-\kappa}+(1-t)^{1-\kappa})\big]-(R-1)\\
&= t(1-t^{-\kappa})+((1-t)-(1-t)^{1-\kappa})-(R-1)\\
&\geq t(1-t^{-\kappa})+(1-2^{\kappa})\,t-(R-1),
\end{aligned}$$

where, applying the convexity/concavity argument, we have used the simple bounds $t^{1-\kappa}+(1-t)^{1-\kappa}\leq 1$ and $(1-t)-(1-t)^{1-\kappa}\geq(1-2^{\kappa})t$. Thus,

$$\psi(t,h)\ \geq\ \frac{t(1-t^{-\kappa})}{-\kappa h}+\frac{(1-2^{\kappa})\,t-(R-1)}{-\kappa h}. \tag{7.8}$$

Now, for h large enough, the last term in (7.8) will be non-negative. Indeed, write the inequality $R-1\leq(1-2^{\kappa})t$ as $1-u(h)\geq g(t)\equiv(1+(1-2^{\kappa})\,t)^{1/\kappa}$. Since g is convex in t and $g(0)=1$, $g(2^{-\kappa})=\frac{1}{2}$, we have $g(t)\leq 1-2^{\kappa-1}t$ in the interval $0\leq t\leq 2^{-\kappa}$. Therefore, the last term in (7.8) is non-negative, as long as

$$u(h)\leq 2^{\kappa-1}t. \tag{7.9}$$

It is time to involve Theorem 5.2. As was already mentioned, the function $f(x)=|x|$ has $\delta_f(\varepsilon)=\frac{2\varepsilon}{1+\varepsilon}\leq 2\varepsilon$. Hence, $4\delta_f(1/h)\leq 1$ for $h\geq 8$, and according to inequality (5.1) with $C=\frac{1}{10}$ (as obtained in the proof), $u(h)\leq(1-\kappa h/20)^{1/\kappa}$. Hence, (7.9) is fulfilled, when $1-\frac{\kappa h}{20}\geq 2^{\kappa^2-\kappa}t^{\kappa}$, that is, when

$$h\geq h_\kappa(t)=20\,\frac{2^{\kappa^2-\kappa}t^{\kappa}-1}{-\kappa}.$$

Note also that in the interval $0<t\leq\frac{1}{2}$, we have $h_\kappa(t)\geq 20\,\frac{2^{\kappa^2-2\kappa}-1}{-\kappa}>40\log 2>8$. Therefore, (7.8) gives

$$\psi(t,h_\kappa(t))\ \geq\ \frac{t(1-t^{-\kappa})}{-\kappa h_\kappa}=\frac{t^{1-\kappa}}{20}\ \frac{t^{\kappa}-1}{2^{\kappa^2-\kappa}t^{\kappa}-1}\geq\frac{2^{-\kappa}-1}{20\,(2^{\kappa^2-2\kappa}-1)}\,t^{1-\kappa}.$$

Thus, the inequality (7.7) with $h=h_\kappa(t)$ yields Theorem 1.2 with $c(\kappa)=\frac{2^{-\kappa}-1}{40\,(2^{\kappa^2-2\kappa}-1)}$, which is positive and continuous, with limit $\lim_{\kappa\uparrow 0}c(\kappa)=\frac{1}{80}$.

Case $0<\kappa<1$. Note that $R\leq 1$. Write (7.7) once more as

$$2m\,I_\mu(t)\ \geq\ \psi(t,h)=\frac{\big(t^{1-\kappa}+(1-t)^{1-\kappa}\big)\,R-1}{\kappa h}. \tag{7.10}$$

For the enumerator we have

$$\begin{aligned}
(t^{1-\kappa}+(1-t)^{1-\kappa})R-1 &= \big[(t^{1-\kappa}+(1-t)^{1-\kappa})-1\big]-(t^{1-\kappa}+(1-t)^{1-\kappa})(1-R)\\
&\geq \big[(t^{1-\kappa}+(1-t)^{1-\kappa})-1\big]-2^{\kappa}(1-R)\\
&= t(t^{-\kappa}-1)+((1-t)^{1-\kappa}-(1-t))-2^{\kappa}(1-R)\\
&\geq t(t^{-\kappa}-1)+(2^{\kappa}-1)\,t-2^{\kappa}(1-R),
\end{aligned}$$

where we have used the bound $t^{1-\kappa} + (1-t)^{1-\kappa} \leq 2^\kappa$ and, applying the concavity argument, the bound $(1-t)^{1-\kappa} - (1-t) \geq (2^\kappa - 1)t$. Thus,

$$\psi(t,h) \, \geq \, t^{1-\kappa}\, \frac{1-t^\kappa}{\kappa h} + \frac{(2^\kappa-1)\, t - 2^\kappa(1-R)}{\kappa h}. \tag{7.11}$$

Now, in order to make the last term in (7.11) be non-negative, we need to require that $1-u(h) \geq \left(1-(1-2^{-\kappa})\, t\right)^{1/\kappa}$. Since the latter may be bounded from above by $1 - \frac{1}{2}\, t$ in the interval $0 \leq t \leq 1$ uniformly over all $\kappa \in (0,1]$, it suffices to require that

$$u(h) \leq \frac{1}{2}\, t. \tag{7.12}$$

Take $h_\kappa(t) = C\, \frac{1-t^\kappa}{\kappa}$ with a positive absolute constant C to be specified later on. Then, under (7.12), the bound (7.11) yields $\psi(t,h_\kappa(t)) \geq \frac{1}{C}\, t^{1-\kappa}$, and we would be done. First suppose $t^\kappa \leq \frac{1}{2}$. Then $h_\kappa(t) \geq \frac{C}{2\kappa}$ and, by Guédon's estimate (5.10), we have $u(h_\kappa(t)) = 0$, as long as $C \geq 8$, which will be assumed. In this case, (7.12) is fulfilled automatically. If $t^\kappa \geq \frac{1}{2}$, that is, $\kappa \log \frac{1}{t} \leq \log 2$, one easily derives

$$h_\kappa(t) \geq \frac{C}{2 \log 2}\, \log \frac{1}{t}.$$

Clearly, $h_\kappa(t) \geq \frac{C}{2} \geq 4$. Now, we apply the Lovász-Simonovits estimate (5.9) to get that

$$u(h_\kappa(t)) \leq 2^{-h_\kappa(t)/2} \, \leq \, 2^{-\frac{C}{4 \log 2}\, \log \frac{1}{t}} \, = \, t^{C/4} \, \leq \, t^2 \leq \frac{1}{2}\, t,$$

where we used the assumption $t \leq \frac{1}{2}$ on the last step. Hence, (7.12) is fulfilled, and therefore the choice $C = 8$ serves for all $\kappa \in (0,1]$. Thus, $\psi(t,h_\kappa(t)) \geq \frac{1}{8}\, t^{1-\kappa}$, and (7.10) with $h = h_\kappa(t)$ yields Theorem 1.2 with a constant function $c(\kappa)$.

Theorem 1.2 is proved.

8 Functional forms. Sobolev-type inequalities

Theorem 1.2 admits the following equivalent formulation, involving the one-dimensional κ-concave probability measures μ_κ with the distribution functions F_κ given in (2.3). As before, let μ be an arbitrary κ-concave probability measure on $\mathbf{R}^n$, $-\infty < \kappa \leq 1$, and let m denote the μ-median for the Euclidean norm, viewed as a random variable on the probability space $(\mathbf{R}^n, \mu)$.

Corollary 8.1. *For any Lipschitz function f on $\mathbf{R}^n$ with $\|f\|_{\rm Lip} \leq 1$, its distribution under μ represents the image of μ_k under a non-decreasing map $T : \mathbf{R} \rightarrow \mathbf{R}$, such that $\|T\|_{\rm Lip} \leq \frac{m}{c(\kappa)}$.*

Let us recall the argument, which is standard. The isoperimetric inequality (1.5) can be "integrated" over the parameter $h > 0$ to yield

$$\mu(A + hB_1) \geq F_\kappa(F_\kappa^{-1}(\mu(A)) + ch), \qquad c = \frac{c(\kappa)}{m}, \tag{8.1}$$

for any Borel set A in $\mathbf{R}^n$, such that $\mu(A) > 0$, where F_κ^{-1} is the inverse function. In turn, letting $h \to 0$ in (8.1), we return to (1.5). For simplicity of notations, let $\kappa \le 0$, so that the supporting interval of μ_k is the whole real line. Given a Lipschitz function f on $\mathbf{R}^n$, the sets $A(a) = \{x : f(x) \le a\}$ satisfy $A(a) + hB_1 \subset A(a+h)$. Hence, (8.1) implies

$$F_\kappa^{-1}(F(a+h)) \ge F_\kappa^{-1}(F(a)) + ch, \qquad h > 0, \tag{8.2}$$

where F is the distribution function of f under μ. Let f be non-constant modulo μ. Then, the interval $\{a : 0 < F(a) < 1\}$ is non-empty, and, by (8.2), F strictly increases on it. Define the "inverse" function by putting

$$F^{-1}(s) = \min\{a \in \mathbf{R} : F(a) \ge s\}, \qquad 0 < s < 1.$$

Since it pushes forward the uniform distribution on (0,1) to the measure with the distribution function F, the map $T = F^{-1}(F_\kappa)$ pushes forward μ_k to the law of f. In terms of $S = F_\kappa^{-1}(F)$, this map may also be defined as $T(x) = \min\{a \in \mathbf{R} : S(a) \ge x\}$, $x \in \mathbf{R}$.

It remains to check that $T(x + ch) \le T(x) + h$, for all $x \in \mathbf{R}$ and $h > 0$. Given $S(a) \ge x$, we need to see that $T(x + ch) \le a + h$. Indeed, by (8.2),

$$\begin{aligned} T(x+ch) &= \min\{b : S(b) \ge x + ch\} \\ &\le \min\{b : S(b) \ge S(a) + ch\} \\ &\le \min\{b : S(b) \ge S(a+h)\} \ \le \ a + h. \end{aligned}$$

Thus, the map T has a Lipschitz semi-norm $\|T\|_{\mathrm{Lip}} \le \frac{1}{c}$, and Corollary 8.1 is proved.

Note that (8.2) is reduced to (8.1) for special Lipschitz functions $f(x) = \mathrm{dist}(A, x)$.

When $\kappa < 0$, it follows from (8.2) that any Lipschitz f with μ-median zero has a distribution with tails satisfying

$$\mu\{f \ge h\} \ \le \ 1 - F_\kappa(ch) \ \le \ \frac{(C_\kappa m)^\alpha}{h^\alpha}, \qquad h > 0, \ \alpha = -\frac{1}{\kappa}.$$

More delicate properties (in comparison with large deviations of Lipschitz functions) may be stated as Sobolev-type inequalities. First, we derive:

Corollary 8.2. *Let $\kappa < 0$ and $0 < q < \frac{1}{1-\kappa}$. For any smooth function f on $\mathbf{R}^n$ with μ-median zero,*

$$\left(\int |f|^q \, d\mu \right)^{1/q} \le Cm \int |\nabla f| \, d\mu, \tag{8.3}$$

where the constant C depends upon q and κ, only.

For the proof we may assume f has a finite Lipschitz semi-norm. First let $f \ge 0$ and $\mu\{f = 0\} \ge \frac{1}{2}$. Applying the generalized coarea inequality (cf. (10), Lemma 3.2) and (1.5), we obtain that

$$\begin{aligned} \int |\nabla f| \, d\mu \ &\ge \ \int_0^{+\infty} \mu^+\{f > h\} \, dh \\ &\ge \ \frac{c(\kappa)}{m} \int_0^{+\infty} \mu\{f > h\}^{1-\kappa} \, dh \ \ge \ \frac{c(\kappa)}{m} \ \sup_{h>0} \left[h \, \mu\{f > h\}^{1-\kappa} \right]. \end{aligned}$$

But for any $f \geq 0$, whenever $0 < q < r < 1$,

$$\int f^q \, d\mu \leq \frac{r}{r-q} \sup_{h>0} \, [h \, \mu\{f > h\}^{1/r}]^q.$$

Applying this to $r = \frac{1}{1-\kappa}$, we arrive at (8.3) with $\frac{1}{C} = c(\kappa)(1 - q(1-\kappa))^{1/q}$. More accurately, a simple approximation yields $\left(\int f^q \, d\mu\right)^{1/q} \leq Cm \int_{\{f>0\}} |\nabla f| \, d\mu$. In the general case, this inequality may be applied to $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$, and then we get (8.3) for f.

When $\kappa \geq 0$, inequality (8.3) is also true with a suitable constant for the critical power $q = \frac{1}{1-\kappa}$, and then it represents an equivalent analytic form for (1.5). However in case $\kappa \leq 0$, when f's approximate indicator functions of Borel sets, (8.3) turns into an isoperimetric inequality, which is close to, but still a little weaker than (1.5). Therefore, it would be interesting to look for other functional forms. As it turns out, there are suitable equivalent analytic inequalities in the class of weak Poincaré-type inequalities for L^1-norm of f. In particular, one may use the following characterization:

Lemma 8.3. *Let μ be a probability measure on $\mathbf{R}^n$. Given $c > 0$ and $p > 1$, the following properties are equivalent:*

a) For all Borel sets A in $\mathbf{R}^n$,

$$\mu^+(A) \geq c \, (\min\{\mu(A), 1 - \mu(A)\})^p.$$

b) For any bounded, smooth function f on $\mathbf{R}^n$ with μ-median zero, and for all $s > 0$,

$$\int |f| \, d\mu \leq \frac{(p-1)^{p-1}}{c \, p^p \, s^{p-1}} \int |\nabla f| \, d\mu + s \operatorname{Osc}_\mu(f).$$

c) For any bounded, smooth function f on $\mathbf{R}^n$ with μ-median zero,

$$\int |f| \, d\mu \leq \frac{1}{c^{1/p}} \left[\int |\nabla f| \, d\mu \right]^{1/p} [\operatorname{Osc}_\mu(f)]^{1-1/p}.$$

Here we use the usual notatation $\operatorname{Osc}_\mu(f) = \operatorname{ess\,sup} f - \operatorname{ess\,inf} f$ for the total oscillation of f with respect to the measure μ.

The statement *c*) follows from *b*) by optimization over $s > 0$, and similarly the converse implication holds. As for the equivalence of *a*) and *b*), one may start more generally from the isoperimetric-type inequality

$$\mu^+(A) \geq I(\mu(A)), \qquad 0 < \mu(A) < 1, \tag{8.4}$$

with a positive, Borel function $I = I(t)$ in $0 < t < 1$, symmetric about $t = \frac{1}{2}$, and look for an optimal function β_1 in

$$\int |f| \, d\mu \leq \beta_1(s) \int |\nabla f| \, d\mu + s \operatorname{Osc}_\mu(f) \tag{8.5}$$

in the same class of functions f as in b). Here, the smoothness requirement may be relaxed to being Lipschitz or locally Lipschitz. Note also, since $\int |f|\, d\mu \le \frac{1}{2}\operatorname{Osc}_\mu(f)$ for any f with median zero, only the values $s \in (0, \frac{1}{2}]$ are of interest in the inequality (8.5).

Again, first assume $f \ge 0$ has a finite Lipschitz seminorm and satisfies $\mu\{f = 0\} \ge \frac{1}{2}$. In view of the homogeneity of (8.5), also let $\operatorname{ess\,sup} f = 1$. An application of the generalized coarea inequality leads to

$$\int_{\{f>0\}} |\nabla f|\, d\mu \ \ge\ \int_0^1 I(\mu\{f > h\})\, dh,$$

while $\int |f|\, d\mu - s\operatorname{Osc}_\mu(f) = \int_0^1 [\mu\{f > h\} - s]\, dh$. Hence, (8.5) would follow from

$$t - s \le \beta_1(s) I(t), \qquad 0 < s < t \le \frac{1}{2}, \tag{8.6}$$

where the optimal function is

$$\beta_1(s) \ =\ \sup_{s<t\le\frac{1}{2}} \frac{t-s}{I(t)}, \qquad 0 < s \le \frac{1}{2}. \tag{8.7}$$

In the general case, we may apply (8.5) to $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$, which are non-negative, Lipschitz, and have median at zero. More accurately, we then have

$$\begin{aligned}
\int_{\{f>0\}} |f|\, d\mu \ &\le\ \beta_1(s) \int_{\{f>0\}} |\nabla f|\, d\mu + s\operatorname{Osc}_\mu(f_+), \\
\int_{\{f<0\}} |f|\, d\mu \ &\le\ \beta_1(s) \int_{\{f<0\}} |\nabla f|\, d\mu + s\operatorname{Osc}_\mu(f_-).
\end{aligned}$$

Adding the two inequalities and making use of $\operatorname{Osc}_\mu(f_+) + \operatorname{Osc}_\mu(f_-) \le \operatorname{Osc}_\mu(f_+)$, we arrive at (8.5) with $\beta_1(s)$, defined in (8.7).

Conversely, once we know that (8.5) holds, one may approximate indicator functions of Borel sets A by functions with finite Lipschitz semi-norm (e.g., as in (10), Lemma 3.5) to derive $\mu(A) \le \beta_1(s)\mu^+(A) + s$. The latter implies the isoperimetric inequality (8.4) with the function I, satisfying the same relation (8.6). Therefore, the optimal choice should be

$$I(t) \ =\ \sup_{0<s<t} \frac{t-s}{\beta_1(s)}, \qquad 0 < t \le \frac{1}{2}. \tag{8.8}$$

Thus, (8.4) and (8.5) are equivalent, if the functions I and β_1 are connected by the dual relations (8.7)-(8.8). For example, for $I(t) = c\,(\min\{t, 1-t\})^p$ as in Lemma 8.3, we have $\beta_1(s) \le \frac{(p-1)^{p-1}}{c\,p^p\, s^{p-1}}$, and for $\beta_1(s) = \frac{(p-1)^{p-1}}{c\,p^p\, s^{p-1}}$, we have $I(t) = ct^p$ in $0 < t \le \frac{1}{2}$.

Now, let us return to an arbitrary κ-concave probability measure μ on $\mathbf{R}^n$ and apply Lemma 8.3 with $p = 1 - \kappa$. Then we arrive at the following equivalent form for (1.5) with some (different) positive, continuous function $c(\kappa)$.

Corollary 8.4. *If $\kappa < 0$, for any bounded, smooth function f on $\mathbf{R}^n$ with μ-median zero, and for all $s > 0$,*

$$\int |f|\,d\mu \le \frac{ms^{\kappa}}{c(\kappa)} \int |\nabla f|\,d\mu + s\,\mathrm{Osc}_{\mu}(f). \tag{8.9}$$

It is easy to see that $c(\kappa)$ may be chosen to be separated from zero, namely, to satisfy $\lim_{\kappa\to 0} c(\kappa) = c > 0$, so (8.9) also includes the case $\kappa = 0$. Letting $s \to 0$ in order to get rid of the oscillation term, we then arrive at the L^1-Poincaré-type inequality

$$\int |f|\,d\mu \le \frac{m}{c} \int |\nabla f|\,d\mu$$

for the class of log-concave measures μ. By Cheeger's argument, the latter is known to be equivalent to the isoperimetric inequality (1.5) with $\kappa = 0$, and is also known to imply the L^2-Poincaré or the spectral gap inequality

$$\mathrm{Var}_{\mu}(f) \le \frac{4m^2}{c^2} \int |\nabla f|^2\,d\mu,$$

where $\mathrm{Var}_{\mu}(f) = \int f^2\,d\mu - (\int f\,d\mu)^2$ denotes the variance of f with respect to μ. The argument may easily be extended to involve more general inequalities such as (8.9). Let $f \ge 0$ be a bounded, smooth function with μ-median zero. Applying (8.5) to f^2 and then Cauchy's inequality, we obtain that L^2-norms of f and $|\nabla f|$ satisfy

$$\|f\|_2^2 \le 2\beta_1(s)\,\|f\|_2\,\|\nabla f\|_2 + s\mathrm{Osc}_{\mu}(f)^2,$$

and hence, $\|f\|_2^2 \le 4\beta_1(s)^2\,\|\nabla f\|_2^2 + 2s\,O(f)^2$. This extends to the case of general f in the form of the weak Poincaré-type inequality

$$\mathrm{Var}_{\mu}(f) \le \beta_2(s) \int |\nabla f|^2\,d\mu + s\,\mathrm{Osc}_{\mu}(f)^2 \tag{8.10}$$

with $\beta_2(s) = 4\beta_1(\frac{s}{2})^2$. Hence, by Corollary 8.4, any κ-concave probability measure μ on $\mathbf{R}^n$ with $\kappa \le 0$ satisfies

$$\mathrm{Var}_{\mu}(f) \le \frac{m^2 s^{2\kappa}}{c(\kappa)} \int |\nabla f|^2\,d\mu + s\,\mathrm{Osc}_{\mu}(f)^2$$

for some positive continuous function $c(\kappa)$ with an arbitrary bounded smooth f.

Remarks. In connection with the slow convergence rates of Markov semigroups $(P_t)_{t\ge 0}$, associated to μ, weak Poincaré-type inequalities have been introduced by M. Röckner and F.-Y. Wang (27). They considered (8.10) as an additive form of inequalities of Nash-type, considered before by T. M. Liggett (21). See also the work by L. Bertini and B. Zegarlinski (6), where generalized Nash inequalities were studied for Gibbs measures.

Inequality (8.10) allows one to quantify the weak spectral gap property (WSGP, for short) in the sense of Kusuoka-Aida. As given in (3), the latter is defined as the property that $f_\ell \to 0$ in μ-probability, as long as $\|f_\ell\|_2 \le 1$, $\int f_\ell\,d\mu = 0$ and $\mathcal{E}(f_\ell, f_\ell) \to 0$, as $\ell \to \infty$, where the Dirichlet form in our particular case is $\mathcal{E}(f,f) = \int |\nabla f|^2\,d\mu$. It is shown in (27) that WSGP is equivalent to (8.10) with $\beta_2(s) < +\infty$, for all $s > 0$. In terms of L^1-convergence of P_t another characterization was given by P. Mathieu (24).

Let us also mention that recently F. Barthe, P. Cattiaux, and C. Roberto have studied relationship between (8.10) and capacitary inequalities on metric probability spaces, and applied them to get concentration inequalities for product measures, involving dependence on the dimension, cf. (5).

9 Compactly supported measures. Localization

If a κ-concave probability measure μ on $\mathbf{R}^n$ is compactly supported and its supporting set K is contained in some ball $B_r(x_0)$ of radius r, (7.1) yields,

$$2r\mu^+(A) \ \geq \ \frac{1-\mu(A)^{1-\kappa}-(1-\mu(A))^{1-\kappa}}{-\kappa}, \qquad \kappa \neq 0, \tag{9.1}$$

$$2r\mu^+(A) \ \geq \ \mu(A)\log\frac{1}{\mu(A)} + (1-\mu(A))\log\frac{1}{1-\mu(A)}, \qquad \kappa = 0. \tag{9.2}$$

In particular, if K has dimension n and μ represents the normalized Lebesgue measure on K, Barthe's estimate (9.1) becomes

$$\mu^+(A) \ \geq \ \frac{n}{2r}\,\Big(\mu(A)^{(n-1)/n} + (1-\mu(A))^{(n-1)/n} - 1\Big). \tag{9.3}$$

The right-hand side has a correct behaviour as $\mu(A)$ is getting small. In the other case, when $\mu(A)$ is of order $\frac{1}{2}$, this isoperimetric inequality may asymptotically be sharpened with respect to the dimension. Namely, by Theorem 1.2, for some universal $c > 0$,

$$\mu^+(A) \ \geq \ \frac{c}{m}\,\min\{\mu(A), 1-\mu(A)\}^{(n-1)/n}, \tag{9.4}$$

where m is the μ-median for the Euclidean norm (the volume radius of K).

For example, for the unit ℓ^1-ball $K = \{x \in \mathbf{R}^n : |x_1| + \cdots + |x_n| \leq 1\}$, we have $r = 1$, and if $\mu(A) = t$ is fixed, the right-hand side of (9.3) is convergent, as $n \to \infty$, to $\frac{1}{2}\,[\,t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t}\,]$. Hence, there is no improvement over the log-concave case as in (9.2). On the other hand, the volume radius of K is of order $\frac{1}{\sqrt{n}}$, so the right-hand side of (9.4) is of order $t\sqrt{n}$.

This difference may be explained by the fact that the minimal ball, containing the supporting set, is not optimal in (7.1). Nevertheless, (9.1) may still be used to recover a number of results such as a Cheeger-type isoperimetric inequality

$$\mu^+(A) \ \geq \ c_\kappa(r)\min\{\,\mu(A), 1-\mu(A)\}. \tag{9.5}$$

Note that, for any $\kappa \in (-\infty, 1]$, the function $u(t) = \frac{1-t^{1-\kappa}-(1-t)^{1-\kappa}}{-\kappa}$ is concave on (0,1), so, $t = \frac{1}{2}$ is critical in the inequality of the form $u(t) \geq c\min\{t, 1-t\}$. Therefore, if μ is κ-concave, (9.1) implies (9.5) with constant

$$c_\kappa(r) = \frac{2^\kappa - 1}{2r\kappa}. \tag{9.6}$$

In particular, for the range $\kappa \geq 0$, we have a uniform bound on the isoperimetric constant, $c_\kappa(r) \geq \frac{\log 2}{2r}$. In the body case, similar bounds in terms of the diameter $d = \mathrm{diam}(K)$, i.e., of the form $I_\mu(t) \geq \frac{c}{d}\,\min\{t, 1-t\}$ were studied by many authors in connection with randomized volume algorithms, cf. e.g. (22), (13).

As for the range $\kappa < 0$, we see that any κ-concave μ also satisfies the Cheeger-type isoperimetric inequality, and the only difference is that the constant in (9.6) is tending to zero for growing $|\kappa|$. Actually, this is not the real case, as the isoperimetric constant can be made independent of κ. Namely, we have:

Theorem 9.1. *Any convex probability measure μ on $\mathbf{R}^n$, concentrated on a bounded convex set K, satisfies for any Borel $A \subset \mathbf{R}^n$,*

$$\mu^+(A) \ \geq \ \frac{1}{\mathrm{diam}(K)} \, \min\{\, \mu(A), 1 - \mu(A)\}. \tag{9.7}$$

In particular, by Maz'ya-Cheeger's theorem, any compactly supported convex μ admits the Poincaré-type inequality

$$\mathrm{Var}_\mu(f) \ \leq \ 4 \, \mathrm{diam}(K)^2 \int |\nabla f|^2 \, d\mu.$$

When μ is the normalized Lebesgue measure on K, this inequality with an optimal absolute factor in front of $\mathrm{diam}(K)^2$ was obtained by L. E. Payne and H. F. Weinberger (26). It should be noted that in the class of log-concave probability measures on $\mathbf{R}^n$ Cheeger-type isoperimetric and Poincaré-type inequalities are in essense equivalent. This has recently been shown by M. Ledoux (20),

The proof of (9.7) requires to apply an additional localization technique in the form, proposed in (26) and developed in (23). First consider the one-dimensional case. Recall that a non-degenerated probability measure μ on the line with supporting interval $K = (a, b)$ is called unimodal, if its distribution function F is convex on K, or concave on K, or if F is convex on (a, x_0) and concave on (x_0, b), for some point $x_0 \in K$. If μ does not have an atom at x_0, it is absolutely continuous and has a positive density p on K. Moreover, the associated function $I(t) = p(F^{-1}(t))$ is non-decreasing on $(0, t_0)$ and non-increasing on $(t_0, 1)$, for some $t_0 \in [0, 1]$. We need the following:

Lemma 9.2. *Any unimodal probability distribution μ on the line, concentrated on a finite interval $K = (a, b)$, satisfies the Cheeger-type isoperimetric inequality* (9.7).

Proof. In dimension one, the best constant in $\mu^+(A) \ \geq \ c \, \min\{\, \mu(A), 1 - \mu(A)\}$ has a simple description as

$$c \ = \ \mathrm{ess\,inf} \, \frac{p(x)}{\min\{F(x), 1 - F(x)\}} \, ,$$

where p is the density of the absolutely continuous component of μ ((10)). Hence, when the associated function $I(t)$ is well defined, $c \ = \ \inf_{0<t<1} \frac{I(t)}{\min\{t, 1-t\}}$. Now let μ be unimodal with a finite supporting interval (a, b). Without loss of generality, we may assume that μ has no atom and that it has a continuous density p, strictly increasing on $(a, x_0]$ and strictly decreasing on $[x_0, b)$, for some $a < x_0 < b$. Hence, the function I is continuous on (0,1), is strictly increasing on $(0, t_0)$ and strictly decreasing on $(t_0, 1)$, for some $0 < t_0 < 1$. Let I^{-1} denote the inverse function for I, restricted to $(t_0, 1)$. By Chebyshev's inequality with respect to the Lebesgue measure λ on $(t_0, 1)$, for any $\varepsilon \in (I(1-), I(t_0))$,

$$\begin{aligned} 1 - I^{-1}(\varepsilon) \ &= \ \lambda\left\{t \in (t_0, 1) : \frac{1}{I(t)} \geq \frac{1}{\varepsilon}\right\} \ \leq \ \varepsilon \, (1 - t_0) \int_{t_0}^1 \frac{d\lambda(t)}{I(t)} \\ &= \ \varepsilon \, (b - F^{-1}(t_0)) \ \leq \ \varepsilon \, (b - a). \end{aligned}$$

Letting $\varepsilon = I(t)$ with $t \in (t_0, 1)$, we get that $I(t) \geq \frac{1-t}{b-a}$. With a similar argument, $I(t) \geq \frac{t}{b-a}$, for any $t \in (0, t_0)$. Hence, in both cases $I(t) \geq \frac{1}{b-a} \min\{t, 1-t\}$ on (0,1), and the lemma follows.

Note that, according to Lemmas 2.1 and 2.2, any convex probability distribution on the line is unimodal. Reduction to the one-dimensional case in Theorem 9.1 may be done with the help of the localization lemma of L. Lovász and M. Simonovits, which is stated below.

Lemma 9.3. (23) *Let u and v be lower-semicontinuous, integrable functions on a convex open set K in $\mathbf{R}^n$, such that*

$$\int_K u(x)\,dx > 0, \qquad \int_K v(x)\,dx > 0. \tag{9.8}$$

Then, for some points $a, b \in K$ and some non-negative affine function ℓ, defined on the interval $\Delta = [a, b]$,

$$\int_\Delta u(x)\,\ell(x)^{n-1}\,dx > 0, \qquad \int_\Delta v(x)\,\ell(x)^{n-1}\,dx > 0. \tag{9.9}$$

The integrals in (9.8) are n-dimensional, while the integrals in (9.9) are taken with respect to the Lebesgue measure on Δ.

Proof of Theorem 9.1. Given a probability measure μ on $\mathbf{R}^n$, the inequality of the form $\mu^+(A) \geq c \min\{\mu(A), 1 - \mu(A)\}$ with fixed $c > 0$, such as (9.7), may equivalently be stated as

$$\mu(A^h) \geq R_h(\mu(A)), \qquad h > 0, \tag{9.10}$$

in the class of all open sets A in $\mathbf{R}^n$. Here $A^h = \{x \in \mathbf{R}^n : \mathrm{dist}(A, x) \leq h\}$ denotes the closed h-neighborhood of A with respect to the Euclidean distance, and $R_h(t) = F(F^{-1}(t) + ch)$ with F being the distribution function of the probability measure on the line with density $\frac{1}{2}\,e^{-|x|}$. It will be convenient to reformulate (9.10) as implication

$$\mu(A) > t \Longrightarrow \mu(A^h) \geq R_h(t), \qquad h > 0,\ 0 < t < 1. \tag{9.11}$$

Now, let μ be a convex, compactly supported probability measure. We may assume μ is absolutely continuous. Then, the supporting convex set K may be chosen to be open, and by Lemma 2.1, the density $p(x)$ of μ may be chosen to be κ-concave on K with $\kappa = -\frac{1}{n}$.

Suppose that (9.7), that is, the property (9.11) with $c = \frac{1}{\mathrm{diam}(K)}$ is not true for some A, t and h, so that $\mu(A) > t$ and $\mu(A^h) < R_h(t)$. This may be written as

$$\int_K (1_A(x) - t)\,p(x)\,dx > 0 \quad \text{and} \quad \int_K (R_h(t) - 1_{A^h}(x))\,p(x)\,dx > 0.$$

Hence, the hypothesis (9.8) of Lemma 9.3 is fulfilled for $u(x) = (1_A(x) - t)p(x)$ and $v(x) = (R_h(t) - 1_{A^h}(x))p(x)$. Note these functions are lower-semicontinuous, since A is open and A^h is closed. Therefore, one can find points $a, b \in K$ and a non-negative affine function ℓ, defined on the interval $\Delta = [a, b]$, such that (9.9) holds. The latter may be written as

$$\nu(A) > t \quad \text{and} \quad \nu(A^h) < R_h(t) \tag{9.12}$$

in terms of the probability measure ν on Δ with density $q(x) = Z\,p(x)\,\ell(x)^{n-1}$ with respect to the Lebesgue measure on Δ, where Z is a normalizing constant. But, by Lemma 4.2, the function q is κ-concave on Δ with $\kappa = -1$, so that by Lemma 2.1 for dimension one, ν is convex on Δ. Therefore, by Lemma 9.2, ν satisfies the Cheeger-type isoperimetric inequality (9.7) and thus $\nu(A^h) \geq R_h(\nu(A))$. On the last step the bound $|\Delta| \leq \mathrm{diam}(K)$ was used. As a result, we obtain a contradiction to (9.12).

Theorem 9.1 is now proved.

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