Electronic Journal of Probability

Vol. 13 (2008), Paper no. 33, pages 980–999.

Journal URL
http://www.math.washington.edu/~ejpecp/

Large-N Limit of Crossing Probabilities, Discontinuity, and Asymptotic Behavior of Threshold Values in Mandelbrot's Fractal Percolation Process

Erik I. Broman [*†] Federico Camia [‡§]

Abstract

We study Mandelbrot's percolation process in dimension $d \geq 2$. The process generates random fractal sets by an iterative procedure which starts by dividing the unit cube $[0,1]^d$ in N^d subcubes, and independently retaining or discarding each subcube with probability p or $1-p$ respectively. This step is then repeated within the retained subcubes at all scales. As p is varied, there is a percolation phase transition in terms of paths for all $d \geq 2$, and in terms of $(d-1)$-dimensional "sheets" for all $d \geq 3$.

For any $d \geq 2$, we consider the random fractal set produced at the path-percolation critical value $p_c(N,d)$, and show that the probability that it contains a path connecting two opposite faces of the cube $[0,1]^d$ tends to one as $N \to \infty$. As an immediate consequence, we obtain that the above probability has a discontinuity, as a function of p, at $p_c(N,d)$ for all N sufficiently large. This had previously been proved only for $d = 2$ (for any $N \geq 2$). For $d \geq 3$, we prove analogous results for sheet-percolation.

In dimension two, Chayes and Chayes proved that $p_c(N,2)$ converges, as $N \to \infty$, to the critical density p_c of site percolation on the square lattice. Assuming the existence of

[*]Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden. E-mail: broman@math.chalmers.se

[†]The work of this author was carried out while at the Department of Mathematics of the Vrije Universiteit Amsterdam.

[‡]Department of Mathematics, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands. E-mail: fede@few.vu.nl

[§]Partially supported by a VENI grant of the NWO.

the correlation length exponent ν for site percolation on the square lattice, we establish the speed of convergence up to a logarithmic factor. In particular, our results imply that $p_c(N,2) - p_c = (\frac{1}{N})^{1/\nu+o(1)}$ as $N \to \infty$, showing an interesting relation with near-critical percolation .

Key words: Fractal percolation, crossing probability, critical probability, enhancement/diminishment percolation, near-critical percolation.

AMS 2000 Subject Classification: Primary 60K35, 60D05, 28A80, 82B43.

Submitted to EJP on January 1, 2007, final version accepted May 30, 2008.

1 Introduction and First Results

In this paper we are concerned with a continuum percolation model, first introduced in [16], which is known as Mandelbrot's fractal percolation process and defined as follows. For any integers $d \geq 2$ and $N \geq 2$, we start by partitioning the unit cube $[0,1]^d \subset \mathbb{R}^d$ into N^d subcubes of equal size. Given $p \in [0,1]$ and a subcube, we then retain the subcube with probability p and discard it with probability $1-p$. This is done independently for every subcube of the partition. This gives us a random set $\mathcal{C}_N^1 = \mathcal{C}_N^1(d,p) \subset [0,1]^d$. Consider any retained (assuming that $\mathcal{C}_N^1 \neq \emptyset$) subcube B in $\mathcal{C}_N^1$. We can repeat the described procedure on a smaller scale by partitioning B into N^d further subcubes, discarding or retaining them as above. We do this for every retained subcube of $\mathcal{C}_N^1$. This will yield a new random set $\mathcal{C}_N^2 \subset \mathcal{C}_N^1$. Iterating the procedure on every smaller scale yields an infinite sequence of random sets $\ldots \subset \mathcal{C}_N^{k+1} \subset \mathcal{C}_N^k \subset \ldots \subset [0,1]^d$. It is easy to see that $\mathcal{C}_N := \cap_{k=1}^{\infty} \mathcal{C}_N^k$ is a well defined random object.

We denote by $CR(\mathcal{C}_N)$ the event that $\mathcal{C}_N$ contains a connected component intersecting both the "left-hand face" $\{0\} \times [0,1]^{d-1}$ and the "right-hand face" $\{1\} \times [0,1]^{d-1}$ of $[0,1]^d$ (we will call such a connected component a path or crossing). We can then define

$$\theta_{N,d}(p) := \mathbb{P}_p(CR(\mathcal{C}_N)). \tag{1}$$

We could of course have used any two opposite faces in this definition. Furthermore, we define

$$p_c(N,d) := \inf\{p : \theta_{N,d}(p) > 0\}.$$

By a standard coupling argument it is easy to see that $\theta_{N,d}(p)$ is non-decreasing in p for every N.

Several authors studied various aspects of Mandelbrot's fractal percolation, including the Hausdorff dimension of the limiting fractal set, as detailed in [10], and the possible existence of paths [6; 11; 17; 7] and $(d-1)$-dimensional "sheets" [7; 21; 18] traversing the unit cube between opposite faces. Dekking and Meester [11] proposed a "morphology" of random Cantor sets comprising several "phases" in which a set can be.

Chayes and Chayes [5] considered the behavior of $p_c(N,2)$ for large N, and proved that it converges to the critical value for Bernoulli site percolation on the square lattice, as $N \to \infty$. This was generalized by Falconer and Grimmett [12; 13] to all dimensions $d \geq 3$, where the limiting value however is not the critical value of site percolation on the (hyper-)cubic lattice, but that of site percolation on a related lattice (see Section 3). Orzechowski [21] proved an analogous results for "sheet" percolation.

Falconer and Grimmett [12; 13] obtained their result as a consequence of proving that, for all $d \geq 2$, $\lim_{N\to\infty} \theta_{N,d}(p) = 1$ for all p strictly larger than the relevant (depending on the dimension) critical value. An interesting question, addressed in Theorem 1.1 below, concerns the large-N limit of the path-crossing probability *at* $p_c(N,d)$. Let us try to motivate that question.

For the sake of this discussion, let us consider the model in two dimensions, where it is known [6; 11] that $\theta_{N,2}(p)$ is discontinuous at $p_c(N,2)$ for all N. Analogously, if we tile the plane with independent copies of the system, the probability that there exists an infinite path intersecting the unit square $[0,1]^2$ has a discontinuity at $p_c(N,2)$ for all N [6]. Such behavior prompted the authors of [6] to call the phase transition in Mandelbrot's fractal percolation first order or discontinuous (although one of the authors later appeared to change his mind on the issue –

see p. 130 of [8]). Regardless of the label, the discontinuity of $\theta_{N,2}(p)$ (and, more generally, of $\theta_{N,d}(p)$ for $d \geq 2$) is an important feature of the phase transition, and it is natural to ask what happens to it as $N \to \infty$.

We point out that the fact that $\lim_{N\to\infty} \theta_{N,d}(p) = 1$ for all p strictly larger than the relevant (depending on the dimension) critical value does not shed much light on the large-N limit of the path-crossing probability *at* the critical point. Consider, for instance, the crossing probability of a square in (ordinary) two-dimensional Bernoulli percolation in the scaling limit (i.e., as the lattice spacing goes to zero) or equivalently in the limit of larger and larger squares. Such a limit is needed to "see" the phase transition and produce a situation that can be compared to the one in Mandelbrot's fractal percolation. However, in Bernoulli percolation, in the scaling limit or in the limit of larger and larger squares, the crossing probability of a square converges to one above the critical point, to zero below it, and to a value strictly between zero and one *at* the critical point, showing a behavior that differs from the one of the next theorem.

Theorem 1.1. *We have that for any $d \geq 2$,*

$$\lim_{N\to\infty} \theta_{N,d}(p_c(N,d)) = 1.$$

When proving Theorem 1.1, in Section 4.1, we will use a result from [1] concerning "enhancement" percolation (see also [14]). We will in fact need a slightly modified version of the result; in particular, we will be looking at crossing probabilities in a "diminishment" percolation model (to be defined precisely in Section 3).

The following corollary is an immediate consequence of the previous theorem. For $d = 2$ and all $N \geq 2$, an elegant and rather elementary proof of this fact can be found in [11].

Corollary 1.2. *For any $d \geq 2$, there exists an $N_0 = N_0(d)$ such that the function $\theta_{N,d}(p)$ is discontinuous at $p_c(N,d)$ for $N \geq N_0$.*

Remark. For $d = 3$ and all $N \geq 2$, a similar result had been proved in [7], but with the probability of crossing a square replaced by that of crossing a $2 \times 2 \times 1$ rectangle between the 2×2 faces. While the result in [7] is more satisfactory because it applies to all $N \geq 2$, it is limited to a three-dimensional system with a special geometry. On the contrary our result is valid in all dimensions and, although we use the unit cube for simplicity, the choice of that particular geometry is not important for our arguments.

When $d \geq 3$, one can consider the existence of $(d-1)$-dimensional "sheets" crossing the unit cube $[0,1]^d$. We denote by $SH(\mathcal{C}_N)$ the event that $\mathcal{C}_N$ contains a surface separating the faces of $[0,1]^d$ perpendicular to the first coordinate direction. We can then define

$$\tilde{\theta}_{N,d}(p) := \mathbb{P}_p(SH(\mathcal{C}_N)). \tag{2}$$

We could of course have used any two opposite faces in this definition. Furthermore, we define

$$\tilde{p}_c(N,d) := \inf\{p : \tilde{\theta}_{N,d}(p) > 0\}.$$

By a standard coupling argument it is easy to see that $\tilde{\theta}_{N,d}(p)$ is non-decreasing in p for every N.

Our next result concerns the large-N limit of the sheet-crossing probability at the critical point, and is analogous to Theorem 1.1.

Theorem 1.3. *We have that for any $d \geq 3$,*

$$\lim_{N\to\infty} \tilde{\theta}_{N,d}(\tilde{p}_c(N,d)) = 1.$$

When proving this theorem, in Section 4.2, we will again use a slight modification of a result from [1] concerning enhancement percolation.

The following corollary is an immediate consequence of the previous theorem.

Corollary 1.4. *For any $d \geq 3$, there exists an $\tilde{N}_0 = \tilde{N}_0(d)$ such that the function $\tilde{\theta}_{N,d}(p)$ is discontinuous at $\tilde{p}_c(N,d)$ for $N \geq \tilde{N}_0$.*

2 Asymptotic Behavior of $p_c(N,2)$

Our last result concerns the asymptotic behavior of $p_c(N,2)$ as $N \to \infty$. It is known [5] that, for $d = 2$, $p_c(N,2)$ tends to the critical density $p_c \equiv p_c(\mathbb{Z}^2, site)$ of (Bernoulli) site percolation (see Section 3 for the exact definition) on the square lattice, as $N \to \infty$. Assuming the existence of the correlation length exponent ν for two-dimensional site percolation on the square lattice, we give bounds on the speed of convergence.

There are several equivalent ways to introduce the concept of correlation length in percolation. In this section we will assume that $p > p_c$, since this is the only case we are interested in. Let $\varphi_N(p)$ be the probability that there is an open crossing in the first coordinate direction of an $N \times N$ square in (Bernoulli) site percolation on the square lattice with density p of open sites (see Section 3 for precise definitions). Following Kesten (see [15], [9], [19] and the references therein), we define

$$L_\delta(p) := \min\{N : \varphi_N(p) \geq 1 - \delta\}. \tag{3}$$

The δ in the definition is unimportant, since for any $\delta, \delta' \in (0, 1/2)$ we have [15; 19] that the ratio $L_\delta(p)/L_{\delta'}(p)$ is bounded away from 0 and ∞ as $p \to p_c$. In view of this, in what follows we will consider δ to be a fixed number smaller than $1/2$. $L_\delta(p)$ is believed to behave like $(p - p_c)^{-\nu}$ as $p \downarrow p_c$, with $\nu = 4/3$ (see, e.g., [15] and references therein). More precisely, taking $\delta < 1/2$, we make the following assumption.

H1: There are two constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that $c_1(p - p_c)^{-4/3} \leq L_\delta(p) \leq c_2(p - p_c)^{-4/3}$ for all $p \in (p_c, 1]$.

Based on that assumption, the following theorem is proved in Section 5.

Theorem 2.1. *Let $d = 2$, and assume that* **H1** *holds. For any $C < \infty$, there exists an $N_C < \infty$ such that for all $N \geq N_C$,*

$$p_c(N,2) \geq p_c + C\left(\frac{1}{N}\right)^{3/4}. \tag{4}$$

Furthermore, there exist constants $D < \infty$ and $N_1 < \infty$ such that, for all $N \geq N_1$,

$$p_c(N,2) \leq p_c + D\left(\frac{\log N}{N}\right)^{3/4}. \tag{5}$$

The limit $N \to \infty$ can be compared to the scaling limit $\varepsilon \to 0$ in ordinary percolation on a lattice with lattice spacing ε. The reason is that it removes the intrinsic scale of the system due to the size of the largest subdivisions in the same way as the limit $\varepsilon \to 0$ in ordinary percolation removes the intrinsic scale of the system due to the lattice. Theorem 2.1 and its proof show an interesting (and at first site maybe surprising) connection between the large-N limit of fractal percolation and the "near-critical" scaling limit of ordinary percolation. The latter corresponds to letting $\varepsilon \to 0$ in a percolation model with lattice spacing ε and density of open sites (or bonds) $p = p_c + \lambda\varepsilon^{3/4}$ for some fixed λ (see, e.g., [2; 3; 4; 20]).

To motivate the connection, roughly speaking, one can argue in the following way. Take N very large, and consider a system with retention probability equal to p_c. After the first iteration the system looks critical, meaning that there is positive probability to see left-to-right crossings, but that such crossings are tenuous and are not able to sustain the remaining iterations. This is not surprising since we know that $p_c(N, 2) > p_c$ for all N. On the other hand, if $p > p_c$, for N sufficiently large, a system with retention probability p looks very supercritical after the first iteration (in other words, $N \gg L_\delta(p)$). Chayes and Chayes [5] used this fact to show that the system has enough connections to sustain all further iterations. As a consequence, one obtains that $p_c(N, 2)$ becomes arbitrarily close to p_c as $N \to \infty$. But what is the speed of convergence? As $N \to \infty$, $p_c(N, 2)$ must be getting close to p_c at the right speed: not too fast, or else after the first iteration the system would look "too critical," and not too slowly, or else it would look "too supercritical." As Theorem 2.1 and its proof show, the right speed is related to near-critical percolation.

Remark. Although **H1** is widely believed to hold (at least) for site and bond percolation on regular lattices, it has so far not been proved for any percolation model. However, a weaker result has been proved [22] for site percolation on the triangular lattice. With the notation introduced above, the latter can be stated as follows.

H2: $L_\delta(p) = (p - p_c)^{-4/3+o(1)}$ as $p \downarrow p_c$.

Assuming **H2** instead of **H1**, our results can be easily adapted to obtain $p_c(N, 2) - p_c = (\frac{1}{N})^{3/4+o(1)}$.

Remark. The definition of correlation length we use in this paper is equivalent to other standard definitions (see, e.g., [15; 9; 19]), including the one that comes from considering the probability $\tau_p^f(x)$ that the origin and x belong to the same finite p-open cluster. The correlation length $\xi(p)$ related to $\tau_p^f(x)$ is defined by

$$\xi(p)^{-1} := \lim_{|x|\to\infty} \left\{ -\frac{1}{|x|} \log \tau_p^f(x) \right\}.$$

In particular, for every $\delta > 0$, there are two constants, $c_1' = c_1'(\delta)$ and $c_2' = c_2'(\delta)$, such that $c_1' L_\delta(p) \le \xi(p) \le c_2' L_\delta(p)$ for all $p > p_c$ (see, e.g., the Appendix of [9]).

Remark: In light of Theorem 2.1, we find the concluding remarks of [5] unclear. On p. L505 of [5], the authors claim that: "In the large-N regime, our proof demonstrates that if N is a

huge (but fixed) multiple of the correlation length of the density-p site model, then $p > p_c(N)$." However, if one accepts hypothesis **H1**, this would imply that, for N large enough, $p_c(N,2) < p_c + \frac{const}{N^{3/4}}$, which contradicts the first statement of Theorem 2.1. If one instead accepts **H2**, then it is not clear what a multiple of the correlation length means.

3 Enhancement and Diminishment Percolation

In this section we introduce and discuss a particular example of enhancement percolation [1], and a closely related example of "diminishment" percolation (see the comment after Theorem 3.16 on p. 65 of [14]). Both models will be used later in the proofs of the main results, where they will be compared (or more precisely, coupled) to the fractal percolation model.

Before we can proceed we need to introduce the two graphs on which the enhancement and diminishment percolation models will be defined. For $d \geq 2$, let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $(x_1, \ldots, x_d) = x \sim y = (y_1, \ldots, y_d)$ if and only if $x \neq y$, $|x_i - y_i| \leq 1$ for all i and $x_i = y_i$ for at least one value of i. Let $\mathbb{M}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $(x_1, \ldots, x_d) = x \sim y = (y_1, \ldots, y_d)$ if and only if $x \neq y$ and $|x_i - y_i| \leq 1$ for all i. For $d = 2$, $\mathbb{L}^2$ coincides with the square lattice and $\mathbb{M}^2$ with its close-packed version, obtained by adding diagonal edges inside the faces of the square lattice. For $d \geq 3$, $\mathbb{M}^d$ contains $\mathbb{L}^d$ as a strict sublattice, which in turns contains the (hyper-)cubic lattice as a strict sublattice.

We consider Bernoulli site percolation on $\mathbb{L}^d$ and $\mathbb{M}^d$ with parameter $p \in [0,1]$: each site is declared open (represented by a 1) with probability p and closed (represented by a 0) with probability $1-p$, independently of all other sites. For any $N \geq 2$, consider the box $\Lambda_N^d := \{1, \ldots, N\}^d \subset \mathbb{Z}^d$.

For $d \geq 2$, we denote by $CR(\Lambda_N^d, \mathbb{L}^d)$ the event that there is an open path in $\mathbb{L}^d$ crossing Λ_N^d in the first coordinate direction. An open path in $\mathbb{L}^d$ is a sequence of open sites such that any two consecutive elements of the sequence are adjacent in $\mathbb{L}^d$. If $CR(\Lambda_N^d, \mathbb{L}^d)$ occurs, we say that there is an open crossing of Λ_N^d. Analogous definitions hold for $d \geq 3$ with $\mathbb{L}^d$ replaced by $\mathbb{M}^d$.

For $d \geq 2$, we define

$$\varphi_N^d(p) := \mathbb{P}_p(CR(\Lambda_N^d, \mathbb{L}^d)) \tag{6}$$

and

$$p_c(\mathbb{L}^d) := \inf\{p : \lim_{N\to\infty} \varphi_N^d(p) > 0\}.$$

As before, we could of course have considered crossings in any direction.

Analogously, for $d \geq 3$, we define

$$\psi_N^d(p) := \mathbb{P}_p(CR(\Lambda_N^d, \mathbb{M}^d)) \tag{7}$$

and

$$p_c(\mathbb{M}^d) := \inf\{p : \lim_{N\to\infty} \psi_N^d(p) > 0\}.$$

In [13] it is proved, for all $d \geq 2$, that $p_c(\mathbb{L}^d) \leq p_c(N,d)$ for all N, and that

$$\lim_{N\to\infty} p_c(N,d) = p_c(\mathbb{L}^d).$$

(The same conclusions had been reached earlier in [5] for $d = 2$.) Similar results are obtained for the sheet percolation problem in dimension $d \geq 3$ in [21], where it is shown that $\tilde{p}_c(N, d) \geq 1 - p_c(\mathbb{M}^d)$ for all N, and that

$$\lim_{N\to\infty} \tilde{p}_c(N, d) = 1 - p_c(\mathbb{M}^d).$$

In fact, the above results were stated for a different definition of $p_c(\mathbb{M}^d)$ and $p_c(\mathbb{L}^d)$, defined using the probability of the existence of an infinite cluster. However, it is a standard result from percolation theory (see, e.g., [14] and references therein) that the latter definition is equivalent to the one given above using limits of crossing probabilities.

We are now ready to define our (stochastic) enhancement percolation model with density s of enhancement. We start by performing Bernoulli site percolation with density $p \in [0, 1]$ on $\mathbb{M}^d$. For a site $x = (x_1, \ldots, x_d) \in \mathbb{M}^d$, if $(x_1 + 1, x_2, \ldots, x_d)$ and $(x_1 - 1, x_2, \ldots, x_d)$ are both open, we make x open with probability s, regardless of its state before the enhancement. Note that this has the effect of "enhancing"the percolation configuration by changing the state of some sites from closed to open. We denote by $\mathbb{P}^{enh}_{p,s}$ the resulting probability measure.

It is immediate to see that our enhancement is essential in the language of [1] (i.e., there exist configurations that do not have doubly-infinite open paths but such that a doubly-infinite open path [the union of two disjoint, infinite open paths starting at neighboring sites] is generated if the enhancement is activated at the origin).

For $s > 0$ fixed, we define

$$\psi^d_N(p, s) := \mathbb{P}^{enh}_{p,s}(CR(\Lambda^d_N, \mathbb{M}^d)).$$

Note that the enhancement breaks the symmetry of the original percolation model, and recall that $CR(\cdot)$ is the event that there is a crossing in the first coordinate direction. In contrast to similar definitions above, choosing the first coordinate direction here is important. Note also that $\psi^d_N(p, 0) = \psi^d_N(p)$, where the right hand side was defined in (7).

The next lemma follows easily from the proof of Theorem 1 of [1].

Lemma 3.1. *Let $d \geq 3$. For fixed $s > 0$, there exists $\delta_1 = \delta_1(s) > 0$ so that, for every p such that $0 < p_c(\mathbb{M}^d) - p \leq \delta_1$,*

$$\lim_{N\to\infty} \psi^d_N(p, s) = 1.$$

The proof of Theorem 1 of [1] is based on a differential inequality. In order to use the same strategy to prove Lemma 3.1, we would need the following differential inequality:

$$\frac{\partial}{\partial s}\psi^d_N(p, s) \geq f_d(p, s)\frac{\partial}{\partial p}\psi^d_N(p, s) \tag{8}$$

with $f_d(p, s)$ continuous and strictly positive on $(0, 1)^2$. This can be easily obtained by applying (a straightforward generalization of) Russo's formula (see [1] and [14]) and the proof of Lemma 2 of [1] to the event $CR(\Lambda^d_N, \mathbb{M}^d)$. Lemma 2 of [1] concerns a different event, namely the existence of an open path connecting the origin to the boundary of the cube of side length $2N$ (say). However, the key ingredient of the proof of the lemma is a local rearrangement of a percolation configuration that involves changing the state of at most a bounded number of sites, and is therefore not sensitive to the "global geometry" of the event.

We do not give a proof of (8) here, since it would look almost identical to the proof of (1.7) of [1], but for the reader's convenience, we explain how to obtain Lemma 3.1 from (8). The latter shows that $\psi_N^d(p,s)$ is a nonincreasing function of t when $(p,s) \equiv (p(t),s(t))$ satisfies

$$\frac{d}{dt}(p,s) = (f_d(p,s), -1).$$

In particular, since f_d is strictly positive, for any $s_0 > 0$ there is some $p_0 < p_c(\mathbb{M}^d)$ such that the forward orbit from (p_0,s_0) crosses the segment $\{p = p_c(\mathbb{M}^d), s \in [0,1]\}$ and reaches some point (p',s') with $p' > p_c(\mathbb{M}^d)$. Then, using the monotonicity in s, $\psi_N^d(p_0,s_0) \geq \psi_N^d(p',s') \geq \psi_N^d(p',0) = \psi_N^d(p')$, which implies the lemma.

Next, we define our (stochastic) diminishment percolation model with density s of diminishment. We start by performing Bernoulli site percolation with density $p \in [0,1]$ on $\mathbb{L}^d$. For a site $x = (x_1,\ldots,x_d) \in \mathbb{L}^d$, if all neighbors of x, except possibly the ones with coordinates $(x_1 \pm 1, x_2, \ldots, x_d)$, are closed, we make x closed with probability s, regardless of its state before the enhancement. Note that this has the effect of "diminishing" the percolation configuration by changing the state of some sites from open to closed. We denote by $\mathbb{P}_{p,s}^{dim}$ the resulting probability measure.

It is immediate to see that our diminishment is essential in the sense that there exist configurations that have a doubly-infinite open path but such that a doubly-infinite open path is not present after the enhancement is activated at the origin.

For $s > 0$ fixed, we define

$$\varphi_N^d(p,s) := \mathbb{P}_{p,s}^{dim}(CR(\Lambda_N^d, \mathbb{L}^d)).$$

As above note that the diminishment breaks the symmetry of the original percolation model so that again it matters that we consider crossings in the first coordinate direction. Note also that $\varphi_N^d(p,0) = \varphi_N^d(p)$, where the right hand side was defined in (6).

The next lemma can again be proved using arguments from [1].

Lemma 3.2. *Let $d \geq 2$. For fixed $s > 0$, there exists $\delta_2 = \delta_2(s) > 0$ so that for every p such that $0 < p - p_c(\mathbb{L}^d) \leq \delta_2$,*

$$\lim_{N\to\infty} \varphi_N^d(p,s) = 0.$$

The lemma can be proved using the following differential inequality:

$$\frac{\partial}{\partial s}\varphi_N^d(p,s) \leq -g_d(p,s)\frac{\partial}{\partial p}\varphi_N^d(p,s), \tag{9}$$

with $g_d(p,s)$ continuous and strictly positive on $(0,1)^2$.

Once again, we do not give a proof of (9), since it would look almost identical to the proof of (1.7) of [1]. To obtain Lemma 3.2 from (9), we can use the same strategy as before. From (9) we see that $\varphi_N^d(p,s)$ is a nondecreasing function of t when $(p,s) \equiv (p(t),s(t))$ satisfies

$$\frac{d}{dt}(p,s) = (-g_d(p,s), -1).$$

In particular, since g_d is strictly positive, for any $s_0 > 0$ there is some $p_0 > p_c(\mathbb{L}^d)$ such that the forward orbit from (p_0,s_0) crosses the segment $\{p = p_c(\mathbb{L}^d), s \in [0,1]\}$ and reaches some point (p',s') with $p' < p_c(\mathbb{L}^d)$. Then, due to the monotonicity in s, $\varphi_N^d(p_0,s_0) \leq \varphi_N^d(p',s') \leq \varphi_N^d(p',0) = \varphi_N^d(p')$, which implies the lemma.

4 Proofs of Theorems 1.1 and 1.3

4.1 Proof of Theorem 1.1

We will use the results from [5] and [12; 13] (for $d = 2$ and $d \geq 3$, respectively) cited in Section 3 and stated in the theorem below for convenience.

Theorem 4.1 ([5],[12; 13]). *For all $d \geq 2$, we have that $p_c(N, d) \geq p_c(\mathbb{L}^d)$ for every $N \geq 2$. Moreover, $\lim_{N\to\infty} p_c(N, d) = p_c(\mathbb{L}^d)$.*

With this, we are now ready to present the actual proof.

Proof of Theorem 1.1. Consider Mandelbrot's fractal percolation process with retention parameter p. Let $\mathcal{A}_k$ denote the event that there is complete retention until level k (as pointed out in [5], "a minor miracle"), i.e., $\mathcal{C}_N^k = [0, 1]^d$. Trivially, for any $k \geq 1$,

$$\theta_{N,d}(p) \leq \mathbb{P}_p(CR(\mathcal{C}_N) \,|\, \mathcal{A}_k).$$

Assume that $\mathcal{A}_{k-1}$ has occurred. We will couple level k of the fractal percolation process, conditioned on $\mathcal{A}_{k-1}$, to a diminishment percolation process on $\mathbb{L}^d$ with density of diminishment $s = 1 - \theta_{N,d}(p)$. We now explain how the coupling works.

If $\mathcal{A}_{k-1}$ occurs, at the next level of the fractal percolation process, the unit cube $[0, 1]^d$ is divided in N^{dk} subcubes of side length $1/N^k$. We identify the subcubes with their centers $\{(\frac{x_1-1/2}{N^k}, \ldots, \frac{x_d-1/2}{N^k}), 1 \leq x_1, \ldots, x_d \leq N^k\}$. Each subcube is retained with probability p and discarded with probability $1 - p$, independently of all other subcubes.

Recall that $\Lambda_N^d = \{1, \ldots, N\}^d \subset \mathbb{Z}^d$. In our diminishment model, sites in $\mathbb{Z}^d \setminus \Lambda_{N^k}^d$ are declared closed, while a site $x = (x_1, \ldots, x_d) \in \Lambda_{N^k}^d$ is declared open if the cube $(\frac{x_1-1/2}{N^k}, \ldots, \frac{x_d-1/2}{N^k})$ is retained at level k of the fractal percolation process, and closed otherwise. This means that the states of sites in $\Lambda_{N^k}^d$ are determined by the fate of the level k subcubes in the fractal percolation process, while all other sites are closed.

On this percolation model we now perform a diminishment according to the following rule. If a site $x = (x_1, \ldots, x_d) \in \Lambda_{N^k}^d$ is open and all its $\mathbb{L}^d$-neighbors except possibly $(x_1 \pm 1, x_2, \ldots, x_d)$ are closed, we look at the effect of the fractal percolation process within the subcube $(\frac{x_1-1/2}{N^k}, \ldots, \frac{x_d-1/2}{N^k})$. If the fractal percolation process inside that subcube does not leave a connection between its two faces perpendicular to the first coordinate direction, we declare site x closed. The result is that, due to the scale invariance of the fractal percolation process, each open site $x = (x_1, \ldots, x_d) \in \Lambda_{N^k}^d$ such that all its $\mathbb{L}^d$-neighbors except possibly $(x_1 \pm 1, x_2, \ldots, x_d)$ are closed is independently declared closed with probability $1 - \theta_{N,d}(p)$. Sites that were already closed simply stay closed. This is obviously a diminishment in the sense of Section 3. We denote by $\mathbb{Q}_{p,1-\theta_{N,d}(p)}^{dim}$ the probability measure corresponding to the diminishment percolation process just defined (with initial configuration chosen as described above, i.e., with all sites outside $\Lambda_{N^k}^d$ closed).

We now make three simple but crucial observations. First of all, recalling the definition of the diminishment percolation model of Section 3 and of the corresponding measure $\mathbb{P}_{p,s}^{dim}$, it is clear that, due to the difference in the initial configurations before activating the enhancement,

$$\mathbb{Q}_{p,s}^{dim}(CR(\Lambda_{N^k}^d, \mathbb{L}^d)) \leq \mathbb{P}_{p,s}^{dim}(CR(\Lambda_{N^k}^d, \mathbb{L}^d)) = \varphi_{N^k}^d(p, s)$$

(where in our case $s = 1 - \theta_{N,d}(p)$).

Secondly, note that any two retained level k subcubes of the fractal percolation process that touch at a single point (a corner) will eventually not touch at all. The reason for this is that at every subsequent step $j > k$ of the fractal construction, the corner at which the two subcubes touch has probability uniformly bounded away from zero to vanish because all the level j subcubes that share that corner are discarded. This means that, if we are interested in the connectivity properties of $\mathcal{C}_N$, any two level k subcubes that touch at a corner can effectively be considered disconnected. (This is the reason why the lattice $\mathbb{L}^d$ is relevant in this context.)

Lastly, from topological considerations and the previous observation, one can easily conclude that (conditioning on $\mathcal{A}_{k-1}$) the absence of a crossing of $\Lambda(\mathbb{L}^d, N^k)$ in the first coordinate direction in the diminishment percolation process described above implies the absence of a left to right crossing of $[0,1]^d$ by the fractal set $\mathcal{C}_N$. (Otherwise stated, if $\mathcal{A}_{k-1}$ happens, then "$CR(\mathcal{C}_N) \subset CR(\Lambda^d_{N^k}, \mathbb{L}^d)$," where the event on the left concerns the fractal percolation process and that on the right the diminishment percolation process, and the inclusion should be interpreted in terms of the coupling between the two processes.)

From the previous discussion we can conclude that

$$\begin{aligned} \theta_{N,d}(p) &\leq \mathbb{P}_p(CR(\mathcal{C}_N) \,|\, \mathcal{A}_{k-1}) \\ &\leq \mathbb{Q}^{dim}_{p,1-\theta_{N,d}(p)}(CR(\Lambda^d_{N^k}, \mathbb{L}^d)) \\ &\leq \mathbb{P}^{dim}_{p,1-\theta_{N,d}(p)}(CR(\Lambda^d_{N^k}, \mathbb{L}^d)) \\ &= \varphi^d_{N^k}(p, 1 - \theta_{N,d}(p)). \end{aligned} \tag{10}$$

We will next use (10) to show that, for any $\epsilon > 0$, there exists an $N_1 = N_1(\epsilon)$ such that, for every $N \geq N_1$, $\theta_N(p_c(N,d) + \tau) > 1 - \epsilon$ for all $\tau > 0$. To do this, we set $s = \epsilon$ in Lemma 3.2, obtain $\delta_2 = \delta_2(\epsilon) > 0$, and then use Theorem 4.1 to find an $N_1 = N_1(\delta_2)$ such that

$$0 \leq p_c(N,d) - p_c(\mathbb{L}^d) < \delta_2/2$$

for all $N \geq N_1$.

Assume now, by contradiction, that

$$\theta_{\hat{N},d}(p_c(\hat{N},d) + \hat{\tau}) \leq 1 - \epsilon,$$

for some $\hat{N} \geq N_1$ and some $\hat{\tau}$ which, because of the monotonicity of $\theta_{N,d}(p)$ in p, we can assume without loss of generality to be smaller than $\delta_2/2$. Using the obvious monotonicity in s of $\varphi^d_{N^k}(p,s)$, from (10) we obtain

$$\theta_{\hat{N},d}(p_c(\hat{N},d) + \hat{\tau}) \leq \varphi^d_{\hat{N}^k}(p_c(\hat{N},d) + \hat{\tau}, \epsilon). \tag{11}$$

Noting that, for any $\tau < \delta_2/2$,

$$0 \leq p_c(\hat{N},d) + \tau - p_c(\mathbb{L}^d) < \delta_2,$$

we can let $k \to \infty$ in (11) and use Lemma 3.2 to obtain

$$\theta_{\hat{N},d}(p_c(\hat{N},d) + \hat{\tau}) = 0. \tag{12}$$

This however is an obvious contradiction, from which it follows that for every $N \geq N_1$,

$$\theta_{N,d}(p_c(N,d)+\tau) > 1-\epsilon \tag{13}$$

for all $\tau > 0$, as claimed above.

We are now ready to conclude the proof. It is easy to see from the construction of $\mathcal{C}_N = \cap_{i=1}^{\infty} \mathcal{C}_N^i$ that the function $\theta_{N,d}(p)$, being the limit of decreasing continuous functions, is right-continuous. Thus, letting $\tau \to 0$ in (13), we obtain

$$\theta_{N,d}(p_c(N,d)) \geq 1-\epsilon$$

for all $N \geq N_1$. Since ϵ was arbitrary, this concludes the proof. QED

4.2 Proof of Theorem 1.3

We will use the following result from [21] cited in Section 3 and stated in the theorem below for convenience.

Theorem 4.2 ([21])**.** *For all $d \geq 3$, we have that $\tilde{p}_c(N,d) \geq 1 - p_c(\mathbb{M}^d)$ for every $N \geq 2$. Moreover, $\lim_{N\to\infty} \tilde{p}_c(N,d) = 1 - p_c(\mathbb{M}^d)$.*

With this, we are now ready to present the actual proof.

Proof of Theorem 1.3. Consider Mandelbrot's fractal percolation process with retention parameter p. Let $\mathcal{A}_k$ denote the event that there is complete retention until level k, i.e., $\mathcal{C}_N^k = [0,1]^d$. Trivially, for any $k \geq 1$,

$$\theta_{N,d}(p) \leq \mathbb{P}_p(CR(\mathcal{C}_N) \,|\, \mathcal{A}_k).$$

Assume that $\mathcal{A}_{k-1}$ has occurred. We will couple level k of the fractal percolation process, conditioned on $\mathcal{A}_{k-1}$, to an enhancement percolation process on $\mathbb{M}^d$ with density of enhancement $s = 1 - \tilde{\theta}_{N,d}(p)$ in a way similar to the proof of Theorem 1.1, as explained below.

If $\mathcal{A}_{k-1}$ occurs, at the next level of the fractal percolation process, the unit cube $[0,1]^d$ is divided in N^{dk} subcubes of side length $1/N^k$. Like in the proof of Theorem 1.1, we identify the subcubes with their centers $\{(\frac{x_1-1/2}{N^k}, \ldots, \frac{x_d-1/2}{N^k}), 1 \leq x_1, \ldots, x_d \leq N^k\}$. Each subcube is retained with probability p and discarded with probability $1-p$, independently of all other subcubes.

Contrary to the proof of Theorem 1.1, in our enhancement model, sites in $\mathbb{Z}^d \setminus \Lambda_{N^k}^d$ are declared open, while a site $x = (x_1, \ldots, x_d) \in \Lambda_{N^k}^d$ is declared closed if the cube $(\frac{x_1-1/2}{N^k}, \ldots, \frac{x_d-1/2}{N^k})$ is retained, and open otherwise. Once again, the states of sites in $\Lambda_{N^k}^d$ are determined by the fate of the level k subcubes of the fractal percolation process. This time however, all sites outside $\Lambda_{N^k}^d$ are open.

On this percolation model we now perform an enhancement according to the following rule. If a site $x = (x_1, \ldots, x_d) \in \Lambda_{N^k}^d$ is closed and $(x_1+1, x_2, \ldots, x_d)$ and $(x_1-1, x_2, \ldots, x_d)$ are both open, we look at the effect of the fractal percolation process within the subcube $(\frac{x_1-1/2}{N^k}, \ldots, \frac{x_d-1/2}{N^k})$. If the fractal percolation process inside that subcube does not leave a $(d-1)$-dimensional surface

separating its two faces perpendicular to the first coordinate direction, we declare site x open. This time the result is that each closed site $x = (x_1, \ldots, x_d) \in \Lambda_{N^k}^d$ such that its two neighbors $(x_1 + 1, x_2, \ldots, x_d)$ and $(x_1 - 1, x_2, \ldots, x_d)$ are both open is independently declared open with probability $1 - \tilde{\theta}_{N,d}(p)$, while sites that were already open simply stay open. This is obviously an enhancement in the sense of Section 3. We denote by $\mathbb{Q}^{enh}_{1-p,1-\tilde{\theta}_{N,d}(p)}$ the probability measure corresponding to the enhancement percolation process just defined (with initial configuration chosen as described above, i.e., with all sites outside $\Lambda_{N^k}^d$ open).

We now make two simple but crucial observations. First of all, recalling the definition of the enhancement percolation model of Section 3 and of the corresponding measure $\mathbb{P}^{enh}_{p,s}$, it is clear that

$$\mathbb{Q}^{enh}_{1-p,s}(CR(\Lambda_{N^k}^d, \mathbb{M}^d)) \geq \mathbb{P}^{enh}_{1-p,s}(CR(\Lambda_{N^k}^d, \mathbb{M}^d)) = \psi^d_{N^k}(1-p, s)$$

(where in this case $s = 1 - \tilde{\theta}_{N,d}(p)$).

Secondly, from topological considerations one can easily conclude that (conditioned on the event $\mathcal{A}_{k-1}$) the presence of an open crossing of $\Lambda_{N^k}^d$ in the first coordinate in the enhancement percolation process described above implies the absence of a sheet crossing of $[0,1]^d$ by the fractal set $\mathcal{C}_N$ separating the two faces perpendicular to the first coordinate direction. (Otherwise stated, if $\mathcal{A}_{k-1}$ happens, then "$SH(\mathcal{C}_N) \subset CR(\Lambda_{N^k}^d, \mathbb{M}^d)^c$," where the event on the left concerns the fractal percolation process and that on the right the enhancement percolation process, the inclusion should be interpreted in terms of the coupling between the two processes, and $CR(\cdot)^c$ denotes the complement of $CR(\cdot)$.)

From the previous discussion we can conclude that

$$\begin{aligned}
\tilde{\theta}_{N,d}(p) &\leq \mathbb{P}_p(SH(\mathcal{C}_N) \,|\, \mathcal{A}_{k-1}) \\
&\leq 1 - \mathbb{Q}^{enh}_{1-p,1-\tilde{\theta}_{N,d}(p)}(CR(\Lambda_{N^k}^d, \mathbb{M}^d)) \\
&\leq 1 - \mathbb{P}^{enh}_{1-p,1-\tilde{\theta}_{N,d}(p)}(CR(\Lambda_{N^k}^d, \mathbb{M}^d)) \\
&= 1 - \psi^d_{N^k}(1-p, 1-\tilde{\theta}_{N,d}(p)). \qquad (14)
\end{aligned}$$

As in the proof of Theorem 1.1, (14) can be used to show that, for any $\epsilon > 0$, there exists an $N_2 = N_2(\epsilon)$ such that, for every $N \geq N_2$, $\tilde{\theta}_N(\tilde{p}_c(N,d) + \tau) > 1 - \epsilon$ for all $\tau > 0$. To do this, we set $s = \epsilon$ in Lemma 3.1, obtain $\delta_1 = \delta_1(\epsilon) > 0$, and then use Theorem 4.2 to find an $N_2 = N_2(\delta_1(\epsilon))$ such that

$$0 \leq \tilde{p}_c(N,d) - (1 - p_c(\mathbb{M}^d)) < \delta_1/2,$$

or equivalently,

$$0 \leq p_c(\mathbb{M}^d) - (1 - \tilde{p}_c(N,d)) < \delta_1/2,$$

for all $N \geq N_2$.

From here, the proof can be concluded in essentially the same way as the proof of Theorem 1.1, with the difference that one needs to use Lemma 3.1 instead of Lemma 3.2. We leave the details to the reader. QED

5 Bounds for $p_c(N, 2)$.

In this section, we consider only two-dimensional fractal percolation, and for simplicity we will not indicate the dimension in the notation.

Recall the definition of correlation length from Section 2:

$$L_\delta(p) := \min\{N : \varphi_N(p) \geq 1 - \delta\},$$

for any $p > p_c = p_c(\mathbb{L}^2) = p_c(\mathbb{Z}^2, site)$ and any $0 < \delta < 1/2$.

Proof of Theorem 2.1. We start by proving (4). One can easily show (see [5]) that $\theta_N(p_c) = 0$. Since $\theta_N(p_c(N)) > 0$ for every $N \geq 2$, we deduce that $p_c(N) > p_c$ for every $N \geq 2$. Assume, by contradiction, that there exists a $C_0 < \infty$ such that

$$p_c(N) < p_c + C_0 \left(\frac{1}{N}\right)^{3/4} \tag{15}$$

for all $N \geq 2$. Fix $\delta > 0$; by assumption **H1** and (15), we have $L_\delta(p_c(N)) \geq c_1(\delta)(p_c(N) - p_c)^{-4/3} > c_1(\delta)C_0^{-4/3}N$, or equivalently, $N < c'L_\delta(p_c(N))$ for all $N \geq 2$ and some constant $c' = c'(\delta)$. It then follows from Theorem 26 of [19] applied to the square lattice (see also Theorem 1 of [15]), together with the Russo-Seymour-Welsh theorem, that $\varphi_N(p_c(N))$ is bounded away from 1 uniformly in N. (Roughly speaking, Theorem 1 of [15] and Theorem 26 of [19] state that crossing probabilities at $p \neq p_c$ are comparable to crossing probabilities at p_c, provided that one considers crossings of domains that are not much larger than the correlation length at p — see Section 8 of [19] concerning the applicability of such results to various regular lattices.) However, that leads to a contradiction since $\varphi_N(p_c(N)) \geq \theta_N(p_c(N))$ and $\lim_{N\to\infty} \theta_N(p_c(N)) = 1$ by Theorem 1.1.

We now proceed to prove (5). Take $N \geq 3m$ for some m, let R_m be a $3m/N \times m/N$ rectangle, and denote by $\mathcal{B}^1$ the event that R_m contains a left-to-right crossing by $\mathcal{C}_N^1$, and both the left and the right third of R_m contain a top-to-bottom crossing by $\mathcal{C}_N^1$ (see Appendix A for more motivation on why we consider this event, and its relation with the functions $f_{N,m}$ and $g_{N,m}$ that will be introduced below). For any $\delta_1 > 0$, one can choose $\delta_0 > 0$ so small that the probability of the event $\mathcal{B}^1$ for any $p > p_c$ is at least $1 - \delta_1$ when m is chosen to be $L_{\delta_0}(p)$. To see this note that, letting $m = L_{\delta_0}(p)$, we have $\varphi_m(p) \geq 1 - \delta_0$. The claim then follows from standard arguments using the Russo-Seymour-Welsh theorem and the Harris-FKG inequality. Observe that the choice of δ_0 is only dependent on δ_1.

For any $N \geq 2$, let

$$p_N := p_c + D \left(\frac{\log N}{N}\right)^{3/4}, \tag{16}$$

where D is a constant yet to be determined. Furthermore, we will let $m = L_{\delta_0}(p_N)$, for a $\delta_0 > 0$ to be chosen later. The definition of p_N and hypothesis **H1** imply that

$$m \leq \frac{c_2(\delta_0)}{D^{4/3}} \frac{N}{\log N}. \tag{17}$$

Following [5], we now introduce the functions

$$f_{N,m}(y) := \frac{c_3}{y(1-gy^{1/4})} N^2 (gy^{1/4})^{N/2m}$$

and

$$g_{N,m}(y) := \frac{c_4}{y(1-gy^{1/4})} \frac{N}{m} (gy^{1/4})^{N/2m} = \frac{c_4}{c_3} \frac{1}{Nm} f_{N,m}(y)$$

where $c_3, c_4 > 0$ are finite constants and $g \leq 3$. The role of these two functions will be clear soon. For completeness and the reader's convenience, their derivation is presented in Appendix A.

Fix a $y_0 > 0$ such that $gy_0^{1/4} < 1$, let $\delta_1 = y_0/2$, and choose $\delta_0 > 0$ so that $\mathbb{P}_p(\mathcal{B}^1) \geq 1 - \delta_1$ for any $p > p_c$. Choose D in the definition of p_N large enough so that

$$h_N(y) := \frac{c_5}{y} N^2 (gy^{1/4})^{\frac{D^{4/3}}{2c_2(\delta_0)} \log N} \to 0, \quad \forall\, 0 < y \leq y_0 \quad \text{as } N \to \infty,$$

where $c_5 = \frac{c_3}{1-gy_0^{1/4}}$. Recalling that $m = L_{\delta_0}(p_N)$, we can then use (17) to conclude that, for all $y \leq y_0$,

$$f_{N,m}(y) \leq h_N(y). \tag{18}$$

Since for $y > 0$, $h_N(y)$ is a positive, continuous function, and $h_N(y) \to 0$ as $N \to \infty$, $\forall y \in (0, y_0]$, it is easy to see that for N sufficiently large, the equation

$$y = \delta_1 + h_N(y) \tag{19}$$

has a solution $y_2(N)$ which converges to δ_1 as $N \to \infty$. Moreover, (18) implies that, whenever (19) has a solution $y_2(N)$,

$$y = \delta_1 + f_{N,m}(y) \tag{20}$$

has a solution $y_1(N)$ such that $y_1(N) \leq y_2(N)$. We pick N_1 such that, for all $N \geq N_1$, (19) has a solution and $h_N(y)$ is increasing in y.

We can now use a bound from [5], rederived in Appendix A (see (23) there), to show that

$$\begin{aligned} \mathbb{P}_{p_N}(CR(\mathcal{C}_N^k)) &\geq 1 - g_{N,m}(\delta_k) \\ &= 1 - \frac{c_4}{c_3} \frac{1}{Nm} f_{N,m}(\delta_k) \\ &\geq 1 - \frac{c_4}{c_3} \frac{h_N(\delta_k)}{N}, \end{aligned} \tag{21}$$

where $\{\delta_k\}_{k \geq 2}$ is some sequence that satisfies the iterative inequality

$$\delta_k \leq \delta_1 + f_{N,m}(\delta_{k-1}). \tag{22}$$

In order for (21) to hold, it suffices that $\mathbb{P}_{p_N}(\mathcal{B}^1) \geq 1 - \delta_1$, which is satisfied for our choice of p_N and $m (= L_{\delta_0}(p_N))$. When (20) has a solution y_1, the sequence $\{\delta_k\}_{k \in \mathbb{N}}$ must have a limit $\delta^* \leq y_1$.

For $N \geq N_1$, letting $k \to \infty$ on both sides of (21), and using (19) and the fact that $\delta^* \leq y_1 \leq y_2$ and $h_N(y)$ is increasing in y, we obtain

$$\begin{aligned} \theta_N(p_N) &\geq 1 - \frac{c_4}{c_3}\frac{h_N(\delta^*)}{N} \\ &\geq 1 - \frac{c_4}{c_3}\frac{h_N(y_2)}{N} \\ &= 1 - \frac{c_4}{c_3}\frac{(y_2 - \delta_1)}{N}. \end{aligned}$$

Since $\lim_{N\to\infty} y_2(N) = \delta_1$, as remarked above, we can conclude that there exists an N_1 (possibly larger than required before) such that

$$\theta_N(p_N) \geq 1 - \frac{c_4}{c_3}\frac{(y_2 - \delta_1)}{N} > 0$$

for all $N \geq N_1$, which concludes the proof. QED

Acknowledgements. It is a pleasure to thank Ronald Meester for many interesting discussions on subjects closely related to the content of this paper. The interest of one of the authors (F.C.) in Mandelbrot's fractal percolation process was started by a question of Alberto Gandolfi during the workshop "Stochastic Processes in Mathematical Physics" at Villa La Pietra, Florence, June 19-23, 2006. The other author (E.B.) thanks everyone at the Vrije Universiteit Amsterdam and Ronald Meester in particular for making his one year stay possible as well as enjoyable and productive.

Appendix A

For completeness and for the reader's convenience, following [5], in this appendix we present the derivation of the functions $f_{N,m}$ and $g_{N,m}$ used in the proof of Theorem 2.1 (note that $g_{N,m}$ is not given a name in [5]).

Let m be a fixed number much smaller than N, and denote by $H_m(x_1, x_2)$ the $3m/N \times m/N$ rectangle $\{y = (y_1, y_2) \in \mathbb{R}^2 : x_1 \leq y_1 \leq x_1 + 3m/N, x_2 \leq y_2 \leq x_2 + m/N\}$, and by $V_m(x_1, x_2)$ the $m/N \times 3m/N$ rectangle $\{y = (y_1, y_2) \in \mathbb{R}^2 : x_1 \leq y_1 \leq x_1 + m/N, x_2 \leq y_2 \leq x_2 + 3m/N\}$. If at level k of the fractal percolation process $H_m(x_1, x_2)$ contains a left-to-right crossing by $\mathcal{C}_N^k$, and both the left and the right third of $H_m(x_1, x_2)$ contain a top-to-bottom crossing by $\mathcal{C}_N^k$, we say that the event $\mathcal{H}^k(x_1, x_2)$ happens. Analogously, if at level k of the fractal percolation process $V_m(x_1, x_2)$ contains a top-to-bottom crossing by $\mathcal{C}_N^k$, and both the top and the bottom third of $V_m(x_1, x_2)$ contain a left-to-right crossing by $\mathcal{C}_N^k$, we say that $\mathcal{V}^k(x_1, x_2)$ happens.

We can partially cover the unit square $[0, 1]^2$ using rectangles $H_m(x_1, x_2)$ and $V_m(x_1, x_2)$ with x_1 taking values $0, 2m/N, 4m/N, \ldots$ and x_2 taking values $0, 2m/N, 4m/N, \ldots$. We call $\mathcal{R}$ the set of overlapping rectangles used for this partial covering of the unit square, and assume for simplicity that N/m is an odd integer (see Figure 1). The readers can easily convince themselves that this assumption can be safely removed in the arguments below. The probabilities of the crossing events $\mathcal{H}^k(\cdot, \cdot)$ and $\mathcal{V}^k(\cdot, \cdot)$ are equal and are the same for all rectangles in $\mathcal{R}$. We therefore denote them π_k.

We identify the $m/N \times m/N$ squares where the rectangles from $\mathcal{R}$ overlap with the sites of a portion of the square lattice (see Fig. 1). On this portion of the square lattice, we consider a family of bond percolation models indexed by $k = 1, 2, \dots$. In the k-th member of the family, we declare a horizontal bond between two adjacent sites open if the event $\mathcal{H}^k(x_1, x_2)$ happens, where (x_1, x_2) is the bottom-left corner of the $m/N \times m/N$ square corresponding to the leftmost of the two adjacent sites. Analogously, we declare a vertical bond between two adjacent sites open if the event $\mathcal{V}^k(x_1, x_2)$ happens, where (x_1, x_2) is the bottom-left corner of the $m/N \times m/N$ square corresponding to the lower site. Note that we obtain a family of dependent bond percolation models with densities $\pi_k, k = 1, 2, \dots$, of open bonds, and that a left-to-right crossing in the k-th bond percolation process implies a left-to-right crossing of the unit square by $\mathcal{C}_N^k$. Note also that for $k = 1$, the event $\mathcal{H}^1(\cdot, \cdot)$ (resp., $\mathcal{V}^1(\cdot, \cdot)$) is simply a crossing event for Bernoulli site percolation on a $3m \times m$ (resp., a $m \times 3m$) rectangular portion of the square lattice.

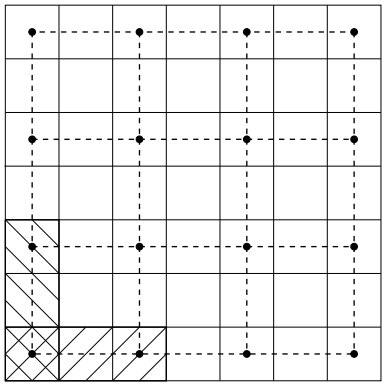

Figure 1: The unit square is partially covered by overlapping $3m/N \times m/N$ and $m/N \times 3m/N$ rectangles like the shaded ones shown in the bottom-left corner. The centers of the squares where the rectangles overlap, indicated by dots, are the sites of an associated portion of the square lattice whose bonds are indicated by broken segments.

Let us now consider for definiteness a given rectangle R_m from $\mathcal{R}$. We denote by $\mathcal{B}^k$ the relevant crossing event (either $\mathcal{H}^k(\cdot, \cdot)$ or $\mathcal{V}^k(\cdot, \cdot)$) inside R_m at the k:th step of the fractal percolation process. We have $\mathbb{P}_p(\mathcal{B}^k) = \pi_k$. Let us assume that $\pi_1 \geq 1 - \delta_1$ for some fixed δ_1. Assuming that $\mathcal{B}^1$ happens, we are going to obtain a lower bound for $\mathbb{P}_p(\mathcal{B}^2)$ in terms of δ_1 (see Fig. 2).

After the second step of the fractal percolation process, R_m is split into $3m^2 \times N^2$ squares of side length $1/N^2$. We can partially cover R_m using rectangles $H_{m/N}(\cdot, \cdot)$ and $V_{m/N}(\cdot, \cdot)$ in the same fashion as we did for the unit square. Each such rectangle contains $3m^2$ squares of side length $1/N^2$. If $R_{m/N}$ is a rectangle contained inside a $1/N \times 1/N$ square that was retained in the first step of the fractal percolation process, the probability to see inside $R_{m/N}$ a scaled copy of the event $\mathcal{B}^1$ is exactly π_1 because of the scale invariance of the fractal percolation process. More generally, as we did above inside the unit square, we can couple step $k + 1$ of the fractal percolation process inside the retained portion of R_m to a dependent bond percolation process with density of open bonds π_k.

We will now use the bond percolation process coupled to the second level of the fractal percolation process to obtain a lower bound for $\mathbb{P}_p(\mathcal{B}^2 \,|\, \mathcal{B}^1)$ in terms of δ_1. One way to ensure the occurrence of $\mathcal{B}^2$ is by a string of open bonds in the coupled bond percolation process. The event that such

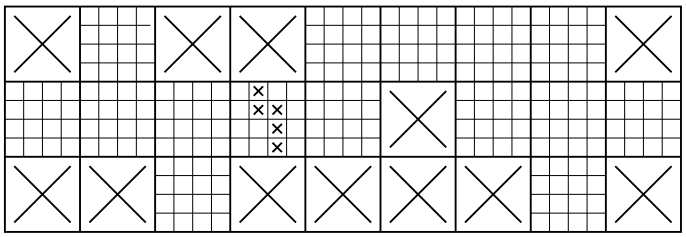

Figure 2: A $3m/N \times m/N$ rectangle where the event $\mathcal{B}^1$ happens but $\mathcal{B}^2$ does not. The larger squares are of size $1/N \times 1/N$. Squares with a cross are not retained. Retained squares are divided in smaller squares of size $1/N^2 \times 1/N^2$. Here, $m = 3$ and $N = 4$.

a string is not present is by duality equivalent to the presence of a string of closed dual bonds, where a dual bond is perpendicular to an ordinary bond and is declared closed (resp., open) if the (unique) ordinary bond it intersects is closed (resp., open). By simple worst-case-scenario counting arguments, one can give an upper bound for the probability of such a string of closed dual bonds, as explained below. Note that we are considering a dependent bond percolation model, but that the dependence range is finite (and in fact very short). We can certainly assume, as in [5], that finding a string consisting of l closed dual bonds implies at least $l/4$ independent trials.

By using trivial upper bounds for the number of strings of length l, we obtain

$$\begin{aligned}
\mathbb{P}_p(\mathcal{B}^2 \,|\, \mathcal{B}^1) &\geq 1 - 3m^2 \left(\frac{4N}{m}\right) \sum_{l \geq \frac{N}{2m}-1} g^l \delta_1^{l/4} \\
&\geq 1 - c_3' Nm \frac{(g\delta_1^{1/4})^{N/2m}}{g\delta_1^{1/4}(1 - g\delta_1^{1/4})} \\
&\geq 1 - c_3 N^2 \frac{(g\delta_1^{1/4})^{N/2m}}{\delta_1(1 - g\delta_1^{1/4})}
\end{aligned}$$

where $g \leq 3$ is an upper bound on the number of possible choices for attaching the next bond to a growing string, and $c_3', c_3 > 0$.

Note that in the first line of the above equation, the factor $\frac{4N}{m}$ comes from the fact that the string of closed dual bonds must start at one of the four sides of a retained $1/N \times 1/N$ square, while $3m^2$ is a trivial upper bound on the number of retained $1/N \times 1/N$ squares, which can certainly be no bigger than the total number of $1/N \times 1/N$ squares in R_m (see Fig. 2). The summation is over values of l large enough so that a string of length l can prevent the event $\mathcal{B}^2$. Introducing the function $f_{N,m}(y) := \frac{c_3}{y(1-gy^{1/4})} N^2 (gy^{1/4})^{N/2m}$, we can write $\mathbb{P}_p(\mathcal{B}^2 \,|\, \mathcal{B}^1) \geq 1 - f_{N,m}(\delta_1)$.

Defining δ_2 via the equation $\pi_2 = \mathbb{P}_p(\mathcal{B}^2) = 1 - \delta_2$, from the discussion above we have

$$1 - \delta_2 = \mathbb{P}_p(\mathcal{B}^2 \,|\, \mathcal{B}^1)\mathbb{P}_p(\mathcal{B}^1) \geq (1 - f_{N,m}(\delta_1))(1 - \delta_1) \geq 1 - \delta_1 - f_{N,m}(\delta_1).$$

Next, we are going to obtain a lower bound for $\mathbb{P}_p(\mathcal{B}^3 \,|\, \mathcal{B}^1)$ in terms of δ_2.

One way to ensure the occurrence of $\mathcal{B}^3$ is by a string of open bonds in the dependent bond percolation process coupled to the third step of the fractal percolation process inside the portion

of R_m retained after the first step of the fractal percolation process. The density of open bonds is now π_2. We can then repeat exactly the same arguments as above, but with π_2 instead of π_1 as density of open bonds. This obviously gives us $\mathbb{P}_p(\mathcal{B}^3 \,|\, \mathcal{B}^1) \geq 1 - f_{N,m}(\delta_2)$ and $1-\delta_3 \geq 1-\delta_1-f_{N,m}(\delta_2)$, where δ_3 is defined via the equation $\pi_3 = \mathbb{P}_p(\mathcal{B}^3) = 1-\delta_3$. Proceeding by induction, we obtain the iterative inequality $1 - \delta_{k+1} \geq 1 - \delta_1 - f_{N,m}(\delta_k)$, i.e.,

$$\delta_{k+1} \leq \delta_1 + f_{N,m}(\delta_k).$$

A way to ensure a left-to-right crossing of the unit square by $\mathcal{C}_N^k$ is by a left-to-right crossing of open bonds in the dependent bond percolation process coupled to step k of the fractal percolation process inside the unit square. Therefore, using arguments analogous to those used above, one can obtain a lower bound for the probability of a left-to-right crossing of the unit square by $\mathcal{C}_N^k$ in terms of δ_k (or $\pi_k = 1 - \delta_k$). Counting arguments and approximations analogous to those used before give

$$\mathbb{P}_p(CR(\mathcal{C}_N^k)) \geq 1 - g_{N,m}(\delta_k), \tag{23}$$

where $g_{N,m}(y) := \frac{c_4}{y(1-gy^{1/4})}\frac{N}{m}(gy^{1/4})^{N/2m}$, with c_4 a positive constant.

References

[1] M. Aizenmann and G. Grimmett, Strict Monotonicity for Critical Points in Percolation and Ferromagnetic Models, *J. Stat. Phys.* **63**, 817–835 (1991). MR1116036

[2] C. Borgs, J. Chayes, H. Kesten and J. Spencer, The Birth of the Infinite Cluster: Finite-Size Scaling in Percolation, *Comm. Math. Phys.* **224**, 153–204 (2001). MR1868996

[3] F. Camia, L. R. Fontes and C. M. Newman, The Scaling Limit Geometry of Near-Critical 2D Percolation, *J. Stat. Phys.* **125**, 1155-1171 (2006). MR2282484

[4] F. Camia, L. R. Fontes and C. M. Newman, Two-Dimensional Scaling Limits via Marked Nonsimple Loops, *Bull. Braz. Math. Soc.* **37**, 537-559 (2006). MR2284886

[5] J.T. Chayes and L. Chayes, The large-N limit of the threshold values in Mandelbrot's fractal percolation process, *J. Phys. A: Math. Gen.* **22**, L501–L506 (1989).

[6] J.T. Chayes, L. Chayes and R. Durrett, Connectivity Properties of Mandelbrot's Percolation Process, *Probab. Theory Relat. Fields* **77**, 307–324 (1988). MR0931500

[7] J.T. Chayes, L. Chayes, E. Grannan and G. Swindle, Phase transitions in Mandelbrot's percolation process in three dimensions, *Probab. Theory Relat. Fields* **90**, 291–300 (1991). MR1133368

[8] L. Chayes, Aspects of the fractal percolation process, *Progress in Probability* **37**, 113–143 (1995). MR1391973

[9] L. Chayes and P. Nolin, Large Scale Properties of the IIIC for 2D Percolation, preprint arXiv:0705.357 (2007).

[10] F.M. Dekking and G.R. Grimmett, Superbranching processes and projections of random Cantor sets, *Probab. Theory Relat. Fields* **78**, 335–355 (1988). MR0949177

[11] F.M. Dekking and R.W.J. Meester, On the structure of Mandelbrot's percolation process and other Random Cantor sets, *J. Stat. Phys.* **58**, 1109–1126 (1990). MR1049059

[12] K.J. Falconer and G.R. Grimmett, The critical point of fractal percolation in three and more dimensions, *J. Phys. A: Math. Gen.* **24**, L491–L494 (1991). MR1117858

[13] K.J. Falconer and G.R. Grimmett, On the geometry of Random Cantor Sets and Fractal Percolation, *J. Theor. Probab.* **5**, 465–485 (1992). MR1176432

[14] G. Grimmett, *Percolation*, Second edition, Springer-Verlag, Berlin (1999). MR1707339

[15] H. Kesten, Scaling Relations for 2D-Percolation, *Commun. Math. Phys.* **109**, 109–156 (1987). MR0879034

[16] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman, San Francisco (1983). MR0665254

[17] R.W.J. Meester, Connectivity in fractal percolation, *J. Theor. Prob.* **5**, 775–789 (1992). MR1182680

[18] M.V. Menshikov, S.Yu. Popov and M. Vachkovskaia, On the connectivity properties of the complementary set in fractal percolation models, *Probab. Theory Relat. Fields* **119**, 176–186 (2001). MR1818245

[19] P. Nolin, Near-critical percolation in two dimensions, preprint arXiv:0711.4948.

[20] P. Nolin, W. Werner, Asymmetry of near-critical percolation interfaces, preprint arXiv:0710.1470.

[21] M.E. Orzechowski, On the Phase Transition to Sheet Percolation in Random Cantor Sets, *J. Stat. Phys.* **82**, 1081–1098 (1996). MR1372435

[22] S. Smirnov and W. Werner, Critical exponents for two-dimensional percolation, *Math. Res. Lett.* **8**, 729–744 (2001). MR1879816

[23] D.G. White, On fractal percolation in $\mathbb{R}^2$, *Statist. Probab. Lett.* **45**, 187–190 (1999). MR1718427