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L_p ESTIMATES FOR SPDE WITH DISCONTINUOUS COEFFICIENTS IN DOMAINS

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Abstract. Stochastic partial differential equations of divergence form with discontinuous and unbounded coefficients are considered in C^1 domains. Existence and uniqueness results are given in weighted L_p spaces, and Hölder type estimates are presented.

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1. Introduction

Let G be an open set in $\mathbb{R}^d$. We consider parabolic stochastic partial differential equations of the form

$$du = (D_i(a^{ij}u_{x^j} + b^i u + f^i) + \bar{b}^i u_{x^i} + cu + \bar{f})\,dt + (\nu^k u + g^k)\,dw_t^k, \quad (1.1)$$

given for $x \in G, t \geq 0$. Here w_t^k are independent one-dimensional Wiener processes, i and j go from 1 to d, and k runs through $\{1, 2, ...\}$. The coefficients a^{ij}, b^i, $\bar{b}^i$, c, ν^k and the free terms $f^i, \bar{f}, g^k$ are random functions depending on $t > 0$ and $x \in G$.

This article is a natural continuation of the article [15], where L_p estimates for the equation

$$du = D_i(a^{ij}u_{x^j} + f^i)\,dt + (\nu^k u + g^k)\,dw_t^k \quad (1.2)$$

with discontinuous coefficients was constructed on $\mathbb{R}^d$.

Our approach is based on Sobolev spaces with or without weights, and we present the unique solvability result of equation (1.1) on $\mathbb{R}^d, \mathbb{R}^d_+$ (half space) and on bounded C^1 domains. We show that L_p-norm of u_x can be controlled by L_p-norms of $f^i, \bar{f}$ and g if p is sufficiently close to 2.

Pulvirenti [13] showed by example that without the continuity of a^{ij} in x one can not fix p even for deterministic parabolic equations. For an L_p theory of linear SPDEs with continuous coefficients on domains, we refer to [1], [2] and [7].

Actually L_2 theory for type (1.1) with bounded coefficients was developed long times ago on the basis of monotonicity method, and an account of it can be found in [14]. But our results are new even for $p = 2$ (and probably even for determistic equation) since, for instance, we are only assuming the functions

$$\rho b^i, \quad \rho\bar{b}^i, \quad \rho^2 c, \quad \rho\nu^k$$

are bounded, where $\rho(x) = \mathrm{dist}(x, \partial G)$. Thus we are allowing our coefficients to blow up near the boundary of G.

An advantage of $L_p(p > 2)$ theory can be found, for instance, in [16], where solvability of some nonlinear SPDEs was presented with the help of L_p estimates for linear SPDEs with discontinuous coefficients. Also we will see that some Hölder type estimates are valid only for $p > 2$ (Corollary 2.5).

We finish the introduction with some notations. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$ and $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$. For $i = 1, ..., d$, multi-indices

$\alpha=(\alpha_1,...,\alpha_d)$, $\alpha_i\in\{0,1,2,...\}$, and functions $u(x)$ we set

$$u_{x^i}=\partial u/\partial x^i=D_iu,\quad D^\alpha u=D_1^{\alpha_1}\cdot...\cdot D_d^{\alpha_d}u,\quad |\alpha|=\alpha_1+...+\alpha_d.$$

2. Main Results

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, $\{\mathcal{F}_t,t\geq 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t\subset\mathcal{F}$, each of which contains all $(\mathcal{F},P)$-null sets. By $\mathcal{P}$ we denote the predictable σ-field generated by $\{\mathcal{F}_t,t\geq 0\}$ and we assume that on Ω we are given independent one-dimensional Wiener processes $w^1_t,w^2_t,...$, each of which is a Wiener process relative to $\{\mathcal{F}_t,t\geq 0\}$.

Fix an increasing function κ_0 defined on $[0,\infty)$ such that $\kappa_0(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$.

Assumption 2.1. The domain $G\subset\mathbb{R}^d$ is of class C^1_u. In other words, there exist constants $r_0,K_0>0$ such that for any $x_0\in\partial G$ there exists a one-to-one continuously differentiable mapping Ψ from $B_{r_0}(x_0)$ onto a domain $J\subset\mathbb{R}^d$ such that

(i) $J_+:=\Psi(B_{r_0}(x_0)\cap G)\subset\mathbb{R}^d_+$ and $\Psi(x_0)=0$;

(ii) $\Psi(B_{r_0}(x_0)\cap\partial G)=J\cap\{y\in\mathbb{R}^d:y^1=0\}$;

(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))}\leq K_0$ and $|\Psi^{-1}(y_1)-\Psi^{-1}(y_2)|\leq K_0|y_1-y_2|$ for any $y_i\in J$;

(iv) $|\Psi_x(x_1)-\Psi_x(x_2)|\leq\kappa_0(|x_1-x_2|)$ for any $x_i\in B_{r_0}(x_0)$.

Assumption 2.2. (i) For each $x\in G$, the functions $a^{ij}(t,x)$, $b^i(t,x)$, $\bar{b}^i(t,x)$, $c(t,x)$ and $\nu^k(t,x)$ are predictable functions of (ω,t).

(ii) There exist constants $\lambda,\Lambda\in(0,\infty)$ such that for any ω,t,x and $\xi\in\mathbb{R}^d$,

$$\lambda|\xi|^2\leq a^{ij}\xi^i\xi^j\leq\Lambda|\xi|^2.$$

(iii) For any x,t and ω,

$$\rho(x)|b^i(t,x)|+\rho(x)|\bar{b}^i(t,x)|+\rho(x)^2|c(t,x)|+\rho(x)|\nu^k(t,x)|_{\ell_2}\leq K.$$

(iv) There is control on the behavior of $b^i,\bar{b}^i,c,\nu$ near ∂G, namely,

$$\lim_{\substack{\rho(x)\to 0\\ x\in G}}\sup_{t,\omega}\rho(x)(|b^i(t,x)|+|\bar{b}^i(t,x)|+\rho(x)|c(t,x)|+|\nu(t,x)|_{\ell_2})=0. \quad (2.1)$$

To describe the assumptions of $f^i,\bar{f}$ and g we use Sobolev spaces introduced in [7], [8] and [12]. If n is a non negative integer, then

$$H^n_p=H^n_p(\mathbb{R}^d)=\{u:u,Du,...,D^\alpha u\in L_p:|\alpha|\leq n\},$$

$$L_{p,\theta}(G):=H^0_{p,\theta}(G)=L_p(G,\rho^{\theta-d}dx),\quad \rho(x):=\text{dist}(x,\partial G),$$

$$H^n_{p,\theta}(G):=\{u:u,\rho u_x,...,\rho^{|\alpha|}D^\alpha u\in L_{p,\theta}(G):|\alpha|\leq n\}. \quad (2.2)$$

In general, by $H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2}L_p$ we denote the space of Bessel potential, where

$$\|u\|_{H^\gamma_p} = \|(1-\Delta)^{\gamma/2}u\|_{L_p},$$

and the weighted Sobolev space $H^\gamma_{p,\theta}(G)$ is defined as the set of all distributions u on G such that

$$\|u\|^p_{H^\gamma_{p,\theta}(G)} := \sum_{n=-\infty}^{\infty} e^{n\theta}\|\zeta_{-n}(e^n\cdot)u(e^n\cdot)\|^p_{H^\gamma_p} < \infty, \tag{2.3}$$

where $\{\zeta_n : n \in \mathbb{Z}\}$ is a sequence of functions $\zeta_n \in C^\infty_0(G)$ such that

$$\sum_n \zeta_n \geq c > 0, \quad |D^m\zeta_n(x)| \leq N(m)e^{mn}.$$

If $G = \mathbb{R}^d_+$ we fix a function $\zeta \in C^\infty_0(\mathbb{R}_+)$ such that

$$\sum_{n\in\mathbb{Z}} \zeta(e^{n+x}) \geq c > 0, \quad \forall x \in \mathbb{R}, \tag{2.4}$$

and define $\zeta_n(x) = \zeta(e^n x)$, then (2.3) becomes

$$\|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n=-\infty}^{\infty} e^{n\theta}\|\zeta(\cdot)u(e^n\cdot)\|^p_{H^\gamma_p} < \infty. \tag{2.5}$$

It is known that up to equivalent norms the space $H^\gamma_{p,\theta}$ is independent of the choice ζ, and $H^\gamma_{p,\theta}(G)$ and its norm are independent of $\{\zeta_n\}$ if G is bounded.

We use above notations for ℓ_2-valued functions $g = (g_1, g_2, ...)$. For instance

$$\|g\|_{H^\gamma_p(\ell_2)} = \||(1-\Delta)^{\gamma/2}g|_{\ell_2}\|_{L_p}.$$

For any stopping time τ, denote $(\!(0,\tau]\!] = \{(\omega,t) : 0 < t \leq \tau(\omega)\}$,

$$\mathbb{H}^\gamma_p(\tau) = L_p((\!(0,\tau]\!], \mathcal{P}, H^\gamma_p), \quad \mathbb{H}^\gamma_{p,\theta}(G,\tau) = L_p((\!(0,\tau]\!], \mathcal{P}, H^\gamma_{p,\theta}(G)),$$

$$\mathbb{H}^\gamma_{p,\theta}(\tau) = L_p((\!(0,\tau]\!], \mathcal{P}, H^\gamma_{p,\theta}), \quad \mathbb{L}_{...}(...) = \mathbb{H}^0_{...}(...).$$

Fix (see [5]) a bounded real-valued function ψ defined in $\bar{G}$ such that for any multi-index α,

$$[\psi]^{(0)}_{|\alpha|} := \sup_G \rho^{|\alpha|}(x)|D^\alpha\psi_x(x)| < \infty$$

and the functions ψ and ρ are comparable in a neighborhood of ∂G. As in [11], by M^α we denote the operator of multiplying by $(x^1)^\alpha$ and $M = M^1$. Define

$$U^\gamma_p = L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_p), \quad U^\gamma_{p,\theta} = M^{1-2/p}L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,\theta}),$$

$$U^\gamma_{p,\theta}(G) = \psi^{1-2/p}L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,\theta}(G)).$$

By $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ we denote the space of all functions $u \in \psi\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)$ such that $u(0,\cdot) \in U^{\gamma}_{p,\theta}(G)$ and for some $f \in \psi^{-1}\mathbb{H}^{\gamma-2}_{p,\theta}(G,\tau)$, $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(G,\tau)$,

$$du = f\,dt + g^k\,dw^k_t, \tag{2.6}$$

in the sense of distributions. In other words, for any $\phi \in C^{\infty}_0(G)$, the equality

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (f(s,\cdot),\phi)\,ds + \sum_0^{\infty}\int_0^t (g^k(s,\cdot),\phi)\,dw^k_s$$

holds for all $t \leq \tau$ with probability 1.

The norm in $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)} = \|\psi^{-1}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)} + \|\psi f\|_{\mathbb{H}^{\gamma-2}_{p,\theta}(G,\tau)}$$
$$+\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(G,\tau)} + \|u(0,\cdot)\|_{U^{\gamma}_{p,\theta}(G)}.$$

It is easy to check that up to equivalent norms the space $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ and its norm are independent of the choice of ψ if G is bounded.

We write $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ if $u \in M\mathbb{H}^{\gamma}_{p,\theta}(\tau)$ satisfies (2.6) for some $f \in M^{-1}\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)$, $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\tau,\ell_2)$, and we define

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(\tau)} = \|M^{-1}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + \|Mf\|_{\mathbb{H}^{\gamma-2}_{p,\theta}(\tau)}$$
$$+\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)} + \|u(0,\cdot)\|_{U^{\gamma}_{p,\theta}}.$$

Similarly we define stochastic Banach space $\mathcal{H}^{\gamma}_p(\tau)$ on $\mathbb{R}^d$ (and its norm) by formally taking $\psi = 1$ and replacing $H^{\gamma}_{p,\theta}(G), U^{\gamma}_{p,\theta}(G)$ by $H^{\gamma}_p, U^{\gamma}_p$, respectively, in the definition of the space $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$.

We drop τ in the notations of appropriate Banach spaces if $\tau \equiv \infty$. Note that if $G = \mathbb{R}^d_+$, then $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ is slightly different from $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ since $\psi(x)$ is bounded. Finally we define

$$\mathfrak{H}^{\gamma}_{p,\theta,0}(...) = \mathfrak{H}^{\gamma}_{p,\theta}(...) \cap \{u : u(0,\cdot) = 0\},$$
$$\mathcal{H}^{\gamma}_{p,0}(...) = \mathcal{H}^{\gamma}_{p}(...) \cap \{u : u(0,\cdot) = 0\}.$$

Some properties of the spaces $H^{\gamma}_{p,\theta}, \mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ and $\mathcal{H}^{\gamma}_p(\tau)$ are collected in the following lemma (see [3],[7], [8] and [12] for detail). From now on we assume that

$$p \geq 2, \quad d-1 < \theta < d-1+p.$$

Lemma 2.3. (i) *The following are equivalent:*

$(a)\, u \in H^{\gamma}_{p,\theta}(G)$,

$(b)\, u \in H^{\gamma-1}_{p,\theta}(G)$ *and* $\psi Du \in H^{\gamma-1}_{p,\theta}(G)$,

$(c)\, u \in H^{\gamma-1}_{p,\theta}(G)$ *and* $D(\psi u) \in H^{\gamma-1}_{p,\theta}(G)$.

In addition, under either of these three conditions

$$\|u\|_{H^{\gamma}_{p,\theta}(G)} \leq N\|\psi u_x\|_{H^{\gamma-1}_{p,\theta}(G)} \leq N\|u\|_{H^{\gamma}_{p,\theta}(G)}, \tag{2.7}$$

$$\|u\|_{H^{\gamma}_{p,\theta}(G)} \leq N\|(\psi u)_x\|_{H^{\gamma-1}_{p,\theta}(G)} \leq N\|u\|_{H^{\gamma}_{p,\theta}(G)}. \tag{2.8}$$

(ii) For any $\nu, \gamma \in \mathbb{R}$, $\psi^{\nu} H^{\gamma}_{p,\theta}(G) = H^{\gamma}_{p,\theta-p\nu}(G)$, and

$$\|u\|_{H^{\gamma}_{p,\theta-p\nu}(G)} \leq N\|\psi^{-\nu}u\|_{H^{\gamma}_{p,\theta}(G)} \leq N\|u\|_{H^{\gamma}_{p,\theta-p\nu}(G)}.$$

(iii) There exists a constant N depending only on d, p, γ, T (and θ) such that for any $t \leq T$,

$$\|u\|^p_{H^{\gamma}_{p,\theta}(G,t)} \leq N\int_0^t \|u\|^p_{\mathfrak{H}^{\gamma+1}_{p,\theta}(G,s)}\,ds \leq Nt\|u\|^p_{\mathfrak{H}^{\gamma+1}_{p,\theta}(G,t)}, \tag{2.9}$$

$$\|u\|^p_{H^{\gamma}_{p}(t)} \leq N\int_0^t \|u\|^p_{\mathcal{H}^{\gamma+1}_{p}(s)}\,ds \leq Nt\|u\|^p_{\mathcal{H}^{\gamma+1}_{p}(t)}. \tag{2.10}$$

(iv) Let $\gamma - d/p = m + \nu$ for some $m = 0, 1, ...$ and $\nu \in (0,1)$, then for any $k \leq m$,

$$|\psi^{k+\theta/p}D^k u|_{C^0} + [\psi^{m+\nu+\theta/p}D^m u]_{C^{\nu}(G)} \leq N\|u\|_{H^{\gamma}_{p,\theta}(G)}.$$

(v) Let

$$2/p < \alpha < \beta \leq 1.$$

Then for any $u \in \mathfrak{H}^{\gamma}_{p,\theta,0}(G,\tau)$ and $0 \leq s < t \leq \tau$,

$$E\|\psi^{\beta-1}(u(t)-u(s))\|^p_{H^{\gamma-\beta}_{p,\theta}(G)} \leq N|t-s|^{p\beta/2-1}\|u\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)}, \tag{2.11}$$

$$E|\psi^{\beta-1}u|^p_{C^{\alpha/2-1/p}([0,\tau],H^{\gamma-\beta}_{p,\theta}(G))} \leq N\|u\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)}. \tag{2.12}$$

Here are our main results.

Theorem 2.4. *Assume G is bounded and $\tau \leq T$. Under the above assumptions, there exist $p_0 = p_0(\lambda, \Lambda, d) > 2$ and $\chi = \chi(p, d, \lambda, \Lambda) > 0$ such that if $p \in [2, p_0)$ and $\theta \in (d-\chi, d+\chi)$, then*

(i) for any $f^i \in \mathbb{L}_{p,\theta}(G,\tau)$, $\bar{f} \in \psi^{-1}\mathbb{H}^{-1}_{p,\theta}(G,\tau)$, $g \in \mathbb{L}_{p,\theta}(G,\tau)$ and $u_0 \in U^1_{p,\theta}(G)$ equation (1.1) admits a unique solution $u \in \mathfrak{H}^1_{p,\theta}(G,\tau)$,

(ii) for this solution

$$\|\psi^{-1}u\|_{\mathbb{H}^1_{p,\theta}(G,\tau)} \leq N(\|f^i\|_{\mathbb{L}_{p,\theta}(G,\tau)} + \|\psi\bar{f}\|_{\mathbb{H}^{-1}_{p,\theta}(G,\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(G,\tau)} + \|u_0\|_{U^1_{p,\theta}(G)}), \tag{2.13}$$

where the constant N is independent of $f^i, \bar{f}, g, u$ and u_0.

Lemma 2.3 (iv) and (v) yield the following results. It is crucial that p is bigger than 2.

Corollary 2.5. *Let $u \in \mathfrak{H}^1_{p,\theta,0}(G,\tau)$ be the solution of (1.1) and*

$$2/p < \alpha < \beta \le 1.$$

(i) Then for any $0 \le s < t \le \tau$,

$$E\|\psi^{\beta-1}(u(t)-u(s))\|^p_{H^{1-\beta}_{p,\theta}(G)} \le N|t-s|^{p\beta/2-1}C(f^i,\bar{f},g,\theta) \tag{2.14}$$

$$E|\psi^{\beta-1}u|^p_{C^{\alpha/2-1/p}([0,\tau],H^{1-\beta}_{p,\theta}(G))} \le NC(f^i,\bar{f},g,\theta), \tag{2.15}$$

where $C(f^i,\bar{f},g,\theta) := \|f^i\|_{\mathbb{L}_{p,\theta}(G,\tau)} + \|\psi\bar{f}\|_{\mathbb{H}^{-1}_{p,\theta}(G,\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(G,\tau)}$.

(ii) If $d \le 2, 1-d/p =: \nu$, then

$$E\int_0^\tau (|\psi^{\theta/p-1}u|_{C^0} + [\psi^{(\theta-d)/p}u]_{C^\nu(G)})dt \le NC(f^i,\bar{f},g,\theta), \tag{2.16}$$

thus if $\theta \le d$, then the function u itself is Hölder continuous in x.

The following corollary shows that if some extra conditions are assumed, then the solutions are Hölder continuous in (t,x) (regardless of the dimension d).

Corollary 2.6. *Let $u \in \mathfrak{H}^1_{p,d,0}(G,T)$ be the solution of (1.1). Assume that $b^i, \bar{b}, c$ are bounded, $\nu = 0$ and*

$$1 - 2/q - d/r > 0, \quad q \ge r > 2,$$

$$f^i, f, g \in L_q(\Omega \times [0,T], \mathcal{P}, L_r(G)).$$

Then there exists $\alpha = \alpha(q,r,d,G) > 0$ such that

$$E|u|^q_{C^\alpha(G\times[0,T])} < \infty. \tag{2.17}$$

Proof. It is shown in [3] that under the conditions of the corollary, there is a solution $v \in \mathfrak{H}^1_{2,d,0}(G,T)$ satisfying (2.17). By the uniqueness result (Theorem 2.4) in the space $\mathfrak{H}^1_{2,d}(G,T)$, we conclude that $u = v$ and thus $v \in \mathfrak{H}^1_{p,d}(G,T)$. □

We will see that the proof of Theorems 2.4 depends also on the following results on $\mathbb{R}^d_+$ and $\mathbb{R}^d$.

Theorem 2.7. *Assume that*

$$x^1|b^i(t,x)| + x^1|\bar{b}^i(t,x)| + (x^1)^2|c(t,x)| + x^1|\nu(t,x)| \le \beta, \quad \forall \omega, t, x.$$

Then there exist $p_0 = p_0(\lambda,\Lambda,d) > 2$, $\beta_0 = \beta_0(p,d,\lambda,\Lambda) \in (0,1)$ and $\chi = \chi(p,d,\lambda,\Lambda) > 0$ such that if

$$\beta \le \beta_0, \quad p \in [2,p_0), \quad d-\chi < \theta < d+\chi, \tag{2.18}$$

then for any $f^i \in \mathbb{L}_{p,\theta}(\tau), \bar{f} \in M^{-1}\mathbb{H}^{-1}_{p,\theta}(\tau)$, $g \in \mathbb{L}_{p,\theta}(\tau)$ *and* $u_0 \in U^1_{p,\theta}$ *equation (1.1) with initial data* u_0 *admits a unique solution* u *in the class* $\mathfrak{H}^1_{p,\theta}(\tau)$ *and for this solution,*

$$\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}(\tau)} \leq N(\|f^i\|_{\mathbb{L}_{p,\theta}(\tau)} + \|M\bar{f}\|_{\mathbb{H}^{-1}_{p,\theta}(\tau)} + \|g\|_{\mathbb{L}_{p,\theta}(\tau)} + \|u_0\|_{U^1_{p,\theta}}), \tag{2.19}$$

where N *depends only* d, p, θ, λ *and* Λ*.*

Theorem 2.8. *Assume that*

$$|b^i(t,x)| + |\bar{b}^i(t,x)| + |c(t,x)| + |\nu(t,x)| \leq K, \quad \forall \omega, t, x.$$

Then there exists $p_0 > 2$ *such that if* $p \leq [2, p_0)$*, then for any* $f^i \in \mathbb{L}_p(\tau), \bar{f} \in \mathbb{H}^{-1}_p(\tau)$*,* $g \in \mathbb{L}_p(\tau)$*,* $u_0 \in U^1_p$ *equation (1.1) with initial data* u_0 *admits a unique solution* u *in the class* $\mathcal{H}^1_p(\tau)$ *and for this solution,*

$$\|u\|_{\mathbb{H}^1_p(\tau)} \leq N(\|f^i\|_{\mathbb{L}_p(\tau)} + \|\bar{f}\|_{\mathbb{H}^{-1}_p(\tau)} + \|g\|_{\mathbb{L}_p(\tau)} + \|u_0\|_{U^1_p}), \tag{2.20}$$

where N *depends only* $d, p, \lambda, \Lambda, K$ *and* T*.*

3. Proof of Theorem 2.7

First we prove the following lemmas.

Lemma 3.1. *Let* $f = (f^1, f^2, ..., f^d), g = (g^1, g^2, ...) \in \mathbb{L}_{2,d}(T)$ *and* $u \in \mathfrak{H}^1_{2,d,0}(T)$ *be a solution of*

$$du = (\Delta u + f^i_{x^i})dt + g^k dw^k_t. \tag{3.1}$$

Then

$$\|u_x\|^2_{\mathbb{L}_{2,d}(T)} \leq \|f\|^2_{\mathbb{L}_{2,d}(T)} + \|g\|^2_{\mathbb{L}_{2,d}(T)}. \tag{3.2}$$

Proof. It is well known (see [11]) that (3.1) has a unique solution $u \in \mathfrak{H}^1_{p,d,0}(T)$ and

$$\|u_x\|^p_{\mathbb{L}_{p,d}(T)} \leq N(p)(\|f\|^p_{\mathbb{L}_{p,d}(T)} + \|g\|^p_{\mathbb{L}_{p,d}(T)}). \tag{3.3}$$

We will show that one can take $N(2) = 1$. Let Θ be the collections of the form

$$f(t,x) = \sum_{i=1}^m I_{(\tau_{i-1},\tau_i]\!]}(t) f_i(x),$$

where $f_i \in C_0^\infty(\mathbb{R}^d_+)$ and τ_i are stopping times, $\tau_i \leq \tau_{i+1} \leq T$. It is well known that the set Θ is dense in $\mathbb{H}^\gamma_{p,\theta}(T)$ for any $\gamma, \theta \in \mathbb{R}$. Also the collection of sequences $g = (g^k)$, such that each $g_k \in \Theta$ and only finitely many of g_k are different from zero, is dense in $\mathbb{H}^\gamma_{p,\theta}(T, \ell_2)$. Thus by considering approximation argument, we may assume that f and g are of this type.

We continue $f(t,x)$ to be an even function and $g(t,x)$ to be an odd function of x^1. Then obviously $f,g \in \mathbb{H}^{\gamma}_p(T)$ for any γ and p. By Theorem 5.1 in [7], equation (3.1) considered in the whole $\mathbb{R}^d$ has a unique solution $v \in \mathcal{H}^1_p$ and $v \in \mathcal{H}^{\gamma}_p$ for any γ. Also by the uniqueness it follows that v is an odd function of x^1 and vanishes at $x^1 = 0$. Moreover remembering the fact that v satisfies

$$dv = \Delta v \, dt$$

outside the support of f and g, we conclude (see the proof of Lemma 4.2 in [10] for detail) that $v \in \mathfrak{H}^{\gamma}_{p,d}$ for any γ.

Thus, both u and v satisfy (3.1) considered in $\mathbb{R}^d_+$ and belong to $\mathfrak{H}^1_{p,d}$. By the uniqueness result (Theorem 3.3 in [11]) on $\mathbb{R}^d_+$, we conclude that $u = v$.

Finally, we see that (3.2) follows from Itô's formula. Indeed (remember that u is infinitely differentiable and vanishes at $x^1 = 0$),

$$|u(t,x)|^2 = \int_0^t (2u\Delta u + 2uf^i_{x^i} + |g|^2_{\ell_2}) \, dt + 2\int_0^t ug^k dw^k_t,$$

therefore

$$\begin{aligned} 0 \le E\int_{\mathbb{R}^d_+} |u(t,x)|^2 \, dx = -2E\int_0^t \int_{\mathbb{R}^d_+} |Du(s,x)|^2 \, dxdt \\ -2E\int_0^t \int_{\mathbb{R}^d_+} f^i D^i u \, dxdt + E\int_0^t \int_{\mathbb{R}^d_+} |g|^2_{\ell_2} \, dxdt \\ \le -E\int_0^t \int_{\mathbb{R}^d_+} |Du(s,x)|^2 \, dxdt \\ +E\int_0^t \int_{\mathbb{R}^d_+} |f|^2 \, dxdt + E\int_0^t \int_{\mathbb{R}^d_+} |g|^2_{\ell_2} \, dxdt. \end{aligned}$$

□

Lemma 3.2. *There exists $p_0 = p_0(\lambda, \Lambda, d) > 2$ such that if $p \in [2, p_0)$ and $u \in \mathfrak{H}^1_{p,d,0}(T)$ is a solution of*

$$du = D_i(a^{ij}u_{x^j} + f^i)dt + g^k dw^k_t, \tag{3.4}$$

then

$$\|u_x\|_{\mathbb{L}_{p,d}(T)} \le N(\|f\|_{\mathbb{L}_{p,d}(T)} + \|g\|_{\mathbb{L}_{p,d}(T)}), \tag{3.5}$$

where N is independent of T.

Proof. We repeat arguments in [15]. Take $N(p)$ from (3.3). By (real-valued version) Riesz-Thorin theorem we may assume that $N(p) \searrow 1$ as $p \searrow 2$. Indeed, consider the operator

$$\Phi : (f^i, g) \to Du,$$

where $u \in \mathfrak{H}^1_{p,d,0}$ is the solution of (3.1). Then for any $r > 2$ and $p \in [2, r]$,

$$\|\Phi\|_p \le \|\Phi\|_2^{1-\alpha} \|\Phi\|_r^\alpha, \quad 1/p = (1-\alpha)/2 + \alpha/r,$$

and (as $p \to 2$)

$$\|\Phi\|_p \le \|\Phi\|_r^\alpha = \|\Phi\|_r^{(1/2-1/p)/(1/2-1/r)} \to 1.$$

Denote $A := (a^{ij}), \kappa := \frac{\lambda+\Lambda}{2}$ and observe that the eigenvalues of $A - \kappa I$ satisfy

$$-(\Lambda-\lambda)/2 = \lambda - \kappa \le \lambda_1 - \kappa \le ... \le \lambda_d - \kappa \le \Lambda - \kappa = (\Lambda-\lambda)/2,$$

and therefore for any $\xi \in \mathbb{R}^d$,

$$|(a^{ij} - \kappa I)\xi| \le \frac{\Lambda - \lambda}{2} |\xi|. \tag{3.6}$$

Assume that $v \in \mathfrak{H}^1_{p,d,0}(T)$ satisfies

$$dv = (\kappa \Delta v + f^i_{x^i})\, dt + g^k dw^k_t.$$

Then $\bar{v}(t,x) := v(t, \sqrt{\kappa}x)$ satisfies

$$d\bar{v} = (\Delta \bar{v} + \bar{f}^i_{x^i})\, dt + \bar{g}^k dw^k_t,$$

where $\bar{f}^i(t,x) = \frac{1}{\sqrt{\kappa}} f^i(t, \sqrt{\kappa}x)$ and $\bar{g}^k(t,x) = g^k(t, \sqrt{\kappa}x)$. Thus by (3.3),

$$\|v_x\|^p_{\mathbb{L}_{p,d}(T)} \le \frac{N(p)}{\kappa^p} \|f\|^p_{\mathbb{L}_{p,d}(T)} + \frac{N(p)}{\kappa^{p/2}} \|g\|^p_{\mathbb{L}_{p,d}(T)}. \tag{3.7}$$

Therefore we conclude that if $u \in \mathfrak{H}^1_{p,d,0}(T)$ is a solution of (3.4), then u satisfies

$$du = (\kappa \Delta u + (f^i + (A - \kappa I)u_{x^j})_{x^i})\, dt + g^k dw^k_t,$$

and

$$\|u_x\|^p_{\mathbb{L}_p(T)} \le \frac{N(p)}{\kappa^p} \|F\|^p_{\mathbb{L}_{p,d}(T)} + \frac{N(p)}{\kappa^{p/2}} \|g\|^p_{\mathbb{L}_{p,d}(T)},$$

where $F^i = (A - \kappa I)u_{x^j} + f^i$. By (3.6)

$$|F|^p \le (1+\epsilon)\frac{(\Lambda-\lambda)^p}{2^p} |u_x|^p + N(\epsilon)|f|^p.$$

Thus, for sufficiently small ϵ, (since $N(p) \searrow 1$ as $p \searrow 2$)

$$\frac{N(p)}{\kappa^p}(1+\epsilon)\frac{(\lambda-\lambda)^p}{2^p} = N(p)(1+\epsilon)\frac{(\Lambda-\lambda)^p}{(\Lambda+\lambda)^p} < 1. \tag{3.8}$$

Obviously the claims of the lemma follow from this. □

Lemma 3.3. *Assume that for any solution $u \in \mathfrak{H}^1_{p,\theta_0}(\tau)$ of (1.1), we have estimate (2.19) for $\theta = \theta_0$, then there exists $\chi = \chi(d, p, \theta_0, \lambda, \Lambda) > 0$ such that for any $\theta \in (\theta_0 - \chi, \theta_0 + \chi)$, estimate (2.19) holds whenever $u \in \mathfrak{H}^1_{p,\theta}(\tau)$ is a solution of (1.1).*

Proof. The lemma is essentially proved in [6] for SPDEs with constant coefficients. By Lemma 2.3, $u \in \mathfrak{H}^1_{p,\theta}(\tau)$ if and only if $v := M^{(\theta-\theta_0)/p}u \in \mathfrak{H}^1_{p,\theta_0}(\tau)$ and the norms $\|u\|_{\mathfrak{H}^1_{p,\theta}(\tau)}$ and $\|v\|_{\mathfrak{H}^1_{p,\theta_0}(\tau)}$ are equivalent. Denote $\varepsilon = (\theta - \theta_0)/p$ and observe that v satisfies

$$dv = (D_i(a^{ij}v_{x^j} + b^i v + \tilde{f}^i) + \bar{b}^i v_{x^i} + cv + \tilde{\bar{f}})dt + (\nu^k v + M^\varepsilon g^k)dw^k_t,$$

where

$$\tilde{f}^i = M^\varepsilon f^i - \varepsilon a^{i1} M^{-1} v,$$
$$\tilde{\bar{f}} = M^\varepsilon \bar{f} - M^{-1}\varepsilon(\bar{b}^1 v + a^{1j}v_{x^j} - a^{11}\varepsilon M^{-1} v + b^1 v + M^\varepsilon f^i).$$

By assumption (remember that Mb^i and $M\bar{b}$ are bounded),

$$\begin{aligned}\|v\|_{\mathfrak{H}^1_{p,\theta_0}(\tau)} &\le N(\|\tilde{f}^i\|_{\mathbb{L}_{p,\theta_0}(\tau)} + \|M\tilde{\bar{f}}\|_{\mathbb{H}^{-1}_{p,\theta_0}(\tau)} + \|M^\varepsilon u_0\|_{U_{p,\theta_0}})\\ &\le N(\|f^i\|_{\mathbb{L}_{p,\theta}(\tau)} + \|M\bar{f}\|_{\mathbb{H}^{-1}_{p,\theta}(\tau)} + \|u_0\|_{U_{p,\theta}})\\ &\quad + N\varepsilon(\|M^{-1}v\|_{\mathbb{L}_{p,\theta_0}(\tau)} + \|v_x\|_{\mathbb{L}_{p,\theta_0}(\tau)}).\end{aligned}$$

Thus it is enough to take ε sufficiently small (see (2.8)). The lemma is proved.

□

Now we come back to our proof. As usual we may assume $\tau \equiv T$ (see [7]), and due to Lemma 3.3, without loss of generality we assume that $\theta = d$.

Take p_0 from Lemma 3.2. The method of continuity shows that to prove the theorem it suffices to prove that if $p \le p_0$, then (2.19) holds true given that a solution $u \in \mathfrak{H}^1_{p,d}(T)$ already exists.

Step 1. We assume that $b^i = \bar{b}^i = c = \nu^k = 0$. By (2.8) (or see Lemma 1.3 (i) in [11])

$$\|u_x\|_{H^\gamma_{p,\theta}} \sim \|M^{-1}u\|_{H^{\gamma+1}_{p,\theta}}.$$

Thus we estimate $\|u_x\|_{\mathbb{L}_{p,d}(T)}$ instead of $\|M^{-1}u\|_{\mathbb{H}^1_{p,d}(T)}$. By Theorem 3.3 in [11] there exists a solution $v \in \mathfrak{H}^1_{p,d}(T)$ of

$$dv = (\Delta v + \bar{f})\,dt, \quad v(0,\cdot) = u_0,$$

and furthermore

$$\|v_x\|_{\mathbb{L}_{p,d}(T)} \le N\|M\bar{f}\|_{\mathbb{H}^{-1}_{p,d}(T)} + N\|u_0\|_{U^1_{p,d}}. \tag{3.9}$$

Observe that $u-v$ satisfies

$$d(u-v) = D_i(a^{ij}(u-v)_{x^j} + \tilde{f}^i)\,dt + g^k\,dw_t^k, \quad (u-v)(0,\cdot) = 0,$$

where $\tilde{f}^i = f^i + (a^{ij} - \delta^{ij})v_{x^j}$. Therefore (2.19) follows from Lemma 3.2 and (3.9).

Step 2(general case). By the result of step 1,

$$\|M^{-1}u\|_{\mathbb{H}^1_{p,d}(T)} \le N\|Mb^iM^{-1}u + f^i\|_{\mathbb{L}_{p,d}(T)} + N\|u_0\|_{U^1_{p,d}}$$

$$+N\|M\bar{b}^iu_{x^i} + M^2cM^{-1}u + M\bar{f}\|_{\mathbb{H}^{-1}_{p,d}(T)} + N\|M\nu M^{-1}u + g\|_{\mathbb{L}_{p,d}(T)}$$

$$\le N\beta(\|M^{-1}u\|_{\mathbb{L}_{p,d}(T)} + \|u_x\|_{\mathbb{L}_{p,d}(T)})$$

$$+N\|u_0\|_{U^1_{p,d}} + N\|f^i\|_{\mathbb{L}_{p,d}(T)} + N\|M\bar{f}\|_{\mathbb{H}^{-1}_{p,d}(T)} + N\|g\|_{\mathbb{L}_{p,d}(T)}.$$

Now it is enough to choose β_0 such that for any $\beta \le \beta_0$,

$$N\beta(\|M^{-1}u\|_{\mathbb{L}_{p,d}(T)} + \|u_x\|_{\mathbb{L}_{p,d}(T)}) \le 1/2\|M^{-1}u\|_{\mathbb{H}^1_{p,d}(T)}.$$

The theorem is proved.

4. Proof of Theorem 2.8

First we need the following result on $\mathbb{R}^d$ proved in [15].

Lemma 4.1. *There exists $p_0 = p_0(\lambda, \Lambda, d) > 2$ such that if $p \in [2, p_0)$ and $u \in \mathcal{H}^1_{p,0}(T)$ is a solution of*

$$du = D_i(a^{ij}u_{x^j} + f^i)dt + g^k dw_t^k, \tag{4.1}$$

then

$$\|u_x\|_{\mathbb{L}_p(T)} \le N(\|f\|_{\mathbb{L}_p(T)} + \|g\|_{\mathbb{L}_p(T)}).$$

Again, to prove the theorem, we only show that the apriori estimate (2.20) holds for $p < p_0$ (also see step 1 below).

As in theorem 5.1 in [7], considering $u-v$, where $v \in \mathcal{H}^1_p(T)$ is the solution of

$$dv = \Delta v dt, \quad v(0,\cdot) = u_0,$$

without loss of generality we assume that $u(0,\cdot) = 0$.

Step 1. Assume that $b^i = \bar{b}^i = c = \nu^k = 0$. By Theorem 5.1 in [7], there exists a solution $v \in \mathcal{H}^1_{p,0}(T)$ of

$$dv = (\Delta v + \bar{f})dt,$$

and it satisfies

$$\|v_x\|_{\mathbb{L}_p(T)} \le N\|\bar{f}\|_{\mathbb{H}^{-1}_p(T)}. \tag{4.2}$$

Observe that $\bar{u} := u - v$ satisfies

$$d\bar{u} = D_i(a^{ij}\bar{u}_{x^j} + \tilde{f}^i)\,dt + g^k dw_t^k,$$

where $\tilde{f}^i = f^i + (A-I)v_{x^j}$. Thus the estimate (2.20) follows from Lemma 4.1 and (4.2).

Step 2. We show that there exists $\epsilon_1 > 0$ such that if $T \leq \epsilon_1$, then all the assertions of the theorem hold true. Thus without loss of generality we assume that $T \leq 1$.

Note that $\bar{b}^i u_{x^i} \in \mathbb{L}_p(T)$ since $u \in \mathbb{H}^1_p(T)$, so by Theorem 5.1 in [7], there exists a unique solution $v \in \mathcal{H}^2_{p,0}(T)$ of

$$dv = (\Delta v + \bar{b}^i u_{x^i})dt,$$

and v satisfies

$$\|v\|^p_{\mathcal{H}^2_p(T)} \leq N\|u_x\|^p_{\mathbb{L}_p(T)}.$$

By (2.10),

$$\|v_x\|^p_{\mathbb{L}_p(T)} \leq N\|v\|^p_{\mathbb{H}^1_p(T)} \leq N(T)\|u_x\|_{\mathbb{L}_p(T)}, \tag{4.3}$$

where $N(T) \to 0$ as $T \to 0$. Observe that $u - v$ satisfies

$$d(u-v) = (D_i(a^{ij}(u-v)_{x^j} + (a^{ij} - \delta^{ij})v_{x^i} + b^i u + f^i) + cu + \bar{f})\,dt$$
$$+(\nu^k u + g^k)\,dw^k_t.$$

By the result of step 1,

$$\|(u-v)_x\|_{\mathbb{L}_p(T)} \leq N(\|(a^{ij} - \delta^{ij})v_{x^i} + b^i u + f^i\|_{\mathbb{L}_p(T)}$$
$$+\|cu + \bar{f}\|_{\mathbb{H}^{-1}_p(T)} + \|\nu^k u + g\|_{\mathbb{L}_p(T)})$$
$$\leq N(\|v_x\|_{\mathbb{L}_p(T)} + \|f^i\|_{\mathbb{L}_p(T)} + \|\bar{f}\|_{\mathbb{H}^{-1}_p(T)} + \|g\|_{\mathbb{L}_p(T)} + \|u\|_{\mathbb{L}_p(T)}),$$

where constants N are independent of T ($T \leq 1$). This and (4.3) yield

$$\|u_x\|_{\mathbb{L}_p(T)} \leq NN(T)\|u_x\|_{\mathbb{L}_p(T)} + N\|f^i\|_{\mathbb{L}_p(T)} + N\|\bar{f}\|_{\mathbb{H}^{-1}_p(T)}$$
$$+N\|g\|_{\mathbb{L}_p(T)} + N\|u\|_{\mathbb{L}_p(T)}.$$

Note that the above inequality holds for all $t \leq T$. Choose ε_1 so that $NN(T) \leq 1/2$ for all $T \leq \varepsilon_1$, then for any $t \leq T \leq \varepsilon_1$ (see Lemma 2.3),

$$\|u\|^p_{\mathcal{H}^1_p(t)} \leq N\|u\|^p_{\mathbb{L}_p(t)} + N(\|f^i\|^p_{\mathbb{H}^{-1}_p(T)} + \|\bar{f}\|^p_{\mathbb{L}_p(T)} + \|g\|^p_{\mathbb{L}_p(T)})$$
$$\leq N\int_0^t \|u\|^p_{\mathcal{H}^1_p(t)}\,dt + N(\|f^i\|^p_{\mathbb{H}^{-1}_p(T)} + \|\bar{f}\|^p_{\mathbb{L}_p(T)} + \|g\|^p_{\mathbb{L}_p(T)}).$$

Gronwall's inequality leads to (2.20).

Step 3. Consider the case $T > \varepsilon_1$. To proceed further, we need the following lemma.

Lemma 4.2. *Let $\tau \le T$ be a stopping and $du(t) = f(t)dt + g^k(t)dw^k_t$.*

(i) Let $u \in \mathcal{H}^{\gamma+2}_{p,0}(\tau)$. Then there exists a unique $\tilde{u} \in \mathcal{H}^{\gamma+2}_{p,0}(T)$ such that $\tilde{u}(t) = u(t)$ for $t \le \tau (a.s)$ and, on $(0,T)$,

$$d\tilde{u} = (\Delta \tilde{u}(t) + \tilde{f}(t))dt + g^k I_{t\le\tau} dw^k_t, \tag{4.4}$$

where $\tilde{f} = (f(t) - \Delta u(t))I_{t\le\tau}$. Furthermore,

$$\|\tilde{u}\|_{\mathcal{H}^{\gamma+2}_p(T)} \le N\|u\|_{\mathcal{H}^{\gamma+2}_p(\tau)}, \tag{4.5}$$

where N is independent of u and τ.

(ii) all the claims in (i) hold true if $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,0}(G,\tau)$ and if one replace the space $\mathcal{H}^{\gamma+2}_p(\tau)$ and $\mathcal{H}^{\gamma+2}_p(T)$ with $\mathfrak{H}^{\gamma+2}_{p,\theta}(G,\tau)$ and $\mathfrak{H}^{\gamma+2}_{p,\theta}(G,T)$, respectively.

Proof. (i) Note $\tilde{f} \in \mathbb{H}^\gamma_p(T), gI_{t\le\tau} \in \mathbb{H}^{\gamma+1}_p(T)$, so that, by Theorem 5.1 in [7], equation (4.4) has a unique solution $\tilde{u} \in \mathcal{H}^{\gamma+2}_{p,0}(T)$ and (4.5) holds. To show that $\tilde{u}(t) = u(t)$ for $t \le \tau$, notice that, for $t \le \tau$, the function $v(t) = \tilde{u}(t) - u(t)$ satisfies the equation

$$v(t) = \int_0^t \Delta v(s)ds, \quad v(0,\cdot) = 0.$$

Theorem 5.1 in [7] shows that $v(t) = 0$ for $t \le \tau\,(a.e)$.

(ii) It is enough to repeat the arguments in (i) using Theorem 2.9 in [1] (instead of Theorem 5.1 in [7]). □

Now, to complete the proof, we repeat the arguments in [4]. Take an integer $M \ge 2$ such that $T/M \le \varepsilon_1$, and denote $t_m = Tm/M$. Assume that, for $m = 1, 2, ..., M-1$, we have the estimate (2.20) with t_m in place of τ (and N depending only on $d, p, \lambda, \Lambda, K$ and T). We are going to use the induction on m. Let $u_m \in \mathcal{H}^1_{p,0}$ be the continuation of u on $[t_m, T]$, which exists by Lemma 4.2(i) with $\gamma = -1$ and $\tau = t_m$. Denote $v_m := u - u_m$, then $(a.s)$ for any $t \in [t_m, T]$, $\phi \in C^\infty_0(G)$ (since $du_m = \Delta u_m dt$ on $[t_m, T]$)

$$(v_m(t), \phi) = -\int_{t_m}^t (a^{ij} v_{mx^j} + b^i v_m + f^i_m, \phi_{x^i})(s)ds$$

$$+\int_{t_m}^t (\bar{b}^i v_{mx^i} + cv_m + \bar{f}_m, \phi)(s)ds + \int_{t_m}^t (\nu^k v_m + g^k_m, \phi)(s)dw^k_s,$$

where

$$f^i_m = (a^{ij} - \delta^{ij})u_{mx^j} + b^i u_m + f^i, \quad \bar{f}_m = \bar{b}^i u_{mx^i} + cu_m + \bar{f},$$

$$g^k_m = \nu^k u_m + g^k.$$

Next instead of random processes on $[0,T]$ one considers processes given on $[t_m,T]$ and, in a natural way, introduce spaces $\mathcal{H}_p^\gamma([t_m,T])$, $\mathbb{L}_p([t_m,t])$, $\mathbb{H}_p^\gamma([t_m,T])$. Then one gets a counterpart of the result of step 2 and concludes that

$$E\int_{t_m}^{t_{m+1}}\|(u-u_m)(s)\|^p_{H^1_p}ds$$
$$\leq NE\int_{t_m}^{t_{m+1}}(\|f^i_m(s)\|^p_{L_p}+\|\bar f_m(s)\|^p_{H^{-1}_p}+\|g_m(s)\|^p_{L_p})ds.$$

Thus by the induction hypothesis we conclude

$$E\int_0^{t_{m+1}}\|u(s)\|^p_{H^1_p}ds\leq NE\int_0^T\|u_m(s)\|^p_{H^1_p}ds$$
$$+NE\int_{t_m}^{t_{m+1}}\|(u-u_m)(s)\|^p_{H^1_p}ds$$
$$\leq N(\|f^i\|^p_{\mathbb{L}_p(t_{m+1})}+\|\bar f\|^p_{\mathbb{H}^{-1}_p(t_{m+1})}+\|g\|^p_{\mathbb{L}_p(t_{m+1})}).$$

We see that the induction goes through and thus the theorem is proved.

5. Proof of Theorem 2.8

As usual we may assume $\tau\equiv T$. It is known (see [1]) that for any $u_0\in U^1_{p,\theta}(G)$ and $(f,g)\in\psi^{-1}\mathbb{H}^{-1}_{p,\theta}(G,T)\times\mathbb{L}_{p,\theta}(G,T)$, there exists $u\in\mathfrak{H}^1_{p,\theta}(G,T)$ such that $u(0,\cdot)=u_0$ and

$$du=(\Delta u+f)\,dt+g^k\,dw^k_t. \tag{5.1}$$

Thus as before, to finish the proof of the theorem, we only need to establish the apriori estimate (2.13) assuming that $u\in\mathfrak{H}^1_{p,\theta}(G,T)$ satisfies (1.1) with initial data $u_0=0$, where $p\in[2,p_0)$ and $\theta\in(d-\chi,d+\chi)$.

To proceed we need the following results.

Lemma 5.1. *Let $u\in\mathfrak{H}^1_{p,\theta,0}(G,T)$ be a solution of (1.1). Then*

(i) there exists $\varepsilon_0\in(0,1)$ (independent of u) such that if u has support in $B_{\varepsilon_0}(x_0),x_0\in\partial G$ then (2.13) holds.

(ii) if u has support on G_ϵ for some $\varepsilon>0$, where $G_\varepsilon:=\{x\in G:dist(x,\partial G)>\varepsilon\}$, then then (2.13) holds.

Proof. The second assertion of the lemma follows from Theorem 2.8 since in this case (see [12]) $u\in\mathcal{H}^1_p(T)$ and

$$\|u\|_{\mathfrak{H}^1_{p,\theta}(G,T)}\sim\|u\|_{\mathcal{H}^1_p(T)}.$$

To prove the first assertion, we use Theorem 2.7. Let $x_0\in\partial G$ and Ψ be a function from Assumption 2.1. It is shown in [5] (or see [1]) that

Ψ can be chosen such that Ψ is infinitely differentiable in $G \cap B_{r_0}(x_0)$ and satisfies

$$[\Psi_x]^{(0)}_{n,B_{r_0}(x_0)\cap G} + [\Psi_x^{-1}]^{(0)}_{n,J_+} < N(n) < \infty \tag{5.2}$$

and

$$\rho(x)\Psi_{xx}(x) \to 0 \quad \text{as} \quad x \in B_{r_0}(x_0) \cap G, \text{and } \ \rho(x) \to 0, \tag{5.3}$$

where the constants $N(n)$ and the convergence in (5.3) are independent of x_0.

Define $r = r_0/K_0$ and fix smooth functions $\eta \in C_0^\infty(B_r), \varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \eta, \varphi \leq 1$, and $\eta = 1$ in $B_{r/2}$, $\varphi(t) = 1$ for $t \leq -3$, and $\varphi(t) = 0$ for $t \geq -1$ and $0 \geq \varphi' \geq -1$. Observe that $\Psi(B_{r_0}(x_0))$ contains B_r. For $m = 1, 2, ...$, $t > 0$, $x \in \mathbb{R}^d_+$ define $\varphi_m(x) = \varphi(m^{-1}\ln x^1)$. Also we denote $\Psi^i_r := D_r\Psi^i, \Psi^i_{rs} := D_rD_s\Psi^i, \Phi^i_r := D_i(\Psi^i_{x^r}(\Psi^{-1}))(\Psi)$,

$$\hat{a}_m := \tilde{a}\eta(x)\varphi_m + (1-\eta\varphi_m)I, \quad \hat{b}_m := \tilde{b}\eta\varphi_m, \quad \hat{\bar{b}}_m := \tilde{\bar{b}}\eta\varphi_m,$$

$$\hat{c}_m := \tilde{c}\eta\varphi_m, \quad \hat{\nu}_m := \tilde{\nu}\eta\varphi_m,$$

where

$$\tilde{a}^{ij}(t,x) = \check{a}^{ij}(t,\Psi^{-1}(x)), \quad \tilde{b}^i(t,x) = \check{b}^i(t,\Psi^{-1}(x)),$$

$$\tilde{\bar{b}}^i(t,x) = \check{\bar{b}}^i(t,\Psi^{-1}(x)), \quad \tilde{c}(t,x) = c(t,\Psi^{-1}(x))$$

$$\tilde{\nu}(t,x) = \nu(t,\Psi^{-1}(x)),$$

$$\check{a}^{ij} = a^{rs}\Psi^i_{x^r}\Psi^j_{x^s}, \quad \check{b}^i = b^r\Psi^i_r,$$

$$\check{\bar{b}}^i = \bar{b}^r\Psi^i_r + a^{rs}\Psi^j_s\Phi^i_r, \quad \check{c} = c + b^r\Phi^i_r.$$

Take β_0 from Theorem 2.7. Observe that $\varphi(m^{-1}\ln x^1) = 0$ for $x^1 \geq e^{-m}$. Also we easily see that (5.3) implies $x^1\Psi_{xx}(\Psi^{-1}(x)) \to 0$ as $x^1 \to 0$. Using these facts and Assumption 2.2(iv), one can find $m > 0$ independent of x_0 such that

$$x^1|\hat{b}_m(t,x)| + x^1|\hat{\bar{b}}_m(t,x)| + (x^1)^2|\hat{c}_m(t,x)| + x^1|\hat{\nu}_m(t,x)| \leq \beta_0,$$

whenever $t > 0, x \in \mathbb{R}^d_+$.

Now we fix a $\varepsilon_0 < r_0$ such that

$$\Psi(B_{\varepsilon_0}(x_0)) \subset B_{r/2} \cap \{x : x^1 \leq e^{-3m}\}.$$

Let's denote $v := u(\Psi^{-1})$ and continue v as zero in $\mathbb{R}^d_+ \setminus \Psi(B_{\varepsilon_0}(x_0))$. Since $\eta\varphi_m = 1$ on $\Psi(B_{\varepsilon_0}(x_0))$, the function v satisfies

$$dv = ((\hat{a}^{ij}_m v_{x^ix^j} + \hat{b}^i_m v + \hat{f}^i)_{x^i} + \hat{\bar{b}}^i_m v_{x^i} + \hat{c}_m v + \hat{\bar{f}})\,dt + (\hat{\nu}^k_m v + \hat{g}^k)\,dw^k_t,$$

where

$$\hat{f}^i = f^i(\Psi^{-1}), \quad \hat{\bar{f}} = \bar{f}(\Psi^{-1}), \quad \hat{g}^k = g^k(\Psi^{-1}).$$

Next we observe that by (5.2) and Theorem 3.2 in [12] (or see [5]) for any $\nu,\alpha\in\mathbb{R}$ and $h\in\psi^{-\alpha}H^{\nu}_{p,\theta}(G)$ with support in $B_{\varepsilon_0}(x_0)$

$$\|\psi^{\alpha}h\|_{H^{\nu}_{p,\theta}(G)}\sim\|M^{\alpha}h(\Psi^{-1})\|_{H^{\nu}_{p,\theta}}. \tag{5.4}$$

Therefore we conclude that $v\in\mathfrak{H}^1_{p,\theta}(T)$. Also by Theorem 2.7 we have

$$\|M^{-1}v\|_{\mathbb{H}^1_{p,\theta}(T)}\leq N\|\hat{f}\|_{\mathbb{L}_{p,\theta}(T)}+N\|M\hat{\bar{f}}\|_{\mathbb{H}^{-1}_{p,\theta}(T)}+N\|\hat{g}\|_{\mathbb{L}_{p,\theta}(T)}.$$

Finally (5.4) leads to (2.13). The lemma is proved. □

Coming back to our proof, we choose a partition of unity $\zeta^m, m=0,1,2,...,N_0$ such that $\zeta^0\in C_0^{\infty}(G)$, $\zeta^{(m)}=\zeta(\frac{2(x-x_m)}{\varepsilon_0})$,$\zeta\in C_0^{\infty}(B_1(0))$, $x_m\in\partial G$, $m\geq 1$, and for any multi-indices α

$$\sup_x\sum\psi^{|\alpha|}|D^{\alpha}\zeta^{(m)}|<N(\alpha)<\infty, \tag{5.5}$$

where the constant $N(\alpha)$ is independent of ε_0 (see section 6.3 in [9]). Thus it follows (see [12]) that for any $\nu\in\mathbb{R}$ and $h\in H^{\nu}_{p,\theta}(G)$ there exist constants N depending only p,θ,ν and $N(\alpha)$ (independent of ε_0) such that

$$\|h\|^p_{H^{\nu}_{p,\theta}(G)}\leq N\sum\|\zeta^m h\|^p_{H^{\nu}_{p,d}(G)}\leq N\|h\|^p_{H^{\nu}_{p,\theta}(G)}, \tag{5.6}$$

$$\sum\|\psi\zeta^m_x h\|^p_{H^{\nu}_{p,\theta}(G)}\leq N\|h\|^p_{H^{\nu}_{p,\theta}(G)}. \tag{5.7}$$

Also,

$$\sum\|\zeta^{(m)}_x h\|^p_{H^{\nu}_{p,\theta}(G)}\leq N(\varepsilon_0)\|h\|^p_{H^{\nu}_{p,\theta}(G)}, \tag{5.8}$$

where the constant $N(\varepsilon_0)$ depends also on ε_0.

Using the above inequalities and Lemma 5.1 we will show

$$\|u_x\|^p_{\mathbb{L}_{p,\theta}(G,t)}\leq N\|u\|^p_{\mathbb{L}_{p,\theta}(G,t)}+\text{appropriate norms of } f^i,\bar{f},g \tag{5.9}$$

and we will drop the term $\|u\|^p_{\mathbb{L}_{p,\theta}(G,t)}$ using (2.9). But as one can see in (5.10) below, one has to handle the term $a^{ij}u_{x^j}\zeta^m_{x^i}$. Obviously if the right side of inequality (5.9) contains the norm $\|u_x\|^p_{\mathbb{L}_{p,\theta}(G,T)}$, then this is useless. The following arguments below are used just to avoid estimating $\|a^{ij}u_{x^j}\zeta^m_{x^i}\|^p_{\mathbb{L}_{p,\theta}(G,T)}$.

Denote $u^m=u\zeta^m$, $m=0,1,...,N_0$. Then u^m satisfies

$$du^m=(D_i(a^{ij}u^m_{x^j}+b^iu^m+f^{m,i})+\bar{b}^iu^m_{x^i}+cu^m+\bar{f}^m-a^{ij}u_{x^j}\zeta^m_{x^i})\,dt$$
$$+(\nu^ku^m+\zeta^mg^k)\,dw^k_t, \tag{5.10}$$

where

$$f^{m,i}=f^i\zeta-a^{ij}u\zeta^m_{x^j},$$
$$\bar{f}^m=-b^iu\zeta^m_{x^i}-f^i\zeta^m_{x^i}-\bar{b}^iu\zeta^m_{x^i}+\bar{f}\zeta^m.$$

Since $\psi^{-1}a^{ij}u_{x^j}\zeta^m_{x^i} \in \psi^{-1}\mathbb{L}_{p,\theta}(G,T)$, by Theorem 2.9 in [1] (or Theorem 2.10 in [5]), there exists unique solution $v^m \in \mathfrak{H}^2_{p,\theta,0}(G,T)$ of

$$dv = (\Delta v - \psi^{-1}a^{ij}u_{x^j}\zeta^m_{x^i})dt,$$

and furthermore

$$\|v^m\|_{\mathfrak{H}^2_{p,\theta}(G,T)} \leq N\|a^{ij}u_{x^j}\zeta^m_{x^i}\|_{\mathbb{L}_{p,\theta}(G,T)}. \tag{5.11}$$

By (2.2) and Lemma 2.3,

$$\|v^m\|_{\mathbb{L}_{p,\theta}(G,T)} + \|\psi v^m_x\|_{\mathbb{L}_{p,\theta}(G,T)} \leq N(T)\|a^{ij}u_{x^j}\zeta^m_{x^i}\|_{\mathbb{L}_{p,\theta}(G,T)}, \tag{5.12}$$

where $N(T) \to 0$ as $T \to 0$.

For $m \geq 1$, define $\eta^m(x) = \zeta(\frac{x-x_m}{\varepsilon_0})$ and fix a smooth function $\eta^0 \in C^\infty_0(G)$ such that $\eta^0 = 1$ on the support of ζ^0. Now we denote $\bar{u}^m := \psi v^m\eta^m$, then $\bar{u}^m \in \mathfrak{H}^2_{p,\theta}(G,T)$ satisfies

$$d\bar{u}^m = (\Delta\bar{u}^m + \tilde{f}^m - a^{ij}u_{x^j}\zeta^m_{x^i})\,dt, \tag{5.13}$$

where $\tilde{f}^m = -2v^m_{x^i}(\eta^m\psi)_{x^i} - v^m\Delta(\eta^m\psi)$. Finally by considering $\tilde{u}^m := u^m - \bar{u}^m$ we can drop the term $a^{ij}u_{x^j}\zeta^m_{x^i}$ in (5.10) because $\tilde{u}^m$ satisfies

$$\begin{aligned} d\tilde{u}^m = (D_i(a^{ij}\tilde{u}^m_{x^j} + b^i\tilde{u}^m + F^{m,i}) + \bar{b}^i\tilde{u}^m_{x^i} + c\tilde{u}^m + \bar{F}_m)\,dt \\ +(\nu^k\tilde{u}^m + G^{m,k})\,dw^k_t, \end{aligned} \tag{5.14}$$

where

$$F^{m,i} = f^i\zeta^m - a^{ij}u\zeta^m_{x^j} + b^i\bar{u}^m + (a^{ij} - \delta^{ij})\bar{u}^m_{x^j},$$

$$\bar{F}^m = \bar{b}^i\bar{u}^m_{x^i} + c\bar{u}^m - b^iu\zeta^m_{x^i} - f^i\zeta^m_{x^i} - \bar{b}^iu\zeta^m_{x^i} + \bar{f}\zeta^m + 2v^m_{x^i}(\eta^m\psi)_{x^i} + v^m\Delta(\eta^m\psi),$$

$$G^{m,k} = \zeta^m g^k + \nu^k\bar{u}^m.$$

By Lemma 5.1, for any $t \leq T$,

$$\|\psi^{-1}\tilde{u}^m\|^p_{\mathbb{H}^1_{p,\theta}(G,t)} \leq N\|F^{m,i}\|^p_{\mathbb{L}_{p,\theta}(G,t)} + N\|\psi\bar{F}^m\|_{\mathbb{H}^{-1}_{p,\theta}(G,t)} + N\|G^m\|^p_{\mathbb{L}_{p,\theta}(G,t)}.$$

Remember that $\psi b^i, \psi\bar{b}, \psi^2c, \psi_x$ and $\psi\psi_{xx}$ are bounded and $\|\cdot\|_{H^{-1}_{p,\theta}} \leq \|\cdot\|_{L_{p,\theta}}$. By (5.6),(5.7) and (5.8),

$$\sum\|\psi\bar{F}^m\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,t)} \leq N(\|\psi\bar{f}\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,t)} + \|f^i\|_{\mathbb{L}_{p,\theta}(G,t)} + \|u\|^p_{\mathbb{L}_{p,\theta}(G,t)})$$

$$+N\sum(\|\bar{u}^m_x\|^p_{\mathbb{L}_{p,\theta}(G,t)} + \|\psi^{-1}\bar{u}^m\|^p_{\mathbb{L}_{p,\theta}(G,t)} + \|\psi v^m_x\|^p_{\mathbb{L}_{p,\theta}(G,t)} + \|v^m\|^p_{\mathbb{L}_{p,\theta}(G,t)})$$

$$\leq N(\|\psi\bar{f}\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,t)} + \|f^i\|_{\mathbb{L}_{p,\theta}(G,t)} + \|u\|^p_{\mathbb{L}_{p,\theta}(G,t)}) + \sum\|v^m\|^p_{\mathbb{H}^1_{p,\theta}(G,t)}).$$

Similarly (actually much easily) the sums

$$\sum\|F^{m,i}\|^p_{\mathbb{L}_{p,\theta}(G,t)}, \quad \sum\|G^m\|^p_{\mathbb{L}_{p,\theta}(G,t)}$$

can be handled. Then one gets for each $t \leq T$ (see (5.12) and note that $\psi^{-1}\bar{u}^m = v^m\eta^m$),

$$\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,t)} \leq N\sum \|\psi^{-1}u_m\|^p_{\mathbb{H}^1_{p,\theta}(G,t)}$$

$$\leq N\sum \|\psi^{-1}\tilde{u}^m\|^p_{\mathbb{H}^1_{p,\theta}(G,t)} + N\sum \|v^m\eta^m\|^p_{\mathbb{H}^1_{p,\theta}(G,t)}$$

$$\leq N\|f^i\|_{\mathbb{L}_{p,\theta}(G,T)} + N\|\psi\bar{f}\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,T)} + N\|g\|_{\mathbb{L}_{p,\theta}(G,T)}$$

$$+N\|u\|_{\mathbb{L}_{p,\theta}(G,t)} + NN(t)\|u_x\|^p_{\mathbb{L}_{p,\theta}(G,t)}.$$

Since $\|u_x\|_{L_{p,\theta}} \leq N\|\psi^{-1}u\|_{H^1_{p,\theta}}$, we can choose $\varepsilon_2 \in (0,1]$ such that

$$NN(t)\|u_x\|^p_{\mathbb{L}_{p,\theta}(G,t)} \leq 1/2\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,t)}, \quad \text{if} \quad t \leq T \leq \varepsilon_2,$$

and therefore

$$\|u\|^p_{\mathfrak{H}^1_{p,\theta}(G,t)} \leq N\int_0^t \|u\|^p_{\mathfrak{H}^1_{p,\theta}(G,s)}\,ds + N\|f^i\|_{\mathbb{L}_{p,\theta}(G,T)}$$

$$+N\|\psi\bar{f}\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,T)} + N\|g\|_{\mathbb{L}_{p,\theta}(G,T)}.$$

This and Gronwall's inequality lead to (2.13) if $T \leq \varepsilon_2$. For the general case, one repeats step 3 in the proof of Theorem 2.8 using Lemma 4.2 (ii) instead of Lemma 4.2 (i). The theorem is proved.

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