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Existence of multi-dimensional infinite volume self-organized critical forest-fire models

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Abstract

Consider the following forest-fire model where the possible locations of trees are the sites of a cubic lattice. Each site has two possible states: 'vacant' or 'occupied'. Vacant sites become occupied according to independent rate 1 Poisson processes. Independently, at each site ignition (by lightning) occurs according to independent rate lambda Poisson processes. When a site is ignited, its occupied cluster becomes vacant instantaneously.

If the lattice is one-dimensional or finite, then with probability one, at each time the state of a given site only depends on finitely many Poisson events; a process with the above description can be constructed in a standard way. If the lattice is infinite and multi-dimensional, in principle, the state of a given site can be influenced by infinitely many Poisson events in finite time.

For all positive lambda, the existence of a multi-dimensional infinite volume forest-fire process with parameter lambda is proven

Key words: forest-fires, self-organized criticality, forest-fire model, existence, well-defined

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1 Introduction

Systems that exhibit self-organized criticality (SOC) have attracted much attention, since they might explain part of the abundance of fractal structures in nature. SOC is based upon the idea that complex behavior can develop spontaneously in certain many-body systems whose dynamics vary abruptly. In [4] H.J. Jensen gives a general overview of and introduction to self-organized criticality. Within the study of SOC, the Drossel-Schwabl forest-fire model has received much attention in the physics literature. See e.g. [6] for current insights.

In contrast to the Drossel-Schwabl forest-fire model, in the forest-fire process studied in this article the time is continuous, the space is infinite and the fire spreads with infinite speed. Informally, it is described as follows: Let $d \geq 1$ and S be a subset of or equal to $\mathbb{Z}^d$. Each site of the set S is either vacant or occupied by a tree. Vacant sites become occupied according to independent rate 1 Poisson processes, the growth processes. Independently, lightning strikes at each site according to independent rate λ Poisson processes, the ignition processes. When an occupied site is ignited, its entire occupied cluster burns down, that is, becomes vacant instantaneously. Here $\lambda > 0$ is the parameter of the model.

In [2] J. van den Berg and A. A. Jàrai study the asymptotic density in a forest-fire model on $\mathbb{Z}^1$. They show that regardless of the initial configuration, already after time of order $\log(1/\lambda)$ the density of vacant sites is of order $1/\log(1/\lambda)$. In [1], J. van den Berg and R. Brouwer let forest-fire processes on $\mathbb{Z}^2$ start with all sites vacant and study, for positive but small λ, the behavior near the 'critical time' t_c; that is, the time after which in the modified system without lightning an infinite occupied cluster would emerge. They show that under a percolation-like assumption, if for fixed $t > t_c$, they let simultaneously λ tend to 0 and m to infinity, the probability that some tree at distance smaller than m from 0 is burnt before time t, does not go to 1.

The subject of this article is the question posed in [3] and [2], whether the multi-dimensional infinite volume forest-fire model is well defined for each parameter $\lambda > 0$.

On a finite set S, with probability 1, the finitely many Poisson processes of growth and ignition at the sites of S are discrete in time. That is, there a.s. exists an enumeration of the growth and ignition events. Given this enumeration, a forest-fire process on S can be constructed recursively. The sketch of the construction of a forest-fire process on a finite set can be found in Section 3.1.

However, if the set S is infinite volume, then such an enumeration almost surely does not exist, and thus a recursive construction is impossible. Only in the special case $S = \mathbb{Z}^1$, suppose that we start with a configuration in which infinitely many sites on the negative and on the positive half line are vacant. Then almost surely there are, at each time t infinitely many sites (on both half lines) that have remained vacant throughout the interval $[0, t]$. These vacant sites divide the infinite line into finite pieces, which enables a graphical representation; see e.g. [5].

To construct a forest-fire process on $\mathbb{Z}^d$, we use in Section 3.2 a sequence of forest-fire processes on the finite sets $B_n^d := \{y \in \mathbb{Z}^d | \, \|y\|_\infty \leq n\}$, $n \geq 1$, tightness, a diagonal sequence argument and Kolmogorov's Extension Theorem. In Sections 3.3 up to 3.5 it is shown that in fact the constructed process satisfies the definition of a forest-fire process on $\mathbb{Z}^d$. That is, for all $d \in \mathbb{N}$ and all $\lambda > 0$, there exists a forest-fire process on $\mathbb{Z}^d$ with parameter λ. Finally, it is shown that in a forest-fire process on $\mathbb{Z}^d$, a.s. there does not exist an infinite cluster.

The formal definition of a forest-fire process and the main results are stated in Section 2.

2 Definition of a forest-fire process and main results

2.1 Definition of a forest-fire process

Definition 1. *For all $F \subseteq \mathbb{Z}^d$ and all $x, y \in \mathbb{Z}^d$, the relation $x \leftrightarrow_F y$ holds, if x and y are connected by a path in F, that is, if there exists a sequence $x = x_0, x_1, \dots, x_n = y$ of distinct sites in F s.t. for all $1 \le i \le n$, the relation $\|x_i - x_{i-1}\|_1 = 1$ holds.*

Definition 2. *Let $S \subseteq \mathbb{Z}^d$ and $(\eta_{t,x})_{t\ge 0, x\in S}$ be a process with values in $\{0,1\}^S$ whose left limits $(\lim_{s\uparrow t} \eta_{s,x})_{t>0} =: (\eta_{t^-,x})_{t>0}$, $x \in S$, exist. For all $t \in \mathbb{R}^+$, we define $F_{t^-} := \{ y \in S \,|\, \eta_{t^-,y} = 1 \}$, and for all $x \in S$, the set*

$$C_{t^-,x} := \left\{ y \in S \,\middle|\, x \leftrightarrow_{F_{t^-}} y \right\}$$

to be the left limit of the cluster at x at time t.

We consider the following forest-fire model where the possible locations of trees are the sites of a subset of the lattice $\mathbb{Z}^d$. Each site has two possible states: 'vacant' or 'occupied'. Vacant sites become occupied (growth of a tree) according to independent rate 1 Poisson processes. Independently, at each site ignition (by lightning) occurs according to independent rate λ Poisson processes. When a site is hit by ignition, its entire occupied cluster burns down, that is, becomes vacant instantaneously.

Definition 3 (Definition of a forest-fire process)**.** *Let $S \subseteq \mathbb{Z}^d$ and $\lambda \in \mathbb{R}^+$. A forest-fire process on S with parameter λ is a process $\bar{\eta}_t = (\bar{\eta}_{t,x})_{x\in S} = (\eta_{t,x},\ G_{t,x},\ I_{t,x})_{x\in S}$ with values in $(\{0,1\} \times \mathbb{N}_0 \times \mathbb{N}_0)^S$, $t \ge 0$, that has the following properties:*

(a) The processes $(G_{t,x})_{t\ge 0}$ and $(I_{t,x})_{t\ge 0}$, $x \in S$, are independent Poisson processes with parameter 1 and λ, respectively;

(b) For all $x \in S$, the process $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\ge 0}$ is càdlàg, i.e., right-continuous with left limits;

(c) For all $t \in \mathbb{R}_0^+$, the increments of the growth and ignition processes after time t, $(G_{t+s,x} - G_{t,x}, I_{t+s,x} - I_{t,x})_{s\ge 0, x\in S}$, are independent of the forest-fire process $(\bar{\eta}_s)_{0\le s\le t}$ up to time t;

(d) For all $x \in S$ and all $t > 0$,

- $\lim_{s\uparrow t} G_{s,x} =: G_{t^-,x} < G_{t,x} \Rightarrow \eta_{t,x} = 1$;
  (Growth of a tree at the site x at time $t \Rightarrow$ The site x is occupied at time t)
- $\eta_{t^-,x} < \eta_{t,x} \Rightarrow G_{t^-,x} < G_{t,x}$;
  (The site x gets occupied at time $t \Rightarrow$ Growth of a tree at the site x at time t)
- $\lim_{s\uparrow t} I_{s,x} =: I_{t^-,x} < I_{t,x} \Rightarrow \forall y \in C_{t^-,x} :\ \eta_{t,y} = 0$;
  (Ignition at the site x at time $t \Rightarrow$ All sites of the cluster at x get vacant at time t)
- $\eta_{t^-,x} > \eta_{t,x} \Rightarrow \exists y \in C_{t^-,x} :\ I_{t^-,y} < I_{t,y}$.
  (The site x gets vacant at time $t \Rightarrow$ The cluster at x must be hit by ignition at time t)

For all $x \in S$, we call $(G_{t,x})_{t\geq 0}$ the *growth process*, $(I_{t,x})_{t\geq 0}$ the *ignition process* and $(\eta_{t,x})_{t\geq 0}$ the *forest-fire process* at the site x. We say that the site $x \in S$ is *occupied* at time t, if $\eta_{t,x} = 1$ holds, and *vacant*, if $\eta_{t,x} = 0$ holds. For all $t \in \mathbb{R}_0^+$, we define $F_t := \{x \in S \mid \eta_{t,x} = 1\}$, i.e., the set of sites that are occupied at time t. We say that the sites x and y are *connected by an occupied path* at time t, if $x \leftrightarrow_{F_t} y$ holds. Maximal connected sets of occupied sites are called clusters. For all $t \in \mathbb{R}_0^+$ and all $x \in S$, we define the *cluster* at x at time t by

$$C_{t,x} := \{y \in S \mid x \leftrightarrow_{F_t} y\}.$$

It is called right continuous if for all $t \geq 0$, there exists an $\epsilon_t > 0$ s.t. for all $t' \in [t, t + \epsilon_t)$, the equality $C_{t,x} = C_{t',x}$ holds.

The events

$$\mathrm{G}_{t',t,x} := \{G_{t',x} < G_{t,x}\} \text{ and } \mathrm{I}_{t',t,x} := \{I_{t',x} < I_{t,x}\}$$

describe the growth of a tree and ignition at the site x in between time t' and $t > t'$, respectively.

$\zeta := (\eta_{0,x})_{x\in S}$ is called the *initial configuration* of the process. We define $F_\zeta := \{z \in S : \zeta_z = 1\}$, and for all $x \in S$,

$$C_{\zeta,x} := \{y \in S \mid x \leftrightarrow_{F_\zeta} y\},$$

i.e., the cluster at x in the initial configuration ζ. We write

$$Z_S^{finite} := \left\{\zeta \in \{0,1\}^S \middle| \forall x \in S : |C_{\zeta,x}| < \infty\right\},$$

to denote the set off all initial configurations that do not contain an infinite cluster.

Given events $(A_i)_{1\leq i\leq n}$, we sometimes write $\{A_1, A_2, \ldots, A_n\} := \cap_{1\leq i\leq n} A_i$ to denote the intersection of the events; we write $A_1 \subseteq_N A_2$, if there exists a null set M s.t. $A_1 \subseteq A_2 \cup M$ holds. The complement of a set A is denoted by $\complement A$. Given a probability space $(\Omega, \mathcal{F}, \mu)$, we write $\hat{\mathcal{F}}$ to denote the completion of the σ-field $\mathcal{F}$.

2.2 Main results

Theorem 1. *For all $d \in \mathbb{N}$, all real numbers $\lambda > 0$ and all $\zeta \in Z_{\mathbb{Z}^d}^{finite}$, there exists a forest-fire process on $\mathbb{Z}^d$ with parameter λ and initial configuration ζ.*

Theorem 2. *Let $d \in \mathbb{N}$ and let $(\bar{\eta}_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda > 0$ and initial configuration $\zeta \in Z_d^{finite}$. Then a.s. there does not exist an infinite cluster in the process $(\bar{\eta}_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$, that is, the set*

$$\left\{\exists x \in \mathbb{Z}^d \; \exists t \in \mathbb{R}_0^+ : \; |C_{t,x}| = \infty\right\}$$

is a null set. Moreover a.s. the left limits of the clusters are finite, that is, the set

$$\left\{\exists x \in \mathbb{Z}^d \; \exists t \in \mathbb{R}^+ : \; |C_{t^-,x}| = \infty\right\}$$

is a null set.

3 Construction of a multi-dimensional infinite volume forest-fire process

The goal is to show that there exists a process that satisfies the definition of a forest-fire process on $\mathbb{Z}^d$. Therefore in Section 3.1, we first sketch the construction of forest-fire processes on finite sets. In Section 3.2 we use a sequence of forest-fire processes on finite boxes to construct a process on $\mathbb{Z}^d$. Sections 3.3 up to 3.5 are used to show that the constructed process a.s. satisfies the definition of a forest-fire process.

3.1 Construction of a forest-fire processes on finite sets

Let $S \subset \mathbb{Z}^d$ be a finite set, and $\zeta \in \{0,1\}^S$ be an initial configuration. The finitely many growth and ignition times at the sites of S are a.s. discrete. That is, there a.s. exists an enumeration (depending on ω) of the growth and ignition events. Given this enumeration, we construct the forest-fire process on S recursively.

To begin, let $(G_{t,x})_{t\geq 0}$, $x \in S$, be i.i.d. Poisson processes with parameter 1, and independently let $(I_{t,x})_{t\geq 0}$, $x \in S$, be i.i.d. Poisson processes with parameter $\lambda > 0$. For all $x \in S$, we denote the time of the n'th jump of the process $(G_{t,x})_{t\geq 0}$ by $g_{n,x}$; that is, the random variable $g_{n,x}$ is the time of the n'th growth of a tree at the site x. The n'th ignition at the site x, that is, the n'th jump of the process $(I_{t,x})_{t\geq 0}$ is denoted by $i_{n,x}$. To describe a growth attempt or an ignition event, we write (t, x, e): t denotes the point in time, x the site and e the type of the event. In case of an ignition $e = 0$, otherwise $e = 1$.

A.s. the random variables $(g_{n,x})_{n\in\mathbb{N}}$ and $(i_{n,x})_{n\in\mathbb{N}}$, $x \in S$, take discrete values in $\mathbb{R}^+$. For all $n \in \mathbb{N}$, we write

$$(t_n, x_n, e_n) \in \bigcup_{(k,x)\in\mathbb{N}\times S} \big\{(g_{k,x}, x, 1) \cup (i_{k,x}, x, 0)\big\},\ t_1 < t_2 < t_3, \ldots$$

to describe the n'th event. Given this enumeration of growth and ignition events, we construct a 'discrete in time' version of the forest-fire process. For all $x \in S$, we define

$$\eta^{discr}_{0,x} := \zeta_x,$$

and recursively for all $j \in \mathbb{N}$,

$$\eta^{discr}_{j,x} := \begin{cases} 1, & \text{if } e_j = 1,\ x = x_j; \\ \eta^{discr}_{j-1,x} & \text{if } e_j = 1,\ x \neq x_j; \\ 0, & \text{if } e_j = 0,\ x \longleftrightarrow_{F_{j-1}} x_j; \\ \eta^{discr}_{j-1,x} & \text{if } e_j = 0,\ x \not\longleftrightarrow_{F_{j-1}} x_j. \end{cases}$$

To explain the construction, note that in the first case, there is the growth of a tree at the site x. Thus the site x is occupied. In the second case, there is the growth of a tree at $x_j \neq x$; the state of the site x remains unchanged. In the third case, there is an ignition at a site that is connected to x by an occupied path; the site x gets vacant. In the last case, the ignition occurs at a site that is not connected to x by an occupied path; the state of the site x remains unchanged.

For all $j \in \mathbb{N}_0$ and all $x \in S$, we define (taking $t_0 = 0$)

$$(\eta_{t,x})_{t_j \le t < t_{j+1}} := \eta^{discr}_{j,x}.$$

Remark 1. Restricted to the complement of a null set, the process

$$(\bar{\eta}_t)_{t \ge 0} := \big(\eta_{t,x}, G_{t,x}, I_{t,x}\big)_{x \in S, t \ge 0}$$

is well defined and satisfies the definition of a forest-fire process on S with parameter λ and initial configuration ζ.

3.2 Construction of a process $\bar{\eta}$ on $\mathbb{Z}^d$

Definition 4. *For all $x \in \mathbb{Z}^d$, all $n \in \mathbb{N}$, let $B^d_{n,x} := \{y \in \mathbb{Z}^d | \; \|x-y\|_\infty \le n\}$ be the hypercube with center x and size $2n$. In case of $x = 0$, we write $B^d_n := B^d_{n,0}$*

First a less formal overview of the construction: To construct a forest-fire process on $\mathbb{Z}^d$, we use the sequence of forest-fire processes on the finite sets $(B^d_n)_{n\ge1}$. We embed these processes into $\mathbb{Z}^d$ and realize them to be canonical processes on probability spaces $\big(E^{\mathbb{R}_0^+ \times \mathbb{Z}^d}, \mathbb{B}(E^{\mathbb{R}_0^+ \times \mathbb{Z}^d}), \mu_n\big)_{n \in \mathbb{N}}$, $E := \{0,1\} \times \mathbb{N}_0 \times \mathbb{N}_0$. If we restrict this sequence of embedded processes to a finite set of time-space points $S \subset \mathbb{Q}_0^+ \times \mathbb{Z}^d$, then by tightness we get the existence of a weakly convergent subsequence. Thus using an appropriate sequence of finite sets of time space-points $S_k \uparrow \mathbb{Q}_0^+ \times \mathbb{Z}^d$, tightness, a diagonal sequence argument and Kolmogorov's Extension Theorem, we get the existence of a process defined for all time-space points in $\mathbb{Q}_0^+ \times \mathbb{Z}^d$, which is closely related to the forest-fire processes on the sets $(B^d_n)_{n\ge1}$. Finally, restricted to the complement of a null set, we define the forest-fire process on $\mathbb{Z}^d$ to be the right limits of the latter process.

$$\begin{tikzcd}
\big(E^{\mathbb{R}_0^+ \times \mathbb{Z}^d}, \mathbb{B}(E^{\mathbb{R}_0^+ \times \mathbb{Z}^d}), \mu_n\big)_{n\in\mathbb{N}} \arrow[r, Leftrightarrow, "\text{canonical processes}"] \arrow[d, "\text{canonical projektions}"] & \big((\eta^n_{t,x}, G^n_{t,x}, I^n_{t,x})_{(t,x)\in\mathbb{R}_0^+\times\mathbb{Z}^d}\big)_{n\in\mathbb{N}} \\
\big(E^{S_k}, \mathbb{B}(E^{S_k}), \mu_{n,k}\big)_{n,k\in\mathbb{N}} \arrow[r, Leftrightarrow, "\text{canonical processes}"] \arrow[d, "\text{weak convergence } n_l\to\infty"] & \big((\eta^{n,k}_{t,x}, G^{n,k}_{t,x}, I^{n,k}_{t,x})_{(t,x)\in S_k}\big)_{n,k\in\mathbb{N}} \\
\big(E^{S_k}, \mathbb{B}(E^{S_k}), \mu_{k\ conv}\big)_{k\in\mathbb{N}} \arrow[r, Leftrightarrow] \arrow[d, "\text{Kolmogorov Extension Theorem}"] & \big((\eta^{k\ conv}_{t,x}, G^{k\ conv}_{t,x}, I^{k\ conv}_{t,x})_{(t,x)\in S_k}\big)_{k\in\mathbb{N}} \\
\big(E^{\mathbb{Q}_0^+ \times \mathbb{Z}^d}, \mathbb{B}(E^{\mathbb{Q}_0^+ \times \mathbb{Z}^d}), \mu_{\mathbb{Q}}\big) \arrow[r, Leftrightarrow, "\text{canonical process}"] \arrow[d, "\text{completion of the } \sigma-\text{field}"] & (\eta^{\mathbb{Q}}_{t,x}, G^{\mathbb{Q}}_{t,x}, I^{\mathbb{Q}}_{t,x})_{(t,x)\in\mathbb{Q}_0^+\times\mathbb{Z}^d} \arrow[d, "(\eta_{t,x},G_{t,x},I_{t,x}):=\lim_{s\downarrow t}(\eta^{\mathbb{Q}}_{s,x},G^{\mathbb{Q}}_{s,x},I^{\mathbb{Q}}_{s,x})"] \\
\big(\Omega, \mathcal{F}, \mu\big) & (\eta_{t,x}, G_{t,x}, I_{t,x})_{(t,x)\in\mathbb{R}_0^+\times\mathbb{Z}^d}
\end{tikzcd}$$

Figure 1: Construction of a forest-fire process on $\mathbb{Z}^d$

To begin the construction let $\lambda > 0$ and $(\zeta_x)_{x\in\mathbb{Z}^d} \in Z^{finite}_{\mathbb{Z}^d}$ be an initial configuration that does not contain an infinite cluster. Let $(G_{t,x})_{t\ge0}$ and $(I_{t,x})_{t\ge0}$, $x \in \mathbb{Z}^d$, be independent Poisson processes with parameter 1 and λ, respectively. For all $n \in \mathbb{N}$, let $(\eta^{(n)}_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in B^d_n}$ be the forest-fire process on B^d_n with initial configuration $(\zeta_x)_{x\in B^d_n}$ and driving growth and ignition processes $(G_{t,x})_{t\geq 0,x\in B^d_n}$ and $(I_{t,x})_{t\geq 0,x\in B^d_n}$.

We embed these processes into $\mathbb{Z}^d$ and realize them to be canonical processes. For all $n \in \mathbb{N}$, let μ_n be the distribution of the process defined by

$$(\tilde{\eta}^n_{t,x})_{t\geq 0, x\in\mathbb{Z}^d} := \begin{cases} (\eta^{(n)}_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0} & \text{if } x \in B^d_n; \\ (0, G_{t,x}, I_{t,x})_{t\geq 0} & \text{if } x \in \mathbb{Z}^d \setminus B^d_n. \end{cases}$$

We define the *forest-fire process on B^d_n embedded into $\mathbb{Z}^d$*, i.e., $(\bar{\eta}^n_{t,x})_{t\geq 0,x\in\mathbb{Z}^d} = (\eta^n_{t,x}, G^n_{t,x}, I^n_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$, to be the canonical process (identity map) on $\big(E^{\mathbb{R}^+_0\times\mathbb{Z}^d}, \mathbb{B}(E^{\mathbb{R}^+_0\times\mathbb{Z}^d}), \mu_n\big)$.

Remark 2. For all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t\geq 0,x\in B^d_n}$ is the distribution of a forest-fire process on B^d_n with parameter λ and initial configuration $(\zeta_x)_{x\in\, B^d_n}$. The distribution of the processes $(G^n_{t,x})_{t\geq 0}$ and $(I^n_{t,x})_{t\geq 0}$, $x \in \mathbb{Z}^d$, is the distribution of independent Poisson processes with parameter 1 and λ, respectively.

Let $(e_n)_{n\in\mathbb{N}}$ be an enumeration of the countable set $\mathbb{Q}^+_0 \times \mathbb{Z}^d$, and set $S_k := \{e_i | 1 \leq i \leq k\}$. For all $k \in \mathbb{N}$, let $(\mu_{n,k})_{n\geq 1}$ be the canonical projection of the measures $(\mu_n)_{n\geq 1}$ onto the set E^{S_k}. Since $|S_k| = k$, we sometimes identify $E^{S_k} \equiv E^k$.

Lemma 1. For all $k \in \mathbb{N}$, the sequence $(\mu_{n,k})_{n\geq 1}$ on $\big(E^{S_k}, \mathbb{B}(E^{S_k})\big)$ is tight.

Proof. Let $k \in \mathbb{N}$. We have to show that for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subset E^{S_k}$ s.t. for all $n \in \mathbb{N}$, the relation $\mu_{n,k}(\complement K_\epsilon) < \epsilon$ holds. Let $\epsilon > 0$. Since the set S_k is finite, we can choose a natural number $m_\epsilon > 0$ s.t. for all $n \in \mathbb{N}$ and all $(t,x) \in S_k$, the relations $\mu_{n,k}\big(G^n(t,x) > m_\epsilon\big) < \frac{\epsilon}{2|S_k|}$ and $\mu_{n,k}\big(I^n(t,x) > m_\epsilon\big) < \frac{\epsilon}{2|S_k|}$ hold. The set $K_\epsilon := (\{0,1\} \times \{0,\ldots,m_\epsilon\} \times \{0,\ldots,m_\epsilon\})^{S_k}$ has the required property. □

Lemma 2. There exist a strictly increasing sequence of natural numbers $(n_l)_{l\in\mathbb{N}}$ and probability measures $(\mu_{k\ conv})_{k\in\mathbb{N}}$ s.t. for all $k \in \mathbb{N}$, the sequence $\big(\mu_{n_l,k}\big)_{l\in\mathbb{N}}$ converges weakly to $\mu_{k\ conv}$.

Proof. By recursion we show that there exist probability measures $(\mu_{k\ conv})_{k\in\mathbb{N}}$, and for all $k \in \mathbb{N}$, a subsequence $(n^{(k+1)}_l)_{l\in\mathbb{N}} \subseteq (n^{(k)}_l)_{l\in\mathbb{N}}$ s.t. for all $1 \leq i \leq k+1$, the sequence $\big(\mu_{n^{(k+1)}_l,i}\big)_{l\in\mathbb{N}}$ converges weakly to $\mu_{i\ conv}$. The result follows if we define $(n_l)_{l\in\mathbb{N}}$ to be the diagonal sequence taken from $\big((n^{(k)}_l)_{l\in\mathbb{N}}\big)_{k\in\mathbb{N}}$.

To begin the recursion, by Lemma 1 the sequence of probability measures $(\mu_{n,1})_{n\in\mathbb{N}}$ is tight. The space E^{S_1} is discrete and countable. We can choose a subsequence $(n^{(1)}_l)_{l\in\mathbb{N}} \subseteq (n)_{n\in\mathbb{N}}$ s.t. the sequence $\big(\mu_{n^{(1)}_l,1}\big)_{l\in\mathbb{N}}$ converges weakly to a probability measure $\mu_{1\ conv}$.

In the recursion step, let $(\mu_{i\ conv})_{1\leq i\leq k}$ be probability measures and $(n^{(k)}_l)_{l\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers s.t. for all $1 \leq i \leq k$, the sequence $\big(\mu_{n^{(k)}_l,i}\big)_{l\in\mathbb{N}}$ converges weakly to $\mu_{i\ conv}$. By Lemma 1, the sequence $\big(\mu_{n^{(k)}_l,k+1}\big)_{l\in\mathbb{N}}$ is tight. The space $E^{S_{k+1}}$ is discrete and countable. We can choose a subsequence $\big(n^{(k+1)}_l\big)_{l\in\mathbb{N}} \subseteq \big(n^{(k)}_l\big)_{l\in\mathbb{N}}$ s.t. for all $1 \leq i \leq k+1$, the sequence $\big(\mu_{n^{(k+1)}_l,i}\big)_{l\in\mathbb{N}}$ converges weakly to a probability measure $\mu_{i\ conv}$.

□

In this article it is not studied, whether the probability measures $(\mu_{k\ conv})_{k\in\mathbb{N}}$ are unique, that is, whether they depend on the choice of the sequence $(n_l)_{l\in\mathbb{N}}$. Therefore from now on, we choose an arbitrary sequence $(n_l)_{l\in\mathbb{N}}$ and probability measures $(\mu_{k\ conv})_{k\in\mathbb{N}}$ that satisfy the property from Lemma 2.

For all $k\in\mathbb{N}$, let $\big(\bar{\eta}^{k\ conv}_{t,x}\big)_{(t,x)\in S_k} := \big(\eta^{k\ conv}_{t,x}, G^{k\ conv}_{t,x}, I^{k\ conv}_{t,x}\big)_{(t,x)\in S_k}$ be the canonical process on $\big(E^{S_k}, \mathbb{B}(E^{S_k}), \mu_{k\ conv}\big)$.

Lemma 3. The sequence of measures $(\mu_{k\ conv})_{k\in\mathbb{N}}$ is consistent.

Proof. Let $k\in\mathbb{N}$. The space E^{S_k} is discrete. By the weak convergence for all $(\omega_j)_{1\le j\le k}\in E^k\equiv E^{S_k}$, we have

$$\begin{aligned}\mu_{k+1\ conv}\big(\{\omega_1\}\times\cdots\times\{\omega_k\}\times E\big) &= \lim_{l\to\infty}\mu_{n_l,k+1}\big(\{\omega_1\}\times\cdots\times\{\omega_k\}\times E\big)\\ &= \lim_{l\to\infty}\mu_{n_l,k}\big(\{\omega_1\}\times\cdots\times\{\omega_k\}\big)\\ &= \mu_{k\ conv}\big(\{\omega_1\}\times\cdots\times\{\omega_k\}\big).\end{aligned}$$

The space E^{S_k} is countable, the result follows. □

For all $k\in\mathbb{N}$, we write π_{S_k} to denote the canonical projection from $E^{\mathbb{Q}_0^+\times\mathbb{Z}^d}$ onto E^{S_k}. As a direct result from Lemma 3 and Kolmogorov's Extension Theorem, we get

Lemma 4. There exists a unique probability measure $\mu_{\mathbb{Q}}$ on $\big(E^{\mathbb{Q}_0^+\times\mathbb{Z}^d}, \mathbb{B}(E^{\mathbb{Q}_0^+\times\mathbb{Z}^d})\big)$ s.t. for all $k\in\mathbb{N}$ and all $A_k\in\mathbb{B}(E^{S_k})$, the equality

$$\mu_{\mathbb{Q}}\bigg(\pi^{-1}_{S_k}(A_k)\bigg) = \mu_{k\ conv}(A_k)$$

holds. (Uniqueness with respect to the measures $(\mu_{k\ conv})_{k\in\mathbb{N}}$.)

Let $\big(\bar{\eta}^{\mathbb{Q}}_{t,x}\big)_{t\in\mathbb{Q}_0^+,x\in\mathbb{Z}^d} := \big(\eta^{\mathbb{Q}}_{t,x}, G^{\mathbb{Q}}_{t,x}, I^{\mathbb{Q}}_{t,x}\big)_{t\in\mathbb{Q}_0^+,x\in\mathbb{Z}^d}$ be the canonical process on the probability space $\big(E^{\mathbb{Q}_0^+\times\mathbb{Z}^d}, \mathbb{B}(E^{\mathbb{Q}_0^+\times\mathbb{Z}^d}), \mu_{\mathbb{Q}}\big)$.

By Remark 2 for all $n\in\mathbb{N}$, the distribution of the processes $(G^n_{t,x})_{t\ge0}$ and $(I^n_{t,x})_{t\ge0}$, $x\in\mathbb{Z}^d$, is the distribution of independent Poisson processes with parameter 1 and λ, respectively. Thus for all $k\in\mathbb{N}$, the distribution of the processes $(G^{k\ conv}_{t,x})_{(t,x)\in S_k}$ and $(I^{k\ conv}_{t,x})_{(t,x)\in S_k}$ is independent of the chosen subsequence $(n_l)_{l\in\mathbb{N}}$, and is equal to the distribution of the processes $(G^1_{t,x})_{(t,x)\in S_k}$ and $(I^1_{t,x})_{(t,x)\in S_k}$. Uniqueness provides that the distributions of $(G^{\mathbb{Q}}_{t,x}, I^{\mathbb{Q}}_{t,x})_{t\in\mathbb{Q}_0^+,x\in\mathbb{Z}^d}$ and $(G^1_{t,x}, I^1_{t,x})_{t\in\mathbb{Q}_0^+,x\in\mathbb{Z}^d}$ must be equal. Poisson processes are right continuous with values in $\mathbb{N}_0$. We get

Lemma 5. Restricted to the complement of a null set, for all $x\in\mathbb{Z}^d$, the processes

$$\big(G_{t,x}\big)_{t\in\mathbb{R}_0^+} := \big(\lim_{s\downarrow t} G^{\mathbb{Q}}_{s,x}\big)_{t\in\mathbb{R}_0^+}$$

and

$$\left(I_{t,x}\right)_{t\in\mathbb{R}_0^+} := \left(\lim_{s\downarrow t} I^{\mathbb{Q}}_{s,x}\right)_{t\in\mathbb{R}_0^+},$$

are well defined. Their distribution is that of independent Poisson processes with parameter 1 and λ, respectively.

To show that a.s. for all $x \in \mathbb{Z}^d$, the process $(\eta_{t,x})_{t\in\mathbb{R}_0^+} := (\lim_{s\downarrow t}\eta^{\mathbb{Q}}_{s,x})_{t\in\mathbb{R}_0^+}$ is well defined, we first show

Lemma 6. Almost surely if a given site is vacant at time t' and occupied at time $t > t'$, then there must have been the growth of at least one tree a the site in the time between. More formally, for all $x \in \mathbb{Z}^d$,

$$\mu_{\mathbb{Q}}\left(\exists t', t \in \mathbb{Q}_0^+,\ t' < t :\ \eta^{\mathbb{Q}}_{t',x} < \eta^{\mathbb{Q}}_{t,x},\ G^{\mathbb{Q}}_{t',x} = G^{\mathbb{Q}}_{t,x}\right) = 0.$$

Proof. Let $x \in \mathbb{Z}^d$ and $t', t \in \mathbb{Q}_0^+$, $t' < t$. By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t\geq 0, x\in B_n^d}$ is the distribution of a forest-fire process on B_n^d. The definition of a forest-fire process implies that a vacant site can only have become occupied, if there has been the growth of a tree at the site. That is, for all $n \in \mathbb{N}$, the set $\left\{\eta^n_{t',x} < \eta^n_{t,x},\ G^n_{t',x} = G^n_{t,x}\right\}$ is a null set.

The relation $S_k \uparrow \mathbb{Q}_0^+ \times \mathbb{Z}^d$ as $k \to \infty$ holds; there must exist $k \in \mathbb{N}$ s.t. (t, x), $(t', x) \in S_k$ holds. By the definition of the measure $\mu_{\mathbb{Q}}$, the weak convergence and since the space E^{S_k} is discrete, we get

$$\begin{aligned}
&\mu_{\mathbb{Q}}\left(\eta^{\mathbb{Q}}_{t',x} < \eta^{\mathbb{Q}}_{t,x},\ G^{\mathbb{Q}}_{t',x} = G^{\mathbb{Q}}_{t,x}\right) = \mu_{k\ conv}\left(\eta^{k\ conv}_{t',x} < \eta^{k\ conv}_{t,x},\ G^{k\ conv}_{t',x} = G^{k\ conv}_{t,x}\right)\\
&= \lim_{l\to\infty}\mu_{n_l,k}\left(\eta^{n_l,k}_{t',x} < \eta^{n_l,k}_{t,x},\ G^{n_l,k}_{t',x} = G^{n_l,k}_{t,x}\right) = \lim_{l\to\infty}\mu_{n_l}\left(\eta^{n_l}_{t',x} < \eta^{n_l}_{t,x},\ G^{n_l}_{t',x} = G^{n_l}_{t,x}\right) = 0.
\end{aligned}$$

The result follows since t' and t run over the countable set $\mathbb{Q}_0^+$. ☐

Lemma 7. Almost surely for all $x \in \mathbb{Z}^d$ and all $t \in \mathbb{R}_0^+$, $\lim_{s\downarrow t}\eta^{\mathbb{Q}}_{s,x}$ exists, and $\lim_{s\uparrow t}\eta^{\mathbb{Q}}_{s,x}$ exists for $t > 0$.

Proof. Let $x \in \mathbb{Z}^d$. If there exists $t \in \mathbb{R}_0^+$ s.t. $\lim_{s\downarrow t}\eta^{\mathbb{Q}}_{s,x}$ does not exist, then there must exist a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive rational numbers with $\liminf_{n\to\infty}\eta^{\mathbb{Q}}_{t_n,x} = 0$ and $\limsup_{n\to\infty}\eta^{\mathbb{Q}}_{t_n,x} = 1$. By Lemma 6, a.s. for all $t', t \in \mathbb{Q}_0^+$, $t' < t$, if $\eta^{\mathbb{Q}}_{t',x} < \eta^{\mathbb{Q}}_{t,x}$ holds, then $G^{\mathbb{Q}}_{t',x} < G^{\mathbb{Q}}_{t,x}$ must hold. Thus restricted to the complement of a null set, the relation $G^{\mathbb{Q}}_{t_0,x} = \infty$ must hold. That is, a.s. there must exists $t_0 \in \mathbb{Q}^+$ s.t. $G^{\mathbb{Q}}_{t_0,x} = \infty$ holds, which is impossible. If there exists $t \in \mathbb{R}^+$ s.t. $\lim_{s\uparrow t}\eta^{\mathbb{Q}}_{s,x}$ does not exist, then a similar arguments yields to the same result. Formally we have for all $x \in \mathbb{Z}^d$,

$$\begin{aligned}
&\left\{\exists t \in \mathbb{R}_0^+ : \lim_{s\downarrow t} \eta^{\mathbb{Q}}_{s,x} \text{ does not exist}\right\} \cup \left\{\exists t \in \mathbb{R}^+ : \lim_{s\uparrow t} \eta^{\mathbb{Q}}_{s,x} \text{ does not exist}\right\} \\
\subseteq &\bigcup_{t_0' \in \mathbb{Q}_0^+} \bigcap_{n\in\mathbb{N}} \left\{\exists (t_i, t_i')_{1\le i\le n} \in \left(\mathbb{Q}_0^+\right)^{2n} \ \forall 1 \le i \le n : \ t_i' < t_i < t_{i-1}', \ \eta^{\mathbb{Q}}_{t_i',x} < \eta^{\mathbb{Q}}_{t_i,x}\right\} \\
\subseteq &N \cup \bigcup_{t_0' \in \mathbb{Q}_0^+} \bigcap_{n\in\mathbb{N}} \left\{G^{\mathbb{Q}}_{t_0',x} \ge n\right\},
\end{aligned}$$

with a null set N by Lemma 6. □

Let $\mathcal{N}$ be a null set s.t. restricted to the complement of the set $\mathcal{N}$,

- For all $x \in \mathbb{Z}^d$ and all $t \in \mathbb{R}_0^+$, $\lim_{s\downarrow t} \eta^{\mathbb{Q}}_{s,x}$ exists, and $\lim_{s\uparrow t} \eta^{\mathbb{Q}}_{s,x}$ exists for $t > 0$;
- The processes $(G_{t,x})_{t\ge 0} := \left(\lim_{s\downarrow t} G^{\mathbb{Q}}_{s,x}\right)_{t\in\mathbb{R}_0^+}$ and $(I_{t,x})_{t\ge 0} := \left(\lim_{s\downarrow t} I^{\mathbb{Q}}_{s,x}\right)_{t\in\mathbb{R}_0^+}$, $x \in \mathbb{Z}^d$, are independent Poisson processes with parameter 1 and λ, respectively. In particular, we require for all $x \in \mathbb{Z}^d$, the processes $(G_{t,x})_{t\ge 0}$ and $(I_{t,x})_{t\ge 0}$ to be càdlàg, increasing with values in $\mathbb{N}_0$, and that the relations $\lim_{t\to\infty} G_{t,x} = \infty$ and $\lim_{t\to\infty} I_{t,x} = \infty$ hold.

By Lemma 5 and 7 such a null set $\mathcal{N}$ exists.

Definition 5. *We define the forest-fire process on $\mathbb{Z}^d$ by*

$$\left(\bar{\eta}_{t,x}\right)_{t\ge 0, x\in\mathbb{Z}^d} := \left(\eta_{t,x}, G_{t,x}, I_{t,x}\right)_{t\ge 0, x\in\mathbb{Z}^d} := \begin{cases} \left(\lim_{s\downarrow t}\left(\eta^{\mathbb{Q}}_{s,x}, G^{\mathbb{Q}}_{s,x}, I^{\mathbb{Q}}_{s,x}\right)\right)_{t\ge 0, x\in\mathbb{Z}^d} & \text{on } \complement\mathcal{N}; \\ (0,0,0) & \text{on } \mathcal{N}. \end{cases}$$

Let μ be the measure $\mu_{\mathbb{Q}}$ associated to the completion of the σ-field $\mathbb{B}\left(E^{\mathbb{Q}_0^+ \times \mathbb{Z}^d}\right)$.

As a direct consequence of the choice of the null set $\mathcal{N}$, we get

Lemma 8. For all $x \in \mathbb{Z}^d$, the process $\left(\eta_{t,x}, G_{t,x}, I_{t,x}\right)_{t\ge 0}$ is càdlàg.

Proof. Let $x \in \mathbb{Z}^d$. The choice of the set $\mathcal{N}$ in Definition 5 implies that it suffices to show that the process $(\eta_{t,x})_{t\ge 0}$ is càdlàg. It is right continuous since formally, the relation

$$\begin{aligned}
&\left\{\exists t \in \mathbb{R}_0^+ \ \forall \epsilon > 0 \ \exists t' \in [t, t+\epsilon) : \ \eta_{t,x} \ne \eta_{t',x}\right\} \\
= &\left\{\exists t \in \mathbb{R}_0^+ \ \forall \epsilon > 0 \ \exists t' \in [t, t+\epsilon) : \ \lim_{s\downarrow t} \eta^{\mathbb{Q}}_{s,x} \ne \lim_{s'\downarrow t'} \eta^{\mathbb{Q}}_{s',x}\right\} \setminus \mathcal{N} \\
\subseteq &\underbrace{\left\{\exists t \in \mathbb{R}_0^+ : \lim_{s\downarrow t} \eta^{\mathbb{Q}}_{s,x} \text{ does not exist}\right\}}_{\subseteq \mathcal{N}} \setminus \mathcal{N} = \emptyset
\end{aligned}$$

holds. The relation

$$\left\{\exists t\in\mathbb{R}^+:\ \lim_{s\uparrow t}\eta_{s,x}\text{ does not exist}\right\}=\left\{\exists t\in\mathbb{R}^+:\ \lim_{s\uparrow t}\lim_{s'\downarrow s}\eta^{\mathbb{Q}}_{s',x}\text{ does not exist}\right\}\setminus\mathcal{N}$$
$$\subseteq\underbrace{\left\{\exists t\in\mathbb{R}^+:\ \lim_{s'\uparrow t}\eta^{\mathbb{Q}}_{s',x}\text{ does not exist}\right\}}_{\subseteq\mathcal{N}}\setminus\mathcal{N}=\emptyset$$

shows that the left limits of the process $(\eta_{t,x})_{t\geq 0}$ exist. □

Theorem 3. Restricted to the complement of a null set, the process $(\bar{\eta}_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ satisfies the definition of a forest-fire process on $\mathbb{Z}^d$ with parameter λ and initial configuration ζ, Definition 3.

Proof. Sections 3.3 up to 3.5 are used to prove the Theorem. □

3.3 The relation of the process $\bar{\eta}$ to the forest-fire processes on the finite boxes

Although we call the process defined in Definition 5, namely the process $(\bar{\eta}_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$, a forest-fire process, up to here it is not clear whether it satisfies the definition of a forest-fire process on $\mathbb{Z}^d$.

To define the process $(\bar{\eta}_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$, we used a sequence of finite volume forest-fire processes which we embedded into $\mathbb{Z}^d$, namely the processes $(\bar{\eta}^n_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$, $n\in\mathbb{N}$. There is a close relation between these finite volume forest-fire processes and the process defined in Definition 5, $(\bar{\eta}_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$. This relation is noted in the following Lemma and will be used several times to show that the process $(\bar{\eta}_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ satisfies the definition of a forest-fire process on $\mathbb{Z}^d$.

Lemma 9. Let A be an event which is described by the configuration of finitely many sites at finitely many points in time. If there exists a natural number N such that for all $n\geq N$, the event A is a.s. impossible in the finite volume forest-fire processes $(\bar{\eta}^n_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$, then the event A is a.s. impossible in the process $(\bar{\eta}_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$. More formally let $S\subset\mathbb{Z}^d$ be a finite set of $m\in\mathbb{N}$ sites, let $h\in\mathbb{N}$ and $A\in\mathbb{B}\big((E^S)^h\big)$. If there exists $N\in\mathbb{N}$ s.t. for all $t_1<t_2<\cdots<t_h\in\mathbb{Q}_0^+$ and all $n\geq N$, the set

$$\left\{\left(\bar{\eta}^n_{t_i,x}\right)_{x\in S,1\leq i\leq h}\in A\right\}$$

is a null set, then the set

$$\left\{\exists\, t_1,t_2,\ldots,t_h\in\mathbb{R}_0^+:\ t_1<t_2<\cdots<t_h,\ \left(\bar{\eta}_{t_i,x}\right)_{x\in S,1\leq i\leq h}\in A\right\}$$

is a null set.

Proof. Let S, m, h and A be as in the statement of the lemma and assume that there is a natural number N with the properties mentioned in the lemma. According to the definition of the process $(\bar\eta_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$, we get

$$\begin{aligned}
&\Big\{\exists\, t_1,t_2,\dots,t_h\in\mathbb{R}_0^+ :\ t_1<t_2<\cdots<t_h,\ \big(\bar\eta_{t_i,x}\big)_{x\in S,1\leq i\leq h}\in A\Big\}\\
\subseteq&\Big\{\exists\, t_1,t_2,\dots,t_h\in\mathbb{R}_0^+ :\ t_1<t_2<\cdots<t_h,\ \big(\lim_{s\downarrow t_i}\bar\eta^{\mathbb{Q}}_{s,x}\big)_{x\in S,1\leq i\leq h}\in A\Big\}\cup\mathcal{N}\\
=&\Big\{\exists\, t_1,t_2,\dots,t_h\in\mathbb{R}_0^+ :\ t_1<t_2<\cdots<t_h,\ \big(\lim_{s\downarrow t_i}\bar\eta^{\mathbb{Q}}_{s,x}\big)_{x\in S,1\leq i\leq h}\in A,\\
&\qquad \forall 1\leq i\leq h\ \exists\epsilon_i>0\ \forall t_i'\in[t_i,t_i+\epsilon_i)\cap\mathbb{Q} :\ \big(\lim_{s\downarrow t_i}\bar\eta^{\mathbb{Q}}_{s,x}\big)_{x\in S}=\big(\bar\eta^{\mathbb{Q}}_{t_i',x}\big)_{x\in S}\Big\}\cup\mathcal{N}\\
\subseteq&\Big\{\exists\, t_1',t_2',\dots,t_h'\in\mathbb{Q}_0^+ :\ t_1'<t_2'<\cdots<t_h',\ \big(\eta^{\mathbb{Q}}_{t_i',x}\big)_{x\in S,1\leq i\leq h}\in A\Big\}\cup\mathcal{N}.
\end{aligned}$$

Let $t_1'<t_2'<\cdots<t_h'\in\mathbb{Q}_0^+$. The relation $S^k\uparrow\mathbb{Q}_0^+\times\mathbb{Z}^d$ as $k\to\infty$ holds. Thus there exists $k\in\mathbb{N}$ s.t. for all $1\leq i\leq h$ and all $x\in S$, the relation $(t_i',x)\in S_k$ holds. By the construction of the measure $\mu_{\mathbb{Q}}$, the weak convergence and since the set E^{S_k} is discrete, we obtain

$$\begin{aligned}
&\mu_{\mathbb{Q}}\Big(\big(\bar\eta^{\mathbb{Q}}_{t_i',x}\big)_{x\in S,1\leq i\leq h}\in A\Big)=\mu_{k\ conv}\Big(\big(\bar\eta^{k\ conv}_{t_i',x}\big)_{x\in S,1\leq i\leq h}\in A\Big)\\
=&\lim_{l\to\infty}\mu_{n_l,k}\Big(\big(\bar\eta^{n_l,k}_{t_i',x}\big)_{x\in S,1\leq i\leq h}\in A\Big)=\lim_{l\to\infty}\mu_{n_l}\Big(\big(\bar\eta^{n_l}_{t_i',x}\big)_{x\in S,1\leq i\leq h}\in A\Big)=0.
\end{aligned}$$

The last equality comes from the assumed property of N. The result follows since $t_1',t_2',\dots,t_h'$ run over a countable set. □

Lemma 10. Almost surely a vacant site cannot get occupied, if there is not the growth of a tree at the site. More formally, for all $x\in\mathbb{Z}^d$, the set

$$\Big\{\exists t\in\mathbb{R}_0^+ :\ \eta_{t-,x}<\eta_{t,x},\ \complement\,\mathrm{G}_{t-,t,x}\Big\}$$

is a null set.

Proof. Let $x\in\mathbb{Z}^d$. By Remark 2 for all $n\in\mathbb{N}$, the distribution of the process $(\bar\eta^n_{t,x})_{t\geq 0,x\in B_n^d}$ is that of a finite volume forest-fire process on B_n^d. We know (see Definition 3) that *finite volume* forest-fire processes have the property that a vacant site can only have become occupied, if there has been the growth of a tree at the site. Thus for all $n\in\mathbb{N}$ and all $t',t\in\mathbb{Q}_0^+$, $t'<t$, the set $\{\eta^n_{t',x}<\eta^n_{t,x},\ \complement\,\mathrm{G}^n_{t',t,x}\}$ must be a null set. Lemma 9 provides that the set

$$\Big\{\exists t\in\mathbb{R}_0^+ :\ \eta_{t-,x}<\eta_{t,x},\ \complement\,\mathrm{G}_{t-,t,x}\Big\}\subseteq\Big\{\exists t',t\in\mathbb{R}_0^+ :\ t'<t,\ \eta_{t'}<\eta_{t,x},\ \complement\,\mathrm{G}_{t',t,x}\Big\}$$

is a null set. □

Definition 6. *The (countable) set of all finite and non-empty connected subsets of $\mathbb{Z}^d$ is*

$$C^f:=\Big\{C\subset\mathbb{Z}^d\,\Big|\,1\leq|C|<\infty,\ \forall x,y\in C :\ x\leftrightarrow_C y\Big\}.$$

Lemma 11. Let $S \in C^f$ be a set of finitely many connected sites. Suppose that the occupied set S is hit by ignition. Almost surely if a site of the set S is occupied after the ignition, then there must have been the growth of a tree at the site. More formally, for all $S \in C^f$, the set

$$\left\{\exists t', t \in \mathbb{R}_0^+ : \ t' < t, \ S \subseteq F_{t'}, \ \exists y \in S : \ \mathrm{I}_{t',t,y}, \ \exists z \in S : \ \eta_{t,z} = 1, \ \complement\, \mathrm{G}_{t',t,z}\right\}$$

is a null set.

Proof. Let $S \in C^f$. By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t \geq 0, x \in B^d_n}$ is that of a finite volume forest-fire process on B^d_n. We know (see Definition 3) that *finite volume* forest-fire processes have the property that if an occupied site is hit by ignition, then the site and the cluster at the site must get vacant. Thus if the occupied and connected set S has been hit by ignition, then a site of the set S must have become vacant. Furthermore *finite volume* forest-fire processes have the property that if a site of the occupied and connected set S gets vacant, then the whole set S must get vacant. Finally, in a *finite volume* forest-fire process a vacant site remains vacant, if there is not the growth of a tree at the site. That is, for all $n \in \mathbb{N}$ and all $t', t \in \mathbb{Q}_0^+$, $t' < t$, the set

$$\left\{ \ S \subseteq F^n_{t'}, \ \exists y \in S : \ \mathrm{I}^n_{t',t,y}, \ \exists z \in S : \ \eta^n_{t,z} = 1, \ \complement\, \mathrm{G}^n_{t',t,z}\right\}$$

is a null set. The result follows by Lemma 9. □

Lemma 12. Suppose that two sites are connected by an occupied path. If one of them has become vacant and there has not been the growth of a tree at the other site, then a.s. the other site must have become vacant, too. More formally, for all $x, y \in \mathbb{Z}^d$, the set

$$\left\{\exists t', t \in \mathbb{R}_0^+ : \ t' < t, \ x \leftrightarrow_{F_{t'}} y, \ \eta_{t,y} = 0, \ \eta_{t,x} = 1, \ \complement\, \mathrm{G}_{t',t,x}\right\}$$

is a null set.

Proof. Let $x, y \in \mathbb{Z}^d$. By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t \geq 0, x \in B^d_n}$ is that of a finite volume forest-fire process on B^d_n. We know (see Definition 3) that *finite volume* forest-fire processes have the property that if an occupied site gets vacant, then the whole occupied and connected set at the site must get vacant. Furthermore *finite volume* forest-fire processes have the property that a vacant site must remain vacant, if there is not the growth of a tree. That is, for all $S \in C^f$, for all $n \in \mathbb{N}$ and all $t', t \in \mathbb{Q}_0^+$, $t' < t$, the set

$$\left\{x, y \in S, \ S \subseteq F^n_{t'}, \ \eta^n_{t,y} = 0, \ \eta^n_{t,x} = 1, \ \complement\, \mathrm{G}^n_{t',t,x}\right\}$$

is a null set. It follows by Lemma 9 that for all $S \in C^f$, the set

$$\left\{\exists t', t \in \mathbb{R}_0^+ : \ t' < t, \ x, y \in S, \ S \subseteq F_{t'}, \ \eta_{t,y} = 0, \ \eta_{t,x} = 1, \ \complement\, \mathrm{G}_{t',t,x}\right\}$$

is a null set. For all $t' \in \mathbb{R}_0^+$, the relation $x \leftrightarrow_{F_{t'}} y = \left\{\exists S \in C^f : \ x, y \in S, \ S \subseteq F_{t'}\right\}$ holds. Thus the result follows since the set C^f is countable. □

Definition 7. *For all $S \subseteq \mathbb{Z}^d$, we define*

$$\partial S := \left\{ x \in \mathbb{Z}^d \setminus S \middle| \exists y \in S : \ \|x - y\|_1 = 1 \right\},$$

i.e., the set of sites next to S. A site can have at most $2d$ neighbors. Thus we have $|\partial S| \leq 2d|S|$, provided that $|S| < \infty$ holds.

Lemma 13. Almost surely if there is no ignition at and not the growth of a tree next to a finite cluster, then the cluster remains unchanged. More formally, for all $S \in C^f$ and all $x \in \mathbb{Z}^d$, the set

$$\left\{ \exists t_0, t_1, t_2 \in \mathbb{R}_0^+ : \ t_0 < t_1 \leq t_2, \ C_{t_0,x} = S, \ \forall y \in S : \ \complement \, \mathrm{I}_{t_0,t_2,y}, \ \forall z \in \partial S : \ \complement \, \mathrm{G}_{t_0,t_2,z}, \ C_{t_1,x} \neq S \right\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$ and $S \in C^f$. By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t \geq 0, x \in B_n^d}$ is that of a finite volume forest-fire process on B_n^d. We know (see Definition 3) that *finite volume* forest-fire processes have the property that if there is no ignition at and not the growth of a tree next to a cluster, then the cluster remains unchanged. Thus for all $n \in \mathbb{Z}^d$ and all $t_0, t_1, t_2 \in \mathbb{Q}_0^+$, $t_0 < t_1 \leq t_2$, the set

$$\left\{ C^n_{t_0,x} = S, \ \forall y \in S : \ \complement \, \mathrm{I}^n_{t_0,t_2,y}, \ \forall z \in \partial S : \ \complement \, \mathrm{G}^n_{t_0,t_2,z}, \ C^n_{t_1,x} \neq S \right\}$$

is a null set. The result follows by Lemma 9. $\square$

Lemma 14. For all $t \in \mathbb{R}_0^+$, the increments of the growth and ignition processes at the sites of $\mathbb{Z}^d$ after time t, $\left(G_{s+t,x} - G_{t,x}, I_{s+t,x} - I_{t,x}\right)_{s \geq 0, x \in \mathbb{Z}^d}$, are independent of the forest-fire process on $\mathbb{Z}^d$ up to time t, $\left(\bar{\eta}_{s,x}\right)_{0 \leq s \leq t, x \in \mathbb{Z}^d}$.

Proof. In the first step, we show that for all $t \in \mathbb{Q}_0^+$ and all $k \in \mathbb{N}$, the σ-fields

$$\mathcal{F}_t^k := \sigma \left\{ \pi_{S_k} \left((\bar{\eta}^{\mathbb{Q}}_{s,x})_{s \leq t, x \in \mathbb{Z}^d} \right) \right\}$$

and

$$\mathcal{Z}_t^k := \sigma \left\{ \pi_{S_k} \left((G^{\mathbb{Q}}_{t+s,x} - G^{\mathbb{Q}}_{t,x}, I^{\mathbb{Q}}_{t+s,x} - I^{\mathbb{Q}}_{t,x})_{s \geq 0, x \in \mathbb{Z}^d} \right) \right\}$$

are independent.

Let $t \in \mathbb{Q}_0^+$ and $k \in \mathbb{N}$. By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t \geq 0, x \in B_n^d}$ is that of a finite volume forest-fire process on B_n^d. We know (see Definition 3) that *finite volume* forest-fire processes have the property that the increments of the growth and ignition processes after time t are independent of the forest-fire process up to time t. That is, for all $A \in \mathcal{F}_t^k$, $B \in \mathcal{Z}_t^k$ and all $n \in \mathbb{N}$, the relation

$$\mu_{n,k}\left(\pi_{S_k}(A \cap B)\right) = \mu_{n,k}\left(\pi_{S_k}(A)\right) \cdot \mu_{n,k}\left(\pi_{S_k}(B)\right)$$

holds. By the definition of the measure $\mu_{\mathbb{Q}}$, the weak convergence and since the space E^{S_k} is discrete, we get

$$\begin{aligned}\mu_{\mathbb{Q}}(A \cap B) &= \mu_{k\ conv}\big(\pi_{S_k}(A \cap B)\big) = \lim_{l\to\infty} \mu_{n_l,k}\big(\pi_{S_k}(A \cap B)\big) \\ &= \lim_{l\to\infty} \mu_{n_l,k}\big(\pi_{S_k}(A)\big) \cdot \lim_{l\to\infty} \mu_{n_l,k}\big(\pi_{S_k}(B)\big) = \mu_{k\ conv}\big(\pi_{S_k}(A)\big) \cdot \mu_{k\ conv}\big(\pi_{S_k}(B)\big) \\ &= \mu_{\mathbb{Q}}(A) \cdot \mu_{\mathbb{Q}}(B).\end{aligned}$$

That is, for all $t \in \mathbb{Q}_0^+$, the σ-fields $\mathcal{F}_t^k$ and $\mathcal{Z}_t^k$ are independent. It follows by a Dynkin argument that for all $t \in \mathbb{Q}_0^+$, the σ-fields

$$\mathcal{F}_t^{\mathbb{Q}} := \sigma\Big\{\bigcup_{k\in\mathbb{N}} \mathcal{F}_t^k\Big\} \text{ and } \mathcal{Z}_t^{\mathbb{Q}} := \sigma\Big\{\bigcup_{k\in\mathbb{N}} \mathcal{Z}_t^k\Big\}$$

are independent.

The definition of the process $(\bar{\eta}_{t,x})_{t\in\mathbb{R}_0^+, x\in\mathbb{Z}^d}$, Definition 5, implies that for all $t \in \mathbb{R}_0^+$,

$$\mathcal{F}_t := \sigma\Big\{\bar{\eta}_{s,x} : 0 \le s \le t, x \in \mathbb{Z}^d\Big\} \subseteq \bigcap_{s>t} \hat{\mathcal{F}}_s^{\mathbb{Q}}.$$

Again a Dynkin argument shows that the σ-field $\sigma\{\cup_{s>t}\mathcal{Z}_s^{\mathbb{Q}}\}$ is independent of $\mathcal{F}_t$.

The processes of growth and ignition are right continuous with values in $\mathbb{N}_0$. For all sets of finitely many sites $S \subset \mathbb{Z}^d$, $|S| < \infty$, the relation

$$\sigma\Big\{G_{t+s,x} - G_{t,x}, I_{t+s,x} - I_{t,x} : s > 0, x \in S\Big\} \subseteq \hat{\sigma}\{\cup_{s>t}\mathcal{Z}_s^{\mathbb{Q}}\}$$

holds, and by a Dynkin argument we obtain

$$\mathcal{Z}_t := \sigma\Big\{G_{t+s,x} - G_{t,x}, I_{t+s,x} - I_{t,x} : s > 0, x \in \mathbb{Z}^d\Big\} \subseteq \hat{\sigma}\{\cup_{s>t}\mathcal{Z}_s^{\mathbb{Q}}\}.$$

That is, for all $t \in \mathbb{R}_0^+$, the σ-fields $\mathcal{F}_t$ and $\mathcal{Z}_t$ are independent. $\square$

3.4 Infinite clusters in the process $\bar{\eta}$

Intuitively, if there exists an infinite cluster, then a.s. it must immediately get hit by ignition; a.s. the sites of an infinite cluster must immediately get vacant. By Lemma 8 for all $x \in \mathbb{Z}^d$, the forest-fire process at the site x, $(\eta_{t,x})_{t\ge0}$, is right continuous. This is a contradiction.

Lemma 15. For any fixed time, almost surely if a site is part of an infinite cluster, then the site immediately gets vacant. More formally, for all $t \in \mathbb{R}_0^+$ and all $x \in \mathbb{Z}^d$, the set

$$\Big\{|C_{t,x}| = \infty,\ \exists\epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1\Big\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$ and $t \in \mathbb{R}_0^+$. The proof uses that if the cluster at x is infinite, then for all $n \in \mathbb{N}$, there must exist an occupied path that connects the site x to a site in $\partial B_{n,x}^d$.

For all $T \subseteq \mathbb{Z}^d$, for all $S \in \{0,1\}^T$, we define $F_S := \{z \in T : \ S_z = 1\}$, and write for all $n \in \mathbb{N}$,

$$\Sigma_{n,x} := \left\{ S \in \{0,1\}^{B_{n+1,x}^d} \middle| \exists y \in \partial B_{n,x}^d : \ x \leftrightarrow_{F_S} y \right\}$$

to denote the set of all configurations of $B_{n+1,x}^d$ in which there exists an occupied path that connects the site x to a site in $\partial B_{n,x}^d$. By Lemma 8, the growth process at the site x is right continuous with values in $\mathbb{N}_0$. Thus we have

$$\begin{aligned}
&\left\{ |C_{t,x}| = \infty, \ \exists \epsilon > 0 \ \forall t' \in [t, t+\epsilon) : \ \eta_{t',x} = 1 \right\} \\
=&\left\{ |C_{t,x}| = \infty, \ \exists \epsilon > 0 \ \forall t' \in [t, t+\epsilon) : \ \eta_{t',x} = 1, \ \complement \mathrm{G}_{t,t',x} \right\} \\
\subseteq & \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \sum_{S \in \Sigma_{n,x}} \underbrace{\left\{ (\eta_{t,z})_{z \in B_{n+1,x}^d} = S, \ \eta_{t+\frac{1}{k},x} = 1, \ \complement \mathrm{G}_{t,t+\frac{1}{k},x} \right\}}_{=:A_{S,k,n,x}}.
\end{aligned}$$

Here $\sum_{i \in I} B_i$ denotes the union of the disjoint sets $(B_i)_{i \in I}$. For all $n \in \mathbb{N}$ and all $S \in \Sigma_{n,x}$, we define

$$C_{S,n,x} := \left\{ y \in B_{n+1,x}^d \middle| \ x \leftrightarrow_{F_S} y \right\},$$

i.e., the set of sites in $B_{n+1,x}^d$ that are connected to x by an occupied path in $B_{n+1,x}^d$. Note that for all $n \in \mathbb{N}$ and all $S \in \Sigma_{n,x}$,the set $C_{S,n,x}$ must contain at least $n+1$ sites.

Let $n, k \in \mathbb{N}$ and $S \in \Sigma_{n,x}$. In Lemma 11 we showed that if a finite connected set of occupied sites is hit by ignition, and a site of the set is occupied after the ignition, then a.s. there must have been the growth of a tree at the site. Thus the relation

$$A_{S,k,n,x} \subseteq_N \left\{ (\eta_{t,z})_{z \in B_{n+1,x}^d} = S, \ \forall y \in C_{S,n,x} : \ \complement \mathrm{I}_{t,t+\frac{1}{k},y} \right\}$$

holds. (Recall the definition of $\subseteq_N$ at the end of Section 2.1.) By Lemma 14, the increments of the ignition processes after time t are independent of the forest-fire process up to time t. We get

$$\begin{aligned}
\mu\Bigg(\sum_{S \in \Sigma_{n,x}} A_{S,k,n,x} \Bigg) &\leq \sum_{S \in \Sigma_{n,x}} \mu\Bigg((\eta_{t,z})_{z \in B_{n+1,x}^d} = S, \ \forall y \in C_{S,n,x} : \ \complement \mathrm{I}_{t,t+\frac{1}{k},y} \Bigg) \\
&= \sum_{S \in \Sigma_{n,x}} \mu\Bigg((\eta_{t,z})_{z \in B_{n+1,x}^d} = S \Bigg) \cdot \mu\Bigg(\forall y \in C_{S,n,x} : \ \complement \mathrm{I}_{t,t+\frac{1}{k},y} \Bigg) \\
&= \sum_{S \in \Sigma_{n,x}} \mu\Bigg((\eta_{t,z})_{z \in B_{n+1,x}^d} = S \Bigg) \cdot \mu\Bigg(\complement \mathrm{I}_{0,\frac{1}{k},x} \Bigg)^{|C_{S,n,x}|} \leq \left(e^{-\lambda \frac{1}{k}}\right)^{n+1}.
\end{aligned}$$

That is, for all $k \in \mathbb{N}$, the relation

$$0 \leq \liminf_{n \to \infty} \Bigg(\sum_{S \in \Sigma_{n,x}} A_{S,k,n,x} \Bigg) \leq \limsup_{n \to \infty} \Bigg(\sum_{S \in \Sigma_{n,x}} A_{S,k,n,x} \Bigg) \leq \lim_{n \to \infty} \left(e^{-\lambda \frac{1}{k}}\right)^{n+1} = 0$$

holds. Thus the set

$$\Big\{|C_{t,x}| = \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1\Big\} \subseteq \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \sum_{S\in\Sigma_{n,x}} A_{S,k,n,x}$$

is a null set. □

Lemma 16. Almost surely if a site is part of an infinite cluster, then the site immediately gets vacant. More formally, for all $x \in \mathbb{Z}^d$, the set

$$\Big\{\exists t \in \mathbb{R}_0^+ :\ |C_{t,x}| = \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1\Big\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$. There are two possibilities if the cluster at x is infinite. The cluster at x can immediately get finite (1), or remain infinite (2). If the infinite cluster at x immediately gets finite, then immediately a site of the cluster at x must get vacant. Formally we have

$$\begin{aligned}
&\Big\{\exists t \in \mathbb{R}_0^+ :\ |C_{t,x}| = \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1\Big\} \\
\subseteq &\Big\{\exists t \in \mathbb{R}_0^+ :\ |C_{t,x}| = \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1, \\
&\qquad\qquad \forall \delta > 0\ \exists y \in C_{t,x}\ \exists t^* \in [t, t+\delta) :\ \eta_{t^*,y} = 0\Big\} \qquad (1) \\
\cup &\Big\{\exists t \in \mathbb{R}_0^+\ \exists \delta > 0\ \forall t' \in [t, t+\delta) :\ |C_{t',x}| = \infty\Big\}. \qquad (2)
\end{aligned}$$

Intuitively, if immediately a site of the cluster at x gets vacant, then the whole cluster, in particular, the site x must get vacant immediately. Formally since the growth process at the site x is right continuous with values in $\mathbb{N}_0$, the event (1) is a subset of or equal to

$$\begin{aligned}
&\Big\{\exists t \in \mathbb{R}_0^+\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1,\ \complement\, \mathrm{G}_{t,t',x},\ \exists y \in C_{t,x}\ \exists t^* \in [t, t+\epsilon) :\ \eta_{t^*,y} = 0\Big\} \\
\subseteq &\Big\{\exists t, t^* \in \mathbb{R}_0^+\ \exists y \in \mathbb{Z}^d :\ t < t^*,\ x \leftrightarrow_{F_t} y,\ \eta_{t^*,x} = 1,\ \complement\, \mathrm{G}_{t,t^*,x},\ \eta_{t^*,y} = 0\Big\},
\end{aligned}$$

which is a null set by Lemma 12.

The set (2) is a subset of or equal to

$$\Big\{\exists t \in \mathbb{Q}_0^+ :\ |C_{t,x}| = \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ \eta_{t',x} = 1\Big\},$$

which is a null set by Lemma 15 and since the set $\mathbb{Q}_0^+$ is countable. □

As direct result of Lemma 16, we get

Lemma 17. Almost surely there exists no infinite cluster. More formally, the set

$$\Big\{\exists x \in \mathbb{Z}^d \; \exists t \in \mathbb{R}_0^+ : \; |C_{t,x}| = \infty\Big\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$. By Lemma 8, the forest-fire process at the site x, $(\eta_{t,x})_{t\geq 0}$, is right continuous with values in $\mathbb{N}_0$. Thus we have

$$\Big\{\exists t \in \mathbb{R}_0^+ : \; |C_{t,x}| = \infty\Big\} = \Big\{\exists t \in \mathbb{R}_0^+ : \; |C_{t,x}| = \infty, \; \exists \epsilon > 0 \; \forall t' \in [t, t+\epsilon) : \; \eta_{t',x} = 1\Big\}.$$

The result follows by Lemma 16. $\square$

3.5 The process $\bar{\eta}$ is a forest-fire process on $\mathbb{Z}^d$

To show the remaining properties of a forest-fire process, we first show that with probability 1, the left limits of the clusters are finite: We give an upper bound $C < 1$ for the probability that a given finite cluster grows before it gets hit by ignition. Then we show that the probability that a given cluster grows n times without being hit by ignition, is smaller than C^n. That is, a cluster a.s. can not grow infinitely often without being hit by ignition. It follows that the left limits of the clusters must be finite.

Lemma 18. The probability that there is the growth of a tree next to a finite set S, before there has been an ignition at the set S, is smaller than or equal to

$$C := 1 - \frac{\lambda}{2d+\lambda} < 1.$$

More formally, for all $t \in \mathbb{R}_0^+$ and all $S \subset \mathbb{Z}^d$, $1 \leq |S| < \infty$,

$$\mu\bigg(\exists t' \in \mathbb{R}_0^+ \; \forall y \in S : \; \complement\, \mathrm{I}_{t,t',y}, \; \exists z \in \partial S : \; \mathrm{G}_{t,t',z}\bigg) \leq C.$$

Proof. Let S be a finite non-empty subset of $\mathbb{Z}^d$. The distribution of the processes $(G_{t,x})_{t\geq 0}$ and $(I_{t,x})_{t\geq 0}$, $x \in S \cup \partial S$, is that of independent Poisson processes with parameter 1 and λ, respectively. Thus the total ignition rate on S is $\lambda|S|$; the total growth rate on ∂S is $|\partial S|$. Hence the probability that there is a growth on ∂S before there is an ignition on S is $|\partial S|/(|\partial S|+\lambda|S|)$, which is at most $1 - \lambda/(2d+\lambda)$. $\square$

Lemma 19. For all $t \in \mathbb{R}_0^+$, let $\mathcal{F}_t := \sigma\big\{\bar{\eta}_{s,x} : \; s \leq t, x \in \mathbb{Z}^d\big\}$ be the σ-field generated by the process $(\bar{\eta}_{s,x})_{s\leq t, x\in\mathbb{Z}^d}$. Let τ be a finite $\mathcal{F}_{\cdot}$-stopping time. Then the increments of the growth and ignition processes after time τ are independent of the forest-fire process on $\mathbb{Z}^d$ up to time τ. The distribution of the increments after time τ equals the distribution after time 0.

More formally, the σ-fields $\mathcal{F}_\tau := \sigma\big\{A \big| \forall t \in \mathbb{R}_0^+ : \; A \cap \{\tau \leq t\} \in \mathcal{F}_t\big\}$ and $\mathcal{Z}_\tau := \sigma\big\{\mathrm{Incr}_\tau\big\}$ are independent. Furthermore for all $\tilde{A} \in \mathbb{B}\big((\mathbb{N}_0 \times \mathbb{N}_0)^{\mathbb{R}_0^+ \times \mathbb{Z}^d}\big)$, the relation

$$\mu\big(\mathrm{Incr}_\tau \in \tilde{A}\big) = \mu\big(\mathrm{Incr}_0 \in \tilde{A}\big) \tag{*}$$

holds. Here we write $\mathrm{Incr}_\tau := \big\{G_{\tau+s,x} - G_{\tau,x}, I_{\tau+s,x} - I_{\tau,x} \; : \; x \in \mathbb{Z}^d, s \geq 0\big\}$ to denote the increments of the growth and ignition processes after time τ.

Proof. In the first step, let $\epsilon > 0$ and τ_ϵ be a $\mathcal{F}_{\cdot}$-stopping time with values in $\epsilon\mathbb{N}_0$. Let $A \in \mathcal{Z}_{\tau_\epsilon}$ and $B \in \mathcal{F}_{\tau_\epsilon}$. By the definition of $\mathcal{Z}_{\tau_\epsilon}$, there exists $\tilde{A} \in \mathbb{B}\big((\mathbb{N}_0 \times \mathbb{N}_0)^{\mathbb{Z}^d \times \mathbb{R}_0^+}\big)$ s.t. $A = \big\{\mathrm{Incr}_{\tau_\epsilon} \in \tilde{A}\big\}$ holds. For all $t \in \mathbb{R}_0^+$, the distribution of the process Incr_t is that of the increments of independent Poisson processes after time t. This provides that for all $k, n \in \mathbb{N}_0$, the relation

$$\mu\big(\mathrm{Incr}_{\epsilon k} \in \tilde{A}\big) = \mu\big(\mathrm{Incr}_{\epsilon n} \in \tilde{A}\big)$$

must hold. In Lemma 14 we showed that for all $k \in \mathbb{N}_0$, the σ-field $\mathcal{F}_{\epsilon k}$ is independent of the σ-field $\mathcal{Z}_{\epsilon k}$. It follows that for all $n \in \mathbb{N}$, the relations

$$\begin{aligned}\mu(A) =& \mu\big(\mathrm{Incr}_{\tau_\epsilon} \in \tilde{A}\big) = \sum_{k \in \mathbb{N}_0} \mu\Big(\underbrace{\{\mathrm{Incr}_{\epsilon k} \in \tilde{A}\}}_{\in \mathcal{Z}_{\epsilon k}} \cap \underbrace{\{\tau_\epsilon = \epsilon k\}}_{\in \mathcal{F}_{\epsilon k}}\Big) \\ &= \sum_{k \in \mathbb{N}_0} \mu\big(\mathrm{Incr}_{\epsilon n} \in \tilde{A}\big) \cdot \mu\big(\tau_\epsilon = \epsilon k\big) = \mu\big(\mathrm{Incr}_{\epsilon n} \in \tilde{A}\big)\end{aligned}$$

and

$$\begin{aligned}\mu\big(A \cap B\big) = \mu\big(\{\mathrm{Incr}_{\tau_\epsilon} \in \tilde{A}\} \cap B\big) &= \sum_{n \in \mathbb{N}_0} \mu\Big(\underbrace{\{\mathrm{Incr}_{\epsilon n} \in \tilde{A}\}}_{\in \mathcal{Z}_{\epsilon n}} \cap \underbrace{B \cap \{\tau_\epsilon = \epsilon n\}}_{\in \mathcal{F}_{\epsilon n}}\Big) \\ &= \sum_{n \in \mathbb{N}_0} \mu\big(A\big) \cdot \mu\big(B \cap \{\tau_\epsilon = \epsilon n\}\big) = \mu\big(A\big) \cdot \mu\big(B\big)\end{aligned}$$

must hold. That is, the σ-fields $\mathcal{F}_{\tau_\epsilon}$ and $\mathcal{Z}_{\tau_\epsilon}$ are independent.

In the second step, let τ be a finite $\mathcal{F}_{\cdot}$-stopping time with values in $\mathbb{R}_0^+$. For all $n \in \mathbb{N}$, we define a stopping time with values in $\frac{1}{n}\mathbb{N}_0$, by $\tau_n := \frac{1}{n}\min\{k \in \mathbb{N}_0 : \ k \geq \tau \cdot n\}$. For all $n \in \mathbb{N}$, the first step shows that the σ-fields $\mathcal{F}_{\tau_n}$ and $\mathcal{Z}_{\tau_n}$ are independent. Moreover for all $n \in \mathbb{N}$, the relation $\tau_n \geq \tau$ holds, and thus we have $\cap_{n \in \mathbb{N}} \mathcal{F}_{\tau_n} \supseteq \mathcal{F}_\tau$. That is, for all $n \in \mathbb{N}$, the σ-field $\mathcal{Z}_{\tau_n}$ is independent of the σ-field $\mathcal{F}_\tau$.

Let S be finite subset of $\mathbb{Z}^d$, and let $\mathrm{Incr}_{\tau,S} := \big\{G_{\tau+s,x} - G_{\tau,x}, I_{\tau+s,x} - I_{\tau,x} : x \in S, s \geq 0\big\}$ be the increments of the growth and ignition processes at the sites of the finite set S after time τ. Let $A_S \in \mathcal{Z}_\tau^S := \sigma\big\{\mathrm{Incr}_{\tau,S}\big\}$, and $\tilde{A}_S \in \mathbb{B}\big((\mathbb{N}_0 \times \mathbb{N}_0)^{\mathbb{R}_0^+ \times S}\big)$ s.t. the relation $A_S = \big\{\mathrm{Incr}_{\tau,S} \in \tilde{A}_S\big\}$ holds. The finitely many growth and ignition processes at the sites of S are right continuous with values in $\mathbb{N}_0$. By the previous step, for all $B \in \mathcal{F}_\tau$, the relations

$$\begin{aligned}\mu\big(A_S \cap B\big) = \mu\big(\{\mathrm{Incr}_{\tau,S} \in \tilde{A}_S\} \cap B\big) &= \lim_{n \to \infty} \mu\Big(\underbrace{\{\mathrm{Incr}_{\tau_n,S} \in \tilde{A}_S\}}_{\in \mathcal{Z}_{\tau_n}} \cap \underbrace{B}_{\in \mathcal{F}_\tau}\Big) \\ &= \lim_{n \to \infty} \mu\big(\mathrm{Incr}_{\tau_n,S} \in \tilde{A}_S\big) \cdot \mu\big(B\big) = \mu\big(A_S\big) \cdot \mu\big(B\big)\end{aligned}$$

and

$$\mu\big(\mathrm{Incr}_{\tau,S} \in \tilde{A}_S\big) = \lim_{n \to \infty} \underbrace{\mu\big(\mathrm{Incr}_{\tau_n,S} \in \tilde{A}_S\big)}_{=\mu\left(\mathrm{Incr}_{0,S} \in \tilde{A}_S\right)} = \mu\big(\mathrm{Incr}_{0,S} \in \tilde{A}_S\big)$$

must hold. That is, the σ-fields $\mathcal{Z}_\tau^S$ and $\mathcal{F}_\tau$ are independent. Thus a Dynkin argument shows that the σ-fields $\mathcal{Z}_\tau = \sigma\big\{\bigcup_{S \subset \mathbb{Z}^d, |S| < \infty} \mathcal{Z}_\tau^S\big\}$ and $\mathcal{F}_\tau$ are independent, and that the relation $(*)$ holds. $\square$

Lemma 20. If a cluster is finite, then it is right continuous. Formally for all $x \in \mathbb{Z}^d$ and all $t \in \mathbb{R}_0^+$, the relation

$$\Big\{|C_{t,x}| < \infty\Big\} = \Big\{|C_{t,x}| < \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ C_{t,x} = C_{t',x}\Big\}$$

holds.

Proof. Let $x \in \mathbb{Z}^d$. Suppose that the cluster at x at time $t \in \mathbb{R}_0^+$ is finite and non-empty. By Lemma 8, the finitely many processes $(\eta_{t,y})_{t\in\mathbb{R}_0^+}$, $y \in C_{t,x} \cup \partial C_{t,x}$, are right continuous with values in $\{0,1\}$. Thus there exists an $\epsilon > 0$ s.t. the cluster at x cannot change within the time between t and $t+\epsilon$. Formally for all $t \in \mathbb{R}_0^+$, the relation

$$\begin{aligned}\Big\{1 \le |C_{t,x}| < \infty\Big\} &= \bigcup_{S\in C^f} \Big\{C_{t,x} = S,\ \forall y \in S \cup \partial S\ \exists \epsilon_y > 0\ \forall t' \in [t, t+\epsilon_y) :\ \eta_{t,y} = \eta_{t',y}\Big\} \\ &= \bigcup_{S\in C^f} \Big\{C_{t,x} = S,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ C_{t,x} = C_{t',x}\Big\} \\ &= \Big\{1 \le |C_{t,x}| < \infty,\ \exists \epsilon > 0\ \forall t' \in [t, t+\epsilon) :\ C_{t,x} = C_{t',x}\Big\}\end{aligned}$$

holds. If the site x is vacant at time t, then the result follows since the forest-fire process at the site x, that is, the process $(\eta_{t,x})_{t\in\mathbb{R}_0^+}$, is right continuous with values in $\{0,1\}$. ☐

Lemma 21. Almost surely a cluster can not grow infinitely often without being hit by ignition. More formally, for all $x \in \mathbb{Z}^d$, the set

$$\Big\{\exists (t_i)_{i\in\mathbb{N}_0} \in \big(\mathbb{R}_0^+\big)^{\mathbb{N}_0}\ \forall i \in \mathbb{N}_0 :\ t_i < t_{i+1},\ |C_{t_i,x}| < |C_{t_{i+1},x}|,\ \forall t' \in [t_i, t_{i+1}) :\ \eta_{t',x} = 1\Big\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$. In Lemma 17, we showed that there a.s. cannot exist an infinite cluster. Lemma 20 provides that if the cluster at x is finite, then it is right continuous. Thus we get

$$\begin{aligned}&\Big\{\exists (t_i)_{i\in\mathbb{N}_0} \in \big(\mathbb{R}_0^+\big)^{\mathbb{N}_0}\ \forall i \in \mathbb{N}_0 :\ t_i < t_{i+1},\ |C_{t_i,x}| < |C_{t_{i+1},x}|,\ \forall t' \in [t_i, t_{i+1}) :\ \eta_{t',x} = 1\Big\} \\ \subseteq_N &\bigcup_{t_0\in\mathbb{Q}_0^+} \underbrace{\Big\{\exists (t_i)_{i\in\mathbb{N}} \in \big(\mathbb{R}^+\big)^{\mathbb{N}}\ \forall i \in \mathbb{N}_0 :\ t_i < t_{i+1},\ |C_{t_i,x}| < |C_{t_{i+1},x}|,\ \forall t' \in [t_i, t_{i+1}) :\ \eta_{t',x} = 1\Big\}}_{=:A_{t_0}}.\end{aligned}$$

Let $t_0 \in \mathbb{Q}_0^+$. (Recall the definition of $\subseteq_N$ at the end of Section 2.1.) We define a sequence of finite $\mathcal{F}_\cdot$-stopping times by $\tau_0 := t_0$, and recursively for all $n \ge 1$,

$$\tau_n := \min\Big\{t > \tau_{n-1} \,\Big|\, \exists y \in C_{\tau_{n-1},x} :\ \mathrm{I}_{\tau_{n-1},t,y}\ \text{ or } \exists z \in \partial C_{\tau_{n-1},x} :\ \mathrm{G}_{\tau_{n-1},t,z}\Big\},$$

if $0 < |C_{\tau_{n-1},x}| < \infty$ holds, and $\tau_n := \tau_{n-1}$ otherwise. That is, the time τ_n is the first time after τ_{n-1} at which there is an ignition at or the growth of a tree next to the cluster at x, if the cluster at x at time τ_{n-1} is finite and non-empty. For all $n \in \mathbb{N}$, the time τ_n is finite, since we defined for all $y \in \mathbb{Z}^d$, the processes $(G_{t,y})_{t\geq 0}$ and $(I_{t,y})_{t\geq 0}$ s.t. the relations $\lim_{t\to\infty} G_{t,y} = \lim_{t\to\infty} I_{t,y} = \infty$ hold.

By induction we show that for all $n \in \mathbb{N}$, if there exist $t_n > t_{n-1} > \cdots > t_1 > t_0$ s.t. the relation $|C_{t_0,x}| < |C_{t_1,x}| < \ldots < |C_{t_n,x}|$ holds, then a.s. the relation $\tau_n \leq t_n$ must hold. In other words, if the cluster at x has grown n times, then there a.s. must have been n times an ignition at the cluster of x or the growth of a tree next to this cluster.

For all $t_1 > t_0$, if the relation $1 \leq |C_{t_0,x}| < |C_{t_1,x}|$ holds, then a site next to $C_{t_0,x}$ must have become occupied at time t_1. Lemma 10 shows that if a vacant site has become occupied, then there a.s. must have been the growth of a tree at the site. Thus the relation $\tau_1 \leq t_1$ must hold. Formally the definition of τ_1 implies that if the relation $\tau_1 > t_0$ holds, then the relation $0 < |C_{t_0,x}| < \infty$, that is, the relation $C_{t_0,x} \in C^f$ must hold. It follows by Lemma 10, that the set

$$\begin{aligned} B_{t_0,1} :=& \left\{ \exists t_1 \in \mathbb{R}_0^+ :\ t_0 < t_1 < \tau_1,\ |C_{t_0,x}| < |C_{t_1,x}| \right\} \\ =& \left\{ \exists t_1 \in \mathbb{R}_0^+\ \exists S \in C^f :\ t_0 < t_1,\ C_{t_0,x} = S,\ |C_{t_0,x}| < |C_{t_1,x}|, \forall y \in \partial S :\ \complement\, \mathrm{G}_{t_0,t_1,y} \right\} \\ \subseteq& \left\{ \exists t_1 \in \mathbb{R}_0^+\ \exists y \in \mathbb{Z}^d :\ \eta_{t_0,y} = 0,\ \complement\, \mathrm{G}_{t_0,t_1,y},\ \eta_{t_1,y} = 1 \right\} \end{aligned}$$

is a null set.

As induction hypothesis, suppose that the set

$$B_{t_0,n} := \left\{ \exists (t_i)_{1\leq i\leq n} \in \left(\mathbb{R}_0^+\right)^n \forall 1 \leq i \leq n :\ t_{i-1} < t_i,\ t_n < \tau_n,\ |C_{t_{i-1},x}| < |C_{t_i,x}| \right\}$$

is a null set. By the definition of $(\tau_n)_{n\geq 0}$ for all $n \in \mathbb{N}_0$, if the relation $\tau_n < \tau_{n+1}$ holds, then the relation $1 \leq |C_{\tau_n,x}| < \infty$, that is, the relation $C_{\tau_n,x} \in C^f$ must hold. Lemma 13 implies that a finite cluster a.s. remains unchanged, if there is no ignition at it and not the growth of a tree next to it; almost surely for all $\tau_n \leq t < t' < \tau_{n+1}$, the relation $C_{\tau_n,x} = C_{t,x} = C_{t',x}$ must hold. Formally if the induction hypothesis holds, then the set

$$\begin{aligned} B_{t_0,n+1} \subseteq_N& \left\{ \exists t_n, t_{n+1} \in \mathbb{R}_0^+ :\ \tau_n \leq t_n < t_{n+1} < \tau_{n+1},\ |C_{t_n,x}| < |C_{t_{n+1},x}| \right\} \\ =& \bigcup_{S\in C^f} \left\{ \exists t_n, t_{n+1} \in \mathbb{R}_0^+ :\ \tau_n \leq t_n < t_{n+1} < \tau_{n+1},\ C_{\tau_n,x} = S,\ |C_{t_n,x}| < |C_{t_{n+1},x}| \right\} \\ \subseteq_N^{L.13}& \bigcup_{S\in C^f} \left\{ \exists t_n, t_{n+1} \in \mathbb{R}_0^+ :\ t_n < t_{n+1},\ C_{t_n,x} = C_{t_{n+1},x} = S,\ |C_{t_n,x}| < |C_{t_{n+1},x}| \right\} = \emptyset \end{aligned}$$

is a null set.

Recall the set A_{t_0}, which has been defined in the beginning of the proof. The induction provides for each t_0, that the relation

$$A_{t_0} \subseteq_N \bigcap_{n\in\mathbb{N}} \Big\{ \exists t_n \in \mathbb{R}^+ : \tau_n \le t_n,\ \forall t' \in [t_0, t_n] :\ \eta_{t',x} = 1 \Big\}$$
$$\subseteq \bigcap_{n\in\mathbb{N}} \Big\{ \forall 0 \le i \le n :\ \eta_{\tau_i,x} = 1 \Big\}$$

must hold.

For all $n \in \mathbb{N}_0$, the definition of τ_{n+1} implies that there is no ignition at and not the growth of a tree next to the cluster at x in the time between τ_n and τ_{n+1}. Thus it follows by Lemma 13 that a.s. the cluster at x cannot change in the time between τ_n and τ_{n+1}. If there is an ignition at time τ_{n+1}, then there a.s. cannot be the growth of a tree at the same time.

Formally we have for all $n \in \mathbb{N}$ and all $S \in C^f$,

$$\Big\{ C_{\tau_n,x} = S,\ \eta_{\tau_{n+1},x} = 1,\ \exists y \in S :\ \mathrm{I}_{\tau_{n+1}^-,\tau_{n+1},y} \Big\}$$
$$\subseteq_N \Big\{ C_{\tau_n,x} = S,\ \eta_{\tau_{n+1},x} = 1,\ \complement\, \mathrm{G}_{\tau_{n+1}^-,\tau_{n+1},x},\ \exists y \in S :\ \mathrm{I}_{\tau_{n+1}^-,\tau_{n+1},y} \Big\}$$
$$\subseteq_N^{L.13} \Big\{ \forall t \in [\tau_n, \tau_{n+1}) :\ C_{t,x} = S,\ \eta_{\tau_{n+1},x} = 1,\ \complement\, \mathrm{G}_{\tau_{n+1}^-,\tau_{n+1},x},\ \exists y \in S :\ \mathrm{I}_{\tau_{n+1}^-,\tau_{n+1},y} \Big\}$$
$$\subseteq \Big\{ \exists t, t' \in \mathbb{R}_0^+ :\ t < t',\ C_{t,x} = S, \eta_{t',x} = 1,\ \complement\, \mathrm{G}_{t,t',x},\ \exists y \in S :\ \mathrm{I}_{t,t',y} \Big\},$$

which is a null set by Lemma 11. That is, the relation

$$\Big\{ C_{\tau_n,x} = S,\ \eta_{\tau_{n+1},x} = 1 \Big\} \subseteq_N \Big\{ C_{\tau_n,x} = S,\ \forall y \in S :\ \complement\, \mathrm{I}_{\tau_{n+1}^-,\tau_{n+1},y} \Big\}$$
$$= \Big\{ C_{\tau_n,x} = S,\ \exists t \in \mathbb{Q}_0^+\ \forall y \in S :\ \complement\, \mathrm{I}_{\tau_n,t,y},\ \exists z \in \partial S :\ \mathrm{G}_{\tau_n,t,z} \Big\}$$

must hold. Lemma 17 shows that a.s. the cluster at x must be finite. Together with Lemma 18 and 19, we get for all $n \in \mathbb{N}_0$,

$$\mu\Big(\forall 0 \le i \le n+1 :\ \eta_{\tau_i,x} = 1 \Big) =^{L.17} \sum_{S\in C^f} \mu\Big(\forall 0 \le i \le n-1 :\ \eta_{\tau_i,x} = 1,\ C_{\tau_n,x} = S,\ \eta_{\tau_{n+1},x} = 1 \Big)$$
$$\le \sum_{S\in C^f} \mu\Big(\underbrace{\forall 0 \le i \le n-1 :\ \eta_{\tau_i,x} = 1,\ C_{\tau_n,x} = S}_{\in \mathcal{F}_{\tau_n}},\ \underbrace{\exists t \in \mathbb{Q}_0^+\ \forall y \in S :\ \complement\, \mathrm{I}_{\tau_n,t,y},\ \exists z \in \partial S :\ \mathrm{G}_{\tau_n,t,z}}_{\in \mathcal{Z}_{\tau_n}} \Big)$$
$$=^{L.19} \sum_{S\in C^f} \mu\Big(\forall 0 \le i \le n-1 :\ \eta_{\tau_i,x} = 1,\ C_{\tau_n,x} = S \Big) \cdot \mu\Big(\exists t \in \mathbb{Q}_0^+\ \forall y \in S :\ \complement\, \mathrm{I}_{0,t,y},\ \exists z \in \partial S :\ \mathrm{G}_{0,t,z} \Big)$$
$$\le^{L.18} C \cdot \sum_{S\in C^f} \mu\Big(\forall 0 \le i \le n-1 :\ \eta_{\tau_i,x} = 1,\ C_{\tau_n,x} = S \Big) = C \cdot \mu\Big(\forall 0 \le i \le n :\ \eta_{\tau_i,x} = 1 \Big),$$

with $C < 1$ as in Lemma 18. We obtain

$$0 \le \lim_{n\to\infty} \mu\Big(\forall 0 \le i \le n :\ \eta_{\tau_i,x} = 1 \Big) \le \lim_{n\to\infty} C^n = 0.$$

That is, the set $A_{t_0} \subseteq_N \bigcap_{n\in\mathbb{N}}\{\forall 0 \le i \le n:\ \eta_{\tau_i,x} = 1\}$ is a null set. The result follows since t_0 runs over the countable set $\mathbb{Q}_0^+$. ☐

Lemma 22. Almost surely the left limits of the clusters are finite. Formally the set

$$\left\{\exists x \in \mathbb{Z}^d\ \exists t \in \mathbb{R}^+ :\ |C_{t^-,x}| = \infty\right\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$. Note that for all $t \in \mathbb{R}^+$, if the left limit of the cluster at x is infinite at time t, then it is impossible that the cluster at x has been bounded before time t. By Lemma 17 a.s. the cluster at x cannot be infinite. It follows that a.s. the cluster at x must have grown infinitely often without being hit by ignition. Formally we have

$$\begin{aligned}
&\left\{\exists t \in \mathbb{R}^+ :\ |C_{t^-,x}| = \infty\right\}\\
=&\left\{\exists t \in \mathbb{R}^+ :\ |C_{t^-,x}| = \infty,\ \forall \delta > 0\ \forall N \in \mathbb{N}\ \exists t^* \in [t-\delta, t) :\ N < |C_{t',x}|\right\}\\
\subseteq_N&\left\{\exists t \in \mathbb{R}^+ :\ |\eta_{t^-,x}| = 1,\ \forall \delta > 0\ \forall N \in \mathbb{N}\ \exists t^* \in [t-\delta, t) :\ N < |C_{t',x}| < \infty\right\}\\
\subseteq&\left\{\exists (t_i)_{i\in\mathbb{N}_0} \in \left(\mathbb{R}_0^+\right)^{\mathbb{N}_0}\ \forall i \in \mathbb{N}_0 :\ t_i < t_{i+1},\ |C_{t_i,x}| < |C_{t_{i+1},x}|,\ \forall t' \in [t_i, t_{i+1}) :\ \eta_{t',x} = 1\right\}.
\end{aligned}$$

The result follows by Lemma 21. ☐

Lemma 23. Almost surely if a site gets vacant, then the cluster at the site must be hit by ignition. Formally for all $x \in \mathbb{Z}^d$, the set

$$\left\{\exists t \in \mathbb{R}^+ :\ \eta_{t^-,x} > \eta_{t,x},\ \forall y \in C_{t^-,x} :\ \complement\mathrm{I}_{t^-,t,y}\right\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$. By Lemma 22, a.s. the left limits of the cluster at x are finite. Thus the relation

$$\begin{aligned}
&\left\{\exists t \in \mathbb{R}^+ :\ \eta_{t^-,x} > \eta_{t,x},\ \forall y \in C_{t^-,x} :\ \complement\mathrm{I}_{t^-,t,y}\right\}\\
\subseteq_N&\bigcup_{S\in C^f}\left\{\exists t \in \mathbb{R}^+ :\ C_{t^-,x} = S,\ \eta_{t^-,x} > \eta_{t,x},\ \forall y \in S :\ \complement\mathrm{I}_{t^-,t,y}\right\}\\
\subseteq&\bigcup_{S\in C^f}\left\{\exists t \in \mathbb{R}^+ :\ \underbrace{x \in S,\ \eta_{t^-,x} > \eta_{t,x},\ \forall y' \in \partial S :\ \eta_{t^-,y'} = 0,\ \forall y \in S :\ \complement\mathrm{I}_{t^-,t,y}}_{=:D_{S,t,x}}\right\}
\end{aligned}$$

holds. Intuitively, the event $D_{S,t,x}$ describes the following situation: The sites next to the set S have been vacant before time t, there is not an ignition at the set S at time t, and the occupied site $x \in S$ gets vacant at time t. There are two possibilities. In the first case, suppose that there is not the growth of a tree next to the set S. Then all sites next to the set S remain vacant and no fire can pass from outside the set S and reach the site x. Furthermore the set S is not hit by ignition, and thus the site x cannot get vacant. This is a contradiction. In the other case, if there is the growth of a tree at a site next to the set S, then there a.s. will not be an ignition at the same time. Provided this, the site x cannot get vacant. This is a contradiction, too.

Let $S \in C^f$. Formally we have

$$\Big\{\exists t \in \mathbb{R}^+ : D_{S,t,x}\Big\} = \underbrace{\Big\{\exists t \in \mathbb{R}^+ : D_{S,t,x},\ \forall z \in \partial S :\ \complement\, \mathrm{G}_{t^-,t,z}\Big\}}_{=:A_{S,x}} \cup \underbrace{\Big\{\exists t \in \mathbb{R}^+ : D_{S,t,x},\ \exists z \in \partial S :\ \mathrm{G}_{t^-,t,z}\Big\}}_{=:B_{S,x}}.$$

We first show that for all $S' \in C^f$, the set $A_{S',x}$ is a null set, and then that the relation $B_{S,x} \subseteq_N \cup_{S' \in C^f} A_{S',x}$ must hold. Provided this, the set $B_{S,x}$ is a null set, and the result follows since S runs over the countable set C^f.

Let $S' \in C^f$. For all $y \in S' \cup \partial S'$, the process $(\eta_{t,y}, G_{t,y}, I_{t,y})_{t \geq 0}$ is a process with values in $\big(\{0,1\} \times \mathbb{N}_0 \times \mathbb{N}_0\big)$ whose left limits exist. Thus since the set $S' \cup \partial S'$ is finite, the relation

$$A_{S',x} \subseteq \Big\{\exists t', t \in \mathbb{R}^+ :\ t' < t,\ x \in S',\ \eta_{t',x} > \eta_{t,x},\ \forall y \in S' :\ \complement\, \mathrm{I}_{t',t,y},\ \forall z \in \partial S' :\ \eta_{t',z} = 0,\ \complement\, \mathrm{G}_{t',t,z}\Big\},$$

holds. By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t \geq 0, x \in B^d_n}$ is that of a finite volume forest-fire process on B^d_n. We know (see Definition 3) that *finite volume* forest-fire processes have the property that if a site is vacant and there is not the growth of a tree at the site, then the site must remain vacant. Provided this it follows that all sites next to the set S' must remain vacant, and that the cluster at the site $x \in S'$ must be a subset of S'. That is, the cluster at x cannot be hit by ignition, if the set S' is not hit by ignition. Finally, in a *finite volume* forest-fire process an occupied site x cannot get vacant, if the cluster at x is not hit by ignition. Formally for all $n \in \mathbb{N}$ and all $t', t \in \mathbb{Q}^+_0$, $t' < t$, the set

$$\Big\{x \in S',\ \eta^n_{t',x} > \eta^n_{t,x},\ \forall y \in S' :\ \complement\, \mathrm{I}^n_{t',t,y},\ \forall z \in \partial S' :\ \eta^n_{t',z} = 0,\ \complement\, \mathrm{G}^n_{t',t,z}\Big\}$$

is a null set. It follows by Lemma 9 that the set $A_{S',x}$ is a null set.

It remains to show that the relation $B_{S,x} \subseteq_N \bigcup_{S' \in C^f} A_{S',x}$ holds. In the situation described by $B_{S,x}$, there is the growth of a tree at a site $z \in \partial S$. We define S' to be the union of the set S, the site z and the left limits of the clusters next to z. In Lemma 22, we showed that a.s. the left limits of the clusters at the $2d$ sites next to z are finite. This provides that a.s. the relation $S' \in C^f$ must hold. Formally the relation

$$\begin{aligned}
B_{S,x} &\subseteq_N \Big\{\exists t \in \mathbb{R}^+ :\ D_{S,t,x},\ \exists z \in \partial S :\ \mathrm{G}_{t^-,t,z},\ \exists S' \in C^f :\ S' = S \cup \{z\} \cup \bigcup_{z' \in \partial\{z\}} C_{t^-,z'}\Big\} \\
&\subseteq \bigcup_{S' \in C^f} \Big\{\exists t \in \mathbb{R}^+ :\ x \in S',\ \eta_{t^-,x} > \eta_{t,x},\ \forall y' \in \partial S' :\ \eta_{t^-,y'} = 0,\ \exists z \in S' :\ \mathrm{G}_{t^-,t,z}\Big\} \\
&\subseteq_N \bigcup_{S' \in C^f} A_{S',x}
\end{aligned}$$

must hold. We used that if there is the growth of a tree at a given site, then a.s. there cannot be an ignition, nor the growth of a tree at another site, at the same time. ∎

Lemma 24. Almost surely if there is the growth of a tree at a site, then the site is occupied. More formally, for all $x \in \mathbb{Z}^d$, the set

$$\left\{ \exists t \in \mathbb{R}^+ : \ \mathrm{G}_{t-,t,x}, \ \eta_{t,x} = 0 \right\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}^d$. If there is the growth of a tree at the site x, then a.s. there will neither be an ignition nor the growth of another tree at the same time. Lemma 10 provides that a.s. a vacant site must remain vacant, if there is not the growth of a tree at the site. By Lemma 23, if an occupied site gets vacant, then a.s. the cluster at the site must be hit by ignition. That is, if there is no ignition, then a.s. all occupied sites must remain occupied. It follows that if there is the growth of a tree at the site x, then a.s. the sites next to x must remain unchanged. Formally we have

$$\begin{aligned}
&\left\{ \exists t \in \mathbb{R}^+ : \ \mathrm{G}_{t-,t,x}, \ \eta_{t,x} = 0 \right\} \\
\subseteq_N &\left\{ \exists t \in \mathbb{R}^+ : \ \mathrm{G}_{t-,t,x}, \ \complement \mathrm{I}_{t-,t,x}, \ \eta_{t,x} = 0, \ \forall y \in \mathbb{Z}^d \setminus \{x\} : \ \complement \mathrm{G}_{t-,t,y}, \ \complement \mathrm{I}_{t-,t,y} \right\} \\
\subseteq_N &\left\{ \exists t \in \mathbb{R}^+ : \ \mathrm{G}_{t-,t,x}, \ \complement \mathrm{I}_{t-,t,x}, \ \eta_{t,x} = 0, \ \forall y \in \partial\{x\} : \ \bar{\eta}_{t-,y} = \bar{\eta}_{t,y} \right\} \\
\subseteq &\left\{ \exists t', t \in \mathbb{R}^+ : \ \mathrm{G}_{t',t,x}, \ \complement \mathrm{I}_{t',t,x}, \ \eta_{t,x} = 0, \ \forall y \in \partial\{x\} : \ \bar{\eta}_{t',y} = \bar{\eta}_{t,y} \right\}.
\end{aligned}$$

By Remark 2 for all $n \in \mathbb{N}$, the distribution of the process $(\bar{\eta}^n_{t,x})_{t \geq 0, x \in B^d_n}$ is that of a finite volume forest-fire process on B^d_n. We know (see Definition 3) that *finite volume* forest-fire processes have the property that if there is the growth of a tree at a given site, then the site gets occupied. The site only gets vacant again, if the site is hit by ignition or if a neighbor of the site burns down. That is, for all $n \geq \|x\|_\infty$ and all $t', t \in \mathbb{Q}^+_0$, $t' < t$, the set

$$\left\{ \mathrm{G}^n_{t',t,x}, \ \complement \mathrm{I}^n_{t',t,x}, \ \eta^n_{t,x} = 0, \ \forall y \in \partial\{x\} : \ \bar{\eta}^n_{t',y} = \bar{\eta}^n_{t,y} \right\}$$

is a null set. The result follows by Lemma 9. $\square$

Lemma 25. Almost surely if a site is hit by ignition, then all sites of the cluster at the site get vacant. More formally, for all $x \in \mathbb{Z}^d$, the set

$$\left\{ \exists t \in \mathbb{R}^+ : \ \mathrm{I}_{t-,t,x}, \ \exists y \in C_{t-,x} : \ \eta_{t,y} = 1 \right\}$$

is a null set.

Proof. Let $x \in \mathbb{Z}$. By Lemma 22, a.s. the left limits of the cluster at x are finite. If there is an ignition, then there a.s. will not be the growth of a tree at the same time. Thus we have

$$\begin{aligned}
&\Big\{\exists t\in\mathbb{R}^+ :\ \mathrm{I}_{t-,t,x},\ \exists y\in C_{t-,x} : \eta_{t,y}=1\Big\}\\
\subseteq_N &\Big\{\exists t\in\mathbb{R}^+ :\ \mathrm{I}_{t-,t,x},\ \exists y\in C_{t-,x} : \eta_{t,y}=1,\ \complement\,\mathrm{G}_{t-,t,y}\Big\}\\
\subseteq_N &\bigcup_{S\in C^f}\Big\{\exists t\in\mathbb{R}_0^+ :\ C_{t-,x}=S,\ \mathrm{I}_{t-,t,x},\ \exists y\in S :\ \eta_{t,y}=1,\ \complement\,\mathrm{G}_{t-,t,y}\Big\}\\
\subseteq &\bigcup_{S\in C^f}\Big\{\exists t',t\in\mathbb{R}_0^+ :\ C_{t',x}=S,\ \mathrm{I}_{t',t,x},\ \exists y\in S :\ \eta_{t,y}=1,\ \complement\,\mathrm{G}_{t',t,y}\Big\},
\end{aligned}$$

which is a null set by Lemma 11 and since the set C^f is countable. □

Lemma 26. Almost surely for all $x\in\mathbb{Z}^d$, the relation $\eta_{0,x}=\zeta_x$ holds. That is, a.s. the initial configuration of the process is ζ.

Proof. Remember that we chose $\zeta\in Z_d^{finite}$. That is, for all $x\in\mathbb{Z}^d$, the set $C_{\zeta,x}$ is finite. Let $x\in\mathbb{Z}^d$. The finitely many growth and ignition processes at and next to the set $C_{\zeta,x}$ are right continuous with values in $\mathbb{N}_0$, the relation

$$\Big\{\eta_{0,x}\neq\zeta_x\Big\}=\Big\{\eta_{0,x}\neq\zeta_x, \exists t\in\mathbb{R}^+\ \forall y\in C_{\zeta,x}\cup\partial C_{\zeta,x}\cup\{x\} :\ G_{t,y}=I_{t,y}=0\Big\}$$

holds. By Remark 2 for all $n\in\mathbb{N}$, the distribution of the process $(\bar\eta^n_{t,x})_{t\geq 0,x\in B_n^d}$ is that of a finite volume forest-fire process on B_n^d with initial configuration $(\zeta_x)_{x\in B_n^d}$. We know (see Definition 3) that *finite volume* forest-fire processes have the property that a vacant site remains vacant, if there is not the growth of a tree at the site. Furthermore in a *finite volume* forest-fire process, if an occupied site gets vacant, then the cluster at the site must be hit by ignition. It follows that for all $n>\|x\|_\infty$ and all $t',t\in\mathbb{Q}_0^+$, $t'<t$, the set

$$\Big\{\eta^n_{t',x}\neq\zeta_x,\ \ \forall y\in C_{\zeta,x}\cup\partial C_{\zeta,x}\cup\{x\} :\ G^n_{t,y}=I^n_{t,y}=0\Big\}$$

is a null set. The result follows by Lemma 9. □

Altogether, Lemma 5, 7, 8, 10, 14, 23, 24, 25 and 26 show that restricted to the complement of a null set, the process $\left(\bar\eta_{t,x}\right)_{x\in\mathbb{Z}^d,t\in\mathbb{R}_0^+}$ defined in Definition 5, satisfies the definition of a forest-fire process on $\mathbb{Z}^d$ with parameter λ and initial configuration ζ, as formalized in Definition 3. This shows Theorem 3; we obtain Theorem 1.

From now on, let $d\in\mathbb{N}$ and let $(\bar\eta_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ be an arbitrary forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda>0$ and initial configuration $\zeta\in Z_d^{finite}$.

Proof of Theorem 2. Note that the definition of a forest-fire process, that is, Definition 3 implies that the process $(\bar\eta_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ satisfies the properties that are shown in Lemma 8, 10, 11, 12, 13 and 14. Thus the proofs and results of Lemma 15, 16, 18, 19, 20, 21 and in particular Lemma 17 and 22, can be taken over directly. This proves the assertion of Theorem 2 . □

Open Problems.
There are several natural questions: Is the infinite volume forest-fire process $(\bar{\eta}_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ adapted to the filtration generated by its driving growth and ignition processes $(G_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ and $(I_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$? Related to this question is the question whether the infinite volume forest-fire process $(\bar{\eta}_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ is uniquely determined by its driving growth and ignition processes and its initial configuration Or, whether the convergence of the finite-volume forest-fire processes which we used to construct a infinite-volume forest-fire process holds in a stronger sense.

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