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BROWNIAN MOTION, BRIDGE, EXCURSION, AND MEANDER CHARACTERIZED BY SAMPLING AT INDEPENDENT UNIFORM TIMES

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Abstract For a random process X consider the random vector defined by the values of X at times $0 < U_{n,1} < ... < U_{n,n} < 1$ and the minimal values of X on each of the intervals between consecutive pairs of these times, where the $U_{n,i}$ are the order statistics of n independent uniform $(0,1)$ variables, independent of X. The joint law of this random vector is explicitly described when X is a Brownian motion. Corresponding results for Brownian bridge, excursion, and meander are deduced by appropriate conditioning. These descriptions yield numerous new identities involving the laws of these processes, and simplified proofs of various known results, including Aldous's characterization of the random tree constructed by sampling the excursion at n independent uniform times, Vervaat's transformation of Brownian bridge into Brownian excursion, and Denisov's decomposition of the Brownian motion at the time of its minimum into two independent Brownian meanders. Other consequences of the sampling formulae are Brownian represention of various special functions, including Bessel polynomials, some hypergeometric polynomials, and the Hermite function. Various combinatorial identities involving random partitions and generalized Stirling numbers are also obtained.

Keywords Alternating exponential random walk, uniform order statistics, critical binary random tree, Vervaat's transformation, random partitions, generalized Stirling numbers, Bessel polynomials, McDonald function, products of gamma variables, Hermite function

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Contents

1 Introduction

Let $(B(t))_{t\geq 0}$ be a standard one-dimensional Brownian motion with $B(0)=0$. For $x\in\mathbb{R}$ let $B^{\mathrm{br},x}:=(B^{\mathrm{br},x}(u))_{0\leq u\leq 1}$ be the *Brownian bridge starting at* 0 *and ending at* x. Let $B^{\mathrm{me}}:=(B^{\mathrm{me}}(u))_{0\leq u\leq 1}$ be a *standard Brownian meander*, and let $B^{\mathrm{me},r}$ be the standard meander conditioned to end at level $r\geq 0$. Informally, these processes are defined as

$$\begin{aligned}
B^{\mathrm{br},x} &\stackrel{d}{=} (B \,|\, B(1)=x)\\
B^{\mathrm{me}} &\stackrel{d}{=} (B \,|\, B(t)>0 \text{ for all } 0<t<1)\\
B^{\mathrm{me},r} &\stackrel{d}{=} (B \,|\, B(t)>0 \text{ for all } 0<t<1, B(1)=r)
\end{aligned}$$

where $\stackrel{d}{=}$ denotes equality in distribution, and $(B\,|\,A)$ denotes $(B(u), 0\leq u\leq 1)$ conditioned on A. In particular

$$\begin{aligned}
B^{\mathrm{br}} &:= B^{\mathrm{br},0} \text{ is the standard Brownian bridge;}\\
B^{\mathrm{ex}} &:= B^{\mathrm{me},0} \text{ is the standard Brownian excursion.}
\end{aligned}$$

These definitions of conditioned Brownian motions have been made rigorous in many ways: for instance by the method of Doob h-transforms [69, 28], by weak limits of suitably scaled and conditioned lattice walks [52, 45, 75], and by weak limits as $\varepsilon\downarrow 0$ of the distribution of B given A_ε for suitable events A_ε with probabilities tending to 0 as $\varepsilon\downarrow 0$, as in [17], for instance

$$(B \,|\, \underline{B}(0,1)>-\varepsilon) \stackrel{d}{\to} B^{\mathrm{me}} \text{ as } \varepsilon\downarrow 0 \tag{1}$$

where $\underline{X}(s,t)$ denotes the infimum of a process X over the interval $[s,t]$, and $\stackrel{d}{\to}$ denotes convergence in distribution on the path space $C[0,1]$ equipped with the uniform topology. For $T>0$ let $G_T:=\sup\{s: s\leq T, B(s)=0\}$ be the last zero of B before time T and $D_T:=\inf\{s: s\geq T, B(s)=0\}$ be first zero of B after time T. It is well known [43, 21, 68] that for each fixed $T>0$, there are the following identities in distribution derived by *Brownian scaling*:

$$\left(\frac{B(uT)}{\sqrt{T}}\right)_{u\geq 0} \stackrel{d}{=} B; \qquad \left(\frac{B(uG_T)}{\sqrt{G_T}}\right)_{0\leq u\leq 1} \stackrel{d}{=} B^{\mathrm{br}} \tag{2}$$

$$\left(\frac{|B(G_T+u(T-G_T))|}{\sqrt{T-G_T}}\right)_{0\leq u\leq 1} \stackrel{d}{=} B^{\mathrm{me}}; \qquad \left(\frac{|B(G_T+u(D_T-G_T))|}{\sqrt{D_T-G_T}}\right)_{0\leq u\leq 1} \stackrel{d}{=} B^{\mathrm{ex}}. \tag{3}$$

Since the distribution of these rescaled processes does not depend on the choice of T, each rescaled process is independent of T if T is a random variable independent of B. It is also known that these processes can be constructed by various other operations on the paths of B, and transformed from one to another by further operations. See [12] for a review of results of this kind. One well known construction of $B^{\mathrm{br},x}$ from B is

$$B^{\mathrm{br},x}(u) := B(u)-uB(1)+ux \qquad (0\leq u\leq 1). \tag{4}$$

Then $B^{\mathrm{me},r}$ can be constructed from three independent standard Brownian bridges $B_i^{\mathrm{br}}, i=1,2,3$ as

$$B^{\mathrm{me},r}(u) := \sqrt{(ru+B_1^{\mathrm{br}}(u))^2+(B_2^{\mathrm{br}}(u))^2+(B_3^{\mathrm{br}}(u))^2} \qquad (0\leq u\leq 1). \tag{5}$$

So $B^{\mathrm{me},r}$ is identified with the *three-dimensional Bessel bridge from* 0 *to* r, and the standard meander is recovered as $B^{\mathrm{me}}:=B^{\mathrm{me},\rho}$, where ρ is independent of the three bridges B_i^{br}, with the *Rayleigh density* $P(\rho\in dx)/dx = xe^{-\frac{1}{2}x^2}$ for $x>0$. Then by construction

$$B^{\mathrm{me}}(1)=\rho=\sqrt{2\Gamma_1} \tag{6}$$

where Γ_1 is a *standard exponential variable*. The above descriptions of $B^{\mathrm{me},r}$ and B^{me} are read from [78, 41]. See also [21, 12, 15, 68] for further background.

This paper characterizes each member X of the *Brownian quartet* $\{B, B^{\mathrm{br}}, B^{\mathrm{me}}, B^{\mathrm{ex}}\}$ in a very different way, in terms of the distribution of values obtained by sampling X at independent random times with uniform distribution on $(0,1)$. The characterization of B^{ex} given here in Theorem 1 is equivalent, via the bijection between plane trees with edge lengths and random walk paths exploited in [56, 57, 48, 46], of Aldous's broken line construction of the random tree derived from B^{ex} by sampling at independent uniform times [7, Corollary 22], [49]. See [4] for details of this equivalence, and related results. This central result in Aldous's theory of the Brownian continuum random tree [5, 6, 7] has recently been applied in [8] to construction of the standard additive coalescent process. See also [10, p. 167] for another application of random sampling of values of a Brownian excursion. Results regarding to the lengths of excursions of B and B^{br} found by sampling at independent uniform times were obtained in [61], and related in [8] to random partitions associated with the additive coalescent. But these results do not provide a full description of the laws of B and B^{br} in terms of sampling at independent uniform times, as provided here in Theorem 1.

The rest of this paper is organized as follows. Section 2 introduces some notation for use throughout the paper, and presents the main results. Except where otherwise indicated, the proofs of these results can be found in Section 3. Following sections contain various further developments, as indicated briefly in Section 2.

2 Summary of Results

For $n = 0, 1, 2, \ldots$ let

$$0 = U_{n,0} < U_{n,1} < \cdots < U_{n,n} < U_{n,n+1} = 1 \tag{7}$$

be defined by $U_{n,i} := S_i/S_{n+1}, 1 \le i \le n+1$ for $S_n := X_1 + \cdots + X_n$ the sum of n independent standard exponential variables. It is well known that the $U_{n,i}, 1 \le i \le n$ are distributed like the order statistics of n independent uniform $(0,1)$ variables [71], and that

$$\textit{the random vector } (U_{n,i}, 1 \le i \le n) \textit{ is independent of } S_{n+1}. \tag{8}$$

Let X be a process independent of these $U_{n,i}$, let $\mu_{n,i}$ be a time in $[U_{n,i-1}, U_{n,i}]$ when X attains its infimum on that interval, so $X(\mu_{n,i}) = \underline{X}(U_{n,i-1}, U_{n,i})$ is that infimum, and define a $\mathbb{R}^{2n+2}$-valued random vector

$$X_{(n)} := (X(\mu_{n,i}), X(U_{n,i}); 1 \le i \le n+1). \tag{9}$$

Let $(T_i, 1 \le i \le n+1)$ be an independent copy of $(S_i, 1 \le i \le n+1)$, let $V_{n,i} := T_i/T_{n+1}$, and let Γ_r for $r > 0$ be independent of the S_i and T_i with the *gamma*(r) density

$$P(\Gamma_r \in dt)/dt = \Gamma(r)^{-1} t^{r-1} e^{-t} \qquad (t > 0) \tag{10}$$

which makes

$$P(\sqrt{2\Gamma_r} \in dx)/dx = \Gamma(r)^{-1} (\tfrac{1}{2})^{r-1} x^{2r-1} e^{-\frac{1}{2}x^2} \qquad (x > 0). \tag{11}$$

Theorem 1 *For each $n = 0, 1, 2, \ldots$ the law of the random vector $X_{(n)}$ is characterized by the following identities in distribution for each of the four processes $X = B, B^{\mathrm{br}}, B^{\mathrm{me}}$ and B^{ex}:*

(i) (Brownian sampling)

$$B_{(n)} \stackrel{d}{=} \sqrt{2\Gamma_{n+3/2}} \left(\frac{S_{i-1} - T_i}{S_{n+1} + T_{n+1}}, \frac{S_i - T_i}{S_{n+1} + T_{n+1}}; 1 \leq i \leq n+1 \right) \tag{12}$$

(ii) (Meander sampling)

$$B^{\mathrm{me}}_{(n)} \stackrel{d}{=} \sqrt{2\Gamma_{n+1}} \left(\frac{S_{i-1} - T_{i-1}}{S_{n+1} + T_n}, \frac{S_i - T_{i-1}}{S_{n+1} + T_n}; 1 \leq i \leq n+1 \,\middle|\, \bigcap_{i=1}^{n} (S_i > T_i) \right) \tag{13}$$

(iii) (Bridge sampling)

$$B^{\mathrm{br}}_{(n)} \stackrel{d}{=} \sqrt{2\Gamma_{n+1}}\, \tfrac{1}{2} \left(U_{n,i-1} - V_{n,i}, U_{n,i} - V_{n,i}; 1 \leq i \leq n+1 \right) \tag{14}$$

(iv) (Excursion sampling)

$$B^{\mathrm{ex}}_{(n+1)} \stackrel{d}{=} \sqrt{2\Gamma_{n+1}}\, \tfrac{1}{2} \left(U_{n,i-1} - V_{n,i-1}, U_{n,i} - V_{n,i-1}; 1 \leq i \leq n+2 \,\middle|\, \bigcap_{i=1}^{n} (U_{n,i} > V_{n,i}) \right) \tag{15}$$

where $U_{n,n+2} := 1$.

The right sides of (12) and (13) could also be rewritten in terms of uniform order statistics $(U_{2n+2,i}, 1, \leq i \leq 2n+2)$ and $(U_{2n+1,i}, 1, \leq i \leq 2n+1)$. So in all four cases the random vector on the right side of the sampling identity is the product of a random scalar $\sqrt{2\Gamma_r}$ for some r, whose significance is spelled out in Corollary 3, and an independent random vector with uniform distribution on some polytope in $\mathbb{R}^{2n+2}$. Ignoring components of $X_{(n)}$ which are obviously equal to zero, the four random vectors $X_{(n)}$ considered in the theorem have joint densities which are explicitly determined by these sampling identities.

To illustrate some implications of the sampling identities, for $n = 1$ the Brownian sampling identity (12) reads as follows: for U with uniform distribution on $(0, 1)$ independent of B, the joint law of the minimum of B on $[0, U]$, $B(U)$, the minimum of B on $[U, 1]$, and $B(1)$ is determined by

$$(\underline{B}(0, U), B(U), \underline{B}(U, 1), B(1)) \stackrel{d}{=} \frac{\sqrt{2\Gamma_{5/2}}}{S_2 + T_2} (-T_1, S_1 - T_1, , S_1 - T_2, S_2 - T_2)\,. \tag{16}$$

The Brownian sampling identity (12) represents the standard Gaussian variable $B(1)$ in a different way for each $n = 0, 1, 2, \ldots$:

$$B(1) \stackrel{d}{=} \sqrt{2\Gamma_{n+3/2}} \frac{S_{n+1} - T_{n+1}}{S_{n+1} + T_{n+1}} \tag{17}$$

as can be checked by a moment computation. To present the case $n = 1$ of the bridge sampling identity (14) with lighter notation, let U and V be two independent uniform$(0, 1)$ variables, with U independent of B^{br}. Then the joint law of the minimum of B^{br} before time U, the value of B^{br} at time U, and the minimum of B^{br} after time U is specified by

$$(\underline{B}^{\mathrm{br}}(0, U), B^{\mathrm{br}}(U), \underline{B}^{\mathrm{br}}(U, 1)) \stackrel{d}{=} \tfrac{1}{2}\sqrt{2\Gamma_2}(-V, U - V, U - 1). \tag{18}$$

In particular, $|B^{\rm br}(U)| \stackrel{d}{=} \frac{1}{2}\sqrt{2\Gamma_2}|U-V|$. It is easily checked that this agrees the observation of Aldous-Pitman [2] that $|B^{\rm br}(U)| \stackrel{d}{=} \frac{1}{2}\sqrt{2\Gamma_1}U$, corresponding to the formula [71, p. 400]

$$P(B^{\rm br}(U) \in dx)/dx = \int_{2|x|}^{\infty} e^{-\frac{1}{2}y^2}dy \quad (x \in \mathbb{R}). \tag{19}$$

See also [62] for various related identities. Corollary 16 in Section 8 gives a sampling identity for $B^{{\rm br},x}$ which reduces for $x=0$ to part (iii) of the previous theorem. There are some interesting unexplained coincidences between features of this description of $B^{{\rm br},x}$ and results of Aldous-Pitman [3] in a model for random trees with edge lengths related to the Brownian continuum random tree.

For any process X with left continuous paths and $X(0)=0$, define a random element $\widehat{X}_{(n)}$ of $C[0,1]$, call it the *nth zig-zag approximation to* X, by linear interpolation between the values at evenly spaced gridpoints determined by components of $X_{(n)}$. That is to say, $\widehat{X}_{(n)}(0) := 0$ and for $1 \le i \le n+1$

$$\widehat{X}_{(n)}\left(\frac{i-1/2}{n+1}\right) := X(\mu_{n,i}); \;\; \widehat{X}_{(n)}\left(\frac{i}{n+1}\right) := X(U_{n,i}) \tag{20}$$

where the $U_{n,i} := S_i/S_{n+1}$ are constructed independent of X, as above. By the law of large numbers, as $n \to \infty$ and $i \to \infty$ with $i/n \to p \in [0,1]$, there is convergence in probability of $U_{n,i}$ to p. It follows that $\widehat{X}_{(n)}$ has an a.s. limit $\widehat{X}$ in the uniform topology of $C[0,1]$ as $n \to \infty$ iff X has continuous paths a.s., in which case $\widehat{X} = X$ a.s.. The same remark applies to $\widetilde{X}_{(n)}$ instead of $\widehat{X}_{(n)}$, where $\widetilde{X}_{(n)}$ is the less jagged approximation to X with the same values $X(U_{n,i})$ at $i/(n+1)$ for $0 \le i \le n+1$, and with linear interpolation between these values, so $\widetilde{X}_{(n)}$ is constructed from the $X(U_{n,i})$ without consideration of the minimal values $X(\mu_{n,i})$. Therefore, Theorem 1 implies:

Corollary 2 *If a process $X := (X(u), 0 \le u \le 1)$ with $X(0)=0$ and left continuous paths is such that for each $n = 1,2,\ldots$ the law of $(X(U_{n,i}); 1 \le i \le n+1)$ is as specified in one of the cases of the previous theorem by ignoring the $X(\mu_{n,i})$, then X has continuous paths a.s. and X is a Brownian motion, meander, bridge, excursion, as the case may be.*

Let $||X_{(n)}||$ denote the total variation of the nth zig-zag approximation to X, that is

$$||X_{(n)}|| := \sum_{i=1}^{n+1}(X(U_{n,i-1}) - X(\mu_{n,i})) + \sum_{i=1}^{n+1}(X(U_{n,i}) - X(\mu_{n,i})) \tag{21}$$

where all $2n+2$ terms are non-negative by definition of the $\mu_{n,i}$. The following immediate consequence of Theorem 1 interprets the various factors $\sqrt{2\Gamma_r}$ in terms of these total variations of zig-zag aproximations. Corollary 14 in Section 6 gives corresponding descriptions of the law of

$$||\widetilde{X}_{(n)}|| := \sum_{i=1}^{n+1} |X(U_{n,i}) - X(U_{n,i-1})| \tag{22}$$

for $X = B$ and $X = B^{\rm br}$. These are a little more complicated, but still surprisingly explicit.

Corollary 3 *For each $n = 0,1,2,\ldots$ the following identities in distribution hold:*

$$||B_{(n)}|| \stackrel{d}{=} \sqrt{2\Gamma_{n+3/2}} \stackrel{d}{=} \sqrt{\Sigma_{i=1}^{2n+3} B_i^2(1)} \tag{23}$$

where the $B_i^2(1)$ are squares of independent standard Gaussian variables, and

$$||B_{(n)}^{\mathrm{me}}|| \stackrel{d}{=} ||B_{(n)}^{\mathrm{br}}|| \stackrel{d}{=} ||B_{(n+1)}^{\mathrm{ex}}|| \stackrel{d}{=} \sqrt{2\Gamma_{n+1}} \stackrel{d}{=} \sqrt{\Sigma_{i=1}^{2n+2} B_i^2(1)}. \tag{24}$$

For $n = 0$, formula (23) is identity of one-dimensional distributions implied by the result of [58] that

$$(B(t) - 2\underline{B}(0,t))_{t\geq 0} \stackrel{d}{=} (R_3(t))_{t\geq 0} \tag{25}$$

where $R_\delta(t) := \sqrt{\sum_{i=1}^{\delta} B_i^2(t)}$ for $\delta = 1, 2, \dots$ is the δ-dimensional Bessel process derived from δ independent copies B_i of B. By (23) for $n = 1$ and Brownian scaling, for U a uniform$(0,1)$ variable independent of B the process

$$(B(t) + 2B(Ut) - 2\underline{B}(0, Ut) - 2\underline{B}(Ut, t))_{t\geq 0} \tag{26}$$

has the same 1-dimensional distributions as $(R_5(t))_{t\geq 0}$. But by consideration of their quadratic variations, these processes do not have the same finite-dimensional distributions.

For $n = 0$ formula (24) reduces to the following well known chain of identities in distribution, which extends (6):

$$B^{\mathrm{me}}(1) \stackrel{d}{=} -2\underline{B}^{\mathrm{br}}(0,1) \stackrel{d}{=} 2B^{\mathrm{ex}}(U) \stackrel{d}{=} \sqrt{2\Gamma_1} \tag{27}$$

where U is uniform $(0,1)$ independent of B^{ex}. The identity $-2\underline{B}^{\mathrm{br}}(0,1) \stackrel{d}{=} \sqrt{2\Gamma_1}$ is just a presentation of Lévy's formula [51, (20)]

$$P(-\underline{B}^{\mathrm{br}}(0,1) > a) = e^{-2a^2} \qquad (a \geq 0).$$

The identity $B^{\mathrm{ex}}(U) \stackrel{d}{=} \sqrt{2\Gamma_1}$ in (27) is also due to Lévy[51, (67)]. This coincidence between the distributions of $-\underline{B}^{\mathrm{br}}(0,1)$ and $B^{\mathrm{ex}}(U)$ is explained by the following corollary of the bridge and excursion sampling identities:

Corollary 4 (Vervaat [75], Biane [13]) *Let $\mu(\omega)$ be first time a path ω attains its minumum on $[0,1]$, let $\Theta_u(\omega)$ for $0 \leq u \leq 1$ be the cyclic increment shift*

$$(\Theta_u(\omega))(t) := \omega(u + t(\mathrm{mod} 1)) - \omega(t), \qquad 0 \leq t \leq 1,$$

and set $\Theta_*(\omega) := \Theta_{\mu(\omega)}(\omega)$. Then there is the identity of laws on $[0,1] \times C[0,1]$

$$(\mu(B^{\mathrm{br}}), \Theta_*(B^{\mathrm{br}})) \stackrel{d}{=} (U, B^{\mathrm{ex}}) \tag{28}$$

for U uniform$(0,1)$ independent of B^{ex}.

In particular, (28) gives

$$-\underline{B}^{\mathrm{br}}(0,1) = \Theta_*(B^{\mathrm{br}})(1 - \mu(B^{\mathrm{br}})) \stackrel{d}{=} B^{\mathrm{ex}}(1-U) \stackrel{d}{=} B^{\mathrm{ex}}(U)$$

in agreement with (27). The identity $||B_{(n)}^{\mathrm{br}}|| \stackrel{d}{=} ||B_{(n+1)}^{\mathrm{ex}}||$ in (24) is also quite easily derived from (28). In fact, the bridge and excursion sampling identities (14) and (15) are so closely related to Vervaat's identity (28) that any one of these three identities is easily derived from the other two. The uniform variable U which appears in passing from bridge to excursion via (28) explains why the excursion sampling identity is most simply compared to its bridge counterpart with a sample size of n for the bridge and $n+1$ for the excursion, as presented in Theorem 1.

As shown by Bertoin [11, Corollary 4] the increments of B^{me} can be made from those of B^{br} by retaining the increments of B^{br} on $[\mu(B^{\mathrm{br}}), 1]$ and reversing and changing the sign of the increments of B^{br} on $[0, \mu(B^{\mathrm{br}})]$. Bertoin's transformation is related to the bridge and meander sampling identities in much the same way as Vervaat's transformation is to the bridge and excursion sampling identities. In particular, Bertoin's transformation allows the identity $||B^{\mathrm{me}}_{(n)}|| \overset{d}{=} ||B^{\mathrm{br}}_{(n)}||$ in (24) to be checked quite easily. According to the bijection between walk paths and plane trees with edge-lengths exploited in [56, 57, 48, 46], the tree constructed by sampling B^{ex} at the times $U_{n,i}$ has total length of all edges equal to $\frac{1}{2}||B^{\mathrm{ex}}_{(n)}||$. So in the chain of identities (24), the link $||B^{\mathrm{ex}}_{(n+1)}|| \overset{d}{=} \sqrt{2\Gamma_{n+1}}$ amounts to the result of Aldous [7, Corollary 22] that the tree derived from sampling B^{ex} at n independent uniform times has total length distributed like $\frac{1}{2}\sqrt{2\Gamma_n}$.

Comparison of the sampling identities for B and B^{me} yields also the following corollary, which is discussed in Section 7. For a random element X of $C[0,1]$, which first attains its minimum at time $\mu := \mu(X)$, define random elements $\mathrm{PRE}_\mu(X)$ and $\mathrm{POST}_\mu(X)$ of $C[0,1]$ by the formulae

$$\mathrm{PRE}_\mu(X) := (X(\mu(1-u)) - X(\mu))_{0\le u\le 1} \tag{29}$$

$$\mathrm{POST}_\mu(X) := (X(\mu + u(1-\mu)) - X(\mu))_{0\le u\le 1}\,. \tag{30}$$

Corollary 5 (Denisov [23]) *For the Brownian motion B and $\mu := \mu(B)$, the the processes $\mu^{-1/2}\mathrm{PRE}_\mu(B)$ and $(1-\mu)^{-1/2}\mathrm{POST}_\mu(B)$ are two standard Brownian meanders, independent of each other and of μ, which has the arcsine distribution on* $[0,1]$.

Theorem 1 is proved in Section 3 by constructing an *alternating exponential walk*

$$0, -T_1, S_1 - T_1, S_1 - T_2, S_2 - T_2, S_2 - T_3, S_3 - T_3, \dots \tag{31}$$

with increments $-X_1, Y_1, -X_2, Y_2, -X_3, Y_3, \dots$ where the X_i and Y_i are standard exponential variables, by sampling the Brownian motion B at the times of points $\tau_1 < \tau_2 < \cdots$ of an independent homogeneous Poisson process of rate $1/2$, and at the times when B attains its minima between these points, as indicated in the following lemma. This should be compared with the different embedding of the alternating exponential walk in Brownian motion of [56, 57, 48], which exposed the structure of critical binary trees embedded in a Brownian excursion. See [4] for further discussion of this point, and [61] for other applications of sampling a process at the times of an independent Poisson process.

Lemma 6 *Let $\tau_i, i = 1, 2, \dots$ be independent of the Brownian motion B with*

$$(\tau_1, \tau_2, \dots) \overset{d}{=} 2(S_1, S_2, \dots) \overset{d}{=} 2(T_1, T_2, \dots)$$

and let μ_i denote the time of the minumum of B on (τ_{i-1}, τ_i), so $B(\mu_i) := \underline{B}(\tau_{i-1}, \tau_i)$, where $\tau_0 = 0$. Then the sequence $0, B(\mu_1), B(\tau_1), B(\mu_2), B(\tau_2), B(\mu_3), B(\tau_3), \dots$ is an alternating exponential random walk; that is

$$(B(\mu_i), B(\tau_i); i \ge 1) \overset{d}{=} (S_{i-1} - T_i, S_i - T_i; i \ge 1). \tag{32}$$

Since $\tau_{n+1} \overset{d}{=} 2\Gamma_{n+1}$, Brownian scaling combines with (32) to give the first identity in the following variation of Theorem 1. The remaining identities are obtained in the course of the proof of Theorem 1 in Section 3.

Corollary 7 *For each $n = 0, 1, 2, \ldots$ and each $X \in \{B, B^{\mathrm{br}}, B^{\mathrm{me}}, B^{\mathrm{ex}}\}$ the distribution of the random vector $X_{(n)}$ defined by (9) is uniquely characterized as follows in terms of the alternating exponential walk, assuming in each case that Γ_r for the appropriate r is independent of X:*

(i) (Brownian sampling)

$$\sqrt{2\Gamma_{n+1}}\, B_{(n)} \stackrel{d}{=} (S_{i-1} - T_i, S_i - T_i; 1 \le i \le n+1) \tag{33}$$

(ii) (Meander sampling) *With $N_- := \inf\{j : S_j - T_j < 0\}$*

$$\sqrt{2\Gamma_{n+1/2}}\, B^{\mathrm{me}}_{(n)} \stackrel{d}{=} (S_{i-1} - T_{i-1}, S_i - T_{i-1}; 1 \le i \le n+1 \,|\, N_- > n) \tag{34}$$

(iii) (Bridge sampling)

$$\sqrt{2\Gamma_{n+1/2}}\, B^{\mathrm{br}}_{(n)} \stackrel{d}{=} (S_{i-1} - T_i, S_i - T_i; 1 \le i \le n+1 \,|\, S_{n+1} - T_{n+1} = 0) \tag{35}$$

(iv) (Excursion sampling) *Let $B^{\mathrm{ex}}_{(n+1)-}$ be the random vector of length $2n+2$ defined by dropping the last two components of $B^{\mathrm{ex}}_{(n+1)}$, which are both equal to 0. Then*

$$\sqrt{2\Gamma_{n+1/2}}\, B^{\mathrm{ex}}_{(n+1)-} \stackrel{d}{=} (S_{i-1} - T_{i-1}, S_i - T_{i-1}; 1 \le i \le n+1 \,|\, N_- = n+1) \tag{36}$$

It will be seen in Section 3 that each identity in Theorem 1 is equivalent to its companion in the above corollary by application of the following lemma:

Lemma 8 (Gamma Cancellation) *Fix $r > 0$, and let Y and Z be two finite-dimensional random vectors with Z bounded. The identity in distribution*

$$Y \stackrel{d}{=} \sqrt{2\Gamma_{r+1/2}}\, Z$$

holds with $\Gamma_{r+1/2}$ independent of Z if and only if the identity

$$\sqrt{2\Gamma_r} Y \stackrel{d}{=} \Gamma_{2r}\, Z$$

holds for Γ_r and Y independent, and Γ_{2r} and Z independent.

By consideration of moments for one-dimensional Y and Z [16, Theorem 30.1], and the Cramér Wold device [16, §29] in higher dimensions, this key lemma reduces to the fact that for independent Γ_r and $\Gamma_{r+1/2}$ variables there is *Wilks identity* [77]:

$$2\sqrt{\Gamma_r\, \Gamma_{r+1/2}} \stackrel{d}{=} \Gamma_{2r} \qquad (r > 0). \tag{37}$$

By evaluation of moments [77, 30], this is a probabilistic expression of Legendre's *gamma duplication formula*:

$$\frac{\Gamma(2r)}{\Gamma(r)} = 2^{2r-1} \frac{\Gamma(r + \frac{1}{2})}{\Gamma(\frac{1}{2})}. \tag{38}$$

See [47],[31], [80, §8.4], [9, §4.7] for other proofs and interpretations of (37). This identity is the simplest case, when $s - r = 1/2$, of the fact that the distribution of $2\sqrt{\Gamma_r \Gamma_s}$ for independent Γ_r and Γ_s is a finite mixture of gamma distributions whenever $s - r$ is half an integer [53, 72]. By a simple density computation reviewed in Section 4, this is a probabilistic expression of the fact that the *McDonald function* or *Bessel function of imaginary argument*

$$K_\nu(x) := \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu}\int_0^\infty t^{\nu-1}e^{-t-(x/2)^2/t}\,dt \tag{39}$$

admits the evaluation

$$K_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}\theta_n(x)x^{-n} \qquad (n = 0, 1, 2, \ldots) \tag{40}$$

with

$$\theta_n(x) = \sum_{m=0}^{n}\frac{(n+m)!}{2^m(n-m)!m!}x^{n-m} \tag{41}$$

the nth *Bessel polynomial* [76, §3.71 (12)],[25, (7.2(40)], [37]. ¿From this perspective, Wilks' identity (37) amounts to the fundamental case of (40)-(41) with $n = 0$. That is, with a more probabilistic notation and the substitution $\xi = \sqrt{2x}$,

$$E\exp\left(-\xi(2\Gamma_{1/2})^{-1}\right) = e^{-\sqrt{2\xi}} \qquad (\xi \geq 0). \tag{42}$$

This is Lévy's well known formula [51] for the Laplace transform of the common distribution of $1/(2\Gamma_{1/2})$, $1/B^2(1)$ and the first hitting time of 1 by B, which is stable with index $\frac{1}{2}$.

Immediately from (39) and (40), there is the following well known formula for the Fourier transform of the distribution of $B(1)/\sqrt{2\Gamma_\nu}$, for Γ_ν independent of the standard Gaussian $B(1)$, and for the Laplace transform of the distribution of $1/\Gamma_\nu$, where the first two identities hold for all real $\nu > 0$ and the last assumes $\nu = n+\frac{1}{2}$ for $n = 0, 1, 2, \ldots$. For all real λ

$$E\exp\left(\frac{i\lambda B(1)}{\sqrt{2\Gamma_\nu}}\right) = E\exp\left(\frac{-\lambda^2}{4\Gamma_\nu}\right) = \frac{2}{\Gamma(\nu)}\left(\frac{\lambda}{2}\right)^\nu K_\nu(\lambda) = \frac{2^n n!}{(2n)!}\theta_n(\lambda)e^{-\lambda} \tag{43}$$

These identities were exploited in the first proof [35] of the infinite divisibility the Student t-distribution of $\sqrt{m}B(1)/\sqrt{2\Gamma_{m/2}}$ for an odd number of degrees of freedom m. Subsequent work [36] showed that the distributions of $1/\Gamma_\nu$ and $B(1)/\sqrt{2\Gamma_\nu}$ are infinitely divisible for all real $\nu > 0$. As indicated in [64] a Brownian proof and interpretation of this fact follow from the result of [29, 64] that for all $\nu > 0$

$$(2\Gamma_\nu)^{-1} \stackrel{d}{=} L_{1,2+2\nu} \tag{44}$$

where $L_{1,\delta}$ denotes the last hitting time of 1 by a Bessel process of dimension δ starting at 0, defined as below (25) for positive integer δ, and as in [64, 68] for all real $\delta > 0$. See also [19] for further results on the infinite divisibility of powers of gamma variables. Formulae (43) and (44) combine to give the identity

$$E\exp\left(-\tfrac{1}{2}\lambda^2 L_{1,2n+3}\right) = \frac{2^n n!}{(2n)!}\,\theta_n(\lambda)\,e^{-\lambda} \text{ for } n = 0, 1, 2\ldots \tag{45}$$

which underlies the simple form of many results related to last exit times of Bessel processes in $2n+3$ dimensions, such as the consequence of William's time reversal theorem [68, Ch. VII, Corollary (4.6)] or of (25) that the distribution of $L_{1,3}$ is stable with index $\frac{1}{2}$. The Bessel polynomials have also found applications in many other branches of mathematics [37], for instance to proofs of the irrationality of π and of e^q for rational q.

Section 4 shows how the structure of Poisson processes embedded in the alternating exponential walk combines with the meander sampling identity to give a probabilistic proof of the representation (40) of $K_{n+1/2}(\lambda)$ for $n = 0, 1, 2, \ldots$, along with some very different interpretations of the Bessel polynomial $\theta_n(\lambda)$, first in terms of sampling from a Brownian meander, then in terms of a Brownian bridge or even a simple lattice walk:

Corollary 9 *Let J_n be the number of $j \in \{1, \dots, n\}$ such that $\underline{B}^{\mathrm{me}}(U_{n,j}, U_{n,j+1}) = \underline{B}^{\mathrm{me}}(U_{n,j}, 1)$. Then*
(i) *the distribution of J_n is determined by any one of the following three formulae:*

$$P(J_n = j) = \frac{j\,(2n-j-1)!\,n!\,2^j}{(n-j)!\,(2n)!} \qquad (1 \le j \le n) \tag{46}$$

$$E\left(\frac{\lambda^{J_n}}{J_n!}\right) = \frac{2^n n!}{(2n)!}\,\lambda\,\theta_{n-1}(\lambda) \qquad (\lambda \in \mathbb{R}) \tag{47}$$

$$E(\lambda^{J_n}) = \frac{\lambda}{(2n-1)}\,F\left(\begin{matrix}1-n, 2\\ 2-2n\end{matrix}\,\middle|\, 2\lambda\right) \qquad (\lambda \in \mathbb{R}) \tag{48}$$

where θ_{n-1} is the $(n-1)$th Bessel polynomial and F is Gauss's hypergeometric function; other random variables J_n with the same distribution are

(ii) [61] *the number of distinct excursion intervals of B^{br} away from 0 discovered by a sample of n points picked uniformly at random from $(0,1)$, independently of each other and of B^{br};*

(iii) [61] *the number of returns to 0 after $2n$ steps of a simple random walk with increments ± 1 conditioned to return to 0 after $2n$ steps, so the walk path has uniform distribution on a set of $\binom{2n}{n}$ possible paths.*

Note that J_n as in (i) is a function of $\widehat{B}^{\mathrm{me}}_{(n)}$, the nth zig-zag approximation to the meander. Parts (ii) and (iii) of the Corollary were established in [61], with the distribution of J_n defined by (46). To explain the connection between parts (i) and (ii), let F^{me} denote the future infimum process derived from the meander, that is

$$F^{\mathrm{me}}(t) := \underline{B}^{\mathrm{me}}(t, 1) \qquad (0 \le t \le 1).$$

The random variable J_n in (i) is the number of j such that $B^{\mathrm{me}}(t) = F^{\mathrm{me}}(t)$ for some $t \in [U_{n,j}, U_{n,j+1}]$, which is the number of distinct excursion intervals of the process $B^{\mathrm{me}} - F^{\mathrm{me}}$ away from 0 discovered by the $U_{n,i}, 1 \le i \le n$. So the equality in distribution of the two random variables considered in (i) and (ii) is implied by the equality of distributions on $C[0,1]$

$$B^{\mathrm{me}} - F^{\mathrm{me}} \stackrel{d}{=} |B^{\mathrm{br}}|. \tag{49}$$

This is read from the consequence of (5) and (25), pointed out by Biane-Yor [15], that

$$(B^{\mathrm{me}}, F^{\mathrm{me}}) \stackrel{d}{=} (|B^{\mathrm{br}}| + L^{\mathrm{br}}, L^{\mathrm{br}}) \tag{50}$$

where $L^{\mathrm{br}} := (L^{\mathrm{br}}(t), 0 \le t \le 1)$ is the usual local time process at level zero of the bridge B^{br}. The identity (49) also allows the results of [61] regarding the joint distribution of the lengths of excursion intervals of B^{br} discovered by the sample points, and the numbers of sample points in these intervals, to be expressed in terms of sampling B^{me} and its future infimum process F^{me}. These results overlap with, but are are not identical to, those which can be deduced by the meander sampling identity from the simple structure of $(W_j, 1 \le j \le n \,|\, N_- > n)$ exposed in the proof of Corollary 9 in Section 4. Going in the other direction, (50) combined with the meander sampling identity (13) gives an explicit description of the joint law of the two random vectors $(|B^{\mathrm{br}}(U_{n,i})|, 1 \le i \le n)$ and $(L^{\mathrm{br}}(U_{n,i}), 1 \le i \le n+1)$, neither of which was considered in [61].

The last interpretation (iii) in Corollary 9 provides a simple combinatorial model for the Bessel polynomials by an exponential generating function derived from lattice path enumerations. Another combinatorial model for Bessel polynomials, based on an ordinary generating function derived from weighted involutions, was proposed by Dulucq and Favreau [24]. Presumably a natural bijection can be constructed which explains the equivalence of these two combinatorial models, but that will not be attempted here. See [63] for further discussion.

As indicated in Section 4, the representation (47) of the Bessel polynomials implies some remarkable formulae for moments of the distribution of J_n defined by this formula. Some of these formulae are the particular cases for $\alpha = \theta = \frac{1}{2}$ of results for a family of probability distributions on $\{1, \dots, n\}$, indexed by (α, θ) with $0 < \alpha < 1, \theta > -\alpha$, obtained as the distributions of the number of components of a particular two-parameter family of distributions for random partitions of $\{1, \dots, n\}$. This model for random partitions was introduced in an abstract setting in [59], and related to excursions of the recurrent Bessel process of dimension $2 - 2\alpha$ in [61, 66]. Section 5 relates the associated two-parameter family of distributions on $\{1, \dots, n\}$ to generalized Stirling numbers and the calculus of finite differences. This line of reasoning establishes a connection between Bessel polynomials and the calculus of finite differences which can be expressed in purely combinatorial terms, though it is established here by an intermediate interpretation involving random partitions derived from Brownian motion.

3 Proofs of the Sampling Formulae

Proof of Lemma 6. By elementary calculations based on of the joint law of $-\underline{B}(0,t)$ and $B(t)$ for $t > 0$, which can be found in [68, III (3.14)], the random variables $-B(\mu_1)$ and $B(\tau_1) - B(\mu_1)$ are independent, with the same exponential(1) distribution as $|B(\tau_1)|$:

$$-B(\mu_1) \stackrel{d}{=} B(\tau_1) - B(\mu_1) \stackrel{d}{=} |B(\tau_1)| \stackrel{d}{=} \Gamma_1. \tag{51}$$

Lemma 6 follows from this observation by the strong Markov property of B. □

As indicated in Section 7, the independence of $-B(\mu_1)$ and $B(\tau_1) - B(\mu_1)$ and the identification (51) of their distribution can also be deduced from the path decomposition of B at time μ_1 which is presented in Proposition 15.

Proof of the Brownian sampling identities (12) and (33). Formula (33) follows from Lemma 6 by Brownian scaling, as indicated in the introduction. Formula (12) follows from (33) by gamma cancellation (Lemma 8), because the random vector on the right side of (33) can be rewritten as

$$\Gamma_{2n+2} \left(\frac{S_{i-1} - T_i}{S_{n+1} + T_{n+1}}, \frac{S_i - T_i}{S_{n+1} + T_{n+1}}; 1 \le i \le n+1 \right) \tag{52}$$

where $\Gamma_{2n+2} := S_{n+1} + T_{n+1}$ is independent of the following random vector, by application of (8) with n replaced by $2n+1$. □

The laws of $B^{\mathrm{me}}, B^{\mathrm{br}}$ and B^{ex} are derived from B by conditioning one or both of $B(1)$ and $\underline{B}(0,1)$ to equal zero, in the sense made rigorous by weak limits such as (1) for the meander. The law of each of the $(2n+2)$-dimensional random vectors $B^{\mathrm{me}}_{(n)}, B^{\mathrm{br}}_{(n)}$ and $B^{\mathrm{ex}}_{(n)}$ is therefore obtained by conditioning $B_{(n)}$ on one or both of $B(1)$ and $\underline{B}(0,1)$. Since $B(1)$ is a component of $B_{(n)}$, and $\underline{B}(0,1)$ is the minimum of $n+1$ components of

$B_{(n)}$, this conditioning could be done by the *naive method* of computations with the joint density of the $2n+3$ independent random variables $S_i, T_i, 1 \leq i \leq n+1$ and $\Gamma_{n+3/2}$ appearing on the right side of the Brownian sampling identity (12). This naive method is used in Section 8 to obtain a more refined sampling identity for the bridge ending at x for arbitrary real x. But as will be seen from this example, computations by the naive method are somewhat tedious, and it would be painful to derive the sampling identities for the meander and excursion this way. The rest of this section presents proofs of the conditioned sampling identities by the simpler approach of first deriving the variants of these identities presented in Corollary 7, all of which have natural interpretations in terms of sampling the Brownian path at the points of an independent Poisson process.

Proof of the bridge sampling identities (35) and (14). Formula (35) follows easily from (33) after checking that

$$(\tau_{n+1} \,|\, B(\tau_{n+1}) = 0) \stackrel{d}{=} 2\Gamma_{n+1/2}. \tag{53}$$

This is obtained by setting $x = 0$ in the elementary formula

$$P(\tau_{n+1} \in dt, B(\tau_{n+1} \in dx) = \frac{1}{n!} t^n (\tfrac{1}{2})^n e^{-t/2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2/t} \, dt \, dx \qquad (t > 0, x \in \mathbb{R}). \tag{54}$$

To pass from (35) to (14), observe that the definitions $U_{n,i} := S_i/S_{n+1}$ and $V_{n,i} := T_i/T_{n+1}$ create four independent random elements, the two random vectors $(U_{n,i}, 0 \leq i \leq n+1)$ and $(V_{n,i}, 0 \leq i \leq n+1)$ and the two random variables S_{n+1} and T_{n+1}. An elementary calculation gives

$$(S_{n+1} \,|\, S_{n+1} - T_{n+1} = 0) \stackrel{d}{=} \tfrac{1}{2}\Gamma_{2n+1}. \tag{55}$$

So the right side of (35) can be replaced by

$$\tfrac{1}{2}\Gamma_{2n+1}(U_{n,i-1} - V_{n,i}, U_{n,i} - V_{n,i}; 1 \leq i \leq n+1). \tag{56}$$

and (14) follows by gamma cancellation (Lemma 8). □

The next lemma is a preliminary for the proofs of the meander and excursion sampling identities. For $\tau_i, i = 1, 2, \ldots$ as in Lemma 6 the points of a Poisson process with rate $1/2$, independent of B, let $(G(\tau_1), D(\tau_1))$ denote the excursion interval of B straddling time τ_1, meaning $G(\tau_1)$ is the time of the last zero of B before time τ_1 and $D(\tau_1)$ is time of the next zero of B after time τ_1. Let $N := \max\{j : \tau_j < D(\tau_1)\}$ be the number of Poisson points which fall in this excursion interval $(G(\tau_1), D(\tau_1))$, so $\tau_N < D(\tau_1) < \tau_{N+1}$. It is now convenient to set $\tau_0 := G(\tau_1)$ rather than $\tau_0 = 0$ as in Lemma 6.

Lemma 10 *The distribution of*

$$(\underline{B}(\tau_{i-1}, \tau_i), B(\tau_i); 1 \leq i \leq N \,|\, B(\tau_1) > 0) \tag{57}$$

is identical to the distribution of

$$(S_{i-1} - T_{i-1}, S_i - T_{i-1}; 1 \leq i \leq N_-) \tag{58}$$

where N_- is the first j such that $S_j - T_j < 0$, and $S_0 = T_0 = 0$. Moreover, for each $n = 0, 1, 2, \ldots$

$$(\tau_{n+1} - G(\tau_1) \,|\, N \geq n+1) \stackrel{d}{=} 2\Gamma_{n+1/2} \stackrel{d}{=} (D(\tau_1) - G(\tau_1) \,|\, N = n+1) \tag{59}$$

and the same identities hold if the conditioning events $(N \geq n+1)$ and $(N = n+1)$ are replaced by $(N \geq n+1, B(\tau_1) > 0)$ and $(N = n+1, B(\tau_1) > 0)$ respectively.

Proof. Using the obvious independence of $|B(\tau_1)|$ and the sign of $B(\tau_1)$, (51) gives

$$(B(\tau_1) \,|\, B(\tau_1) > 0) \stackrel{d}{=} S_1. \tag{60}$$

The identification of the distribution of the sequence in (57) given $B(\tau_1) > 0$ now follows easily from the strong Markov property, as in Lemma 6. The identities in (59) are consequences of the well known result of Lévy[68, XII (2.8)] that the Brownian motion B generates excursion intervals of various lengths, which are the jumps of the stable subordinator of index $\frac{1}{2}$ which is the inverse local time process of B at 0, according to the points of a homogeneous Poisson process with intensity measure Λ such that $\Lambda(s,\infty) = \sqrt{2/(\pi s)}$ for $s > 0$. Thus the intensity of intervals in which there is an $(n+1)$th auxiliary Poisson point in ds is $P(\tau_{n+1} \in ds)\Lambda(s,\infty)$, which is proportional to $s^n s^{-1/2} e^{-s/2} ds$. This is the density of $2\Gamma_{n+1/2}$ up to a constant, which gives the first identity in (59). A similar calculation gives the second identity in (59), which was observed already in [61]. The results with further conditioning on $(B(\tau_1) > 0)$ follow from the obvious independence of this event and $(N, G(\tau_1), D(\tau_1), \tau_1, \tau_2, \ldots)$. $\square$

Proof of the meander sampling identities (13) and (34). In the notation of Lemma 10, it is easily seen that

$$\left(\frac{\tau_i - G(\tau_1)}{\tau_{n+1} - G(\tau_1)}, 1 \le i \le n+1 \,\middle|\, B(\tau_1) > 0, N \ge n+1\right) \stackrel{d}{=} (U_{n,i}, 1 \le i \le n+1)$$

and that the random vector on the left side above is independent of $\tau_{n+1} - G(\tau_1)$ given $B(\tau_1) > 0$ and $N \ge n+1$. Given also $\tau_{n+1} - G(\tau_1) = t$ the process $(B(G(\tau_1)+s), 0 \le s \le t)$ is a Brownian meander of length t, with the same law as $(\sqrt{t}B^{\mathrm{me}}(s/t), 0 \le s \le t)$. Formula (34) can now be read from Lemma 10, and (13) follows from (34) by the gamma cancellation lemma. $\square$

Proof of the excursion sampling identities (36) and (15). The proof of (36) follows the same pattern as the preceding proof of (34), this time conditioning on $B(\tau_1) > 0, N = n+1$ and $D(\tau_1) - G(\tau_1) = t$ to make the process $(B(G(\tau_1)+s), 0 \le s \le t)$ a Brownian excursion of length t. The passage from (36) to (15) uses the easily verified facts that given $N_- = n+1$ the sum S_{n+1} is independent of the normalized vector $S_{n+1}^{-1}(T_i, S_i; 1 \le i \le n)$ with

$$(S_{n+1} \,|\, N_- = n+1) \stackrel{d}{=} \tfrac{1}{2}\Gamma_{2n+1} \tag{61}$$

$$(S_{n+1}^{-1}(T_i, S_i; 1 \le i \le n) \,|\, N_- = n+1) \stackrel{d}{=} (V_{n,i}, U_{n,i}; 1 \le i \le n \,|\, \cap_{i=1}^n (V_{n,i} < U_{n,i})),$$

so the proof is completed by another application of gamma cancellation. $\square$

Proof of Corollary 4 (Vervaat's identity). For any process X which like B^{br} has cyclically stationary increments, meaning $\Theta_u(X) \stackrel{d}{=} X$ for every $0 < u < 1$, and which attains its minimum a.s. uniquely at time $\mu(X)$, it is easily seen that the nth zig-zag approximation $\widehat{X}_{(n)}$ attains its minimum at time $\mu(\widehat{X}_{(n)})$ which has uniform distribution on $\{(i-1/2)/n+1), 1 \le i \le n+1\}$, and that $\mu(\widehat{X}_{(n)})$ is independent of $\Theta_*(\widehat{X}_{(n)})$. For $X = B^{\mathrm{br}}$ inspection of the bridge and excursion sampling identities shows further that

$$\Theta_*(\widehat{B}^{\mathrm{br}}_{(n)}) \stackrel{d}{=} \left(\widehat{B}^{\mathrm{ex}}_{(n+1)}\left(\frac{\frac{1}{2} + u(n+1)}{n+2}\right), 0 \le u \le 1\right) \tag{62}$$

where the linear time change on the right side of (62) just eliminates the initial and final intervals, each of length $\frac{1}{2}/(n+2)$ where $\widehat{B}^{\mathrm{ex}}_{(n+1)}$ equals 0. As $n \to \infty$, $\mu(\widehat{B}^{\mathrm{br}}_{(n)}) \to \mu(B^{\mathrm{br}})$ and $\Theta_*(\widehat{B}^{\mathrm{br}}_{(n)}) \to \Theta_*(B^{\mathrm{br}})$ with respect to the uniform metric on $C[0,1]$ almost surely, and the zig-zag process on the right side of (62) converges uniformly a.s. to B^{ex}. $\square$

4 Bessel polynomials, products of independent gamma variables, and the meander

By an elementary change of variables, for independent Γ_r and Γ_s with $r, s > 0$, the joint density of Γ_r and $2\sqrt{\Gamma_r \Gamma_s}$ is given for $x, z > 0$ by the formula

$$\frac{P(\Gamma_r \in dx, 2\sqrt{\Gamma_r \Gamma_s} \in dz)}{dx\, dz} = \frac{2^{1-2s}}{\Gamma(r)\Gamma(s)} x^{r-s-1} z^{2s-1} e^{-x-(z/2)^2/x}. \tag{63}$$

Hence by integrating out x and applying (39), there is the formula [53],[72]: for $z > 0$

$$P\left(2\sqrt{\Gamma_r \Gamma_s} \in dz\right)/dz = \frac{2^{2-r-s}}{\Gamma(r)\Gamma(s)} z^{r+s-1} K_{r-s}(z). \tag{64}$$

As remarked in [70, p. 96], (63) and (64) identify the conditional law of Γ_r given $\Gamma_r \Gamma_s = w$ as the *generalized inverse Gaussian distribution,* which has found many applications [70].

As noted in [53, 72], the classical expression (40) for $K_{n+1/2}(x)$ in terms of the Bessel polynomial $\theta_n(x)$ as in (41) implies that for each $n = 0, 1, \ldots$ the density of $2\sqrt{\Gamma_r \Gamma_{r+n+1/2}}$ is a finite linear combination of gamma densities with positive coefficients. That is to say, the distribution of $2\sqrt{\Gamma_r \Gamma_{r+n+1/2}}$ is a probabilistic mixture of gamma distributions. After some simplifications using the gamma duplication formula (38), it emerges that for each $n = 1, 2, \ldots$ and each $r > 0$ there is the identity in distribution

$$2\sqrt{\Gamma_r \Gamma_{r+n-1/2}} \stackrel{d}{=} \Gamma_{2r+J_{n,r}-1} \tag{65}$$

where on the left the gamma variables Γ_r and $\Gamma_{r+n-1/2}$ are independent, and on the right $J_{n,r}$ is a random variable assumed independent of gamma variables $\Gamma_{2r+j-1}, j = 1, 2, \ldots$, with the following distribution on $\{1, \ldots, n\}$:

$$P(J_{n,r} = j) = \frac{(2n-j-1)!\,(2r)_{j-1}}{(n-j)!\,(j-1)!\,2^{2n-j-1}\,(r+\frac{1}{2})_{n-1}} \qquad (1 \le j \le n) \tag{66}$$

where

$$(x)_n := x(x+1)\cdots(x+n-1) = \Gamma(x+n)/\Gamma(x). \tag{67}$$

The probability generating function of $J_{n,r} - 1$ is found to be

$$E(\lambda^{J_{n,r}-1}) = \frac{(\frac{1}{2})_{n-1}}{(r+\frac{1}{2})_{n-1}} F\left(\begin{matrix} 1-n, 2r \\ 2-2n \end{matrix} \middle| \, 2\lambda \right) \tag{68}$$

where F is Gauss's hypergeometric function

$$F\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{69}$$

with the understanding that for $a = 1-n$ as in (68) the series terminates at $k = n-1$.

In the particular case $r = 1$, the identity (65) for all $n = 1, 2, \ldots$ can be read from the Rayleigh distribution (6) of $B^{\mathrm{me}}(1)$, the form (34) of the meander sampling identity and the interpretation of the right side of (65) for $r = 1$ implied by the following lemma. The proof of this lemma combined with the meander sampling identity yields also Corollary 9, formulae (46) and (48) being the particular cases $r = 1$ of (66) and (68). Moreover the previous argument leading to (65) can be retraced to recover first the classical identities (40)-(41) for all $n = 0, 1, \ldots$, then (65) for all $r > 0$ and all $n = 1, 2, \ldots$.

Lemma 11 *Let*

$$W_n := S_n - T_n = \sum_{i=1}^{n} (X_i - Y_i)$$

be symmetric random walk derived from independent standard exponential variables X_i, Y_i, and let $N_- := \inf\{n : W_n < 0\}$ be the hitting time of the negative half-line for this walk. For each $n = 1, 2, \ldots$ the distribution of W_n given the event

$$(N_- > n) := (W_i > 0 \text{ for all } 1 \le i \le n)$$

is determined by

$$(W_n \,|\, N_- > n) \stackrel{d}{=} S_{J_n} \tag{70}$$

where J_n is a random variable independent of $S_1, S_2, \ldots$ with the distribution displayed in (46), which is also the distribution of $J_{n,1}$ defined by (66).

Proof. Let $1 \le N_1 < N_2 < \ldots$ denote the sequence of *ascending ladder indices* of the walk (W_n), that is the sequence of successive j such that $W_j = \max_{1 \le i \le j} W_i$. It is a well known consequence of the memoryless property of the exponential distribution that

$$(W_{N_1}, W_{N_2}, \ldots) \stackrel{d}{=} (S_1, S_2, \ldots) \tag{71}$$

and that the $W_{N_1}, W_{N_2}, \ldots$ are independent of $N_1, N_2 - N_1, N_3 - N_2, \ldots$, which are independent, all with the same distribution as N_-; that is for $i \ge 1$ and $n \ge 0$, with $N_0 := 0$,

$$P(N_i - N_{i-1} > n + 1) = P(N_- > n) \frac{(\frac{1}{2})_n}{n!} = \binom{2n}{n} 2^{-2n}, \tag{72}$$

See Feller [27, VI.8,XII.7]. For each $n \ge 1$ let $R_{n,1} = 1, \ldots, R_{n,L_n} = n$ be the successive indices j such that $W_j = \min_{j \le i \le n} W_i$. By consideration of the walk with increments $X_i - Y_i, 1 \le i \le n$ replaced by $X_{n+1-i} - Y_{n+1-i}, 1 \le i \le n$, as in [27, XII.2], for each $n = 1, 2, \ldots$ there is the identity in distribution

$$((L_n; R_{n,1}, \ldots, R_{n,L_n}) \,|\, N_1 > n) \stackrel{d}{=} ((K_n; N_1, \ldots, N_{K_n}) \,|\, A_n) \tag{73}$$

where

$$A_n := (N_i = n \text{ for some } i\ ) = (W_n = \max_{0 \le i \le n} W_i) \tag{74}$$

is the event that n is an ascending ladder index, and

$$K_n := \sum_{m=1}^{n} 1(A_m) \tag{75}$$

is the number of ascending ladder indices up to time n. Moreover, the two conditioning events have the same probability $P(A_n) = P(N_- > n)$ as in (72), and from (71) given $N_1 > n$ and $L_n = \ell$ the $W_{R_{n,i}}$ for $1 \leq i \leq \ell$ are independent of the $R_{n,i}$ for $1 \leq i \leq \ell$, where $R_{n,\ell} = n$, with

$$((W_{R_{n,1}}, \ldots, W_{R_{n,\ell}}) \,|\, L_n = \ell) \stackrel{d}{=} (S_1, \ldots, S_\ell)$$

The distribution of the random vectors in (73) was studied in [61]. As indicated there, and shown also by Lemma 12 and formula (85) below, the distribution of K_n given A_n is that of $J_{n,1}$ described by (66). □

According to the above argument, two more random variables with the distribution of $J_n = J_{n,1}$ described in Corollary 9 can be defined as follows in terms of the walk (W_j):

(iv) *the number* L_n *of* $j \in \{1, \ldots, n\}$ *with* $W_j = \min_{j \leq i \leq n} W_i$, *given* $N_- > n$;

(v) *as the number* K_n *of* $j \in \{1, \ldots, n\}$ *with* $W_j = \max_{1 \leq i \leq j} W_i$, *given* $W_n = \max_{1 \leq i \leq n} W_i$.

The discussion in [27, XII.2,XII.7] shows that the same distribution is obtained from either of these constructions for any random walk (W_j) with independent increments whose common distribution is continuous and symmetric.

The fact that formula (66) defines a probability distribution on $\{1, \ldots, n\}$ for every $r > 0$ can be reformulated as follows. Replace $n-1$ by m and use (68) for $\lambda = 1$ to obtain the following identity, the case $r = \frac{1}{2}$ of which appears in [32, (5.98)]:

$$F\left(\begin{matrix} -m, 2r \\ -2m \end{matrix} \,\middle|\, 2\right) = \frac{(r+\frac{1}{2})_m}{(\frac{1}{2})_m} \quad (m = 0, 1, 2, \ldots). \tag{76}$$

Put another way, there is the following formula for moments of $J_n = J_{n,1}$: for all $n = 1, 2, \ldots$ and all $r > 0$

$$E\left(\frac{(2r)_{J_n - 1}}{J_n!}\right) = \frac{(r+\frac{1}{2})_{n-1}}{(\frac{3}{2})_{n-1}}. \tag{77}$$

In particular, for $r = 1 + k/2$ for positive integer k this reduces to the following expression for the rising factorial moments of $J_n + 1$:

$$E(J_n + 1)_k = \frac{(k+1)!(\frac{1}{2}(k+3))_{n-1}}{(\frac{3}{2})_{n-1}} \qquad (k = 0, 1, 2, \ldots) \tag{78}$$

which for even k can be further reduced by cancellation. If J_n is defined as in Corollary 9 to be the number of $j \in \{1, \ldots, n\}$ such that $\underline{B}^{\mathrm{me}}(U_{n,j}, U_{n,j+1}) = \underline{B}^{\mathrm{me}}(U_{n,j}, 1)$, then by application of [61, Proposition 4] and (50),

$$J_n/\sqrt{2n} \to B^{\mathrm{me}}(1) \text{ almost surely as } n \to \infty. \tag{79}$$

Straightforward asymptotics using (78) and (6) show that the convergence (79) holds also in kth mean for every $k = 1, 2, \ldots$.

It is natural to look for interpretations of the distribution of $J_{n,r}$ appearing in (65) and (68) for other values of r. For $r = 1/2$ it can be verified by elementary computation that for $W_n := S_n - T_n$, the difference of two independent gamma(n) variables there is the following companion of (70):

$$|W_n| \stackrel{d}{=} S_{J_{n,1/2}} \tag{80}$$

where $J_{n,1/2}$ is independent of $S_1, S_2, \ldots$. If W_n is embedded in Brownian motion as in Lemma 6 as $W_n = B(\tau_n)$, this can also be derived as follows from the decomposition of the Brownian path at the time λ_n of its last exit from 0 before time τ_n:

$$|W_n| \stackrel{d}{=} \sqrt{\tau_n A} B^{\mathrm{me}}(1) = 2\sqrt{\Gamma_n A \Gamma_1} \tag{81}$$

where $A := (\tau_n - \lambda_n)/\tau_n$ is an arcsine variable independent of $\tau_n = 2\Gamma_n$ and $B^{\mathrm{me}}(1) = \sqrt{2\Gamma_1}$. It is elementary that $A\Gamma_1 \stackrel{d}{=} \Gamma_{1/2}$, so (80) follows from (81) and (65) for $r = \frac{1}{2}$. Formula (80) can also be interpreted directly in terms of the alternating exponential walk. Let

$$\Lambda_n := \max\{j \leq n : W_j W_n \leq 0\},$$

and let

$$u_n := P(N_- > n) = (\tfrac{1}{2})_n / n!$$

as in Lemma 11. Elementary calculations based on (80) and Lemma 11 show that

$$J_{n,1/2} \stackrel{d}{=} J_{n-\Lambda_n} \tag{82}$$

where given $\Lambda_n = j$ the distribution of $J_{n-\Lambda_n}$ is that of J_{n-j}, and

$$P(\Lambda_n = j) = \begin{cases} 2u_n & \text{if } j = 0 \\ u_j u_{n-j} & \text{if } 1 \leq j \leq n-1 \end{cases} \tag{83}$$

This argument shows also that $J_{n,1/2}$ can be constructed from the walk $(W_j, 0 \leq j \leq n)$ as the number of $k > \Lambda_n$ such that $|W_k| = \min_{k \leq j \leq n} |W_j|$. Let $G_{n,r}(\lambda) := E\lambda^{J_{n,r}}$ as determined by (68). Then (82) and (83) imply the identity of hypergeometric polynomials

$$\sum_{j=1}^{n-1} u_j u_{n-j} G_{n-j,1}(\lambda) = G_{n,1/2}(\lambda) - 2u_n G_{n,1}(\lambda) \tag{84}$$

which is easily checked for small n. A question left open here is whether the identity (65) has any natural interpretation in terms of Brownian motion or the alternating exponential random walk for values of r other than the values $r = 1$ and $r = \frac{1}{2}$ discussed above.

5 Combinatorial identities related to random partitions

By evaluation of moments on both sides, and using the special case $n = 1$, the identity (65) for $n \geq 1$ is seen after some elementary calculations to amount to the formula

$$(\tfrac{1}{2}t)_n = \sum_{k=1}^{n} a(n, k, \tfrac{1}{2})(t)_k \tag{85}$$

where

$$a(n, k, \tfrac{1}{2}) := \frac{2^{k-2n}(2n-k-1)!}{(k-1)!(n-k)!} = P(J_n = k)\,\frac{(\frac{1}{2})_n}{k!} \tag{86}$$

for $J_n \stackrel{d}{=} J_{n,1}$ as in the previous section. The identity (85) with $a(n,k,\frac{1}{2})$ given by the the first equality in (86) can be read from formulae for finite difference operators due to Toscano [73, (17),(122)],[74, (2.3),(2.11)]. In a similar vein, (76) reduces to the following identity: for all $r > 0$ and $m = 0, 1, 2, \ldots$

$$(r + \tfrac{1}{2})_m = \sum_{j=0}^{m} 2a(m+1, j+1, \tfrac{1}{2})(2r)_j. \tag{87}$$

For each m both sides are polynomials in r of degree m, so this holds also an identity of polynomials. Many similar identities can be found in Toscano's paper [73].

As shown by Toscano, for arbitrary real α the coefficients $a(n,k,\alpha)$ in the formula

$$(t\alpha)_n = \sum_{k=1}^{n} a(n,k,\alpha)(t)_k \tag{88}$$

define polynomials in α, known as *generalized Stirling numbers* [73, 74, 20, 38] which can be expressed as

$$a(n,k,\alpha) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^j \binom{k}{j}(-j\alpha)_n = \sum_{r=k}^{n}(-1)^{n-r}\begin{bmatrix} n \\ r \end{bmatrix}\begin{Bmatrix} r \\ k \end{Bmatrix}\alpha^r \tag{89}$$

where $\left[{n \atop r}\right]$ and $\left\{{r \atop k}\right\}$ are the unsigned Stirling numbers of the first and second kinds respectively, as defined in [32]. The following probabilistic interpretation of the $a(n,k,\alpha)$ for $0 < \alpha < 1$ gives a generalization of (86).

Lemma 12 *Let $0 < N_1 < N_2, \ldots$ be the ascending ladder indices of a random walk $W_n = Z_1 + \cdots + Z_n$ where the Z_i independent and identically distributed, such that $P(W_n > 0) = \alpha$ for all n for some $\alpha \in (0,1)$, let K_n be the number of ascending ladder indices up to time n, and A_n the event that n is an ascending ladder index. Then for $1 \le k \le n$*

$$P(K_n = k, A_n) = \text{coefficient of } x^n \text{ in } (1-(1-x)^\alpha)^k = \frac{k!}{n!}a(n,k,\alpha) \tag{90}$$

$$P(A_n) = \frac{(\alpha)_n}{n!} \tag{91}$$

and hence

$$P(K_n = k \,|\, A_n) = \frac{k!}{(\alpha)_n}a(n,k,\alpha) \qquad (1 \le k \le n) \tag{92}$$

Proof. According to Feller [27, XII.7], the generating function of N_1 is $E(x^{N_1}) = 1 - (1-x)^\alpha$. This gives the first equality in (90) because $(K_n = k, A_n) = (N_k = n)$ where N_k is the sum of k independent copies of N_1. The second equality in (90) reduces to the first equality in (89) by binomial expansions. Formula (91), which appears in [61, (19)], is obtained from a standard generating function computation [26]:

$$\sum_{n=0}^{\infty} P(A_n)x^n = (1 - E(x^{N_1}))^{-1} = (1-(1-(1-x)^\alpha))^{-1} = (1-x)^{-\alpha}.$$

□

In the particular case $\alpha = 1/2$, a classical formula of Lambert [32, (5.70)] gives the expansion

$$(1-(1-x)^{1/2})^k = \left(\frac{x}{2}\right)^k \sum_{m=0}^{\infty} \binom{2m+k}{m} \frac{k}{2m+k} \left(\frac{x}{4}\right)^m$$

for $k > 0$ which together with (90) yields the two expressions for $a(n,k,1/2)$ in (86), hence formula (46) for the distribution of J_n.

Note that Lemma 12 implies

$$a(n,k,\alpha) > 0 \text{ for all } 1 \leq k \leq n \text{ and } 0 < \alpha < 1, \tag{93}$$

which does not seem obvious from either of the combinatorial formulae (89). According to [61, Prop. 4 and Lemma 7], as $n \to \infty$ the conditional distribution of K_n/n^α given A_n converges weakly to the distribution on $(0,\infty)$ with density $\Gamma(\alpha+1)xg_\alpha(x)$, where $g_\alpha(x)$ is the density of the Mittag-Leffler distribution determined by the moments

$$\int_0^\infty x^p g_\alpha(x)dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)} \qquad (p > -1). \tag{94}$$

That is [67]

$$g_\alpha(x) = \frac{1}{\pi\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \Gamma(\alpha k+1) x^{k-1} \sin(\pi\alpha k). \tag{95}$$

Known local limit approximations [39] to the distribution of N_k for large k by a stable density with index α yield a corresponding local limit approximation to the distribution of K_n given A_n. This translates via (92) into the following asymptotic formula for the generalized Stirling numbers $a(n,k,\alpha)$ which does not seem to appear in the combinatorial literature: for each $0 < \alpha < 1$ and $0 < x < \infty$,

$$a(n,k,\alpha) \sim \frac{\Gamma(n)}{\Gamma(k)} n^{-\alpha} \alpha\, g_\alpha(x) \qquad \text{as } n \to \infty \text{ with } k \sim xn^\alpha. \tag{96}$$

As observed in [61], the distribution of K_n given A_n in (92), depending on a parameter $\alpha \in (0,1)$, is identical to the distribution of the number of components $\kappa(\Pi_n)$ of a random partition Π_n governed by $P_{\alpha,\alpha}$, where $P_{\alpha,\theta}$ for $\alpha \in (0,1)$ and $\theta > 0$ governs Π_n according to the following probability distribution on the finite set of all partitions of $\{1,\ldots,n\}$, introduced in [59]:

$$P_{\alpha,\theta}(\Pi_n = \pi) = \frac{\alpha^k(\theta/\alpha)_k}{(\theta)_n} \prod_{i=1}^k (1-\alpha)_{n_i-1} \tag{97}$$

for each partition π with k components of sizes $n_i \geq 1$ with with $\sum_{i=1}^k n_i = n$. Easily from (97), by summing over all π with $\kappa(\pi) = k$,

$$P_{\alpha,\theta}(\kappa(\Pi_n) = k) = P_{\alpha,\alpha}(\kappa(\Pi_n) = k) \frac{(\theta/\alpha)_k}{k!} \frac{(\alpha)_n}{(\theta)_n}. \tag{98}$$

The identity $\sum_{k=1}^n P_{\alpha,\theta}(\kappa(\Pi_n) = k) = 1$ for all $\theta > 0$, applied to $\theta = t\alpha$ gives

$$(t\alpha)_n = \sum_{k=1}^n \frac{(\alpha)_n}{k!} P_{\alpha,\alpha}(\kappa(\Pi_n) = k)(t)_k$$

which allows either of (88) and (92) to be deduced immediately from the other, without passing via (89). The above argument yields also

$$P_{\alpha,\theta}(\kappa(\Pi_n)=k)=\frac{(\theta/\alpha)_k}{(\theta)_n}a(n,k,\alpha). \tag{99}$$

As indicated in [59], if formula (98) is modified to avoid dividing by 0 when either α or θ equals 0, the modified formula defines a probability distribution on partitions for all (α,θ) with $0\le\alpha<1$ and $\theta>-\alpha$. Similar remarks apply to (99), which in the limit case as $\alpha\downarrow 0$ reduces to a well known result for the Ewens sampling formula [44, (41.5)]. Easily from (99) and (96), for all (α,θ) with $0<\alpha<1$ and $\theta>-\alpha$, the $P_{\alpha,\theta}$ distribution of $\kappa(\Pi_n)/n^\alpha$ converges as $n\to\infty$ to the distribution with density $(\Gamma(\theta+1)/\Gamma(\theta/\alpha+1))x^{\theta/\alpha}g_\alpha(x)$. See also [66, 60] for further results related to this two-parameter family of random partitions.

6 Distributions of some total variations

In view of the simple exact formulae provided by Corollary 3 for the distributions of $||X_{(n)}||$ for $X=B,B^{\mathrm{br}},B^{\mathrm{ex}},B^{\mathrm{me}}$, it is natural to look for corresponding descriptions of the laws of the variations $||\widetilde{X}_{(n)}||:=\sum_{i=1}^{n+1}|X(U_{n,i})-X(U_{n,i-1})|$ for these processes. As an immediate consequence of the form (33) of the Brownian sampling identity and the elementary identity $|S_1-T_1|\overset{d}{=}S_1$, the distribution of $||\widetilde{B}_{(n)}||$ is such that

$$\sqrt{2\Gamma_{n+1}}\,||\widetilde{B}_{(n)}||\overset{d}{=}\Gamma_{n+1}. \tag{100}$$

So for each $n=0,1,\ldots$ the distribution of $||\widetilde{B}_{(n)}||$ is uniquely determined by the moments

$$E||\widetilde{B}_{(n)}||^p=2^{-p/2}\,\frac{\Gamma(n+1+p)}{\Gamma(n+1+p/2)}\qquad (n+1+p>0). \tag{101}$$

This formula is easily verified for $p=1$ and $p=2$ by conditioning on the U_{ni}. The same method shows that the formula for any positive integer p reduces to an evaluation of a sum over partititions of p which is a known evaluation of a Bell polynomial [22, p. 135, 3h]. A more explicit description of the distribution of $||\widetilde{B}_{(n)}||$, and a corresponding result for $||\widetilde{B}^{\mathrm{br}}_{(n)}||$, will now be obtained by application of the following lemma. This leaves open the problem of finding analogous results for B^{me} and B^{ex}.

Let $\beta_{a,b}$ denote a random variable with the *beta*(a,b) distribution of $\Gamma_a/(\Gamma_a+\Gamma'_b)$ for independent gamma variables Γ_a and Γ'_b.

Lemma 13 *For each pair of real parameters r and s with $0\le s\le 2r-1$ the identity in distribution*

$$\sqrt{2\Gamma_r}\,Y\overset{d}{=}\Gamma_s \tag{102}$$

holds for Γ_r independent of Y iff

$$Y\overset{d}{=}\beta_{s,2r-1-s}\sqrt{2\Gamma_{r-1/2}} \tag{103}$$

where the beta and gamma variables are independent.

Proof. By Wilks' identity (37) and the elementary fact that $\beta_{a,b}:=\Gamma_a/(\Gamma_a+\Gamma'_b)$ is independent of $\Gamma_a+\Gamma'_b\overset{d}{=}\Gamma_{a+b}$,

$$\sqrt{2\Gamma_r}\,\beta_{s,2r-1-s}\sqrt{2\Gamma_{r-1/2}}=\beta_{s,2r-1-s}\sqrt{2\Gamma_{r-1/2}}\sqrt{2\Gamma_r}\overset{d}{=}\beta_{s,2r-1-s}\Gamma_{2r-1}\overset{d}{=}\Gamma_s.$$

So (103) implies (102). The converse holds because the distribution of Y in (102) is obviously determined by its moments. □

By conditioning on $\sqrt{2\Gamma_c}$, whose density is displayed in (11), for $a, b, c > 0$ the density $f_{a,b,c}$ of $\beta_{a,b}\sqrt{2\Gamma_c}$ in (103) is given by the formula

$$f_{a,b,c}(y) = \frac{\Gamma(a+b)\, y^{a-1}}{\Gamma(a)\Gamma(b)\Gamma(c)2^{c-1}} \int_y^\infty t^{2c-a-b}(t-y)^{b-1}e^{-\frac{1}{2}t^2}\, dt. \tag{104}$$

In particular, for $c = (a+b)/2$ the density $f_{a,b} := f_{a,b,(a+b)/2}$ of $\beta_{a,b}\sqrt{2\Gamma_{(a+b)/2}}$ reduces by the gamma duplication formula (38) to

$$f_{a,b}(y) = \frac{2^{(a+b+1)/2}\Gamma((a+b+1)/2)}{\Gamma(a)\Gamma(b)}\, y^{a-1}\phi_b(y) \tag{105}$$

where the function ϕ_b is defined as follows for $b > 0$ in terms of Z with the standard Gaussian density $\phi(z) := 1/\sqrt{2\pi}e^{-\frac{1}{2}z^2}$:

$$\phi_b(y) := E((Z-y)^{b-1}1(Z>y)) = \int_y^\infty (z-y)^{b-1}\phi(z)dz. \tag{106}$$

By comparison of (106) with [50, (10.5.2)], the function $\phi_b(y)$ can be expressed in terms of the either the *Hermite function* $H_\nu(z)$ of [50] or the *confluent Hypergeometric function* $U(a,b,z)$ of [1] as

$$\phi_b(y) = \phi(y)\Gamma(b)2^{b/2}H_{-b}(y/\sqrt{2}) = \phi(y)\Gamma(b)2^{-b/2}U(\tfrac{1}{2}b, \tfrac{1}{2}, \tfrac{1}{2}y^2). \tag{107}$$

As indicated in [55, 50], this function can also be expressed in terms of one of Weber's parabolic cylinder functions. Easily from (106) there are the formulae

$$\phi_1(y) = P(Z>y);\ \phi_2(y) = \phi(y) - y\phi_1(y);\quad \phi_3(y) = -y\phi(y) + (1+y^2)\phi_1(y) \tag{108}$$

and so on, according to the recursion

$$\phi_b(y) = (b-2)\phi_{b-2}(y) - y\phi_{b-1}(y) \qquad (b>2) \tag{109}$$

which follows from (106) by integration by parts. Hence for $m = 0, 1, 2, \ldots$

$$\phi_{m+1}(y) = \phi(y)\left(-\frac{d}{dy}\right)^m \frac{\phi_1(y)}{\phi(y)} = (-1)^m[h_m^*(y)\phi_1(y) - k_m^*(y)\phi(y)] \tag{110}$$

where h_m^* and k_m^* are variations of the Hermite polynomials, with non-negative integer coefficients, of degrees m and $m-1$ respectively, determined by the same recurrence $p_{m+1}(y) = yp_m(y) + mp_{m-1}(y)$ with different initial conditions [55, p. 74-77].

Corollary 14 *For each $n = 1, 2, \ldots$ the distribution of*

$$||\widetilde{B}_{(n)}|| := \sum_{i=1}^{n+1} |B(U_{n,i}) - B(U_{n,i-1})|$$

is that of $\beta_{n+1,n}\sqrt{2\Gamma_{n+1/2}}$ *derived from independent beta and gamma variables: for* $y > 0$

$$P(||\widetilde{B}_{(n)}|| \in dy) = \frac{2^{n+1}}{\Gamma(n)}\, y^n \phi_n(y)\, dy \tag{111}$$

while the corresponding distribution for B^{br} *instead of* B *is that of* $\beta_{n,n}\sqrt{2\Gamma_n}$*, with*

$$P(||\widetilde{B}^{\mathrm{br}}_{(n)}|| \in dy) = \frac{2^{n+1/2}\Gamma(n+\frac{1}{2})}{\Gamma(n)^2} y^{n-1}\, \phi_n(y)\, dy. \tag{112}$$

As $n \to \infty$ *the distributions of both* $||\widetilde{B}_{(n)}|| - \sqrt{n/2}$ *and* $||\widetilde{B}^{\mathrm{br}}_{(n)}|| - \sqrt{n/2}$ *converge to the normal distribution with mean* 0 *and variance* $3/8$.

Proof. Formula (111) is read from (100), Lemma 13, and (105). The result for the bridge is obtained similarly using the following property of the random walk $W_n := \sum_{i=1}^n (X_i - Y_i)$. If $\Gamma_n := \sum_{i=1}^n |X_i - Y_i|$, then

$$(\Gamma_{n+1} \,|\, W_{n+1} = 0) \stackrel{d}{=} \Gamma_n. \tag{113}$$

Combined with the bridge sampling identity (35) this gives

$$\sqrt{2\Gamma_{n+1/2}}\, ||\widetilde{B}^{\mathrm{br}}_{(n)}|| \stackrel{d}{=} \Gamma_n \tag{114}$$

and (112) now follows from Lemma 13 and (105). To check (113), observe first that

$$(\Gamma_{n+1}, W_{n+1}) \stackrel{d}{=} (\Gamma_{n+1}, \Gamma_{n+1} Q_{n+1})$$

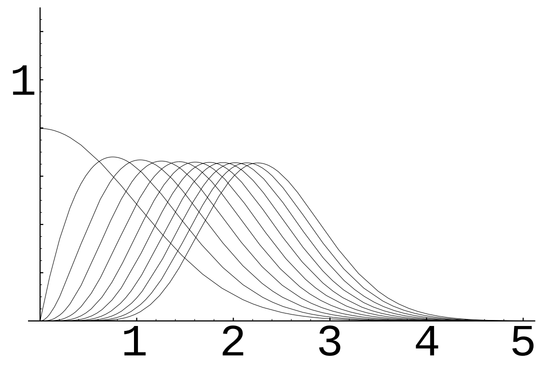
for a random variable Q_{n+1} with values in $[-1,1]$ which is independent of Γ_{n+1}, with a distribution which has atoms at ± 1 and a density q_{n+1} on $(-1,1)$ which is strictly positive and continuous, but whose precise form is irrelevant. Thus for $|w| < t$

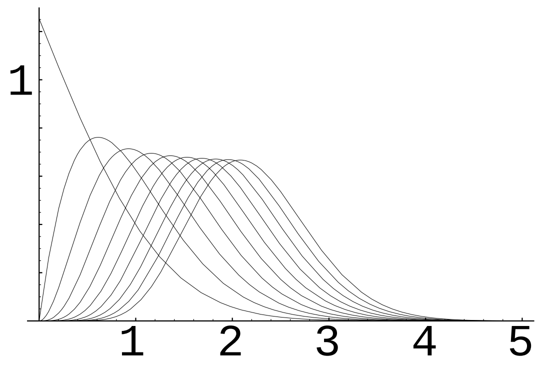
$$P(W_{n+1} \in dw \,|\, \Gamma_{n+1} = t) = t^{-1} q_{n+1}(w/t) dw$$

which implies (113). The asymptotic normality of $||\widetilde{B}_{(n)}||$ and $||\widetilde{B}^{\mathrm{br}}_{(n)}||$ follows easily from the representations in terms of beta and gamma variables, and the well known normal approximations of these variables. The limiting variance can be checked using the moment formulae (101) and (115) below, and standard asymptotics for the gamma function [1, 6.1.47]. □

The following graphs display the densities found in (111)-(112):

Density of $||\widetilde{B}_{(n)}||$ for $n = 0, 1, 2 \ldots, 10$ Density of $||\widetilde{B}^{\mathrm{br}}_{(n)}||$ for $n = 1, 2 \ldots, 10$





According to (111)-(112), for each $n = 1, 2, \ldots$ the distribution of $||\widetilde{B}_{(n)}||$ is the size-biased distribution of $||\widetilde{B}^{\rm br}_{(n)}||$. This is reminiscent of Lévy's result that for $L_1^0(\omega)$ the local time at 0 of a path ω up to time 1, the Rayleigh distribution of $L_1^0(B^{\rm br})$ is the sized biased distribution $L_1^0(B)$, where $L_1^0(B) \stackrel{d}{=} |B(1)| = ||\widetilde{B}_{(0)}||$. See [14, 65] for some developments of the latter result.

The above description (114) of the law of $||\widetilde{B}^{\rm br}_{(n)}||$ amount to the following moment formula: for $p > 0$

$$E||\widetilde{B}^{\rm br}_{(n)}||^p = 2^{-p/2}\,\frac{\Gamma(n+p)}{\Gamma(n)}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{1}{2}+\frac{p}{2})}. \tag{115}$$

For $p = 1$ this is easily checked by conditioning on the U_{ni}, using the fact that the exchangeability of increments of the bridge and the exchangeability of the spacings $U_{n,i} - U_{n,i-1}$ for $1 \le i \le n+1$, implies that the random variables $B^{\rm br}(U_{n,i}) - B^{\rm br}(U_{n,i-1})$ are exchangeable, with the same distribution as $\sqrt{U_{n,1}(1-U_{n,1})}B(1)$ for $B(1)$ standard Gaussian independent of $U_{n,1}$. For $p = 2$ formula (115) reduces to

$$E||\widetilde{B}^{\rm br}_{(n)}||^2 = n(n+1)/(2n+1).$$

This can be checked by the same method, but not so easily. As shown in [42], this formula is related via some remarkable integral identities to algebraic evaluations of some Euler integrals and to duplication formulae for Appell's hypergeometric function F_1.

7 Decomposition at the minimum

The following proposition can be read from the work of Williams [79], Millar [54] and Greenwood-Pitman [34].

Proposition 15 [79, 54, 34]. *For τ_1 independent of B with $\tau_1 \stackrel{d}{=} 2\Gamma_1$, and μ_1 the a.s. unique time that B attains its minimum on $[0, \tau_1]$, the processes*

$$(B(\mu_1 - t) - B(\mu_1))_{0 \le t \le \mu_1} \text{ and } (B(\mu_1 + t) - B(\mu_1))_{0 \le t \le \tau_1 - \mu_1} \tag{116}$$

are two independent copies of

$$(\sqrt{\mu_1}B^{\rm me}(t/\mu_1))_{0 \le t \le \mu_1} \tag{117}$$

for a standard meander $B^{\rm me}$ that is independent of μ_1, and $\mu_1 \stackrel{d}{=} 2\Gamma_{1/2}$.

Proof. To briefly review the derivation of this result via [79, 54, 34], the identity in distribution between the second process in (116) and that in (117), with μ_1 in (117) replaced by $\tau_1 - \mu_1$, follows from Lévy's theorem that $(B(t) - \underline{B}(0,t))_{t\geq 0}$ has the same distribution as $(|B(t)|)_{t\geq 0}$ and the first identity in (3). As indicated in [34], the independence of the two fragments in (116) amounts to the last exit decomposition of $(B(t) - \underline{B}(0,t))_{t\geq 0}$ at the time μ_1 of its last zero before time τ_1, which can be deduced from Itô's excursion theory, and the identical distribution of the two fragments in (116) is explained by a duality argument using time reversal. That $\mu_1 \stackrel{d}{=} 2\Gamma_{1/2}$ is evident because the sum of two independent copies of μ_1 is $\tau_1 \stackrel{d}{=} 2\Gamma_1$. $\square$

The strong Markov property of B shows that the common distribution of $-B(\mu_1)$ and $B(\tau_1) - B(\mu_1)$ has the memoryless property, hence is exponential with rate λ for some $\lambda > 0$. The claim of (51) that $\lambda = 1$ follows easily, since the decomposition gives

$$2\mathrm{Var}(-B(\mu_1)) = E[B(\tau_1)]^2 = E\tau_1 = 2.$$

Note the implication of Proposition 15 and (51) that

$$\sqrt{2\Gamma_{1/2}}B^{\mathrm{me}}(1) \stackrel{d}{=} \Gamma_1 \tag{118}$$

for $\Gamma_{1/2}$ independent of $B^{\mathrm{me}}(1)$. Thus the identification (6) of the distribution of $B^{\mathrm{me}}(1)$ amounts to the particular case $r = 1/2$ of Wilks' identity (37).

It is easily seen that Proposition 15 is equivalent to Denisov's decomposition (Corollary 5) by application of Brownian scaling. The following argument shows how these decompositions can be deduced from the sampling identities for Brownian motion and Brownian meander.

Proof of Corollary 5 (Denisov's decomposition). Consider the symmetric random walk $W_j := S_j - T_j, j \geq 0$, with $N_- := \min\{j : W_j < 0\}$ as before. Let M_n be the a.s. unique index m in $\{0, \ldots, n\}$ such that $W_m = \min_{0\leq j\leq n} W_j$. As observed by Feller [27, XII.8] and Denisov [23, (3)], it is elementary that

$$P(M_n = m) = P(N_- > m)\, P(N_- > n - m) \qquad (m = 0, \ldots, n) \tag{119}$$

and that for each m the fragments $(W_{m-j} - W_m)_{0\leq j\leq m}$ and $(W_{m+j} - W_m)_{0\leq j\leq n-m}$ are conditionally independent given $(M_n = m)$ with

$$((W_{m-j} - W_m)_{0\leq j\leq m} \,|\, M_n = m) \stackrel{d}{=} ((W_j)_{0\leq j\leq m} \,|\, N_- > m) \tag{120}$$

$$((W_{m+j} - W_m)_{0\leq j\leq n-m} \,|\, M_n = m) \stackrel{d}{=} ((W_j)_{0\leq j\leq n-m} \,|\, N_- > n - m). \tag{121}$$

Recall that $\widetilde{X}_{(n)} \in C[0,1]$ denotes the zig-zag approximation to $X \in C[0,1]$ defined by $\widetilde{X}_{(n)}(i/(n+1)) := X(U_{n,i})$ and linear interpolation between these values. The sampling identities (33) and (34) allow the above decomposition for the walk (W_j) to be expressed in Brownian terms as follows: for each $n \geq 2$, there is the identity in distribution $\mu(\widetilde{B}_{(n-1)}) \stackrel{d}{=} M_n/n$, and given $\mu(\widetilde{B}_{(n-1)}) = m/n$ for each $1 \leq m \leq n-1$ the random vectors $\sqrt{2\Gamma_n}\mathrm{PRE}_\mu(\widetilde{B}_{(n-1)})$ and $\sqrt{2\Gamma_n}\mathrm{POST}_\mu(\widetilde{B}_{(n-1)})$ are conditionally independent given $\mu(\widetilde{B}_{(n-1)}) = m/n$ with

$$\left(\sqrt{2\Gamma_n}\mathrm{PRE}_\mu(\widetilde{B}_{(n-1)})\,\middle|\, \mu(\widetilde{B}_{(n-1)}) = \frac{m}{n}\right) \stackrel{d}{=} \sqrt{2\Gamma_{m-1/2}}\, \widetilde{B}^{\mathrm{me}}_{(m-1)}$$

and

$$\left(\sqrt{2\Gamma_n}\mathrm{POST}_\mu(\widehat{B}_{(n-1)})\,\middle|\, \mu(\widetilde{B}_{(n-1)}) = \frac{m}{n}\right) \stackrel{d}{=} \sqrt{2\Gamma_{n-m-1/2}}\, \widetilde{B}^{\mathrm{me}}_{(n-m-1)}$$

Because B attains its minimum at a unique time a.s., there is a.s. convergence of $\mu(\widetilde{B}_{(n-1)})$ to $\mu(B)$, whose distribution is therefore arcsine [27, XII.8]. Since Γ_r/r converges in probability to 1 as $r \to \infty$, Denisov's decomposition follows easily. $\square$

The difference between the above argument and Denisov's is that the passage from the random walk decomposition to the Brownian one is done by an easy strong approximation with a particular walk rather than by the more general weak convergence result of Iglehart [40]. Iglehart's argument can be simplified as indicated by Bolthausen [18] to avoid painful discussions of tightness, but the above derivation of Denisov's decomposition seems even simpler.

Identities for generating functions. According to the decomposition of the walk $(W_j)_{0\le j\le n}$ at the time M_n that it achieves its minimum value, described around (119),(120), (121), the conditional distribution of W_n given $M_n = m$ is that of $Z_{n-m} - Z'_m$, where Z_{n-m} and Z'_m are independent, and distributed like W_{n-m} given $N_- > n-m$ and W_m given $N_- > m$ respectively. Combined with the description of the distribution of W_n given $N_- > n$ provided by Lemma 11, and the formula (72) for $P(N_- > n)$, the implied decomposition of the distribution of W_n by conditioning on M_n can be expressed as follows in terms of moment generating functions: for $|\theta| < 1$

$$\left(\frac{1}{1-\theta^2}\right)^n = \sum_{m=0}^{n} \binom{2m}{m}\binom{2n-2m}{n-m} 2^{-2n} G_m\left(\frac{1}{1+\theta}\right) G_{n-m}\left(\frac{1}{1-\theta}\right) \tag{122}$$

where $G_0(\lambda) = 1$ and $G_n(\lambda)$ for $n \ge 1$ is the ordinary probability generating function of J_n as in (48). Multiply (122) by z^n and sum over n to see that (122) is equivalent to the following expression of the Wiener-Hopf factorization associated with the random walk (W_n), which can be interpreted probabilistically for $0 < z < 1$ by stopping the walk at an independent random time with geometric$(1-z)$ distribution [33]:

$$\frac{1-\theta^2}{1-\theta^2-z} = H(\theta,z)H(-\theta,z) \text{ where } H(\theta,z) := \sum_{n=0}^{\infty}\binom{2n}{n} 2^{-2n} z^n G_n\left(\frac{1}{1+\theta}\right). \tag{123}$$

To check (123), consider an arbitrary sequence of independent positive integer valued random variables $N_1, N_2 - N_1, N_3 - N_2, \ldots$ with common distribution with generating function $F(s) := E(s^{N_1})$. Let $A_n := (N_i = n \text{ for some } i)$, let $J_n := \sum_{m=1}^{n} 1(A_m)$, and $G_n(x) := E(x^{J_n} \mid A_n)$. Then by an easy extension of a standard formula of renewal theory ([26, XIII.2] which is the special case $x = 1$)

$$\sum_{n=0}^{\infty} G_n(x) P(A_n) s^n = (1 - xF(s))^{-1}. \tag{124}$$

In the case at hand, $F(s) = 1 - \sqrt{1-s}$, $P(A_n) = \binom{2n}{n}2^{-2n}$, so (123) is easily confirmed using (124).

The continuous analog of (122), which arises in the Brownian limit, is the following formula which is valid for all real θ and all $t > 0$:

$$e^{\frac{1}{2}\theta^2 t} = \frac{1}{\pi}\int_0^t \frac{\Psi_2(\theta\sqrt{w})\Psi_2(-\theta\sqrt{w})}{\sqrt{t}\sqrt{t-w}} dw \tag{125}$$

where

$$\Psi_2(\theta) := \int_0^\infty t e^{-\frac{1}{2}t^2 - \theta t} dt = 1 - \theta\phi_1(\theta)/\phi(\theta)$$

for ϕ_1 and ϕ as in (106) and (108). Since $\Psi_2(\theta) = E(e^{-\theta\rho})$ for ρ with the Rayleigh distribution (6) of $B^{\mathrm{me}}(1)$, for $t = 1$ this is the disintegration of the m.g.f. of $B(1)$ implied by Denisov's decomposition (Corollary 5).

The formula for general $t > 0$ follows by scaling. The formula can also be checked as follows by an argument which parallels the discrete case discussed above. After taking a Laplace transform, formula (125) reduces to the following Wiener-Hopf factorization, which can be interpreted probabilistically via Proposition 15: for $\lambda > \frac{1}{2}\theta^2$

$$\left(\lambda - \tfrac{1}{2}\theta^2\right)^{-1} = L(\theta, \lambda)L(-\theta, \lambda) \tag{126}$$

where

$$L(\theta, \lambda) = \int_0^\infty \frac{\Psi_2(\theta\sqrt{w})}{\sqrt{\pi}\sqrt{w}}\, e^{-\lambda w}\, dw = \frac{\sqrt{2}}{\sqrt{2\lambda} + \theta}, \tag{127}$$

the final identity being an analytic expression of (6) and (118).

8 Bridges with arbitrary end point

The following corollary of the Brownian sampling identity establishes a result for the bridge $B^{\mathrm{br},x}$ with an arbitrary endpoint x.

Corollary 16 *For all real x and all $n = 0, 1, 2, \ldots$*

$$B^{\mathrm{br},x}_{(n)} \stackrel{d}{=} \tfrac{1}{2}(U_{n,i-1,x} - V_{n,i,x}, U_{n,i,x} - V_{n,i,x}; 1 \le i \le n+1) \tag{128}$$

where $(U_{n,i,x}, V_{n,i,x}; 1 \le i \le n+1)$ is constructed from the uniform order statistics $U_{n,i}$ and $V_{n,i}$ and a family of independent positive random variables $L_{n,r}$ as

$$U_{n,i,x} := U_{n,i}(L_{n,|x|} + 2x) \text{ and } V_{n,i,x} := V_{n,i}L_{n,|x|} \tag{129}$$

where $L_{n,r}$ for $r \ge 0$ is assumed to have the probability density

$$P(L_{n,r} \in dy)/dy = \frac{1}{n!2^n} y^n (y+r)(y+2r)^n e^{-\frac{1}{2}y^2 - ry} \qquad (y > 0). \tag{130}$$

In particular, with notation as in (21)

$$||B^{\mathrm{br},x}_{(n)}|| \stackrel{d}{=} |x| + L_{n,|x|} \tag{131}$$

Proof. According to the Brownian sampling identity (12),

$$B(1) \stackrel{d}{=} \xi_n(2\beta_n - 1) \text{ where } \xi_n := \sqrt{2\Gamma_{n+3/2}} \text{ and } \beta_n := \frac{S_{n+1}}{S_{n+1} + T_{n+1}}. \tag{132}$$

By construction of $B^{\mathrm{br},x}$ as $(B \mid B(1) = x)$, the proof of (128) is just a matter of checking that for each x the random vector on the right side of (128) is distributed according to the conditional density of the random vector on the right side of the Brownian sampling identity (12) given $\xi_n(2\beta_n - 1) = x$, as defined by an elementary calculation with ratios of densities. Because ξ_n and $(2\beta_n - 1)$ are independent,

$$P(\xi_n \in dw \mid \xi_n(2\beta_n - 1) = x)/dw = \frac{f_{\xi_n}(w) f_{2\beta_n - 1}(|x|/w)}{w f_{B_1}(x)} \tag{133}$$

where $f_Y(y)$ denotes the density of a random variable Y at y. Since $|(2\beta_n - 1)| \leq 1$ the density in (133) vanishes except if $w \geq |x|$, so we can define the distribution of a positive random variable $L_{n,r}$ for $r \geq 0$ by

$$(\xi_n \,|\, \xi_n(2\beta_n - 1) = x) \stackrel{d}{=} |x| + L_{n,|x|}. \tag{134}$$

By an elementary computation using (132), (133), $\xi_n = \sqrt{2\Gamma_{n+3/2}}$, the beta$(n+1, n+1)$ density of β_n, and the gamma duplication formula (38), the density of $L_{n,r}$ is given by (130). Now

$$\frac{\sqrt{2\Gamma_{n+3/2}}S_i}{S_{n+1} + T_{n+1}} = \xi_n \beta_n U_{n,i} \text{ and } \frac{\sqrt{2\Gamma_{n+3/2}}T_i}{S_{n+1} + T_{n+1}} = \xi_n (1 - \beta_n) V_{n,i} \tag{135}$$

where the $U_{n,i} := S_i/S_{n+1}$ and $V_{n,i} := T_i/T_{n+1}$ are independent of β_n and ξ_n, and the conclusion follows. $\square$

For $x > 0$ let $f_{n,r}(y) := P(L_{n,r} \in dy)/dy$ as given by the right side of formula (130). A byproduct of the above argument is the fact that $f_{n,r}(y)$ is a probability density in y for each $n \geq 1$ and each $r \geq 0$. This was shown by two entirely different methods in Aldous-Pitman [3], where this family of densities arose from the distribution of the total length of edges in a natural model for random trees with $n + 2$ labelled vertices and some random number of unlabeled vertices. The work of Neveu-Pitman [56, 57] and Aldous [7] relates Brownian excursions to certain kinds of binary trees, with no junctions of degree higher than 3. But in the trees constructed by the model of [3] there is no limit on the degrees of some vertices. So the coincidence invites a deeper explanation.

Note the consequence of (131) and (23), that if Z is a standard normal variable independent of the family $L_{n,r}, r \geq 0$ then $|Z| + L_{n,|Z|} \stackrel{d}{=} \sqrt{2\Gamma_{n+3/2}}$. In terms of densities, for $y \geq 0$

$$\int_0^y f_{n,r}(y - r)\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}r^2}dr = \frac{(\frac{1}{2})^{n+1/2}}{\Gamma(n + \frac{3}{2})}\, y^{2n+2}e^{-\frac{1}{2}y^2} \tag{136}$$

as can be checked by reduction to a beta integral. Is there is a natural construction of the family $(L_{n,r}, r \geq 0)$ as a stochastic process which might give another interpretation of this identity?

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1965.

[2] D. Aldous and J. Pitman. Brownian bridge asymptotics for random mappings. *Random Structures and Algorithms*, 5:487–512, 1994.

[3] D. Aldous and J. Pitman. A family of random trees with random edge lengths. Technical Report 526, Dept. Statistics, U.C. Berkeley, 1998. Available via http://www.stat.berkeley.edu/users/pitman.

[4] D. J. Aldous and J. Pitman. Brownian trees and excursions. In preparation, 1999.

[5] D.J. Aldous. The continuum random tree I. *Ann. Probab.*, 19:1–28, 1991.

[6] D.J. Aldous. The continuum random tree II: an overview. In M.T. Barlow and N.H. Bingham, editors, *Stochastic Analysis*, pages 23–70. Cambridge University Press, 1991.

[7] D.J. Aldous. The continuum random tree III. *Ann. Probab.*, 21:248–289, 1993.

[8] D.J. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.*, 26:1703–1726, 1998.

[9] D.J. Aldous and J. Pitman. Tree-valued Markov chains derived from Galton-Watson processes. *Ann. Inst. Henri Poincaré*, 34:637–686, 1998.

[10] L. Alili, C. Donati-Martin, and M. Yor. Une identité en loi remarquable pour l'excursion brownienne normalisée. In M. Yor, editor, *Exponential functionals and principal values related to Brownian motion*, pages 155–180. Biblioteca de la Revista Matemática Ibero-Americana, 1997.

[11] J. Bertoin. Décomposition du mouvement brownien avec dérive en un minimum local par juxtaposition de ses excursions positives et negatives. In *Séminaire de Probabilités XXV*, pages 330–344. Springer-Verlag, 1991. Lecture Notes in Math. 1485.

[12] J. Bertoin and J. Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bull. Sci. Math. (2)*, 118:147–166, 1994.

[13] Ph. Biane. Relations entre pont et excursion du mouvement Brownien réel. *Ann. Inst. Henri Poincaré*, 22:1–7, 1986.

[14] Ph. Biane, J. F. Le Gall, and M. Yor. Un processus qui ressemble au pont brownien. In *Séminaire de Probabilités XXI*, pages 270–275. Springer, 1987. Lecture Notes in Math. 1247.

[15] Ph. Biane and M. Yor. Quelques précisions sur le méandre brownien. *Bull. Sci. Math.*, 112:101–109, 1988.

[16] P. Billingsley. *Probability and Measure.* Wiley, New York, 1995. 3rd ed.

[17] R. M. Blumenthal. Weak convergence to Brownian excursion. *Ann. Probab.*, 11:798–800, 1983.

[18] E. Bolthausen. On a functional central limit theorem for random walks conditioned to stay positive. *Ann. Probab.*, 4:480–485, 1976.

[19] L. Bondesson. *Generalized gamma convolutions and related classes of distributions and densities.* Springer-Verlag, 1992.

[20] Ch. A. Charalambides and J. Singh. A review of the Stirling numbers, their generalizations and statistical applications. *Commun. Statist.-Theory Meth.*, 17:2533–2595, 1988.

[21] K. L. Chung. Excursions in Brownian motion. *Arkiv fur Matematik*, 14:155–177, 1976.

[22] L. Comtet. *Advanced Combinatorics.* D. Reidel Pub. Co., Boston, 1974. (translated from French).

[23] I. V. Denisov. A random walk and a Wiener process near a maximum. *Theor. Prob. Appl.*, 28:821–824, 1984.

[24] S. Dulucq and L. Favreau. A combinatorial model for Bessel polynomials. In *Orthogonal polynomials and their applications (Erice, 1990)*, pages 243–249. Baltzer, Basel, 1991.

[25] A. Erdélyi et al. *Higher Transcendental Functions*, volume II of *Bateman Manuscript Project.* McGraw-Hill, New York, 1953.

[26] W. Feller. *An Introduction to Probability Theory and its Applications,* Vol 1, 3rd ed. Wiley, New York, 1968.

[27] W. Feller. *An Introduction to Probability Theory and its Applications,* Vol 2, 2nd ed. Wiley, New York, 1971.

[28] P. Fitzsimmons, J. Pitman, and M. Yor. Markovian bridges: construction, Palm interpretation, and splicing. In E. Çinlar, K.L. Chung, and M.J. Sharpe, editors, *Seminar on Stochastic Processes, 1992*, pages 101–134. Birkhäuser, Boston, 1993.

[29] R. K. Getoor. The Brownian escape process. *Ann. Probab.*, 7:864–867, 1979.

[30] L. Gordon. Bounds for the distribution of the generalized variance. *The Annals of Statistics*, 17:1684 – 1692, 1989.

[31] L. Gordon. A stochastic approach to the gamma function. *Amer. Math. Monthly*, 101:858–865, 1994.

[32] R. L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics: a foundation for computer science. 2nd ed.* Addison-Wesley, Reading, Mass., 1989.

[33] P. Greenwood and J. Pitman. Fluctuation identities for random walk by path decomposition at the maximum. *Advances in Applied Probability*, 12:291–293, 1978.

[34] P. Greenwood and J. Pitman. Fluctuation identities for Lévy processes and splitting at the maximum. *Advances in Applied Probability*, 12:893–902, 1980.

[35] E. Grosswald. The Student t-distribution for odd degrees of freedom is infinitely divisible. *Ann. Probability*, 4(4):680–683, 1976.

[36] E. Grosswald. The Student t-distribution of any degree of freedom is infinitely divisible. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 36(2):103–109, 1976.

[37] E. Grosswald. *Bessel polynomials.* Springer, Berlin, 1978.

[38] L. C. Hsu and P. J.-S. Shiue. A unified approach to generalized Stirling numbers. *Adv. in Appl. Math.*, 20(3):366–384, 1998.

[39] I.A. Ibragimov and Yu. V. Linnik. *Independent and Stationary Sequences of Random Variables.* Gröningen, Wolthers-Noordhof, 1971. [Translated from original in Russian (1965), Nauka, Moscow].

[40] D.L. Iglehart. Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probab.*, 2:608–619, 1974.

[41] J. P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.*, 21:500–510, 1984.

[42] M. E. H. Ismail and J. Pitman. Algebraic evaluations of some Euler integrals, duplication formulae for Appell's hypergeometric function F_1, and Brownian variations. Technical Report 554, Dept. Statistics, U.C. Berkeley, 1999. Available via http://www.stat.berkeley.edu/users/pitman.

[43] K. Itô and H. P. McKean. *Diffusion Processes and their Sample Paths.* Springer, 1965.

[44] N.L. Johnson, S. Kotz, and N. Balakrishnan. *Discrete Multivariate Distributions.* Wiley, New York, 1997.

[45] W.D. Kaigh. An invariance principle for random walk conditioned by a late return to zero. *Ann. Probab.*, 4:115–121, 1976.

[46] G. Kersting. Symmetry properties of binary branching trees. Preprint, Fachbereich Mathematik, Univ. Frankfurt, 1997.

[47] S. Kullback. An application of characteristic functions to the distribution problem of statistics. *Ann. Math. Statist.*, 5:264–305, 1934.

[48] J. F. Le Gall. Marches aléatoires, mouvement Brownien et processus de branchement. In *Séminaire de Probabilités XXIII*, pages 258–274. Springer, 1989. Lecture Notes in Math. 1372.

[49] J. F. Le Gall. The uniform random tree in a Brownian excursion. *Probab. Th. Rel. Fields*, 96:369–383, 1993.

[50] N. N. Lebedev. *Special Functions and their Applications.* Prentice-Hall, Englewood Cliffs, N.J., 1965.

[51] P. Lévy. Sur certains processus stochastiques homogènes. *Compositio Math.*, 7:283–339, 1939.

[52] T. M. Liggett. An invariance principle for conditioned sums of random variables. *J. Math. Mech.*, 18:559–570, 1968.

[53] H. J. Malik. Exact distribution of the product of independent generalized gamma variables with the same shape parameter. *The Annals of Mathematical Statistics*, 39:1751 – 1752, 1968.

[54] P. W. Millar. Zero-one laws and the minimum of a Markov process. *Trans. Amer. Math. Soc.*, 226:365–391, 1977.

[55] J. C. P. Miller. *Tables of Weber parabolic cylinder functions.* Her Majesty's Stationery Office, London, 1955.

[56] J. Neveu and J. Pitman. The branching process in a Brownian excursion. In *Séminaire de Probabilités XXIII*, pages 248–257. Springer, 1989. Lecture Notes in Math. 1372.

[57] J. Neveu and J. Pitman. Renewal property of the extrema and tree property of a one-dimensional Brownian motion. In *Séminaire de Probabilités XXIII*, pages 239–247. Springer, 1989. Lecture Notes in Math. 1372.

[58] J. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. *Advances in Applied Probability*, 7:511–526, 1975.

[59] J. Pitman. Exchangeable and partially exchangeable random partitions. *Probab. Th. Rel. Fields*, 102:145–158, 1995.

[60] J. Pitman. Some developments of the Blackwell-MacQueen urn scheme. In T.S. Ferguson et al., editor, *Statistics, Probability and Game Theory; Papers in honor of David Blackwell*, volume 30 of *Lecture Notes-Monograph Series*, pages 245–267. Institute of Mathematical Statistics, Hayward, California, 1996.

[61] J. Pitman. Partition structures derived from Brownian motion and stable subordinators. *Bernoulli*, 3:79–96, 1997.

[62] J. Pitman. The distribution of local times of Brownian bridge. Technical Report 539, Dept. Statistics, U.C. Berkeley, 1998. To appear in *Séminaire de Probabilités XXXIII*. Available via http://www.stat.berkeley.edu/users/pitman.

[63] J. Pitman. A lattice path model for the Bessel polynomials. Technical Report 551, Dept. Statistics, U.C. Berkeley, 1999. Available via http://www.stat.berkeley.edu/users/pitman.

[64] J. Pitman and M. Yor. Bessel processes and infinitely divisible laws. In *Stochastic Integrals*, pages 285–370. Springer, 1981. Lecture Notes in Math. 851.

[65] J. Pitman and M. Yor. Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc. (3)*, 65:326–356, 1992.

[66] J. Pitman and M. Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.*, 25:855–900, 1997.

[67] H. Pollard. The compleletly monotonic character of the Mittag-Leffler function $E_\alpha(-x)$. *Bull. Amer. Math. Soc.*, 54:1115–6, 1948.

[68] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer, Berlin-Heidelberg, 1999. 3rd edition.

[69] P. Salminen. Brownian excursions revisited. In *Seminar on Stochastic Processes 1983*, pages 161–187. Birkhäuser Boston, 1984.

[70] V. Seshadri. *The inverse Gaussian distribution.* The Clarendon Press, Oxford University Press, New York, 1993.

[71] G. R. Shorack and J. A. Wellner. *Empirical processes with applications to statistics.* John Wiley & Sons, New York, 1986.

[72] B. M. Steece. The distribution of the product and quotient for powers of two independent generalized gamma variates. *Metron*, 33:227 – 235, 1975.

[73] L. Toscano. Numeri di Stirling generalizzati operatori differenziali e polinomi ipergeometrici. *Commentationes Pontificia Academica Scientarum*, 3:721–757, 1939.

[74] L. Toscano. Some results for generalized Bernoulli, Euler, Stirling numbers. *Fibonacci Quart.*, 16(2):103–112, 1978.

[75] W. Vervaat. A relation between Brownian bridge and Brownian excursion. *Ann. Probab.*, 7:143–149, 1979.

[76] G.N. Watson. *A treatise on the theory of Bessel functions.* Cambridge: University Press, 1966.

[77] S. S. Wilks. Certain generalizations in the analysis of variance. *Biometrika*, 24:471–494, 1932.

[78] D. Williams. Decomposing the Brownian path. *Bull. Amer. Math. Soc.*, 76:871–873, 1970.

[79] D. Williams. Path decomposition and continuity of local time for one dimensional diffusions I. *Proc. London Math. Soc. (3)*, 28:738–768, 1974.

[80] M. Yor. *Some Aspects of Brownian Motion, Part I: Some Special Functionals.* Lectures in Math., ETH Zürich. Birkhaüser, 1992.